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On positive solutions for a class of quasilinear elliptic equations

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Abstract. This paper studies the existence, nonexistence and uniqueness of positive solutions for a class of quasilinear equations. We also analyze the behavior of these solutions with respect to two parameters κ and λ that appear in the equation. The proof of our main results relies on bifurcation techniques, the sub- and supersolution method and a construction of an appropriate large solution.

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1. Introduction

The main goal of this paper is to study the existence, nonexistence, uniqueness and asymptotic behavior of positive solutions for the quasilinear elliptic problem

$$\begin{cases} -\Delta u - \kappa\Delta(u^2)u = \lambda u - b(x)u^p & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{$\mathcal{P}_\kappa$}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a smooth bounded domain, $p > 1$ is a constant, κ and λ are positive parameters, and the weight function $b(x)$ satisfies certain regularity conditions.

Problem ($\mathcal{P}_\kappa$) with $\kappa = 0$ becomes the classical semilinear elliptic problem

$$\begin{cases} -\Delta u = \lambda u - b(x)u^p & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{$\mathcal{P}_0$}$$

whose positive solutions are *equilibria* or *stationary* solutions of the following reaction diffusion problem of logistic type

$$\begin{cases} u_t - \Delta u = \lambda u - b(x)u^p & \text{in} \quad \Omega, \quad t > 0 \\ u = 0 & \text{on} \quad \partial\Omega, \quad t > 0 \\ u(0) = u_0 \geq 0, \end{cases}$$

see, for instance, [8,27] and references therein. We mention that problem ($\mathcal{P}_0$) has been subject of intense study by many authors. If $\lambda \leq \lambda_1$ (λ_1 is the principal eigenvalue of $(-\Delta, H_0^1(\Omega))$, then problem ($\mathcal{P}_0$) can only have the trivial solution, see [2]. In [4], the authors prove that if $\lambda > \lambda_1$, then the problem has two nontrivial solutions of constant sign (one positive and the other negative). Soon thereafter, the results are improved in [34], where the author proved that if $\lambda > \lambda_2$, then problem ($\mathcal{P}_0$) has three nontrivial solutions. Subsequently, in [3], the authors slightly improved the work [34] and they also presented an approach based on Morse theory.

The study of quasilinear equations involving the operator $L_\kappa u := -\Delta u - \kappa\Delta(u^2)u$ arises in various branches of mathematical physics. It is well known that nonlinear Schrödinger equations of the form

$$i\partial_t\psi = -\Delta\psi + V(x)\psi - \kappa\Delta(|\psi|^2)\psi - h(|\psi|^2)\psi, \tag{1}$$

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where $\psi : \mathbb{R} \times \Omega \to \mathbb{C}$, $V = V(x)$ is a given potential, κ is a real constant and h is a real function, have been studied in relation with some mathematical models in physics (see for instance [33]). It was shown that a system describing the self-trapped electron on a lattice can be reduced in the continuum limit to (1) and numerical results on this equation are obtained in [11]. In [30], motivated by the nanotubes and fullerene-related structures, it was proposed and shown that a discrete system describing the interaction of a 2-dimensional hexagonal lattice with an excitation caused by an excess electron can be reduced to (1) and numerical results have been done on domains of disk type, cylinder type and sphere type.

Setting $\psi(t,x) = \exp(-iFt)u(x),\ F \in \mathbb{R}$, into the equation (1), we obtain the stationary equation

$$-\Delta u - \kappa\Delta(u^2)u = g(u) - V(x)u \quad \text{in} \quad \Omega, \tag{2}$$

where we have renamed $V(x) - F$ to be $V(x)$ and $g(u) = h(u^2)u$.

When $\Omega = \mathbb{R}^N$, the quasilinear equation (2) has received special attention in the past several years, see, for instance, [14,16,24,33] and references therein. In these papers, the authors obtain the existence by performing a change of variable, which transforms the quasilinear equation into a new semilinear equation, and they used variational approach. Here, we apply bifurcation techniques and the sub- and supersolution method in order to analyze ($\mathcal{P}_\kappa$).

In addition to the studies involving the operator $L_\kappa u$, another important motivation to study problem ($\mathcal{P}_\kappa$) is the fact that many papers have been devoted to study quasilinear and semilinear equations involving logistic terms, which appear naturally in several contexts. For instance, when $\kappa = 0$, problem ($\mathcal{P}_\kappa$) becomes the classical logistic equation with linear diffusion and refuge, where $u(x)$ describes the density of the individuals of species at the location $x \in \Omega$ and the nonlinearity $g(x,u) := \lambda u - b(x)u^p$ is the well-known logistic reaction term. There are several papers available in the literature dedicated to the analysis of ($\mathcal{P}_0$). See, for instance, the pioneering paper [19] which deals with the logistic equation in a more general setting. We also refer to [12,27,31,32] and references therein.

It is worth mentioning that problem ($\mathcal{P}_\kappa$) can be seen as a quasilinear perturbation of the classical equation ($\mathcal{P}_0$), specially when $\kappa \simeq 0$. As we shall see in Theorems 1.1 and 1.3, the presence of this quasilinear term breaks the blowup (5) that occurs with the positive solutions of ($\mathcal{P}_0$). Moreover, when $\kappa \downarrow 0$, the positive solutions of ($\mathcal{P}_\kappa$) tend to the positive solutions of ($\mathcal{P}_0$).

In order to study the positive solutions of problem ($\mathcal{P}_\kappa$), we will assume the following assumptions on $b(x)$:

(b_0) The function $b : \overline{\Omega} \to [0,\infty)$ belongs to $C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$;

(b_1) The open set $\Omega_+ := \{x \in \Omega;\ b(x) > 0\}$ satisfies $\overline{\Omega}_+ \subset \Omega$ and there is a finite number of smooth components Ω_+^j, $j = 1,\dots n$, such that $\overline{\Omega_+^j} \cap \overline{\Omega_+^i} = \emptyset$ if $i \neq j$. Moreover, the open set

$$\Omega_{b,0} := \Omega \backslash \overline{\Omega}_+$$

is connected. It should be noted that $\partial\Omega_+ \subset \Omega$ and $\partial\Omega_{b,0} = \partial\Omega \cup \partial\Omega_+$.

Before stating our main results, let us recall some notations. Throughout this paper, for any function $V \in L^\infty(\Omega)$ called *potential*, we will denote by $\lambda_1[-\Delta + V]$ the principal eigenvalue of the operator $-\Delta + V$ in Ω under homogeneous Dirichlet boundary conditions. By simplicity, we also use the convention $\lambda_1 := \lambda_1[-\Delta]$. Moreover, we will denote by $\lambda_{b,0}$ the principal eigenvalue of $-\Delta$ in $\Omega_{b,0}$ under homogeneous Dirichlet boundary conditions, when $\Omega_{b,0} \neq \emptyset$ and $\lambda_{b,0} = \infty$ when $\Omega_{b,0} = \emptyset$ (i. e., when $b(x) > 0$, for all $x \in \Omega$).

We are now in a position to state our first main result that deals with existence, nonexistence, uniqueness and asymptotic behavior of positive solutions of ($\mathcal{P}_\kappa$) with respect to λ.

Theorem 1.1. *Let $p > 1$, $\kappa > 0$ and assume (b_0). Then, problem ($\mathcal{P}_\kappa$) has a positive solution if and only if $\lambda > \lambda_1$. Moreover, if $p \geq 3$ or $b(x) \equiv b > 0$ is a constant, it is unique if it exists and it will be denoted by $\Psi_{\lambda,\kappa}$. In addition, the map $\lambda \in (\lambda_1, +\infty) \mapsto \Psi_{\lambda,\kappa} \in \mathcal{C}_0^1(\overline{\Omega})$ is increasing, in the sense that $\Psi_{\lambda,\kappa} > \Psi_{\mu,\kappa}$ if $\lambda > \mu > \lambda_1$. Furthermore,*

$$\lim_{\lambda\downarrow\lambda_1}\|\Psi_{\lambda,\kappa}\|_\infty = 0 \tag{3}$$

and, for any compact $K\subset\overline{\Omega}_{b,0}\backslash\partial\Omega$*, one has*

$$\lim_{\lambda\to+\infty}\Psi_{\lambda,\kappa}=\infty \quad \textit{uniformly in } K. \tag{4}$$

Note that we do not assume the hypothesis (b_1) in this theorem. Moreover, it should be noted that our assumptions on the weight function $b(x)$ include the case $b\equiv 0$, which has been treated in the paper [17], where the authors proved that $(\mathcal{P}_\kappa)$ has only the trivial solution if $\lambda<\lambda_1$ by using variational approach. Thus, Theorem 1.1 improves their results.

To state our main result with respect to the behavior of the (unique) positive solution of $(\mathcal{P}_\kappa)$, when $\kappa\downarrow 0$, let us recall some important properties of the positive solutions of $(\mathcal{P}_0)$ (see, for instance, Theorem 1.1 in [15] and references therein).

Theorem 1.2. *Assume* (b_0), (b_1) *and* $p>1$*. Then, the following assertions hold:*

(a) *The problem* $(\mathcal{P}_0)$ *has a positive solution if and only if* $\lambda\in(\lambda_1,\lambda_{b,0})$*. Moreover, it is unique if it exists and it will be denoted by* Θ_λ*. In addition,* Θ_λ *is a nondegenerate solution of* $(\mathcal{P}_0)$ *and the map* $\lambda\in(\lambda_1,\lambda_{b,0})\mapsto\Theta_\lambda\in\mathcal{C}^1_0(\overline{\Omega})$ *is increasing, in the sense that* $\Theta_\lambda>\Theta_\mu$ *if* $\lambda_{b,0}>\lambda>\mu>\lambda_1$*. Furthermore, for each compact* $K\subset\overline{\Omega}_{b,0}\backslash\partial\Omega$*,*

$$\lim_{\lambda\to\lambda_{b,0}}\Theta_\lambda=\infty \quad \textit{uniformly in } K \tag{5}$$

and, for each compact $K\subset\Omega_+$*,*

$$\lim_{\lambda\to\lambda_{b,0}}\Theta_\lambda=M_{\lambda_{b,0}} \quad \textit{uniformly in } K, \tag{6}$$

where $M_{\lambda_{b,0}}$ *stands for the minimal positive classical solution of the singular boundary value problem*

$$\begin{cases} -\Delta u=\lambda u-b(x)u^p & \textit{in} \quad \Omega_+,\\ u=\infty & \textit{on} \quad \partial\Omega_+, \end{cases} \tag{7}$$

with $\lambda=\lambda_{b,0}$*.*

(b) *Problem* (7) *possesses a minimal positive solution for each* $\lambda\in\mathbb{R}$ *and it will be denoted by* M_λ*.*

Since for $p\ge 3$ or $b(x)\equiv b>0$, $(\mathcal{P}_\kappa)$ has a unique positive solution (denoted by $\Psi_{\lambda,\kappa}$, according to Theorem 1.1), we have the following result concerning the asymptotic behavior of $\Psi_{\lambda,\kappa}$ with respect to the parameter κ:

Theorem 1.3. *Suppose* (b_0), (b_1) *and* $p\ge 3$ *or* $b(x)\equiv b>0$*. The following assertions hold:*

(a) *If* $\lambda\in(\lambda_1,\lambda_{b,0})$*, then* $\lim_{\kappa\downarrow 0}\Psi_{\lambda,\kappa}=\Theta_\lambda$ *in* $\mathcal{C}^1_0(\overline{\Omega})$*;*

(b) *If* $\lambda_{b,0}<+\infty$ *and* $\lambda\ge\lambda_{b,0}$*, then for any compact* $K\subset\overline{\Omega}_{b,0}\backslash\partial\Omega$

$$\lim_{\kappa\downarrow 0}\Psi_{\lambda,\kappa}=+\infty \quad \textit{uniformly in } K; \tag{8}$$

(c) *Suppose in addition that* $p>3$*. If* $\lambda_{b,0}<+\infty$*, and* $\lambda\ge\lambda_{b,0}$ *then, for any compact* $K\subset\Omega_+$*,*

$$\lim_{\kappa\downarrow 0}\Psi_{\lambda,\kappa}=M_\lambda \quad \textit{uniformly in } K, \tag{9}$$

where M_λ *stands for the minimal positive classical solution of the singular boundary value problem* (7)*.*

It should be noted that this theorem means that effect of adding the quasilinear term is regularizing the minimal metasolutions of (7). Indeed, by Theorem 1.3 (b) and (c), the unique positive regular solution of $(\mathcal{P}_\kappa)$ approximates to the minimal metasolution as $\kappa\downarrow 0$. It is a similar phenomenon given by [25, Theorem 1.3]. However, we highlight that our quasilinear perturbation is more sophisticated than the perturbation of [25].

Finally, we would like to mention that, in the process of conclusion of this work, we found out about the paper [20], where the authors study a problem related to ($\mathcal{P}_\kappa$). Moreover, we can use some results of [20] to present a proof of the behavior of the positive solutions of ($\mathcal{P}_\kappa$) when $\kappa \to +\infty$. Specifically, we have:

Theorem 1.4. *Let $p > 1$, $\kappa > 0$ and assume (b_0). For each $\lambda > \lambda_1$ fixed, if u_κ is a positive solution of ($\mathcal{P}_\kappa$), then*

$$\lim_{\kappa\to\infty} \|u_\kappa\|_\infty = 0.$$

Note that, in this case, it was not necessary the uniqueness of positive solution for ($\mathcal{P}_\kappa$).

The outline of this paper is as follows. In Sect. 2, we introduce the dual approach of ($\mathcal{P}_\kappa$) and we prove the first results which will be playing an important role in our analyses. In Sect. 3, we show the existence and uniqueness of positive solutions for ($\mathcal{P}_\kappa$). Section 4 is devoted to prove a pivotal a priori bounds, and in Sect. 5, we will use theses estimates to study the asymptotic behavior of the positive solution of ($\mathcal{P}_\kappa$) with respect to the parameter κ. In the Final Remarks, we prove a stability result for (10).

2. An auxiliary problem

In this section, we introduce the dual approach developed in the papers [14,24] to deal with ($\mathcal{P}_\kappa$). Specifically, we convert the quasilinear equation ($\mathcal{P}_\kappa$) into a semilinear one by using a suitable change of variable. To this end, we argue as follows. For each $\kappa \geq 0$, let $f_\kappa : \mathbb{R} \to \mathbb{R}$ denote the solution of the Cauchy problem

$$f_\kappa'(t) = \frac{1}{(1 + 2\kappa f_\kappa^2(t))^{1/2}}, \quad f_\kappa(0) = 0.$$

By the standard theory of ODE, we obtain that f_κ is uniquely determined, invertible and of class $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$. Moreover, it is well known that the inverse function of f is given by

$$f_\kappa^{-1}(t) := \int_0^t (1 + 2\kappa s^2)^{1/2} \mathrm{d}s, \quad \forall\, t \geq 0.$$

Thus, by performing the change of variable $u = f_\kappa(v)$ and setting $g(x, s) = \lambda s - b(x)s^{p-1}$ if $s \geq 0$, $x \in \Omega$ and $g(x, s) = 0$ for $s < 0$, $x \in \Omega$, we obtain that problem ($\mathcal{P}_\kappa$) is equivalent to the following semilinear elliptic equation:

$$\begin{cases} -\Delta v = \lambda f_\kappa(v) f_\kappa'(v) - b(x)(f_\kappa(v))^p f_\kappa'(v) & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{10}$$

Furthermore, we can see that v is a classical positive solution of (10) if and only if $u = f_\kappa(v)$ is a classical positive solution of ($\mathcal{P}_\kappa$) (see [14,24]). Thus, we will analyze the auxiliary problem (10).

Firstly, we recall some useful properties of $f_\kappa(t)$.

Lemma 2.1. *Let $\kappa > 0$ and $t \geq 0$. Then,*

(i) $0 \leq f_\kappa(t) \leq t$;
(ii) $0 \leq f_\kappa'(t) \leq 1$;
(iii) $f_\kappa(t) f_\kappa'(t) \leq 1/\sqrt{2\kappa}$;
(iv) $f_\kappa''(t) = -2\kappa f_\kappa(t)(f_\kappa'(t))^4 = [(f_\kappa'(t))^4 - (f_\kappa'(t))^2]/f_\kappa(t)$;
(v) $\frac{1}{2} f_\kappa(t) \leq t f_\kappa'(t) \leq f_\kappa(t)$;
(vi) $\lim_{t\to 0^+} f_\kappa(t)/t = 1$;
(vii) *The map* $t \in (0, \infty) \mapsto f_\kappa(t)/t^{1/2}$ *is nondecreasing;*
(viii) $[f_\kappa(t) f_\kappa'(t)]' = (f_\kappa'(t))^2 - 2\kappa(f_\kappa(t))^2(f_\kappa'(t))^4 = (f_\kappa'(t))^4$.
(ix) $[f_\kappa(t)^p f_\kappa'(t)]' = f_\kappa^{p-1}(t)[(p-1)(f_\kappa'(t))^2 + (f_\kappa'(t))^4]$.

Proof. The properties (i)–(vii) are well known in the literature (see, for instance, [5,7,14]). To prove $(viii)$, by a direct calculation and using (iv), we get

$$\begin{aligned}[f_\kappa(t)f_\kappa'(t)]' &= (f_\kappa'(t))^2 + f_\kappa(t)f_\kappa''(t)\\ &= (f_\kappa'(t))^2 - 2\kappa(f_\kappa(t))^2(f_\kappa'(t))^4 = (f_\kappa'(t))^4\end{aligned}$$

Similarly, we obtain (ix) as follows

$$\begin{aligned}[f_\kappa(t)^p f_\kappa'(t)]' &= [f_\kappa^{p-1}(t)(f_\kappa(t)f_\kappa'(t))]'\\ &= (p-1)f_\kappa^{p-2}(t)f_\kappa'(t)(f_\kappa(t)f_\kappa'(t)) + f_\kappa^{p-1}(t)(f_\kappa'(t))^4\\ &= f_\kappa^{p-1}(t)[(p-1)(f_\kappa'(t))^2 + (f_\kappa'(t))^4].\end{aligned}$$

□

As a consequence of Lemma 2.1, we also have the following properties:

Lemma 2.2. *Assume that $\kappa > 0$ and $p > 1$. Then*

(i) The map $t \in (0, +\infty) \mapsto f_\kappa(t)f_\kappa'(t)/t$ is of class $\mathcal{C}^1$, decreasing and it verifies

$$f_\kappa(t)f_\kappa'(t) \le t, \quad \forall\, t \ge 0, \tag{11}$$

$$\lim_{t\to 0^+} \frac{f_\kappa(t)f_\kappa'(t)}{t} = 1 \tag{12}$$

and

$$\lim_{t\to \infty} \frac{f_\kappa(t)f_\kappa'(t)}{t} = 0; \tag{13}$$

(ii) For $p \ge 3$, the map $t \in (0, \infty) \mapsto f_\kappa^p(t)f_\kappa'(t)/t$ is of class $\mathcal{C}^1$, increasing and it verifies

$$\lim_{t\to 0^+} \frac{f_\kappa^p(t)f_\kappa'(t)}{t} = 0. \tag{14}$$

Proof. Let us prove that $t \in (0, +\infty) \mapsto f_\kappa(t)f_\kappa'(t)/t$ is decreasing. By a direct calculation and using Lemma 2.1 (iv), we obtain

$$\begin{aligned}\left(\frac{f_\kappa(t)f_\kappa'(t)}{t}\right)' &= \frac{[(f_\kappa'(t))^2 + f_\kappa(t)f_\kappa''(t)]t - f_\kappa(t)f_\kappa'(t)}{t^2}\\ &= \frac{[(f_\kappa'(t))^2 - 2(f_\kappa(t))^2(f_\kappa'(t))^5]t - f_\kappa(t)f_\kappa'(t)}{t^2}, \quad \forall t > 0\end{aligned}$$

Thus, $(f_\kappa(t)f_\kappa'(t)/t)' < 0$ for all $t > 0$ if and only if

$$tf_\kappa'(t) < 2t(f_\kappa(t))^2(f_\kappa'(t))^4 + f_\kappa(t).$$

which is true, thanks to Lemma 2.1 (i), (ii) and (v). The inequality (11) is a direct consequence of Lemma 2.1 (i) and (ii). The limit (12) is obtained by combining Lemma 2.1 (vi) and using that

$$\lim_{t\to 0^+} f_\kappa'(t) = \lim_{t\to 0^+} \frac{1}{(1 + 2\kappa f_\kappa^2(t))^{1/2}} = 1.$$

The limit (13) follows from Lemma 2.1 (iii).

Now, suppose that $p \ge 3$. To prove that the map $t \in [0, \infty) \mapsto f_\kappa^p(t)f_\kappa'(t)/t$ is increasing, we observe that, using Lemma 2.1 (iv), for all $t > 0$, we have

$$\begin{aligned}\left(\frac{f_\kappa^p(t)f_\kappa'(t)}{t}\right)' &= \frac{[p(f_\kappa(t))^{p-1}(f_\kappa'(t))^2 + f_\kappa^p(t)f_\kappa''(t)]t - f_\kappa^p(t)f_\kappa'(t)}{t^2}\\ &= \frac{[p(f_\kappa(t))^{p-1}(f_\kappa'(t))^2 + (f_\kappa(t))^{p-1}((f_\kappa'(t))^4 - (f_\kappa'(t))^2)]t - f_\kappa^p(t)f_\kappa'(t)}{t^2}.\end{aligned}$$

Thus, $(f_\kappa^p(t)f_\kappa'(t)/t)' > 0$ if and only if $[p(f_\kappa'(t))^2 + (f_\kappa'(t))^4 - (f_\kappa'(t))^2]t - f_\kappa(t)f_\kappa'(t) > 0$, that is,

$$t(f_\kappa'(t))^4 + (p-1)t(f_\kappa'(t))^2 > f_\kappa(t)f_\kappa'(t). \tag{15}$$

On the other hand, since $p \geq 3$, it follows from Lemma 2.1 (v) that

$$tf_\kappa'(t) \geq \frac{f_\kappa(t)}{2} \geq \frac{f_\kappa(t)}{p-1}, \quad \forall\, t \geq 0.$$

Now, using Lemma 2.1 (ii) and the fact that $t(f_\kappa'(t))^4 > 0$ for all $t > 0$, we conclude that (15) is true. Finally, (14) is an easy consequence of (12) and $\lim_{t\to 0^+} f^{p-1}(t) = f^{p-1}(0) = 0$. □

With respect to the map $\kappa \in (0,\infty) \mapsto f_\kappa(t)$ (for each $t > 0$ fixed), we have the following lemma:

Lemma 2.3. *For each $t > 0$ fixed, the function $\kappa \in (0,\infty) \mapsto f_\kappa(t)$ is continuous and decreasing.*

Proof. The continuity of the map $\kappa \in (0,\infty) \mapsto f_\kappa(t)$ follows from the standard theory of ordinary differential equations. To prove that it is decreasing, we argue as follows. Let κ_1, κ_2 be constants such that $0 < \kappa_1 < \kappa_2$. We need to prove that $f_{\kappa_2}(t) < f_{\kappa_1}(t)$ for all $t > 0$. Since for each $t > 0$, the function $\kappa \mapsto f_\kappa^{-1}(t) = \int_0^t (1 + 2\kappa s^2)^{1/2}\mathrm{d}s$ is increasing, it suffices to prove that

$$f_{\kappa_2}^{-1}(f_{\kappa_2}(t)) < f_{\kappa_2}^{-1}(f_{\kappa_1}(t)), \tag{16}$$

which is equivalent to $t < \int_0^{f_{\kappa_1}(t)} (1 + 2\kappa_2 s^2)^{1/2}\mathrm{d}s$. To this, consider the function defined by

$$h(t) = \int_0^{f_{\kappa_1}(t)} (1 + 2\kappa_2 s^2)^{1/2}\mathrm{d}s - t, \quad t \geq 0$$

and notice that $h(0) = 0$. We claim that $h'(t) > 0$ for all $t > 0$ which implies that $h(t) > 0$ and hence (16) holds. Indeed, observe that $h'(t) = (1 + 2\kappa_2 f_{\kappa_1}^2(t))^{1/2} f_{\kappa_1}'(t) - 1 > 0$ if and only if

$$\frac{1}{(1 + 2\kappa_1 f_{\kappa_1}^2(t))^{1/2}} = f_{\kappa_1}'(t) > \frac{1}{(1 + 2\kappa_2 f_{\kappa_1}^2(t))^{1/2}},$$

which holds if $\kappa_1 < \kappa_2$ and this completes the proof. □

We finish this section by deriving an *a priori* estimate for positive solutions of (10) in the particular case $b(x) \equiv b > 0$. This estimate will be useful to prove an uniqueness result in the next section.

Lemma 2.4. *Let $v \in \mathcal{C}^2(\overline{\Omega})$ be a positive solution of (10) with $b(x) \equiv b > 0$ constant. Then*

$$bf_\kappa^{p-1}(v(x)) \leq \lambda, \quad \forall\, x \in \Omega. \tag{17}$$

Proof. Let v be a classical positive solution of (10). Since the maximum value of v in $\overline{\Omega}$ is attained in Ω, let $x_0 \in \Omega$ be such that $v(x_0) = \max_{x\in\overline{\Omega}} v(x)$. Thus,

$$0 \leq -(\Delta v)(x_0) = \lambda f_\kappa(v(x_0))f_\kappa'(v(x_0)) - bf_\kappa^p(v(x_0))f_\kappa'(v(x_0))$$

and as $f_\kappa(v(x_0))f_\kappa'(v(x_0)) > 0$, the previous inequality is equivalent to $bf_\kappa^{p-1}(v(x_0)) \leq \lambda$. Using that $f_\kappa(t)$ is increasing for $t > 0$, we obtain $bf_\kappa^{p-1}(v(x)) \leq bf_\kappa^{p-1}(v(x_0)) \leq \lambda$ for all $x \in \Omega$, and this completes the proof. □

3. Existence, nonexistence and uniqueness of positive solution

In this section, we will study the existence, nonexistence and uniqueness of positive solution for (10). We begin by establishing a necessary condition for existence of positive solution for (10) (and hence for ($\mathcal{P}_\kappa$)).

Lemma 3.1. (Nonexistence). *If (b_0) holds, then problem (10) does not have positive solutions for $\lambda \leq \lambda_1$. In particular, if $b(x) \equiv 0$, then problem (10) does not have positive solutions for $\lambda \leq \lambda_1$.*

Proof. Suppose that $v > 0$ is a solution of (10) with $\lambda \le \lambda_1$. Then, it satisfies

$$\begin{cases} -\Delta v + \widetilde{b}(x)v = 0 & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{18}$$

where

$$\widetilde{b}(x) := b(x)\frac{f_\kappa^p(v(x))f_\kappa'(v(x))}{v(x)} - \lambda\frac{f_\kappa(v(x))f_\kappa'(v(x))}{v(x)}.$$

Thanks to Lemma 2.2, the maps $t \in (0,\infty) \mapsto f_\kappa(t)f_\kappa'(t)/t$ and $t \in (0,\infty) \mapsto f_\kappa^p(t)f_\kappa'(t)/t$ are of class $\mathcal{C}^1$. Thus, since $v > 0$ is of class $C^{2,\gamma}$, we get the $\widetilde{b} \in \mathcal{C}^\gamma(\Omega)$ and the principal eigenvalue $\lambda_1[-\Delta + \widetilde{b}(x)]$ is well defined. Moreover, by (18), $\lambda_1[-\Delta+\widetilde{b}(x)] = 0$. Since $b(x)f_\kappa^p(v(x))f_\kappa'(v(x))/v(x) \ge 0$, using the monotonicity properties of the principal eigenvalue combined with (11), we conclude that $0 > \lambda_1[-\Delta - \lambda] = \lambda_1 - \lambda$, which is a contradiction and this ends the proof. □

The next proposition shows an uniqueness result of positive solution for (10).

Proposition 3.2. *Suppose $p \ge 3$ or $b(x) \equiv b > 0$. Then, the problem (10) admits at most a positive solution.*

Proof. First, we will consider the case $p \ge 3$. By the classical Brezis–Oswald result (see [10]), it is sufficient to prove that the function

$$q(x,t) := \lambda\frac{f_\kappa(t)f_\kappa'(t)}{t} - b(x)\frac{f_\kappa^p(t)f_\kappa'(t)}{t}$$

is decreasing in $t > 0$, for each $x \in \Omega$. Thus, the monotonicity of $q(x,t)$ follows by Lemma 2.2.

Now, assume that $b(x) \equiv b > 0$ is constant. We will argue by contradiction. Suppose that $v_1 > 0$ and $v_2 > 0$ are solutions of (10) with $v_1 \ne v_2$. Denoting, by simplicity, $g_i = f_\kappa(v_i)$ and $g_i' = f_\kappa'(v_i)$ $(i = 1,2)$, we have

$$-\Delta(v_1 - v_2) = \lambda(g_1g_1' - g_2g_2') - b(g_1^pg_1' - g_2^pg_2') \quad \text{in} \quad \Omega. \tag{19}$$

Define $W : \Omega \to \mathbb{R}$ by

$$W(x) = \begin{cases} \dfrac{-\lambda[g_1(x)g_1'(x) - g_2(x)g_2'(x)] + b[g_1^p(x)g_1'(x) - g_2^p(x)g_2'(x)]}{v_1(x) - v_2(x)} & \text{if} \;\; v_1(x) \ne v_2(x), \\ 0 & \text{if} \;\; v_1(x) = v_2(x). \end{cases}$$

and consider $h(t) := \lambda f_\kappa(t)f_\kappa'(t) - bf_\kappa^p(t)f_\kappa'(t)$ for $t \ge 0$. Note that h is differentiable and for $x \in \Omega$ a simple calculation shows that

$$\int_0^1 h'(sv_2(x) + (1-s)v_1(x))\mathrm{d}s = \begin{cases} \dfrac{-\lambda[g_1(x)g_1'(x) - g_2(x)g_2'(x)] + b[g_1^p(x)g_1'(x) - g_2^p(x)g_2'(x)]}{v_1(x) - v_2(x)} & \text{if} \;\; v_1(x) \ne v_2(x), \\ h'(v_1(x)) & \text{if} \;\; v_1(x) = v_2(x). \end{cases}$$

Therefore,

$$|W(x)| \le \left|\int_0^1 h'(sv_2(x) + (1-s)v_1(x))\mathrm{d}s\right| \le \max_{t\in[0,d]} |h'(t)| \quad \forall\, x \in \Omega,$$

where $d = \max_{x\in\overline{\Omega}} v_1(x) + \max_{x\in\overline{\Omega}} v_2(x)$ and this implies that $W \in L^\infty(\Omega)$. Thus, it follows from (19) that

$$-\Delta(v_1 - v_2) + W(x)(v_1 - v_2) = 0 \quad \text{in} \quad \Omega.$$

Since $v_1 \neq v_2$, we have $W \neq 0$ and hence $\lambda_j[-\Delta + W(x)] = 0$ for some $j \geq 1$, where $\lambda_j[-\Delta + W(x)]$ stands for an eigenvalue of $-\Delta + W(x)$ in Ω under homogeneous Dirichlet boundary conditions. By the dominance of the principal eigenvalue, we get

$$0 = \lambda_j[-\Delta + W(x)] \geq \lambda_1[-\Delta + W(x)]. \tag{20}$$

On the other hand, since v_1 is a positive solution of (10), we have

$$\lambda_1\left[-\Delta - \lambda\frac{g_1 g_1'}{v_1} + b\frac{g_1^p g_1'}{v_1}\right] = 0. \tag{21}$$

We claim that

$$-\lambda\frac{g_1 g_1'}{v_1} + b\frac{g_1^p g_1'}{v_1} \leq W \quad \text{in} \quad \Omega, \tag{22}$$

with strict inequality in an open subset of Ω. If (22) holds, then the proof is completed because we can combine (21)-(22) and the monotonicity properties of the principal eigenvalue to obtain

$$0 = \lambda_1\left[-\Delta - \lambda\frac{g_1 g_1'}{v_1} + b\frac{g_1^p g_1'}{v_1}\right] < \lambda_1[-\Delta + W(x)],$$

which contradicts (20). Now, we will prove (22). If $v_1(x) = v_2(x)$ then $W(x) = 0$ and (22) is equivalent to

$$-\lambda\frac{g_1(x)g_1'(x)}{v_1(x)} + b\frac{g_1^p(x)g_1'(x)}{v_1(x)} \leq 0$$

that is, $bg_1^{p-1} \leq \lambda$, which occurs thanks to Lemma 2.4. If $v_1 > v_2$, then $v_1 - v_2 > 0$ and (22) is equivalent to

$$-\lambda g_1 g_1' + bg_1^p g_1' + \lambda\frac{g_1 g_1'}{v_1}v_2 - b\frac{g_1^p g_1'}{v_1}v_2 \leq -\lambda g_1 g_1' + \lambda g_2 g_2' + bg_1^p g_1' - bg_2^p g_2' \quad \text{in} \quad \{x \in \Omega;\ v_1(x) > v_2(x)\},$$

that is,

$$[\lambda - bg_1^{p-1}]\frac{g_1 g_1'}{v_1} \leq [\lambda - bg_2^{p-1}]\frac{g_2 g_2'}{v_2} \quad \text{in} \quad \{x \in \Omega;\ v_1(x) > v_2(x)\}. \tag{23}$$

Since the map $t \in [0, \infty) \mapsto f_\kappa(t)f_\kappa'(t)/t$ is decreasing, we have

$$0 \leq \frac{g_1 g_1'}{v_1} < \frac{g_2 g_2'}{v_2}. \tag{24}$$

On the other hand, since the map $t \in [0, \infty) \mapsto f_\kappa(t)$ is increasing and $v_1 > v_2$, we get $g_1 > g_2$. Thus, we can infer that

$$0 \leq \lambda - bg_1^{p-1} \leq \lambda - bg_2^{p-1} \quad \text{in} \quad \{x \in \Omega;\ v_1(x) > v_2(x)\}. \tag{25}$$

Therefore, (24) and (25) imply that (23) is true, showing that (22) holds for $v_1 > v_2$. The case $v_1 < v_2$ is analogous and this ends the proof. □

Now, we will show that λ_1 is the unique bifurcation point of positive solutions of (10) from the trivial solution. For this, let e_1 be the unique positive solution of

$$\begin{cases} -\Delta v = 1 & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and let E be the space consisting of all $u \in \mathcal{C}(\overline{\Omega})$ for which there exists $\gamma = \gamma_u > 0$ such that

$$-\gamma e_1(x) \leq u(x) \leq \gamma e_1(x) \quad \forall\ x \in \Omega,$$

endowed with the norm $\|u\|_E := \inf\{\gamma > 0;\ -\gamma e_1(x) \leq u(x) \leq \gamma e_1(x),\ \forall x \in \Omega\}$ and the natural pointwise order. It is well known that E is an ordered Banach space whose positive cone, say P, is normal and has nonempty interior (see [1]). Thus, consider the map $\mathfrak{F} : \mathbb{R} \times E \longrightarrow E$ defined by

$$\mathfrak{F}(\lambda, v) = v - (-\Delta)^{-1}[\lambda f_\kappa(v)f_\kappa'(v) - b(x)f_\kappa^p(v)f_\kappa'(v)],$$

where $(-\Delta)^{-1}$ is the inverse of the Laplacian operator under homogeneous Dirichlet boundary conditions. We can see that the application $\mathfrak{F}$ is of $\mathcal{C}^1$ class and (10) can be written in the form

$$\mathfrak{F}(\lambda, v) = 0.$$

Moreover, by the Strong Maximum Principle, any nonnegative and nontrivial solution of (10) (resp. (3)) is in fact strictly positive. Indeed, if v is a nonnegative and nontrivial solution of (10), then it satisfies

$$\begin{cases} -\Delta v + \widetilde{a}(x)v = \lambda f_\kappa(v) f'_\kappa(v) > 0 & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{26}$$

where $\widetilde{a}(x) := b(x) f^p_\kappa(v(x)) f'_\kappa(v(x))/v(x)$ if $v(x) > 0$ and $a(x) := 0$ if $v(x) = 0$. Thus, $\widetilde{a} \in L^\infty(\Omega)$ and $\widetilde{a}(x) \geq 0$. Consequently, $\lambda_1[-\Delta + \widetilde{a}(x)] > 0$ and the operator $-\Delta + \widetilde{a}(x)$ satisfies the Strong Maximal Principle (see, for instance, [28, Theorem 2.1] or [26]). In view of (26) and since $v \neq 0$, we conclude that $v(x) > 0$ in Ω.

Let $\mathcal{S} \subset \mathbb{R} \times E$ be the set of nontrivial solutions of (10) plus all possible bifurcation points from the trivial solution $(\lambda, 0)$. Thus, we have:

Proposition 3.3. *The number λ_1 is a bifurcation point of (10) from the trivial solution to a continuum of positive solutions of (10). Moreover, it is the unique bifurcation point of positive solutions from $(\lambda, 0)$. If $\Sigma_0 \subset \mathcal{S}$ denotes the component of positive solutions of (10) emanating from $(\lambda, 0)$, then Σ_0 is unbounded in $\mathbb{R} \times E$.*

Proof. Observe that (3) can be written as $\mathcal{L}(\lambda)v + \mathcal{N}(\lambda, v) = 0$ where $\mathcal{L}(\lambda) = I_E - \lambda(-\Delta)^{-1}$ and

$$\mathcal{N}(\lambda, v) = -(-\Delta)^{-1}[\lambda(f_\kappa(v) f'_\kappa(v) - v) - b(x) f^p_\kappa(v) f'_\kappa(v)].$$

Moreover, thanks to (12) and (14), we have

$$\lim_{t \to 0^+} \frac{\lambda(f_\kappa(t) f'_\kappa(t) - t) - b(x) f^p_\kappa(t) f'_\kappa(t)}{t} = 0,$$

and, hence, $\mathcal{N}(\lambda, v) = o(\|v\|_E)$ as $\|v\|_E \to 0$. Therefore, we can apply the unilateral bifurcation theorem for positive operators, see [29, Theorem 6.5.5], to conclude the result. □

Next, we are ready to complete the proof of Theorem 1.1. Actually, it will be an immediate consequence of the following result:

Theorem 3.4. *Let $p > 1$, $\kappa > 0$ and assume (b_0). Then, problem (10) possesses a positive solution if and only if $\lambda > \lambda_1$. Moreover, if $p \geq 3$ or $b(x) \equiv b > 0$ is a constant, it is unique if it exists and it will be denoted by $\Theta_{\lambda,\kappa}$. In addition, the map $\lambda \in (\lambda_1, +\infty) \mapsto \Theta_{\lambda,\kappa} \in \mathcal{C}^1_0(\overline{\Omega})$ is increasing, in the sense that $\Theta_{\lambda,\kappa} > \Theta_{\mu,\kappa}$, if $\lambda > \mu > \lambda_1$. Furthermore, $\lim_{\lambda \downarrow \lambda_1} \|\Theta_{\lambda,\kappa}\|_\infty = 0$ and for any compact $K \subset \overline{\Omega}_{b,0} \backslash \partial\Omega$,*

$$\lim_{\lambda \to +\infty} \Theta_{\lambda,\kappa} = \infty \quad \textit{uniformly in } K.$$

Proof. By Proposition 3.3, λ_1 is a bifurcation point of (10) from the trivial solution and it is the only one for positive solutions. Moreover, there exists an unbounded continuum Σ_0 of positive solutions emanating from $(\lambda_1, 0)$. In order to prove the existence of a positive solution for every $\lambda > \lambda_1$, it suffices to show that, for every $\lambda_* > \lambda_1$, there exists a constant $C = C(\lambda_*) > 0$ such that

$$\|v\|_\infty \leq C, \quad \forall\ (\lambda, v) \in \Sigma_0 \ \text{ and } \ \lambda \leq \lambda_*. \tag{27}$$

Indeed, by the global nature of Σ_0, this estimate implies that $\text{Proj}_{\mathbb{R}}\Sigma_0 = (\lambda_1, \infty)$, where $\text{Proj}_{\mathbb{R}}\Sigma_0$ is the projection of Σ_0 into $\mathbb{R}$. To prove (27), we will build a family $\overline{W}(\lambda)$ of supersolutions of (10) and we will apply Theorem 2.2 of [21]. Thus, we consider the continuous map $\overline{W} : [\lambda_1, \lambda_*] \to \mathcal{C}^2_0(\overline{\Omega})$ defined by

$\overline{W}(\lambda) = K(\lambda)e$, where $K(\lambda)$ is a positive constant to be chosen later and e is the unique positive solution of

$$\begin{cases} -\Delta v = 1 & \text{in} \quad \widehat{\Omega}, \\ v = 0 & \text{on} \quad \partial\widehat{\Omega}, \end{cases} \tag{28}$$

for some regular domain $\Omega \subset\subset \widehat{\Omega}$. Then, $\overline{W}(\lambda) = K(\lambda)e$ is a supersolution of (10) if

$$1 \geq \lambda \frac{f_\kappa(Ke) f'_\kappa(Ke)}{Ke} e - b(x) \frac{f^p_\kappa(Ke) f'_\kappa(Ke)}{Ke} e \quad \text{in} \quad \Omega.$$

According to Proposition 2.2, $\lim_{t\to\infty} f_\kappa(t) f'_\kappa(t)/t = 0$. Consequently, for $K = K(\lambda) > 0$ large enough, $\overline{W}(\lambda) = K(\lambda)e$ is a supersolution (but not a solution) of (10), for every $\lambda \in [\lambda_1, \lambda_*]$ and $W(\lambda_1) = K(\lambda_1)e > 0$ in Ω. Thus, by Theorem 2.2 of [21], it follows (27).

To prove that $\Theta_{\lambda,\kappa} > \Theta_{\mu,\kappa}$ if $\lambda > \mu > \lambda_1$, just note that $\Theta_{\mu,\kappa}$ is a (strict) subsolution of (10) if $\mu \in (\lambda_1, \lambda)$. By the uniqueness of positive solution of (10), we conclude the result.

The convergence (3) is an immediate consequence of Proposition 3.3.

Now, in order to prove (4), let $\varphi_{b,0} > 0$ be the eigenfunction associated with $\lambda_{b,0}$ such that $\|\varphi_{b,0}\|_\infty = 1$ and consider

$$\Psi = \begin{cases} \varphi_{b,0} & \text{in} \quad \Omega_{b,0}, \\ 0 & \text{in} \quad \Omega \backslash \overline{\Omega}_{b,0}. \end{cases}$$

It is clear that $\Psi \in H^1_0(\Omega)$. We will show that for $\lambda > \lambda_{b,0}$, $\varepsilon(\lambda)\Psi$ is a subsolution of (10) (in the sense of [9]) for a constant $\varepsilon(\lambda) > 0$ to be chosen. Indeed, since $b \equiv 0$ in $\Omega_{b,0}$ and $\Psi = 0$ in $\Omega\backslash\overline{\Omega}_{b,0}$, it suffices to verify that

$$\lambda_{b,0}\varepsilon\varphi_{b,0} = -\Delta(\varepsilon\varphi_{b,0}) \leq \lambda f_\kappa(\varepsilon\varphi_{b,0}) f'_\kappa(\varepsilon\varphi_{b,0}) \quad \text{in} \quad \Omega_{b,0},$$

that is,

$$\frac{\lambda_{b,0}}{\lambda} \leq \frac{f_\kappa(\varepsilon\varphi_{b,0}) f'_\kappa(\varepsilon\varphi_{b,0})}{\varepsilon\varphi_{b,0}} \quad \text{in} \quad \Omega_{b,0}.$$

According to Lemma 2.2, the map $t \in [0, \infty) \mapsto h_\kappa(t) := f_\kappa(t) f'_\kappa(t)/t$ is decreasing and, hence, is invertible. Then, the above inequality is equivalent to $h^{-1}_\kappa(\lambda_{b,0}/\lambda) \geq \varepsilon\varphi_{b,0}$. Once that $\|\varphi_{b,0}\|_\infty = 1$, choosing $\varepsilon(\lambda) := h^{-1}_\kappa(\lambda_{b,0}/\lambda)$ we obtain that $\varepsilon(\lambda)\varphi_{b,0}$ is a subsolution of (10). Moreover, it follows from (12) that $\lim_{t\to 0} h^{-1}_\kappa(t) = +\infty$ and therefore

$$\lim_{\lambda\to\infty} \varepsilon(\lambda) = \lim_{\lambda\to\infty} h^{-1}_\kappa\left(\frac{\lambda_{b,0}}{\lambda}\right) = +\infty. \tag{29}$$

Lastly, the previous arguments establish that $K(\lambda)e$ is a supersolution of (10) for all K large enough. Thus, since $\min_{x\in\overline{\Omega}} e(x) > 0$, we can choose K such that $\varepsilon(\lambda)\varphi_{b,0} \leq K(\lambda)e$. Therefore, by the method of sub and supersolution and the uniqueness of positive solution for (10), we can infer that $\varepsilon(\lambda)\varphi_{b,0} \leq \Theta_{\lambda,\kappa}$. Consequently, by (29), we obtain (4) and this complete the proof. □

Note that, as a direct consequence of this result, the proof of Theorem 1.1 follows by setting $\Psi_{\lambda,\kappa} := f_\kappa(\Theta_{\lambda,\kappa})$.

4. A priori bounds in Ω_+

This section is devoted to obtain an *a priori* estimate for positive solutions of (10), uniform in $\kappa > 0$, $\kappa \simeq 0$ in any compact subset of Ω_+. It is a crucial step to prove Theorem 1.3 (*c*). As we will see below, to obtain these estimates, we will assume $p > 3$. To this aim, we need to study the following auxiliary problem

$$\begin{cases} -\Delta v = \lambda v - b_0 g(v) & \text{in} \quad B_r, \\ v = \infty & \text{on} \quad \partial B_r, \end{cases} \tag{30}$$

where $b_0 > 0$ is a constant, $B_r := B_r(x_0) = \{x \in \mathbb{R}^N;\ |x - x_0| < r\}$ is an open ball in $\mathbb{R}^N$ centered in $x_0 \in \mathbb{R}^N$ and

$$g(t) := \frac{f_1^{p+1}(t)}{t}, \quad \forall\, t > 0. \tag{31}$$

First, we will prove some important properties of g.

Lemma 4.1. *The map $g : (0,\infty) \to (0,\infty)$ defined in (31) is increasing and it satisfies $g(0) := \lim_{t\to 0^+} g(t) = 0$. Moreover, there exists a constant $C > 0$ such that*

$$g(t) \geq C t^{(p-1)/2}, \quad \forall\, t \geq 1. \tag{32}$$

Furthermore,

$$f_\kappa^p(t) f_\kappa'(t) \leq g(t), \quad \forall\, t > 0 \quad \text{and} \quad 0 < \kappa < 1. \tag{33}$$

Proof. In order to prove that g is increasing, note that, by Lemma 2.1 (iii), we have

$$t f_1(t) \geq \frac{f_1(t)}{2} > \frac{f_1(t)}{p+1}, \quad \forall\, t > 0,$$

since $p > 1$. Thus,

$$g'(t) = \left(\frac{f_1^{p+1}(t)}{t}\right)' = \frac{(p+1) f_1^p(t) t - f_1^{p+1}(t)}{t^2} > 0, \quad \forall\, t > 0.$$

To conclude the proof of inequality (32), observe that for each $t > 0$, one has

$$\frac{g(t)}{t^{(p-1)/2}} = \left(\frac{f_1(t)}{t^{1/2}}\right)^{p+1}.$$

By Lemma 2.1 (vii), $t \mapsto g(t)/t^{(p-1)/2}$ is nondecreasing and thus $\frac{g(t)}{t^{(p-1)/2}} \geq g(1)$ for all $t \geq 1$. Choosing $C = g(1)$, we obtain (32). Moreover, $\lim_{t\to 0^+} f_\kappa^p(t)(f_\kappa(t)/t) = g(0) = 0$. Finally, combining the monotonicity of $\kappa \mapsto f_\kappa(\cdot)$ with Lemma 2.1 (v), we get

$$f_\kappa^p(t) f_\kappa'(t) \leq \frac{f_\kappa^{p+1}(t)}{t} < \frac{f_1^{p+1}(t)}{t} = g(t), \quad \forall\, t > 0.$$

Therefore, the inequality (33) holds. □

Now, we will establish an existence result for (7). We recall that there are many results about the existence, uniqueness and blow-up rate of large solution of problems related to (30), see, for instance, [13,18,22,23] and references therein. The following lemma is a consequence of these works.

Lemma 4.2. *(i) Let λ, b_0, M be positive constants and consider the following nonlinear boundary value problem*

$$\begin{cases} -\Delta v = \lambda v - b_0 g(v) & \text{in} \quad B_r, \\ v = M & \text{on} \quad \partial B_r. \end{cases} \tag{34}$$

Then, (34) has an unique positive solution denoted by $\Theta_{[\lambda,b_0,M,B_r]}$.

(ii) Suppose $p > 3$. For each $x \in B_r$, the point-wise limit

$$\Theta_{[\lambda,b_0,\infty,B_r]}(x) := \lim_{M\uparrow\infty} \Theta_{[\lambda,b_0,M,B_r]}(x)$$

is well defined and it is a classical minimal positive solution of (30).

Proof. The existence of positive solution for (34) can be easily obtained by the method of sub and supersolution and the uniqueness follows from similar arguments used in Sect. 3.

To prove (ii), we will apply Theorem 1.1 of [13]. Thus, it is sufficient to show that $g \in \mathcal{C}^1([0,\infty))$, $g \geq 0$, the map $t \in (0,+\infty) \mapsto g(t)/t$ is increasing and it verifies the Keller–Osserman condition, i.e.,

$$\int_1^{\infty} \frac{dt}{\sqrt{G(t)}} < \infty, \quad \text{where} \quad G(t) := \int_0^t g(s)\mathrm{d}s. \tag{35}$$

Indeed, the regularity and positivity of g is given by Lemma 4.1. To prove that $t \in (0,+\infty) \mapsto g(t)/t$ is increasing, note that

$$\left(\frac{g(t)}{t}\right)' = \left(\frac{f_1^{p+1}(t)}{t^2}\right)' = \frac{(p+1)f_1^p(t)f_1'(t)t^2 - 2tf_1^{p+1}}{t^2} > 0,$$

if and only if, $(p+1)tf_1'(t) > 2f_1(t)$. Since $p > 3$, it follows from Lemma 2.1 (v) that

$$(p+1)tf_1'(t) \geq \frac{(p+1)}{2}f_1(t) > 2f_1(t),$$

showing that $g(t)/t$ is increasing. Finally, observe that (32) is a sufficient condition for (35) to occur. This completes the proof. □

It should be pointed out that the Lemma 4.2 (ii) also can be obtained by adapting the arguments of [22].

Now, we are able to prove the main result of this section.

Proposition 4.3. *Suppose $p > 3$. For each compact $K \subset \Omega_+ = \{x \in \Omega;\ b(x) > 0\}$, there exists a constant $C = C(\lambda, K) > 0$ such that $\|\Theta_{\lambda,\kappa}\|_{\mathcal{C}(K)} \leq C$ for all $\kappa \in (0,1)$, where $\Theta_{\lambda,\kappa}$ stands for the unique positive solution of* (10).

Proof. Let $B_r := B_r(x_0) \subset\subset \Omega_+$. In particular, $b_K := \min_{x\in B_r} b(x) > 0$. By (11) and (33), for all $0 < \kappa < 1$, $\lambda > \lambda_1$, $\Theta_{\lambda,\kappa}$ satisfies

$$-\Delta\Theta_{\lambda,\kappa} = \lambda f_\kappa(\Theta_{\lambda,\kappa})f_\kappa'(\Theta_{\lambda,\kappa}) - b(x)f_\kappa^p(\Theta_{\lambda,\kappa})f_\kappa'(\Theta_{\lambda,\kappa}) \leq \lambda\Theta_{\lambda,\kappa} - b_K g(\Theta_{\lambda,\kappa}) \quad \text{in} \quad B_r.$$

Thus, $\Theta_{\lambda,\kappa}$ is a subsolution of (34) for all $M \geq \max_{B_r}\Theta_{\lambda,\kappa}$. Since large constants are positive supersolutions of (34), by the sub- and supersolution method combined with the uniqueness of positive solution of (34), we can infer that

$$\Theta_{\lambda,\kappa} \leq \Theta_{[\lambda,M,b_K,B_r]} \quad \text{in} \quad B_r, \quad \forall\ M \geq \max_{B_r}\Theta_{\lambda,\kappa},\ 0 < \kappa < 1.$$

Letting $M \to \infty$ in the above inequality, we get $\Theta_{\lambda,\kappa} \leq \Theta_{[\lambda,\infty,b_K,B_r]}$ in B_r and for all $0 < \kappa < 1$. In particular,

$$\Theta_{\lambda,\kappa} \leq \Theta_{[\lambda,\infty,b_K,B_r]} \quad \text{in} \quad B_{r/2}; \quad 0 < \kappa < 1.$$

Consequently, setting $C := \max_{B_{r/2}}\Theta_{[\lambda,\infty,b_K,B_r]}$, we obtain $\|\Theta_{\lambda,\kappa}\|_{\mathcal{C}(B_{r/2})} \leq C$. Observe that C depends on $b_K := \min_{x\in B_r} b(x)$, B_r and λ. Finally, since K can be covered by a finite union of such balls, the proof is complete. □

5. Behavior of the positive solutions with respect to κ

In this section, we will prove Theorems 1.3 and 1.4. First, we will establish the behavior of the solutions of $(\mathcal{P}_\kappa)$ when $\kappa \to 0$. Some arguments used here are inspired in [15]. We point out that we will prove the results for the unique positive solution $\Theta_{\lambda,\kappa}$ of (10) and therefore we obtain a similar result for the unique positive solution $\Psi_{\lambda,\kappa} = f_\kappa(\Theta_{\lambda,\kappa})$ of $(\mathcal{P}_\kappa)$.

Proof of Theorem 1.3. To prove (a), we will apply the Implicit Function Theorem. Suppose $\lambda \in (\lambda_1, \lambda_{b,0})$. Note that, for $\delta > 0$ small enough, $\kappa \in [0,\delta) \mapsto f_\kappa(\cdot)$ is a continuous map and $f'_\kappa = 1/(1+2\kappa f_\kappa^2)^{1/2}$, $\kappa \in [0,\delta) \mapsto f'_\kappa(\cdot)$ is also continuous. Therefore, we can consider a continuous extension of f_κ and f'_κ for $(-\delta,\delta)$. Define $\mathcal{F} : (-\delta,\delta) \times \mathcal{C}_0^1(\overline{\Omega}) \to \mathcal{C}_0^1(\overline{\Omega})$ by

$$\mathcal{F}(\kappa, v) = v - (-\Delta)^{-1}[\lambda f_\kappa(v) f'_\kappa(v) - b f_\kappa^p(v) f'_\kappa(v)].$$

Thus, $\mathcal{F}(\kappa, v)$ is continuous in κ and of class $\mathcal{C}^1$ in v. Moreover, the zeros of $\mathcal{F}$ provide us the positive solution of (10) if $\kappa > 0$ and the positive solution of classical logistic equation $(\mathcal{P}_0)$ if $\kappa = 0$, since $f_0(t) = t$, $t \geq 0$. Differentiating with respect to v at $(0, \Theta_\lambda)$, we obtain

$$D_v\mathcal{F}(0,\Theta_\lambda)v = v - (-\Delta)^{-1}[\lambda v - pb\Theta_\lambda^{p-1} v], \quad \forall\, v \in \mathcal{C}_0^1(\overline{\Omega}).$$

Since Θ_λ is a nondegenerate positive solution of $(\mathcal{P}_0)$, the operator $\mathcal{F}(0,\Theta_\lambda)$ is an isomorphism. Thus, it follows from the Implicit Function Theorem that, for $\delta > 0$ small, there exists a continuous map $\kappa \in (-\delta,\delta) \mapsto v(\kappa) \in \mathcal{C}_0^1(\overline{\Omega})$ such that $v(0) = \Theta_\lambda$ and $\mathcal{F}(\kappa, v(\kappa)) = 0$ for each $\kappa \in (-\delta,\delta)$. Observe that $v(\kappa)$ is a positive solution of (10) for $\kappa > 0$ and $\kappa \simeq 0$, since Θ_λ lies in the interior of the positive cone of $\mathcal{C}_0^1(\overline{\Omega})$. Consequently, by the uniqueness of positive solution of (10), we obtain that $v(\kappa) = \Theta_{\lambda,\kappa}$. In particular, $\lim_{\kappa\downarrow 0}\Theta_{\lambda,\kappa} = \lim_{\kappa\downarrow 0} v(\kappa) = v(0) = \Theta_\lambda$, completing the proof of item (a).

Now, we will prove (b). Suppose $\lambda \geq \lambda_{b,0}$. By the monotonicity of $\lambda \mapsto \Theta_{\lambda,\kappa}$, for each $\varepsilon > 0$ small enough, we have $\Theta_{\lambda_{b,0}-\varepsilon,\kappa} < \Theta_{\lambda,\kappa}$. Using part (a), we can infer that

$$\Theta_{\lambda_{b,0}-\varepsilon} = \lim_{\kappa\downarrow 0}\Theta_{\lambda_{b,0}-\varepsilon,\kappa} < \liminf_{\kappa\downarrow 0}\Theta_{\lambda,\kappa}.$$

Taking into account (5), we conclude that

$$+\infty = \lim_{\varepsilon\to 0^+}\Theta_{\lambda_{b,0}-\varepsilon} \leq \liminf_{\kappa\downarrow 0}\Theta_{\lambda,\kappa} \quad \text{uniformly in compact subsets of } \overline{\Omega}_{b,0}\backslash\partial\Omega.$$

and therefore, $\lim_{\kappa\downarrow 0}\Theta_{\lambda,\kappa} = +\infty$ uniformly in compact subsets of $\overline{\Omega}_{b,0}\backslash\partial\Omega$, which proves (8). Conversely, $M_{\lambda_{b,0}} \leq \liminf_{\kappa\downarrow 0}\Theta_{\lambda,\kappa}$ in $\overline{\Omega}_+$, where $M_{\lambda_{b,0}}$ stands for the minimal positive solution of (7) with $\lambda = \lambda_{b,0}$, since $\lim_{\varepsilon\to 0^+}\Theta_{\lambda_{b,0}-\varepsilon} = M_{\lambda_{b,0}}$ in $\overline{\Omega}_+$. In particular, $\lim_{\kappa\downarrow 0}\Theta_{\lambda,\kappa} = \infty$ on $\partial\Omega_+$. By a rather standard compactness argument combined with Proposition 4.3 (see for instance [27, Proposition 3.3]), we obtain that the point-wise limit

$$M_\lambda(x) := \lim_{\kappa\downarrow 0}\Theta_{\lambda,\kappa}(x)$$

provide us a classical positive solution of (7) and this finishes the proof. □

Finally, we conclude this section by establishing the behavior of the solution of $(\mathcal{P}_\kappa)$ when $\kappa \to \infty$.

Proof of Theorem 1.4. For each $\lambda > \lambda_1$, let u_κ be a positive solution of $(\mathcal{P}_\kappa)$ and $v_\kappa = f_\kappa^{-1}(u_\kappa)$ the respective solution of (10). By Theorem 1.1 of [20], the problem

$$\begin{cases} -\Delta w = \lambda f_\kappa(w) f'_\kappa(w) & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{36}$$

has a unique positive solution, say w_κ, and it satisfies

$$\lim_{\kappa\to\infty}\|w_\kappa\|_\infty = 0. \tag{37}$$

Moreover, for all constant $K = K(\lambda) > 0$ large enough, Ke is a supersolution of (36), where e is the unique positive solution of

$$\begin{cases} -\Delta v = 1 & \text{in} \quad \widehat{\Omega}, \\ v = 0 & \text{on} \quad \partial\widehat{\Omega}, \end{cases}$$

for some regular domain $\Omega \subset\subset \widehat{\Omega}$. On the other hand, using that $b(x) \geq 0$ we get

$$-\Delta v_\kappa = \lambda f_\kappa(v_\kappa) f'_\kappa(v_\kappa) - b(x) f_\kappa^p(v_\kappa) f'_\kappa(v_\kappa) \leq \lambda f_\kappa(v_\kappa) f'_\kappa(v_\kappa) \quad \text{in} \quad \Omega,$$

that is, v_κ is a subsolution of (36). In addition, we can take K sufficiently large such that Ke is a supersolution of (36) and $v_\kappa \leq Ke$ in Ω. By the sub and supersolution method, there exists a positive solution of (36) between v_κ and Ke. Since (36) has a unique positive solution, necessarily the solution obtained is w_κ and consequently $v_\kappa \leq w_\kappa \leq \|w_\kappa\|_\infty$. This inequality, together with (37), implies that $\lim_{\kappa\to\infty}\|v_\kappa\|_\infty = 0$. Thus, in view of Lemma 2.1, the positive solution $u_\kappa = f_\kappa(v_\kappa)$ of $(\mathcal{P}_\kappa)$ satisfies $\|u_\kappa\|_\infty = \|f_\kappa(v_\kappa)\|_\infty \leq \|v_\kappa\|_\infty \to 0$, as $\kappa \to \infty$, and the proof is complete. □

6. Final remarks

In this section, we show a stability result for (10) with the additional assumption that $p \geq 3$. We recall that the stability of a positive solution (λ_0, v_0) of (10) as a steady state of the associated parabolic equation is given by the spectrum of the linearized operator of (10), which is

$$\mathcal{L}(\lambda_0, v_0) := -\Delta - \lambda_0[f_\kappa(v_0)f'_\kappa(v_0)]' + b(x)[f^p_\kappa(v_0)f'_\kappa(v_0)]',$$

subject to homogeneous Dirichlet boundary conditions on $\partial\Omega$. Thus, (λ_0, v_0) is said to be linearly asymptotically stable if $\lambda_1[\mathcal{L}(\lambda_0, v_0)] > 0$.

First, we present a result that relates the linearized operators of (10) and $(\mathcal{P}_\kappa)$. To this end, since $\Delta(u^2)u = 2u|\nabla u|^2 + 2u^2\Delta u$, problem $(\mathcal{P}_\kappa)$ can be rewritten as

$$\begin{cases} -(1+2\kappa u^2)\Delta u - 2\kappa u|\nabla u|^2 = \lambda u - b(x)u^p & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{$\mathcal{P}_\kappa$}$$

Hence, the linearized operator of $(\mathcal{P}_\kappa)$ at (λ_0, u_0) is given by

$$\overline{\mathcal{L}}(\lambda_0, u_0) := -(1+2\kappa u_0^2)\Delta - 4\kappa u_0\nabla u_0\nabla - (4\kappa u_0\Delta u_0 + 2\kappa|\nabla u_0|^2) + (b(x)pu_0^{p-1} - \lambda_0).$$

With these considerations, we have:

Lemma 6.1. *Let (λ_0, u_0) be a classical positive solution of $(\mathcal{P}_\kappa)$ and $v_0 = f^{-1}(u_0)$ the respective solution of the dual problem (10). For each $\phi \in W^{2,p}(\Omega)$, $p > 1$, if we define $\psi = \sqrt{1+2\kappa u_0^2}\phi$, then*

$$\mathcal{L}(\lambda_0, v_0)\psi = \frac{1}{\sqrt{1+2\kappa u_0}}\overline{\mathcal{L}}(\lambda_0, u_0)\phi. \tag{38}$$

Proof. The proof is similar to [6, Lemma 2.3], so we will be brief. By a direct calculation,

$$\Delta\psi = \sqrt{1+2ku_0^2}\Delta\phi + \frac{4\kappa u_0}{\sqrt{1+2ku_0^2}}\nabla u_0\nabla\phi + \frac{2\kappa|\nabla u_0|^2}{\sqrt{1+2ku_0^2}}\phi + \frac{2\kappa u_0}{\sqrt{1+2ku_0^2}\Delta u_0}\phi. \tag{39}$$

On the other hand, it follows from Lemma 2.1 $(viii)$ and (ix) that

$$[f_\kappa(v_0)f'_\kappa(v_0)]'\,\psi = (f'_\kappa(v_0))^4\psi = \frac{1}{(\sqrt{1+2ku_0^2})^3}\phi \tag{40}$$

$$[f^p_\kappa(v_0)f'_\kappa(v_0)]'\,\psi = u_0^{p-1}\left[\frac{(p-1)}{\sqrt{1+2ku_0^2}} + \frac{1}{(\sqrt{1+2ku_0^2})^3}\right]\phi. \tag{41}$$

Thus, combining (39), (40) and (41), we get

$$\begin{aligned}\mathcal{L}(\lambda_0, u_0)\psi &= -\sqrt{1+2ku_0^2}\Delta\phi - \frac{4\kappa u_0}{\sqrt{1+2ku_0^2}}\nabla u_0\nabla\phi - \frac{2\kappa|\nabla u_0|^2}{\sqrt{1+2ku_0^2}}\phi - \frac{2\kappa u_0}{\sqrt{1+2ku_0^2}\Delta u_0}\phi \\ &\quad - \frac{\lambda_0}{(\sqrt{1+2ku_0^2})^3}\phi + b(x)u_0^{p-1}\left[\frac{(p-1)}{\sqrt{1+2ku_0^2}} + \frac{1}{(\sqrt{1+2ku_0^2})^3}\right]\phi \\ &= \frac{1}{\sqrt{1+2ku_0^2}}\left[\mathcal{L}(\lambda_0, u_0)\phi + \frac{2\kappa u_0}{1+2ku_0^2}((1+2ku_0^2)\Delta u_0 + 2\kappa u_0|\nabla u_0|^2 + \lambda_0 u_0 - b(x)u_0^p)\right].\end{aligned}$$

Since (λ_0, u_0) is a solution of $(\mathcal{P}_\kappa)$, it follows that (38) holds. □

As a consequence, we have the following result:

Corollary 6.2. *Let (λ_0, u_0) be a classical positive solution of $(\mathcal{P}_\kappa)$ and let $v_0 = f^{-1}(u_0)$ denote the respective classical solution of the dual problem (10). Then*

(i) *A function $\phi \in W^{2,p}(\Omega)$, $p > 1$, is a strict supersolution of $\overline{\mathcal{L}}(\lambda_0, u_0)$ if, and only if, $\psi := \sqrt{1 + 2\kappa u_0^2}\phi$ is a strict supersolution of $\mathcal{L}(\lambda_0, u_0)$;*

(ii) *$\lambda_1[\overline{\mathcal{L}}(\lambda_0, u_0)] > 0$ if, and only if, $\lambda_1[\mathcal{L}(\lambda_0, v_0)] > 0$;*

(iii) *(λ_0, u_0) is a nondegenerate solution of $(\mathcal{P}_\kappa)$ if, and only if, (λ_0, v_0) is a nondegenerate positive solution of (10).*

Proof. (i) If $\phi \in W^{2,p}(\Omega)$ is a strict supersolution of $\overline{\mathcal{L}}(\lambda_0, u_0)$ then $\psi := \sqrt{1 + 2\kappa u_0^2}\phi > 0$ and, by (38), it satisfies

$$\mathcal{L}(\lambda_0, v_0)\psi = \frac{1}{\sqrt{1 + 2\kappa u_0}} \overline{\mathcal{L}}(\lambda_0, u_0)\phi > 0.$$

Hence, ψ is a strict supersolution of $\mathcal{L}(\lambda_0, v_0)$. The converse is analogous.

(ii) By the characterization of the Maximum Principle, see, for instance, [28, Theorem 2.1] or [26], $\lambda_1[\overline{\mathcal{L}}(\lambda_0, u_0)] > 0$ (respectively, $\lambda_1[\mathcal{L}(\lambda_0, v_0)] > 0$) if and only if, there exists a positive strict supersolution of $\overline{\mathcal{L}}(\lambda_0, u_0)$ (respectively, $\mathcal{L}(\lambda_0, v_0)$). Thus, (i) implies (ii).

(iii) Just note that, by (38), $\mathrm{Ker}\,[\overline{\mathcal{L}}(\lambda_0, u_0)] = 0$ if and only if, $\mathrm{Ker}\,[\mathcal{L}(\lambda_0, u_0)] = 0$.

□

According to the previous corollary, in order to show that a solution of $(\mathcal{P}_\kappa)$ is nondegenerate, it is sufficient to analyze the linearized operator of the dual problem (10). With respect to the sign of $\lambda_1[\mathcal{L}(\lambda, \Theta_{\lambda,\kappa})]$, we have the following result:

Proposition 6.3. *Suppose $p \geq 3$. Then, for each $\lambda > \lambda_1$ and $\kappa > 0$, the unique positive solution $(\lambda, \Theta_{\lambda,\kappa})$ of (10) is linearly asymptotically stable, that is,*

$$\lambda_1[\mathcal{L}(\lambda, \Theta_{\lambda,\kappa})] > 0.$$

Proof. To simplify the notation, we shall denote $f = f_\kappa(\Theta_{\lambda,\kappa})$ and $f' = f'_\kappa(\Theta_{\lambda,\kappa})$. By the characterization of the Maximum Principle, in order to prove that $\lambda_1[\mathcal{L}(\lambda, \Theta_{\lambda,\kappa})] > 0$, it is sufficient to show that there exists a positive strict supersolution of $\mathcal{L}(\lambda, \Theta_{\lambda,\kappa})$. Let us prove that $\Theta_{\lambda,\kappa}$ is a strict supersolution of $\mathcal{L}(\lambda, \Theta_{\lambda,\kappa})$. Indeed, since $\Theta_{\lambda,\kappa}$ is a positive solution of (10), we have $-\Delta\Theta_{\lambda,\kappa} = \lambda f f' - b(x) f^p f'$. Thus, using Lemma 2.1 $(viii)$ and (ix), we find that

$$\begin{aligned}
\mathcal{L}(\lambda, \Theta_{\lambda,\kappa})\Theta_{\lambda,\kappa} &= -\Delta\Theta_{\lambda,\kappa} - \lambda[f f']'\Theta_{\lambda,\kappa} + b(x)[f^p f']'\Theta_{\lambda,\kappa} \\
&= \lambda f f' - b(x) f^p f' - \lambda[(f')^2 - 2\kappa f^2 (f')^4]\Theta_{\lambda,\kappa} \\
&\quad + b(x) f^{p-1}[(p-1)(f')^2 + (f')^4]\Theta_{\lambda,\kappa} \\
&= \lambda(f - f'\Theta_{\lambda,\kappa}) f' + b(x) f^{p-1} f'((p-1) f'\Theta_{\lambda,\kappa} - f) \\
&\quad + 2\lambda\kappa f^2 (f')^4 \Theta_{\lambda,\kappa} + b(x) f^{p-1} (f')^4 \Theta_{\lambda,\kappa}.
\end{aligned} \tag{42}$$

Since $p \geq 3$, it follows from Lemma 2.1 (v) that

$$\begin{aligned}
f - f'\Theta_{\lambda,\kappa} = f(\Theta_{\lambda,\kappa}) - f'(\Theta_{\lambda,\kappa})\Theta_{[\lambda,\kappa]} > 0 \quad \text{and} \\
(p-1) f'\Theta_{\lambda,\kappa} - f = (p-1) f'(\Theta_{\lambda,\kappa})\Theta_{\lambda,\kappa} - f(\Theta_{\lambda,\kappa}) > 0.
\end{aligned} \tag{43}$$

Moreover, since $b(x) f^{p-1} f'\Theta_{\lambda,\kappa} \geq 0$, $f' \geq 0$ and $2\lambda\kappa f^2 (f')^4 \Theta_{\lambda,\kappa} \geq 0$, we can infer from (42) and (43) that $\mathcal{L}(\lambda, \Theta_{\lambda,\kappa})\Theta_{\lambda,\kappa} > 0$, which establishes that $\Theta_{\lambda,\kappa} > 0$ is a strict positive supersolution of $\mathcal{L}(\lambda, \Theta_{\lambda,\kappa})$. This completes the proof. □

As a direct consequence of this proposition, we obtain:

Corollary 6.4. *Suppose $p \geq 3$. Then*

(i) *For each $\lambda > \lambda_1$, $(\lambda, \Theta_{\lambda,\kappa})$ is a nondegenerate positive solution of (10);*
(ii) *The map $\lambda \in (\lambda_1, +\infty) \mapsto \Theta_{\lambda,\kappa} \in \mathcal{C}_0^1(\Omega)$ is of class $\mathcal{C}^\infty$.*

Proof. The proof of (i) is standard and once that $t \in [0, +\infty) \mapsto f_\kappa(t)$ is of class $\mathcal{C}^\infty$, (ii) follows from Implicit Function Theorem applied to the operator

$$\mathfrak{F}(\lambda, u) := u - (-\Delta)^{-1}[\lambda f_\kappa(u) f'_\kappa(u) - b(x) f^p_\kappa(u) f'_\kappa(u)].$$

□

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