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Traveling waves in the Kermack–McKendrick epidemic model with latent period

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Abstract. We study traveling waves for a diffusive susceptible–infected–recovery model, due to Kermack and McKendrick, of an epidemic with standard incidence and latent period included. In contrast to the classical case where the mass action incidence is employed, the total population is varied in the present model. It turns out that the governing equation for the recovery species cannot be decoupled from the other two equations for the susceptible and the infected species, and hence that the present model cannot be reduced to a two-component system as the classical one does. The existence of traveling waves of the model in this study can be completely characterized by the basic reproduction number of the system of ordinary differential equations associated with the present model. The model admits a continuum of traveling waves parameterized by wave speed c when waves do exist. Our approach is based on the fixed point theory and a delicately designed pair of super-/sub-solutions. This set of super-/sub-solutions also allows us to completely answer two unsolved questions in the existing literatures where the latent period is zero: (i) the existence of the minimal-speed wave which is believed to play a key role in the evolution of epidemic diseases and (ii) the existence of traveling waves does not depend on the relative ratio of the diffusivity of the infected species to the one of the recovery species.

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1. Introduction

Ross [28], in his pioneering work on epidemic modeling, introduced ordinary differential equations to describe the disease transmission between susceptible and infected individuals. Since then, there are a large number of works on epidemics modeling. Among them, the Kermack–McKendrick model [17–19] was successfully proposed to explain the rapid rise and fall in the number of infected individuals in the community during the course of epidemics [16, Chapter 2]. The Kermack–McKendrick model is a three-component system of ordinary differential equations:

$$\begin{aligned} S' &= -\kappa(I,N)S, \\ I' &= \kappa(I,N)S - \gamma I, \\ R' &= \gamma I, \end{aligned} \tag{1.1}$$

where the prime denotes the differentiation with respect to time t, S, I, and R are the number of susceptible individuals, the number of infected individuals, and the number of recovered individuals, respectively, at time t, $N = S + I + R$ is the total population size, and γ is the recovery rate. Here the function $\kappa(I,N)$ is termed as the *force of infection* in epidemiology, and the function $\kappa(I,N)S$ is called *incidence* which is the number of individuals becoming infected per unit of time [4,6,8,16,23]. Depending on the nature of disease-relevant contact within the population, there are two plausible types of incidence [11,16,23,24]: (i) the mass action incidence ($\kappa(I,N)S = \beta SI$) and (ii) the standard incidence ($\kappa(I,N)S = \beta SI/N$). Here $\beta > 0$ is a constant. Mass action incidence is used in the case where the number of disease-relevant contact increases as the population size increases (e.g., influenza), while standard incidence is employed in the case where the number of disease-relevant contact cannot increase indefinitely even if the population size increases indefinitely (e.g., sexually transmitted diseases) [11,16,23]. Note that system (1.1) with the mass action incidence is a special case of the model formulated by Kermack and McKendrick in [17].

The Kermack–McKendrick model (1.1) focuses on the behavior of an epidemic only in a single spatial location and ignores the possible movement into and out of the population. However, there is an increasing interest to use spatially dependent models such as reaction–diffusion models to study disease transmission [5,23,25]. Moreover, due to the well-developed modern transportation, the mobility of populations seems to be essential in the study of epidemics since it allows for the rapid transfer of diseases. The second missing factor in (1.1) is the latent period [14]. It is reported that many types of infectious diseases have so-called latent period. That is, it takes time for an individual who gets infection to become infective for the other individuals. Thus, the *standard incidence* assumption, the mobility of the population, and the existence of latent period together lead us to the consideration of the following spatially dependent Kermack–McKendrick model:

$$\begin{aligned} \frac{\partial S}{\partial t}(t,x) &= d_1 \frac{\partial^2 S}{\partial x^2}(t,x) - \frac{\beta I(t-\tau,x)}{N(t,x)} \cdot S(t,x), \\ \frac{\partial I}{\partial t}(t,x) &= d_2 \frac{\partial^2 I}{\partial x^2}(t,x) + \frac{\beta I(t-\tau,x)}{N(t,x)} \cdot S(t,x) - (\gamma+\delta) I(t,x), \\ \frac{\partial R}{\partial t}(t,x) &= d_3 \frac{\partial^2 R}{\partial x^2}(t,x) + \gamma I(t,x), \end{aligned} \tag{1.2}$$

where $S(t,x)$, $I(t,x)$, and $R(t,x)$ denote the density of susceptible individuals, the density of infected individuals, and the density of recovery individuals, respectively, at time $t \geq 0$ and spatial position $x \in \mathbb{R}$, and $N(t,x) := S(t,x) + I(t,x) + R(t,x)$ is the total population size at time t and spatial location x. Here the constants d_1, d_2, and d_3 are the diffusion rates of the susceptible individual, the infected individual, and the recovered individual, respectively. The parameter β is the transmission rate constant, $\gamma > 0$ is the recovery rate, $\delta \geq 0$ is the death/quarantine rate of infected individuals, and $\tau \geq 0$ is the latent period of the disease.

As explained in [5], a number of epidemic models in the setting of reaction–diffusion equations admit a continuum of traveling waves parameterized by the associated wave speed. Traveling waves are important in epidemiology since they can describe the spread of diseases into uninfected regions. Moreover, the so-called spreading speed of diseases is closely related to the minimal wave speed (see [5, Chapter 8] and [20,21]). A traveling wave solution of system (1.2) is a solution of system (1.2) in the form $(S,I,R)(x-ct)$ where the functions S, I, and R, termed as wave profiles, are the triple of nonnegative smooth functions defined on $\mathbb{R}$, and the constant c is the wave speed. Substituting $(S,I,R)(x-ct)$ into system (1.2), one can see that the wave solution $(S(z), I(z), R(z))(z = x - ct)$ solves the following system of ordinary differential equations:

$$\begin{aligned} & d_1 S''(z) + cS'(z) - \frac{\beta I(z+c\tau) S(z)}{S(z)+I(z)+R(z)} = 0, \\ & d_2 I''(z) + cI'(z) + \frac{\beta I(z+c\tau) S(z)}{S(z)+I(z)+R(z)} - (\gamma+\delta) I(z) = 0, \\ & d_3 R''(z) + cR'(z) + \gamma I(z) = 0. \end{aligned} \tag{1.3}$$

Next, we turn to discuss the boundary conditions for solutions of system (1.3). For this, we follow the discussion of Kermack and McKendrick [17, p. 701] to formulate the boundary conditions. Indeed, as Kermack and McKendrick [17] suggested, if a group of infected individuals are introduced into the population where all of individuals are initially susceptible to the disease and a disease outbreak is assumed, then one of the important questions in epidemiology is to decide which of the following two cases can occur when the termination of epidemics is reached: (i) no susceptible individuals are left or (ii) many susceptible individuals are still present in the affected population. Mathematically, we can formulate these two cases as follows: Given $S_\infty > 0$, determine constants $S_{-\infty} \in [0, S_\infty)$ and $R_{-\infty} \geq 0$ such that system (1.3) admits a solution (S, I, R) satisfying the following boundary conditions

$$\begin{aligned} &S(-\infty) = S_{-\infty},\ S(+\infty) = S_\infty,\\ &I(\pm\infty) = 0,\\ &R(-\infty) = R_{-\infty},\ R(+\infty) = 0. \end{aligned} \tag{1.4}$$

In this setting, for a wave to be biologically acceptable, the wave speed c must be positive (i.e., the wave must propagate from the left to the right). Hence, a traveling wave solution of (1.2) is a solution of (1.2) connecting the initial disease-free equilibrium $(S_\infty, 0, 0)$ to another disease-free equilibrium $(S_{-\infty}, 0, R_{-\infty})$.

Now we briefly review previous studies on traveling wave solutions of the variants of system (1.2). First, for the case of mass action incidence, Hosono and Ilyas [13] showed that the corresponding system admits a continuum of traveling wave solutions with the minimal speed if and only if the corresponding reproduction number is larger than one (see also [12] for the extreme case of non-diffusive infected individuals, [15] for the other extreme case of non-diffusive susceptible individuals, and [1] for the case where the latent period is included). Second, for the case of saturate incidence ($\kappa(I, N) = \beta I/(1 + \alpha I)$ with positive constants β and α), Xu [31] (see also [9] for the existence of minimal-speed wave, and [22] for a relevant system) proved that similar results also hold for the corresponding system. Third, to incorporate the effect of non-local interaction and latent period on the spread of diseases, Wang and Wu [32] studied the Kermack and McKendrick model with non-local delayed transmission and showed that similar results hold for their system. We remark that for the aforementioned cases, the governing equation for the recovery individual is decoupled from the other two equations, and so the governing system for traveling waves is in fact a two-component system, not a three-component system as it is in this study.

For system (1.2) with the zero latent period ($\tau = 0$), Wang and Wang in [29] (see also [30] for a simplified two-component system) showed that (i) if the basic reproduction number $\mathcal{R}_0 := \beta/(\gamma + \delta) > 1$ and $d_3 < 2d_2$, then for each given $c > c_\flat := 2\sqrt{d_2(\beta - (\gamma + \delta))}$ and $S_\infty > 0$, system (1.3) with boundary conditions (1.4) and $\tau = 0$ admits a solution, and (ii) if $(\mathcal{R}_0, c) \in (1, +\infty) \times (0, c_\flat) \cup (0, 1] \times \mathbb{R}$, then there are no nonnegative solutions for system (1.3) with the boundary conditions (1.4) and $\tau = 0$. We remark that the existence of traveling wave solutions with the minimal speed $c_\flat$ is unsolved in [29] and that whether the assumption $d_3 < 2d_2$ is a technical assumption for the existence of traveling waves is not understood there. Note that the existence of the minimal-speed wave is believed to play an important role in the determination of spreading speed of epidemic models [5]. Motivated by the above discussion, in this study we will give a complete characterization on the existence of traveling wave solutions for (1.2) with the latent period ($\tau \geq 0$) and any positive diffusion coefficients $d_i (i = 1, 2, 3)$, and, in particular, the existence of minimal-speed wave. Thus, our results also solve the aforementioned unsolved problem for the case that $\tau = 0$.

Main results

It is known that traveling wave solutions of the Kermack–McKendrick model are closely related to their behaviors around their tails. Hence, before stating our main results, we first give an intuitive description about the decaying rates of traveling wave solutions. Indeed, linearizing system (1.3) at disease-free equilibrium $(S_\infty, 0, 0)$, the I-component of the solution (S, I, R) of the resulting linearized system satisfies the following equation

$$d_2 I''(z) + cI'(z) + \beta I(z + c\tau) - (\gamma + \delta)I(z) = 0.$$

A direct computation indicates that for a given $c \in \mathbb{R}$, the decaying rate $\lambda > 0$ of a solution $\mathrm{e}^{-\lambda z}$ of the above linear equation is a zero of the corresponding characteristic function $f(\cdot, c)$ defined by

$$f(\lambda, c) = d_2\lambda^2 - c\lambda + \beta \mathrm{e}^{-\lambda c\tau} - (\gamma + \delta).$$

One can show (see Lemma 2.1) that if the basic reproduction number $\mathcal{R}_0 := \frac{\beta}{\gamma+\delta} > 1$, then there exists a unique $c_* > 0$ such that for $c = c_*$, $f(\cdot, c)$ has a unique zero $\lambda_* > 0$; for $c > c_*$, $f(\cdot, c)$ has exactly two zeros $0 < \lambda_1 = \lambda_1(c) < \lambda_* < \lambda_2 = \lambda_2(c)$; and for $c \in [0, c_*)$, $f(\cdot, c)$ has no real zeros.

In this paper, our main results can be stated as follows.

Theorem 1.1. (Traveling wave solutions of system (1.2) for $\mathcal{R}_0 \in (0, 1]$)
Assume that $\mathcal{R}_0 \in (0, 1]$. Then there are no nonnegative solutions of problem (1.3)-(1.4).

Theorem 1.2. (Traveling wave solutions of system (1.2) for $\mathcal{R}_0 > 1$)
Assume that $\mathcal{R}_0 > 1$. Then the following hold:

(I) *For each $c \in (0, c_*)$, there are no nonnegative solutions of problem (1.3)-(1.4).*

(II) *Let $S_\infty > 0$ be given. Then for each $c \geq c_*$, there exist constants $S_{-\infty} \in [0, S_\infty)$ and $R_{-\infty} > 0$ such that (1.3) and (1.4) admit a nonnegative solution (S^c, I^c, R^c) with the following properties:*

(i)

$$R_{-\infty} = \frac{\gamma}{\gamma+\delta}(S_\infty - S_{-\infty}), \tag{1.5a}$$

$$\int_{\mathbb{R}} (\gamma+\delta) I^c(y)\mathrm{d}y = \int_{\mathbb{R}} \frac{\beta I^c(y+c\tau)S^c(y)}{S^c(y)+I^c(y)+R^c(y)}\mathrm{d}y = c(S_\infty - S_{-\infty}). \tag{1.5b}$$

(ii) $0 < I^c \leq (\mathcal{R}_0 - 1)S_\infty$*,* $(S^c)' > 0$ *and* $(R^c)' < 0$ *in* $\mathbb{R}$.

(iii) *The function $I^c(z)$ has the following asymptotical behavior:*

$$I^c(z) = \mathcal{O}(\mathrm{e}^{-\lambda_1 z}) \text{ (resp. } \mathcal{O}(z\mathrm{e}^{-\lambda_* z})), \text{ as } z \to +\infty, \text{ for } c > c_* \text{ (resp. for } c = c_*).$$

We make remarks before proceeding further. First, a careful examination of the argument (the proofs of Lemmas 2.13 and 2.14 and the asymptotical behavior of the functions $R^c_\pm$ given in Sect. 2.1) reveals that the obtained wave solution is sandwiched between a pair of sub-/super-solutions, and the R-component of the sub-solution (resp. super-solution) is of order $\mathcal{O}(\mathrm{e}^{-(\max\{\lambda_1, c/d_3\})z})$ (resp. $\mathcal{O}(\mathrm{e}^{-(\min\{\lambda_1, c/d_3\})z})$ as $z \to \infty$, where $\lambda_1 = \lambda_1(c)$ is defined as above. Thus, we cannot deduce the exact asymptotical behavior of the R-component of wave solutions from those of this pair of sub-/super-solutions. However, for the case that $\lambda_1 < c/d_3$ (e.g., $d_3 < 2d_2/(1+\beta\tau)$ meets this condition), a suitable modification of the above sub-/super-solutions can be made in the way that the R-components of the resulting pair of sub-/super-solutions have the same asymptotical behavior (see Remark 2.7), and thus, the asymptotical behavior for the R-components of wave solutions can be deduced. However, for the case that $\lambda_1 \geq c/d_3$, this pair of modified sub-/super-solutions fail to have the same asymptotical behavior of their R-components, and thus, the asymptotical behavior for the R-components of wave solutions cannot be deduced. Second, the characterization of the minimal wave speed c_* implies that the minimum wave speed depends on the diffusion rate of infected individuals, but neither on that of susceptible individuals nor on that of recovery individuals. Moreover, our results indicate that traveling wave solutions of system (1.2) with $\tau = 0$ can exist for the parameter region $\{d_3 \geq 2d_2\}$, and hence that the assumption $d_3 < 2d_2$ made in [29] is a technical assumption for the existence of traveling waves for system (1.2) with $\tau = 0$. Also our results confirm the existence of minimal-speed wave for system (1.2) with $\tau = 0$, which is unsolved in [29].

Finally, the remaining of this paper is organized as follows: Sect. 2 is devoted to the proof of Theorem 1.2(II), while Sect. 3 is devoted to the proofs of Theorems 1.1 and 1.2(I).

2. Existence of traveling waves

In this section, we will establish the existence of traveling waves of system (1.2). The proof follows the framework of Berestycki et al. [3]. Indeed, we first apply the Schauder fixed point theorem and a suitable super-/sub-solution triple (see Sect. 2.1) to construct approximated wave solutions on the finite interval

$[-n, n]$ for each $n \in \mathbb{N}$ (see Sect. 2.2). Then we pass to the limit as $n \to +\infty$ to obtain wave solutions on $\mathbb{R}$ (see Sect. 2.3). *In the remaining of this section, we always assume that* $\mathcal{R}_0 > 1$.

2.1. Construction of super-/sub-solutions

We begin with the following lemma which characterizes the distribution of the zeros of the function $f(\cdot, c)$ for each given $c \geq c_*$. The proof is standard (e.g., see [31]), and so we omit it.

Lemma 2.1. *Assume that* $\mathcal{R}_0 > 1$. *There exists a unique* $c_* > 0$ *such that for* $c = c_*$, $f(\cdot, c)$ *has a unique zero* $\lambda_* > 0$; *for* $c > c_*$, $f(\cdot, c)$ *has exactly two zeros* $0 < \lambda_1(c) < \lambda_* < \lambda_2(c)$; *and for* $c \in [0, c_*)$, $f(\cdot, c)$ *has no real zeros. Moreover, we have*

$$\frac{\partial f}{\partial \lambda}(\lambda_1(c), c) < 0 \quad \text{and} \quad \frac{\partial f}{\partial \lambda}(\lambda_2(c), c) > 0 \quad \text{for} \quad c > c_*, \tag{2.1}$$

$$\frac{\partial f}{\partial \lambda}(\lambda_*, c_*) = 2d_2\lambda_* - c_* - \beta c_* \tau \mathrm{e}^{-\lambda_* c_* \tau} = 0. \tag{2.2}$$

2.1.1. The component S_+^c of the super-solution (S_+^c, I_+^c, R_+^c). In the following, we shall construct the super-/sub-solutions of (1.3) for each $c \geq c_*$. We first define the component S_+^c of the super-solution (S_+^c, I_+^c, R_+^c) as follows. Set

$$S_+^c \equiv S_\infty \ \text{ on } \mathbb{R} \text{ for } c \geq c_*.$$

2.1.2. The component I_+^c of the super-solution (S_+^c, I_+^c, R_+^c). Now we define the component I_+^c of the super-solution (S_+^c, I_+^c, R_+^c) as follows. For $c > c_*$, set

$$I_+^c(z) = \begin{cases} L_0, & z \leq z_0^c, \\ \mathrm{e}^{-\lambda_1 z}, & z > z_0^c, \end{cases}$$

where λ_1 is defined in Lemma 2.1, $L_0 = \left(\frac{\beta}{\gamma+\delta} - 1\right) S_\infty$ and $z_0^c = -\frac{\ln L_0}{\lambda_1}$. For $c = c_*$, set

$$I_+^{c_*}(z) = \begin{cases} L_0, & z \leq z_0^{c_*}, \\ L_1^{c_*} z \mathrm{e}^{-\lambda_* z}, & z > z_0^{c_*}, \end{cases}$$

where λ_* is defined in Lemma 2.1, $L_1^{c_*} = \mathrm{e}\lambda_* L_0$ and $z_0^{c_*} = \lambda_*^{-1}$. Note that $I_+^c(z) \leq L_0$ for $z \in \mathbb{R}$ and $c > c_*$. A direct computation yields

$$\max_{z \geq z_0^{c_*}} z\mathrm{e}^{-\lambda_* z} = z\mathrm{e}^{-\lambda_* z}\big|_{z = z_0^{c_*}} = (\mathrm{e}\lambda_*)^{-1}.$$

Together with the relation $L_1^{c_*} = \mathrm{e}\lambda_* L_0$, we thus have

$$I_+^c(z) \leq L_0 \ \text{ for } z \in \mathbb{R} \ \text{ and } \ c \geq c_*. \tag{2.3}$$

We remark that the form of $I_+^{c_*}$ is motivated by [9].

Lemma 2.2. *For* $c \geq c_*$ *and any nonnegative function* $R(z)$ *defined on* $\mathbb{R}$, *the function* $I_+^c(z)$ *satisfies the inequality*

$$d_2(I_+^c)''(z) + c(I_+^c)'(z) + \frac{\beta I_+^c(z + c\tau) S_+^c(z)}{S_+^c(z) + I_+^c(z) + R(z)} - (\gamma + \delta) I_+^c(z) \leq 0 \ \text{ for } z \neq z_0^c.$$

Proof. To begin with, let $R(z)$ be a nonnegative function on $\mathbb{R}$. We first consider the case that $c > c_*$. Then a direction computation indicates that for $z > z_0^c$, we have

$$\begin{aligned} &d_2(I_+^c)''(z) + c(I_+^c)'(z) + \frac{\beta I_+^c(z + c\tau) S_+^c(z)}{S_+^c(z) + I_+^c(z) + R(z)} - (\gamma + \delta) I_+^c(z) \\ &\quad \leq d_2(I_+^c)''(z) + c(I_+^c)'(z) + \beta I_+^c(z + c\tau) - (\gamma + \delta) I_+^c(z) = f(\lambda_1, c)\mathrm{e}^{-\lambda_1 z} = 0. \end{aligned}$$

For $z < z_0^c$, we have $I_+^c(z) = L_0$, and, by (2.3), $I_+^c(z + c\tau) \le L_0$. Then it follows that

$$d_2(I_+^c)''(z) + c(I_+^c)'(z) + \frac{\beta I_+^c(z+c\tau)S_+^c(z)}{S_+^c(z) + I_+^c(z) + R(z)} - (\gamma+\delta)I_+^c(z) \le \left[\frac{\beta S_\infty}{S_\infty + L_0} - (\gamma+\delta)\right]L_0 = 0,$$

where we have used the definition of L_0 in the last equality.

Next, we turn to the case that $c = c_*$. We first consider the case that $z > z_0^{c_*}$. For this case, $I_+^{c_*}(z) = L_1^{c_*} z\mathrm{e}^{-\lambda_* z}$. Then for any $R(z) \ge 0$, we have

$$\begin{aligned} &d_2(I_+^{c_*})''(z) + c_*(I_+^{c_*})'(z) + \frac{\beta I_+^{c_*}(z+c_*\tau)S_+^{c_*}(z)}{S_+^{c_*}(z) + I_+^{c_*}(z) + R(z)} - (\gamma+\delta)I_+^{c_*}(z) \\ &\le\ d_2(I_+^{c_*})''(z) + c_*(I_+^{c_*})'(z) + \beta I_+^{c_*}(z+c_*\tau) - (\gamma+\delta)I_+^{c_*}(z) \\ &\le\ L_1^{c_*}\left[f(\lambda_*, c_*)z\mathrm{e}^{-\lambda_* z} - (2d_2\lambda_* - c_* - \beta c_*\tau\mathrm{e}^{-\lambda_* c_*\tau})\mathrm{e}^{-\lambda_* z}\right] = 0 \quad \text{(by (2.2))}. \end{aligned}$$

Finally, for the case that $c = c_*$ and $z < z_0^{c_*}$, the proof follows the same lines for the case that $c > c_*$ and $z < z_0^c$. Thus, the proof of this lemma is completed. □

2.1.3. The component R_+^c of the super-solution (S_+^c, I_+^c, R_+^c). For each $c \ge c_*$, fix a $\lambda_1^R = \lambda_1^R(c) \in (0, \min\{\lambda_1, c/d_3\})$ close to $\min\{\lambda_1, c/d_3\}$. From the choice of λ_1^R, we have $c\lambda_1^R - d_3(\lambda_1^R)^2 > 0$ for $c \ge c_*$. Recall that

$$\lambda_1 < \lambda_* < \lambda_2 \ \text{ for } \ c > c_* \ \text{ and } \ \lambda_* = \lambda_1 = \lambda_2 \ \text{ for } \ c = c_*.$$

For $c > c_*$ and $z \in \mathbb{R}$, define

$$R_+^c(z) = \frac{\gamma K_0^c}{c\lambda_1^R - d_3(\lambda_1^R)^2}\cdot \mathrm{e}^{-\lambda_1^R z} \ \text{ with } \ K_0^c = \max\left\{1, L_0^{1-(\lambda_1^R/\lambda_1)}\right\}.$$

For $c = c_*$ and $z \in \mathbb{R}$, set

$$R_+^{c_*}(z) = \frac{\gamma K_0^{c_*}}{c_*\lambda_1^R - d_3(\lambda_1^R)^2}\cdot \mathrm{e}^{-\lambda_1^R z} \ \text{ with } \ K_0^{c_*} = \max\left\{L_1^{c_*}\max_{z\ge z_0^{c_*}} z\mathrm{e}^{-(\lambda_* - \lambda_1^R)z},\ L_0\mathrm{e}^{\lambda_1^R/\lambda_*}\right\},$$

where $z_0^{c_*}$ is defined in Lemma 2.2.

Lemma 2.3. *For each $c \ge c_*$, the function R_+^c satisfies the inequality*

$$d_3(R_+^c)''(z) + c(R_+^c)'(z) + \gamma I_+^c(z) \le 0 \ \textit{ for } z \in \mathbb{R}.$$

Proof. First, we consider the case that $c > c_*$. Indeed, one can verify that $K_0^c \ge \mathrm{e}^{(\lambda_1^R - \lambda_1)z}$ for $z \ge z_0^c$, and $K_0^c \ge L_0\mathrm{e}^{\lambda_1^R z}$ for $z \le z_0^c$. Then it follows from the definition of $I^+(z)$ that it holds

$$d_3(R_+^c)''(z) + c(R_+^c)'(z) + \gamma I_+^c(z) = -\gamma\mathrm{e}^{-\lambda_1^R z}\left[K_0^c - \mathrm{e}^{\lambda_1^R z}I_+^c(z)\right] \le 0,\ z \in \mathbb{R}.$$

Next, we turn to the case that $c = c_*$. We first treat the case that $z > z_0^{c_*}$. For this case, we have $I_+^{c_*}(z) = L_1^{c_*} z\mathrm{e}^{-\lambda_* z}$. Then a direct computation gives

$$\begin{aligned} d_3(R_+^{c_*})''(z) + c_*(R_+^{c_*})'(z) + \gamma I_+^{c_*}(z) &= -\gamma\mathrm{e}^{-(\lambda_1^R)z}\left[K_0^{c_*} - L_1^{c_*} z\mathrm{e}^{-(\lambda_* - \lambda_1^R)z}\right] \\ &\le -\gamma\mathrm{e}^{-(\lambda_1^R)z}\left[K_0^{c_*} - L_1^{c_*}\max_{z\ge z_0^{c_*}} z\mathrm{e}^{-(\lambda_* - \lambda_1^R)z}\right] \le 0, \end{aligned}$$

which establishes the assertion for $z > z_0^{c_*}$. Now, we treat the case that $z \le z_0^{c_*}$. For this case, we have $I_+^{c_*}(z) = L_0$. Using the relation that $z_0^{c_*} = 1/\lambda_*$, a direct computation yields

$$\begin{aligned} d_3(R_+^{c_*})''(z) + c_*(R_+^{c_*})'(z) + \gamma I_+^{c_*}(z) &= -\gamma\mathrm{e}^{-(\lambda_1^R)z}\left[K_0^{c_*} - L_0\mathrm{e}^{(\lambda_1^R)z}\right] \\ &\le -\gamma\mathrm{e}^{-(\lambda_1^R)z}\left[K_0^{c_*} - L_0\mathrm{e}^{(\lambda_1^R)z_0^{c_*}}\right] \le 0. \end{aligned}$$

This proves the assertion for $z \le z_0^{c_*}$, thereby completing the proof of this lemma. □

2.1.4. The component S_-^c of the sub-solution (S_-^c, I_-^c, R_-^c). Now, for any given $c \geq c_*$, we choose an $\epsilon_0^c \in (0, \min\{\lambda_1, c/d_1\})$. Recall that $\lambda_1(c_*) = \lambda_*$. Thus, both $\mathrm{e}^{-(\lambda_1-\epsilon_0^c)z}$ and $z\mathrm{e}^{-(\lambda_*-\epsilon_0^{c_*})z}$ tend to 0 as $z \to +\infty$. For $c > c_*$, we choose a large $z_1^c > z_0^c$ such that $\beta \mathrm{e}^{-\lambda_1 z_1^c} \leq S_\infty \epsilon_0^c (c - d_1\epsilon_0^c)$. Set $M_0^c = \mathrm{e}^{\epsilon_0^c z_1^c}$. Note that $-\lambda_1 + \epsilon_0^c < 0$. Then it follows

$$\beta \mathrm{e}^{(-\lambda_1+\epsilon_0^c)z}\mathrm{e}^{-\lambda_1 c\tau} \leq M_0^c \beta \mathrm{e}^{-\lambda_1 z_1^c} \leq M_0^c S_\infty \epsilon_0^c (c - d_1\epsilon_0^c), \quad z \geq z_1^c. \tag{2.4}$$

Similarly, for $c = c_*$, we choose a large $z_1^{c_*} > z_0^{c_*}$ such that

$$z\mathrm{e}^{-(\lambda_*-\epsilon_0^{c_*})z} \quad \text{is decreasing in } z \geq z_1^{c_*}, \tag{2.5}$$

and that $\beta L_1^{c_*} z_1^{c_*} \mathrm{e}^{-\lambda_* z_1^{c_*}} \leq S_\infty \epsilon_0^{c_*}(c_* - d_1\epsilon_0^{c_*})$. Set $M_0^{c_*} = \mathrm{e}^{\epsilon_0^{c_*} z_1^{c_*}}$. Then we have

$$\beta L_1^{c_*} z\mathrm{e}^{(-\lambda_*+\epsilon_0^{c_*})z} \leq \beta L_1^{c_*} z_1^{c_*} \mathrm{e}^{-\lambda_* z_1^{c_*}} \mathrm{e}^{\epsilon_0^{c_*} z_1^{c_*}} \leq M_0^{c_*} S_\infty \epsilon_0^{c_*}(c_* - d_1\epsilon_0^{c_*}), \quad z \geq z_1^{c_*}. \tag{2.6}$$

For $c \geq c_*$, define

$$S_-^c(z) = \begin{cases} 0, & z \leq z_1^c, \\ S_\infty(1 - M_0^c \mathrm{e}^{-\epsilon_0^c z}), & z > z_1^c. \end{cases}$$

Lemma 2.4. *For each $c \geq c_*$ and each pair of nonnegative functions (I, R) defined on $\mathbb{R}$ such that $I + R > 0$ on $(-\infty, z_1^c]$, the function S_-^c satisfies the inequality*

$$d_1(S_-^c)''(z) + c(S_-^c)'(z) - \frac{\beta I_+^c(z + c\tau) S_-^c(z)}{S_-^c(z) + I(z) + R(z)} \geq 0 \ \ \textit{for} \ \ z \neq z_1^c. \tag{2.7}$$

Proof. Note that $S_-^c(z) = 0$ for $c \geq c_*$ and $z \leq z_1^c$. Thus, it suffices to prove that (2.7) holds for $z > z_1^c$ and $R(z) + I(z) > 0$ on $(-\infty, z_1^c]$.

We first consider the case that $c > c_*$. Recall that $I_+^c(z) = \mathrm{e}^{-\lambda_1 z}$ for $z > z_1^c$. Then we have

$$\begin{aligned} d_1(S_-^c)''(z) + c(S_-^c)'(z) - \frac{\beta I_+^c(z + c\tau) S_-^c(z)}{S_-^c(z) + I(z) + R(z)} &\geq d_1(S_-^c)''(z) + c(S_-^c)'(z) - \beta I_+^c(z + c\tau) \\ &= \mathrm{e}^{-\epsilon_0^c z}\left[M_0^c S_\infty \epsilon_0^c (c - d_1\epsilon_0^c) - \beta \mathrm{e}^{-\lambda_1 c\tau + (-\lambda_1+\epsilon_0^c)z}\right] \\ &\geq 0 \quad (\text{by } (2.4)). \end{aligned}$$

Next we consider the case that $c = c_*$. Using (2.5), it follows that for $z > z_1^{c_*}$,

$$I_+^{c_*}(z) = L_1^{c_*} z\mathrm{e}^{-(\lambda_*-\epsilon_0^{c_*})z}\mathrm{e}^{-\epsilon_0^{c_*} z} \geq L_1^{c_*}(z + c_*\tau)\mathrm{e}^{-(\lambda_*-\epsilon_0^{c_*})(z+c_*\tau)}\mathrm{e}^{-\epsilon_0^{c_*}(z+c_*\tau)} = I_+^{c_*}(z + c_*\tau).$$

Together with (2.6) and the fact that $I(z) + R(z) \geq 0$ for $z > z_1^{c_*}$, we have that for $z > z_1^{c_*}$,

$$\begin{aligned} &d_1(S_-^{c_*})''(z) + c_*(S_-^{c_*})'(z) - \frac{\beta I_+^{c_*}(z + c_*\tau) S_-^{c_*}(z)}{S_-^{c_*}(z) + I(z) + R(z)} \\ &\geq \mathrm{e}^{-\epsilon_0^{c_*} z}\left[M_0^{c_*} S_\infty \epsilon_0^{c_*}(c_* - d_1\epsilon_0^{c_*}) - \beta L_1^{c_*} z\mathrm{e}^{(-\lambda_*+\epsilon_0^{c_*})z}\right] \geq 0. \end{aligned}$$

Hence, (2.7) holds for $z > z_1^{c_*}$. This completes the proof of this lemma. □

2.1.5. The component I_-^c of the sub-solution (S_-^c, I_-^c, R_-^c). Next, we construct the I-component of the sub-solution of (1.3). To begin with, we set some auxiliary quantities. Set

$$\eta^c := S_\infty(1 - \mathrm{e}^{-\epsilon_0^c z_1^c}) > 0 \ \ \text{for each} \ \ c \geq c_*. \tag{2.8}$$

For any given $c > c_*$, by Lemma 2.1, there exists a small $\epsilon_1^c \in (0, \min\{\lambda_1^R, \lambda_* - \lambda_1(c)\})$ such that $f(\lambda_1 + \epsilon_1^c, c) < 0$. Then we choose a large $z_2^c > \max\{1, 2z_1^c\}$ such that

$$-f(\lambda_1 + \epsilon_1^c, c) \geq \frac{\beta \mathrm{e}^{-(\lambda_1^R)z}}{\eta^c} \cdot \frac{(\gamma K_0^c + c\lambda_1^R - d_3(\lambda_1^R)^2)}{c\lambda_1^R - d_3(\lambda_1^R)^2} \ \ \text{for} \ \ z \geq z_2^c. \tag{2.9}$$

To determine $z_2^{c_*}$, set

$$\varphi(z) := \frac{\beta e^{-\lambda_* c_* \tau}}{\eta^{c_*}}\Big[(z + c_*\tau)z^{\frac{3}{2}}\big(L_1^{c_*} z e^{-\lambda_* z} + \frac{\gamma K_0^{c_*}}{c_*\lambda_1^R - d_3(\lambda_1^R)^2}\cdot e^{-(\lambda_1^R)z}\big)\Big]$$

for $z \geq 0$. Since $\varphi(z) \to 0$ as $z \to +\infty$, we can choose a large $z_2^{c_*} > \max\{1, 2z_1^{c_*}\}$ such that

$$\frac{d_2}{4} > \varphi(z) \;\; \text{for} \;\; z \geq z_2^{c_*}. \tag{2.10}$$

Set $M_1^c = e^{\epsilon_1^c z_2^c}$ $(c > c_*)$. Now, for $c > c_*$, we define

$$I_-^c(z) = \begin{cases} 0, & z \leq z_2^c, \\ (1 - M_1^c e^{-\epsilon_1^c z})e^{-\lambda_1 z}, & z > z_2^c, \end{cases}$$

and for $c = c_*$, we set

$$I_-^{c_*}(z) = \begin{cases} 0, & z \leq z_2^{c_*}, \\ L_1^{c_*}[z - \sqrt{z_2^{c_*}}\sqrt{z}\,]e^{-\lambda_* z}, & z > z_2^{c_*}. \end{cases}$$

Lemma 2.5. *For each $c \geq c_*$, there exists $z_2^c > z_1^c$ such that the function I_-^c satisfies the inequality*

$$d_2(I_-^c)''(z) + c(I_-^c)'(z) + \frac{\beta I_-^c(z + c\tau)S_-^c(z)}{S_-^c(z) + I_-^c(z) + R_+^c(z)} - (\gamma + \delta)I_-^c(z) \geq 0 \;\; \text{for } z \neq z_2^c. \tag{2.11}$$

Proof. Since $I_-^c \equiv 0$ in $(-\infty, z_2^c)$, it suffices to show that (2.11) holds for $z > z_2^c$ and $c \geq c_*$. To this end, note that for $c \geq c_*$, the left-hand side of (2.11) can be rewritten as follows:

$$\begin{aligned} &d_2(I_-^c)''(z) + c(I_-^c)'(z) + \frac{\beta I_-^c(z + c\tau)S_-^c(z)}{S_-^c(z) + I_-^c(z) + R_+^c(z)} - (\gamma + \delta)I_-^c(z) \\ &\quad= \big[d_2(I_-^c)''(z) + c(I_-^c)'(z) + \beta I_-^c(z + c\tau) - (\gamma + \delta)I_-^c(z)\big] \\ &\qquad+ \Big[\frac{\beta I_-^c(z + c\tau)S_-^c(z)}{S_-^c(z) + I_-^c(z) + R_+^c(z)} - \beta I_-^c(z + c\tau)\Big] := \mathrm{A}^c + \mathrm{B}^c. \end{aligned}$$

Recall that $f(\lambda_1, c) = 0$ for $c \geq c_*$. Then for $c > c_*$, we have

$$\begin{aligned} \mathrm{A}^c &= -M_1^c e^{-(\lambda_1+\epsilon_1^c)z}\big[d_2(\lambda_1 + \epsilon_1^c)^2 - c(\lambda_1 + \epsilon_1^c) + \beta e^{-(\lambda_1+\epsilon_1^c)c\tau} - \gamma - \delta\big] \\ &= -M_1^c e^{-(\lambda_1+\epsilon_1^c)z} f(\lambda_1 + \epsilon_1^c, c). \end{aligned} \tag{2.12}$$

For $c = c_*$, using the fact that $f_\lambda(\lambda_*, c_*) = 0$ (see (2.2)) and the definition of $I_-^{c_*}(z)$, we have

$$\begin{aligned} \mathrm{A}^{c_*} &= \frac{d_2(\sqrt{z_2^{c_*}}L_1^{c_*})}{4}\cdot\frac{e^{-\lambda_* z}}{z^{\frac{3}{2}}} + \beta(\sqrt{z_2^{c_*}}L_1^{c_*})e^{-\lambda_*(z+c_*\tau)}\Big[c_*\tau\cdot\frac{\sqrt{z + c_*\tau} - \sqrt{z}}{2\sqrt{z}(\sqrt{z + c_*\tau} + \sqrt{z})}\Big] \\ &\geq \frac{d_2 L_1^{c_*}}{4}\cdot\frac{e^{-\lambda_* z}}{z^{\frac{3}{2}}}. \end{aligned} \tag{2.13}$$

Now, we write the term B^c as follows:

$$\mathrm{B}^c = -\frac{\beta I_-^c(z + c\tau)(I_-^c(z) + R_+^c(z))}{S_-^c(z) + I_-^c(z) + R_+^c(z)} = -\frac{\beta I_-^c(z + c\tau)(I_-^c(z) + R_+^c(z))}{N^c(z)}$$

with $N^c(z) := S_-^c(z) + I_-^c(z) + R_+^c(z)$. Recall that η_c is defined by (2.8). Then using the fact that $z_2^c > 2z_1^c$ and that $S_-^c(z)$ is increasing in z, we have that for any given $c \geq c_*$,

$$N^c(z) \geq S_-^c(z) \geq S_-^c(z_2^c) \geq S_-^c(2z_1^c) = S_\infty(1 - e^{-\epsilon_0^c z_1^c}) = \eta^c, \; z \geq z_2^c. \tag{2.14}$$

Next, we estimate $\mathrm{A}^c + \mathrm{B}^c$. Recall that $\lambda_1 > \lambda_1^R > \epsilon_1^c$ and that $M_1^c = \mathrm{e}^{\epsilon_1^c z_2^c}$ for $c > c_*$. Hence, for $c > c_*$, it follows from (2.12) and the definitions of I_-^c and R_+^c that

$$\begin{aligned}
\mathrm{A}^c + \mathrm{B}^c \geq & - M_1^c \mathrm{e}^{-(\lambda_1+\epsilon_1^c)z} f(\lambda_1+\epsilon_1^c, c) - \frac{\beta \mathrm{e}^{-(\lambda_1+\lambda_1^R)z}}{N^c(z)}\left(1 + \frac{\gamma K_0^c}{c\lambda_1^R - d_3(\lambda_1^R)^2}\right)\\
\geq & \,\mathrm{e}^{-(\lambda_1+\epsilon_1^c)z}\left[-M_1^c f(\lambda_1+\epsilon_1^c, c) - \frac{\beta \mathrm{e}^{-(\lambda_1^R-\epsilon_1^c)z}}{\eta^c}\cdot\frac{\left(\gamma K_0^c + c\lambda_1^R - d_3(\lambda_1^R)^2\right)}{c\lambda_1^R - d_3(\lambda_1^R)^2}\right] \quad (\text{by } (2.14))\\
\geq & \,\mathrm{e}^{-(\lambda_1+\epsilon_1^c)z+\epsilon_1^c z_2^c}\Big[-f(\lambda_1+\epsilon_1^c, c) - \frac{\beta \mathrm{e}^{-\lambda_1^R z_2^c}}{\eta^c}\cdot\frac{\left(\gamma K_0^c + c\lambda_1^R - d_3(\lambda_1^R)^2\right)}{c\lambda_1^R - d_3(\lambda_1^R)^2}\Big] \geq 0 \ (\text{by } (2.9))
\end{aligned}$$

for all $z > z_2^c$. Thus, for $c > c_*$, the function I_-^c satisfies the inequality (2.11) for $z > z_2^c$.

Now, we will show that $A^{c_*} + B^{c_*} \geq 0$ for $z > z_2^{c_*}$. Indeed, using (2.13) and (2.14) and the definitions of I_-^c and R_+^c, it follows that for $z \geq z_2^{c_*}$, it holds

$$\begin{aligned}
&\mathrm{A}^{c_*} + \mathrm{B}^{c_*}\\
&\geq \frac{d_2 L_1^{c_*}}{4}\cdot\frac{\mathrm{e}^{-\lambda_* z}}{z^{\frac{3}{2}}} - \frac{\beta}{N^{c_*}(z)}\left[L_1^{c_*}(z+c_*\tau)\mathrm{e}^{-\lambda_*(z+c_*\tau)}\right]\cdot\left(L_1^{c_*} z\mathrm{e}^{-\lambda_* z} + \frac{\gamma K_0^{c_*}}{c_*\lambda_1^R - d_3(\lambda_1^R)^2}\mathrm{e}^{-(\lambda_1^R)z}\right)\\
&\geq L_1^{c_*}\cdot\frac{\mathrm{e}^{-\lambda_* z}}{z^{\frac{3}{2}}}\left\{\frac{d_2}{4} - \frac{\beta \mathrm{e}^{-\lambda_* c_* \tau}}{\eta^{c_*}}\left[(z+c_*\tau)z^{\frac{3}{2}}\left(L_1^{c_*} z\mathrm{e}^{-\lambda_* z} + \frac{\gamma K_0^{c_*}}{c_*\lambda_1^R - d_3(\lambda_1^R)^2}\cdot\mathrm{e}^{-(\lambda_1^R)z}\right)\right]\right\}\\
&= L_1^{c_*}\cdot\frac{\mathrm{e}^{-\lambda_* z}}{z^{\frac{3}{2}}}\left[\frac{d_2}{4} - \varphi(z)\right] > 0 \quad (\text{by } (2.10)).
\end{aligned}$$

Thus, the function $I_-^{c_*}$ satisfies (2.11) for $z > z_2^{c_*}$. This completes the proof of this lemma. □

2.1.6. The component R_-^c of the sub-solution (S_-^c, I_-^c, R_-^c). We first set up some auxiliary quantities. For each $c \geq c_*$, fix a $\lambda_2^R = \lambda_2^R(c) > \max\{\lambda_1, c/d_3\}$ close to $\max\{\lambda_1, c/d_3\}$. Note that $d_3(\lambda_2^R)^2 - c\lambda_2^R > 0$ for $c \geq c_*$. Next, for each $c \geq c_*$, choose a small $\epsilon_2^c \in (0, \lambda_2^R - \lambda_1)$. Then we have $d_3(\lambda_2^R+\epsilon_2^c)^2 - c(\lambda_2^R+\epsilon_2^c) > 0$ for $c \geq c_*$. Now for each $c \geq c_*$, in view of the relation $\lambda_2^R - \lambda_1 > 0$, we can fix a large $z_3^c > z_2^c$ such that for $c > c_*$ and $z \geq z_3^c$, it holds

$$\left[d_3\left(\lambda_2^R + \epsilon_2^c\right)^2 - c\left(\lambda_2^R + \epsilon_2^c\right)\right]\mathrm{e}^{-(\lambda_2^R-\lambda_1)z} + M_1^c \mathrm{e}^{-\epsilon_1^c z} < 1, \tag{2.15}$$

whereas for $c = c_*$ and $z \geq z_3^{c_*}$, it holds

$$L_1^{c_*} z > L_1^{c_*}\sqrt{z_2^{c_*}}\sqrt{z} + \left[d_3\left(\lambda_2^R + \epsilon_2^{c_*}\right)^2 - c_*\left(\lambda_2^R + \epsilon_2^{c_*}\right)\right]\mathrm{e}^{-(\lambda_2^R-\lambda_*)z}. \tag{2.16}$$

Next, for each $c \geq c_*$, define the auxiliary function

$$\tilde{R}_-^c(z) = \begin{cases} \gamma(1 - M_2^c \mathrm{e}^{-\epsilon_2^c z})\mathrm{e}^{-(\lambda_2^R)z}, & z > z_3^c,\\ 0, & z \leq z_3^c,\end{cases}$$

where $M_2^c = \mathrm{e}^{\epsilon_2^c z_3^c}$. For $c \geq c_*$, since $\tilde{R}_-^c > 0$ in $(z_3^c, +\infty)$ and $\tilde{R}_-^c(+\infty) = \tilde{R}_-^c(z_3^c) = 0$, the absolute maximal value of the function $\tilde{R}_-^c$ on $[z_3^c, +\infty)$ exists. Hence, for each $c \geq c_*$, we can choose a $z_4^c > z_3^c$ such that $\tilde{R}_-^c(z_4^c) = \max_{z \geq z_3^c} \tilde{R}_-^c(z) := \Lambda^c$. With a direct computation, one can verify that the function $\tilde{R}_-^c(z)$ takes its maximal value $\max_{z\in\mathbb{R}} \tilde{R}_-^c(z)$ at the unique point z_4^c.

Now for each $c \geq c_*$, we define the component R_-^c of the sub-solution (S_-^c, I_-^c, R_-^c) as follows:

$$R_-^c(z) = \begin{cases} \gamma(1 - M_2^c \mathrm{e}^{-\epsilon_2^c z})\mathrm{e}^{-(\lambda_2^R)z}, & z > z_4^c,\\ \Lambda^c, & z \leq z_4^c.\end{cases}$$

Lemma 2.6. *For $c \geq c_*$, the function $R^c_-(z)$ satisfies the inequality*

$$d_3(R^c_-)''(z) + c(R^c_-)'(z) + \gamma I^c_-(z) \geq 0 \;\; \textit{for} \;\; z \neq z^c_4. \tag{2.17}$$

Proof. For $c \geq c_*$, since $R^c_- \equiv \Lambda^c$ and $I^c_- \geq 0$ in $(-\infty, z^c_4)$, (2.17) holds for $z < z^c_4$. Hence, it remains to show that the inequality (2.17) holds for $z > z^c_4$.

We first consider the case that $c > c_*$. Recall that for $c > c_*$, $I^c_-(z) = (1 - M^c_1 e^{-\epsilon^c_1 z})e^{-\lambda_1 z}$ for $z > z^c_2$. Then for $z > z^c_4$, using (2.15), $M^c_2 = e^{\epsilon^c_2 z^c_3}$ and $z^c_4 > z^c_3$, we have

$$\begin{aligned} & d_3(R^c_-)''(z) + c(R^c_-)'(z) + \gamma I^c_-(z) \\ & = \gamma e^{-\lambda_1 z}\Big\{\Big[\big(d_3(\lambda^R_2)^2 - c\lambda^R_2\big)\, e^{-(\lambda^R_2 - \lambda_1)z} \\ & \quad - M^c_2\left(d_3(\lambda^R_2 + \epsilon^c_2)^2 - c(\lambda^R_2 + \epsilon^c_2)\right) e^{-(\lambda^R_2 - \lambda_1 + \epsilon^c_2)z}\Big] + (1 - M^c_1 e^{-\epsilon^c_1 z})\Big\} \\ & \geq \gamma e^{-\lambda_1 z}\left\{1 - M^c_1 e^{-\epsilon^c_1 z^c_3} - M^c_2\left[d_3(\lambda^R_2 + \epsilon^c_2)^2 - c(\lambda^R_2 + \epsilon^c_2)\right] e^{-(\lambda^R_2 - \lambda_1 + \epsilon^c_2) z^c_3}\right\} \geq 0. \end{aligned}$$

Hence, for $c > c_*$, the function R^c_- satisfies the inequality (2.17) for $z > z^c_4$.

Now we treat the case that $c = c_*$. Recall that $I^{c_*}_-(z) = L^{c_*}_1[z - \sqrt{z^{c_*}_2}\sqrt{z}\,]e^{-\lambda_* z}$ for $z > z^{c_*}_2$. Then for $z > z^{c_*}_4$, using (2.16), $M^{c_*}_2 = e^{\epsilon^{c_*}_2 z^{c_*}_3}$ and $z^{c_*}_4 > z^{c_*}_3$, we have

$$\begin{aligned} & d_3(R^{c_*}_-)''(z) + c_*(R^{c_*}_-)'(z) + \gamma I^{c_*}_-(z) \\ & = \gamma e^{-\lambda_* z}\Big\{\Big[\big(d_3(\lambda^R_2)^2 - c_*\lambda^R_2\big)\, e^{-(\lambda^R_2 - \lambda_*)z} \\ & \quad - M^{c_*}_2\,\left(d_3(\lambda^R_2 + \epsilon^{c_*}_2)^2 - c_*(\lambda^R_2 + \epsilon^{c_*}_2)\right) e^{-(\lambda^R_2 - \lambda_* + \epsilon^{c_*}_2)z}\Big] + L^{c_*}_1[z - \sqrt{z^{c_*}_2}\sqrt{z}\,]\Big\} \\ & \geq \gamma e^{-\lambda_* z}\left\{L^{c_*}_1[z - \sqrt{z^{c_*}_2}\sqrt{z}\,] - M^{c_*}_2\left[d_3(\lambda^R_2 + \epsilon^{c_*}_2)^2 - c_*(\lambda^R_2 + \epsilon^{c_*}_2)\right] e^{-(\lambda^R_2 - \lambda_* + \epsilon^{c_*}_2)z}\right\} \geq 0. \end{aligned}$$

Thus, for $c = c_*$, $R^{c_*}_-$ satisfies (2.17) for $z > z^{c_*}_4$. This completes the proof of this lemma. □

Remark 2.7. Now we have constructed a pair of super-/sub-solution triples $(S^c_\pm, I^c_\pm, R^c_\pm)$ for the system (1.3). In the coming sections, we will show that the traveling waves (S^c, I^c, R^c) are sandwiched between this pair of super-/sub-solutions, and that for a given small positive ϵ, up to a multiplicative constant, the function R^c is sandwiched between $e^{-(\max\{\lambda_1, c/d_3\}+\epsilon)z}$ and $e^{-(\min\{\lambda_1, c/d_3\}-\epsilon)z}$ as $z \to +\infty$. It is worth remarking that for the case that $\lambda_1 < c/d_3$ (e.g., we can assume that $d_3 < 2d_2/(1+\beta\tau)$) and $c > c_*$, there exists a small $\hat{\epsilon}^c_2 \in (0, \min\{\epsilon^c_1, c/d_3 - \lambda_1\})$ with ϵ^c_1 defined in Sect. 2.1.5 and $\hat{z}^c_3 > 0$ such that it holds

$$\frac{d_3(\lambda_1 + \hat{\epsilon}^c_2)^2 - c(\lambda_1 + \hat{\epsilon}^c_2)}{d_3\lambda_1^2 - c\lambda_1} e^{-\hat{\epsilon}^c_2 z} - M^c_1 e^{-\epsilon^c_1 z} > 0, \;\; z \geq \hat{z}^c_3.$$

We can further define

$$\hat{R}^c_+(z) = \frac{\gamma}{c\lambda_1 - d_3\lambda_1^2} \cdot e^{-\lambda_1 z}, \; z \in \mathbb{R}; \quad \hat{R}^c_-(z) = \begin{cases} \dfrac{\gamma}{c\lambda_1 - d_3\lambda_1^2}(1 - \hat{M}^c_2 e^{-\hat{\epsilon}^c_2 z})e^{-\lambda_1 z}, & z > \hat{z}^c_4, \\ \hat{\Lambda}^c, & z \leq \hat{z}^c_4, \end{cases}$$

where $\hat{M}^c_2 = e^{\hat{\epsilon}^c_2 \hat{z}^c_3}$, and $\hat{z}^c_4 (> \hat{z}^c_3)$ and $\hat{\Lambda}^c$ are constants taken as similar to M^c_2, z^c_4 and Λ^c of R^c_- in Sect. 2.1.6. Moreover, one can check that for $\lambda_1 < c/d_3$, $(S^c_\pm, I^c_\pm, \hat{R}^c_\pm)$ are super-/sub-solutions for system (1.3) with $c > c_*$, which would in turn imply that I^c and R^c have the same decaying rates $\mathcal{O}(e^{-\lambda_1 z})$ as $z \to +\infty$ (see also [29] for $\tau = 0$). Finally, we note that this modified pair of super-/sub-solutions only works for $\lambda_1 < c/d_3$.

2.2. Approximating wave solutions (S_n^c, I_n^c, R_n^c) on the finite internal $[-n, n]$

In this subsection, for each $n \in \mathbb{N}$, we consider the approximated wave solutions problem on the finite interval $[-n, n]$. For this, consider the following truncated problem:

$$\begin{aligned}
&d_1 S''(z) + cS'(z) - \frac{\beta I(z+c\tau)S(z)}{S(z)+I(z)+R(z)} = 0,\\
&d_2 I''(z) + cI'(z) - (\gamma+\delta)I(z) + \frac{\beta I(z+c\tau)S(z)}{S(z)+I(z)+R(z)} = 0, \quad z \in (-n, n)\\
&d_3 R''(z) + cR'(z) + \gamma I(z) = 0,
\end{aligned} \tag{2.18}$$

with the boundary conditions

$$(S, I, R)(-n) = (S_-^c, I_-^c, R_-^c)(-n), \;\; (S, I, R)(n) = (S_-^c, I_-^c, R_-^c)(n). \tag{2.19}$$

Here we set $n > z_4^c$. Note that for $n > z_4^c$, we have

$$(S_-^c, I_-^c, R_-^c)(-n) = (0, 0, \Lambda^c) \;\text{ and }\; (S_-^c, I_-^c, R_-^c)(n) > (0, 0, 0).$$

Next, for any fixed $c \geq c_*$, we introduce a closed and convex space

$$\begin{aligned}
\Gamma_{c,n} = &\Big\{(S, I, R) \in C(\mathbb{R})^3 : S_-^c \leq S \leq S_\infty,\; I_-^c \leq I \leq I_+^c \text{ and } R_-^c \leq R \leq R_+^c\Big\}\\
&\bigcap \Big\{(S, I, R) \in C(\mathbb{R})^3 : (S, I, R)(\pm z) = (S_-^c, I_-^c, R_-^c)(\pm n) \;\text{ for } z \geq n\Big\}
\end{aligned}$$

in the Banach space $C(\mathbb{R})^3 = C(\mathbb{R}) \times C(\mathbb{R}) \times C(\mathbb{R})$ equipped with the norm $\|(S, I, R)\|_{C(\mathbb{R})^3} = \|S\|_{C(\mathbb{R})} + \|I\|_{C(\mathbb{R})} + \|R\|_{C(\mathbb{R})}$. Now consider the map $\mathcal{F}$ defined by

$$\mathcal{F} :\; \Gamma_{c,n} \ni (S_0, I_0, R_0) \longrightarrow C(\mathbb{R})^3 \ni (S, I, R),$$

where the triple of functions (S, I, R) solves the boundary value problem on $[-n, n]$,

$$d_1 S''(z) + cS'(z) - \frac{\beta I_0(z+c\tau)}{S(z)+I_0(z)+R_0(z)} S(z) = 0, \tag{2.20}$$

$$d_2 I''(z) + cI'(z) - (\gamma+\delta)I(z) + \frac{\beta S_0(z) I_0(z+c\tau)}{S_0(z)+I(z)+R_0(z)} = 0, \tag{2.21}$$

$$d_3 R''(z) + cR'(z) + \gamma I_0(z) = 0 \tag{2.22}$$

with the boundary conditions

$$(S, I, R)(-n) = (S_-^c, I_-^c, R_-^c)(-n), \;\; (S, I, R)(n) = (S_-^c, I_-^c, R_-^c)(n) \tag{2.23}$$

and for each $z \geq n$, (S, I, R) satisfies $(S, I, R)(\pm z) = (S_-^c, I_-^c, R_-^c)(\pm n)$.

One can observe that each fixed point of the mapping $\mathcal{F}$ is a solution of truncated problem (2.18) and (2.19). Hence, to solve truncated problem (2.18) and (2.19), it suffices to show that the mapping $\mathcal{F}$ has a fixed point. We will employ the Schauder fixed point theorem to establish this.

2.2.1. $\mathcal{F}$ maps $\Gamma_{c,n}$ into itself. In this subsection, we will show that the mapping $\mathcal{F}$ is well defined as stated in the following lemma.

Lemma 2.8. *For each $c \geq c_*$ and $n > z_4^c$, one has $\mathcal{F}(\Gamma_{c,n}) \subseteq \Gamma_{c,n}$.*

Precisely, for a given $(S_0, I_0, R_0) \in \Gamma_{c,n}$, we will prove that problem (2.20)–(2.23) admits a unique solution $(S, I, R) \in \Gamma_{c,n}$. Since each equation in system (2.20)–(2.22) is decoupled from the others, it suffices to establish the existence and uniqueness of solutions for each equation in system (2.20)–(2.22) with the corresponding boundary conditions. We divide the proof into the following three lemmas: Lemmas 2.9, 2.10, and 2.11.

Lemma 2.9. *For each $c \geq c_*$ and $n > z_4^c$, Eq. (2.22) with the boundary condition $R(\pm n) = R_-^c(\pm n)$ admits a unique solution R. Moreover, the solution R satisfies*

$$R_-^c(z) \leq R(z) \leq R_+^c(z) \ \text{ for } z \in [-n, n].$$

Proof. Firstly, the existence and uniqueness of R follow from the standard ODE theory. Since $R(\pm n) \geq 0$, the strong maximum principle asserts that $R(z) > 0$ in $(-n, n)$.

Secondly, we prove that $R \geq R_-^c$ on $[-n, n]$. To see this, due to $I_0 \geq I_-^c$ on $[-n, n]$, we have

$$d_3 R''(z) + cR'(z) + \gamma I_-^c(z) \leq d_3 R''(z) + cR'(z) + \gamma I_0^c(z) = 0, \ \ z \in (-n, n).$$

Set $w_R^- := R - R_-^c$. Then the above inequality and (2.17) give $d_3(w_R^-)'' + c(w_R^-)' \leq 0$ on $(-n, n)\backslash\{z_4^c\}$ for $c \geq c_*$. In addition, $w_R^-(\pm n) = 0$ by the boundary conditions (2.23). Note that $R \in C^2((-n, n))$ due to $I_0 \in C([-n, n])$. From the construction of R_-^c, $R_-^c \in C^1((-n, n)) \cap C^2\big((-n, n)/\{z_4^c\}\big)$. Thus, we have $w_R^- \in C^1\big((-n, n)\big) \cap C^2((-n, n)/\{z_4^c\}\big)$. Together with the comparison lemma from [27, Theorem 4.1], it follows that $w_R^- \geq 0$ on $[-n, n]$, and hence that $R \geq R_-^c$ on $[-n, n]$.

Thirdly, we show that $R_+^c \geq R$ on $[-n, n]$. Let $w_R^+ := R_+^c - R$ on $[-n, n]$. Recall that $I_+^c \geq I_0$ on $[-n, n]$. Then for $c \geq c_*$, Lemma 2.3 asserts that $w_R^+(z)$ satisfies $d_3(w_R^+)'' + c(w_R^+)' \leq 0$ in $(-n, n)$. Further, conditions (2.23) imply that $w_R^+(\pm n) > 0$. Again, from the comparison lemma in [27, Theorem 4.1], it follows that $w_R^+ \geq 0$ on $[-n, n]$. This completes the proof of this lemma. □

Lemma 2.10. *For each $c \geq c_*$ and $n > z_4^c$, Eq. (2.21) with the boundary condition $I(\pm n) = I_-^c(\pm n)$ admits a unique solution I. Moreover, the solution I satisfies*

$$I_-^c(z) \leq I(z) \leq I_+^c(z) \ \text{ for } z \in [-n, n]. \tag{2.24}$$

Proof. The proof develops into three steps.

Step 1: Existence of solutions. For this, consider the boundary value problem

$$\begin{aligned} \mathbf{L}[I] &= d_2 I'' + f(z, I, I') \\ &= d_2 I'' + cI' + g(z, I) \\ &:= d_2 I'' + cI' + \Big[\frac{\beta S_0(z) I_0(z + c\tau)}{S_0(z) + I(z) + R_0(z)} - (\gamma + \delta) I\Big] = 0, \ \ z \in (-n, n), \\ I(-n) &= I_-^c(-n), \ \ I(n) = I_-^c(n). \end{aligned} \tag{2.25}$$

Recall from (2.3) that $I_+^c(z) \leq L_0$ for $z \in \mathbb{R}$. Also note that $S_0(z) \leq S_+^c(z) = S_\infty$ for $z \in \mathbb{R}$. Then the argument for Lemma 2.2 gives that

$$\mathbf{L}[L_0] = -(\gamma + \delta) L_0 + \frac{\beta S_0(z) I_0(z + c\tau)}{S_0(z) + L_0 + R_0(z)} < 0 \text{ in } (-n, n).$$

Note that $\mathbf{L}[0] = \frac{\beta S_0(z) I_0(z + c\tau)}{S_0(z) + R_0(z)} \geq 0$ for $z \in (-n, n)$. Hence, L_0 and 0 are super-solution and sub-solution of problem (2.25), respectively. Next, since (S_0, I_0, R_0) lies between the super-/sub-solution pairs $(S_\pm^c, I_\pm^c, R_\pm^c)$, there exists $c_1 > 0$ such that $|f(z, I, y)| \leq h(|y|) := c_1(1 + |y|)$ for any $I \in [0, L_0]$ and $y \in \mathbb{R}$. One can verify that $\int_{L_0/(2n)}^{+\infty} s/h(s)\mathrm{d}s = +\infty$. Thus, $f(z, I, I')$ satisfies the Nagumo's condition on $[-n, n]$ relative to the pair of super-/sub-solutions $(0, L_0)$ (see [26]). Taken together, the existence theorem via super-/sub-solutions [2, Theorem 1.5.1] asserts that problem (2.25) admits a solution $I \in C^2([-n, n])$ such that $0 \leq I(z) \leq L_0$ for $z \in [-n, n]$.

Step 2: Uniqueness. First, one can verify that any solution $I \in C([-n, n])$ of problem (2.25) satisfies $f_1(z) := S_0(z) + I(z) + R_0(z) > 0$ for $z \in [-n, n]$. This follows from continuity of f_1 and $f_1(\pm n) > 0$. Note that the function $g(z, I)$ defined in (2.25) satisfies

$$\frac{\partial g}{\partial I}(z, I) = -\frac{\beta S_0(z) I_0(z + c\tau)}{(S_0(z) + I + R_0(z))^2} - (\gamma + \delta) < 0, \ z \in [-n, n].$$

This, together with the comparison argument, implies the uniqueness of solutions of problem (2.25).

Step 3: Verification of (2.24). First we establish that $I \ge I_-^c$ on $[-n,n]$. Indeed, using the fact that $S_0 \ge S_-^c$, $I_0 \ge I_-^c$ and $R_+^c \ge R_0$ on $[-n,n]$, it follows from the first equation of (2.25) that

$$d_2 I''(z) + cI'(z) - (\gamma+\delta)I(z) + \frac{\beta S_-^c(z) I_-^c(z+c\tau)}{S_-^c(z)+I(z)+R_+^c(z)} \le 0,\ z \in (-n,n).$$

Let $w_I^- := I - I_-^c$ on $[-n,n]$. Then from Eq. (2.11) in Lemma 2.5, it follows that for each $c \ge c_*$, the function w_I^- satisfies

$$\begin{aligned} d_2(w_I^-)''(z) + c(w_I^-)'(z) - (\gamma+\delta)w_I^-(z) - g^-(z)w_I^-(z) \le 0, \quad z \in [z_2^c, n], \\ g^-(z) = \frac{\beta S_-^c(z) I_-^c(z+c\tau)}{\big(S_-^c(z)+I(z)+R_+^c(z)\big)\big(S_-^c(z)+I_-^c(z)+R_+^c(z)\big)} \ge 0, \quad z \in [-n,n]. \end{aligned}$$

Note that $w_I^-(n) = 0$ and $w_I^-(z_2^c) = I(z_2^c) - I_-^c(z_2^c) = I(z_2^c) \ge 0$. Taken together, using the maximum principle we have $w_I^- \ge 0$ on $[z_2^c, n]$, and so $I \ge I_-^c$ on $[z_2^c, n]$. Since $I_-^c \equiv 0$ on $[-n, z_2^c]$, it follows that $I \ge I_-^c$ on $[-n,n]$. Thus, the leftmost inequality in (2.24) holds.

Now we show that $I \le I_+^c$ on $[-n,n]$. Indeed, using the fact that $S_0 \le S_+^c$ and $I_0 \le I_+^c$ on $[-n,n]$, it follows from the first equation of (2.25) that

$$d_2 I''(z) + cI'(z) - (\gamma+\delta)I(z) + \frac{\beta S_+^c(z) I_+^c(z+c\tau)}{S_+^c(z)+I(z)+R_0(z)} \ge 0,\ z \in (-n,n).$$

Then using Lemma 2.2, and employing similar arguments as for w_R^- in Lemma 2.9, we can conclude that $I_+^c \ge I$ on $[-n,n]$ and $c \ge c_*$. The proof of this lemma is thus completed. □

Lemma 2.11. *For each $c \ge c_*$ and $n > z_4^c$, Eq.* (2.20) *with the boundary condition $S(\pm n) = S_-^c(\pm n)$ admits a unique solution S. Moreover, the solution S satisfies $S' > 0$ in $(-n,n)$, and*

$$S_-^c(z) \le S(z) \le S_+^c(z) \ \ \textit{for } z \in [-n,n]. \tag{2.26}$$

Proof. The proof is divided into two parts.

Step 1: Existence and uniqueness of solutions. For this, consider the initial value problem

$$\begin{aligned} & d_1 S''(z) + cS'(z) - \frac{\beta S(z) I_0(z+c\tau)}{S(z)+I_0(z)+R_0(z)} = 0, \\ & S(-n) = 0, \quad S'(-n) = m \ge 0. \end{aligned} \tag{2.27}$$

Recall that $I_0, R_0 \ge 0$ in $(-n,n)$. By applying the standard ODE theories, we conclude that

(i) if $m = 0$, problem (2.27) admits a unique solution $S(\cdot;m) \equiv 0$ on $[-n,n]$;

(ii) if $m > 0$, there exists a $\delta_0 > 0$ such that problem (2.27) admits a unique solution $S(\cdot;m)$ defined on $[-n, -n+\delta_0)$. Moreover, multiplying the first equation of problem (2.27) by $\mathrm{e}^{-(c/d_1)z}$, and then integrating the resulting equation from $-n$ to z, we obtain

$$S'(z;m) = m\mathrm{e}^{-\frac{c}{d_1}(z+n)} + \frac{\beta}{d_1}\int_{-n}^{z} \frac{I_0(y+c\tau)S(y;m)}{S(y;m)+I_0(y)+R_0(y)}\,\mathrm{e}^{\frac{c}{d_1}(y-z)}\mathrm{d}y \tag{2.28}$$

for $z \in [-n, -n+\delta_0)$. From Eq. (2.28), we see that

$$0 < m\mathrm{e}^{-\frac{c}{d_1}(z+n)} \le S'(z;m) \le m + \frac{\beta}{d_1}\int_{-n}^{n} I_0(y+c\tau)\mathrm{d}y \tag{2.29}$$

as long as $S(z;m) > 0$. Hence, for each $m > 0$, (2.27) admits a unique solution $S(\cdot;m)$ on $[-n,n]$. Moreover, $S(\cdot;m)$ satisfies (2.29) in $(-n,n)$. Next, integrating the leftmost inequality of (2.29) from $-n$ to n, we find that $S(n;m) \ge \frac{md_1}{c}(1-\mathrm{e}^{-2cn/d_1})$ for $m > 0$, and so $S(n;m) \to +\infty$ as $m \to +\infty$.

From the continuity of $S(n;m)$ in m, we can choose a $m^*>0$ such that $S(n;m^*)=S^c_-(n)>0$. Thus, $S(\cdot):=S(\cdot;m^*)$ is a solution of Eq. (2.20) satisfying $S(\pm n)=S^c_-(\pm n)$. The uniqueness of solutions follows from (2.28) and that the function $s/(s+I_0(y)+R_0(y))$ is increasing in $s\geq 0$.

Step 2: Verification of (2.26). Since $S(\cdot)=S(\cdot;m^*)$ satisfies (2.29) in $(-n,n)$, we have $S(z)\leq S(n)=S^c_-(n)\leq S_\infty=S^c_+(z)$ for $z\in[-n,n]$. Now we show that $S\geq S^c_-$ on $[-n,n]$. Since $S(z)\geq 0=S^c_-(z)$ for $z\in[-n,z^c_1]$, it remains to show that $S\geq S^c_-$ on $[z^c_1,n]$. Note that S satisfies

$$d_1S''(z)+cS'(z)=\frac{\beta S(z)I_0(z+c\tau)}{S(z)+I_0(z)+R_0(z)}\leq\frac{\beta S(z)I^c_+(z+c\tau)}{S(z)+I_0(z)+R_0(z)},\quad z\in[z^c_1,n].$$

Together with the inequality (2.7) in Lemma 2.4 (with $I(z)=I_0(z)$ and $R(z)=R_0(z)$), the function $w^-_S(z)=S(z)-S^c_-(z)$ satisfies

$$d_1(w^-_S)''(z)+c(w^-_S)'(z)-h(z)w^-_S(z)\leq 0,\ \ z\in(z^c_1,n),\ c\geq c_*,$$
$$h(z)=\frac{\beta I^c_+(z+c\tau)(I_0(z)+R_0(z))}{\big(S(z)+I_0(z)+R_0(z)\big)\big(S^c_-(z)+I_0(z)+R_0(z)\big)}\geq 0,\ z\in[-n,n].$$

Since $w^-_S(z^c_1)>0$ and $w^-_S(n)=0$, the maximum principle then asserts that $w^-_S\geq 0$ on $[z^c_1,n]$, and so $S(z)\geq S^c_-(z)$ on $[-n,n]$. The proof of this lemma is completed. □

2.2.2. The precompactness and continuity of $\mathcal{F}$.

Lemma 2.12. *The mapping $\mathcal{F}:\Gamma_{c,n}\to\Gamma_{c,n}$ is continuous and precompact for each $c\geq c_*$ and $n>z^c_4$.*

The proof of this lemma is standard, and so is omitted.

2.2.3. Approximating wave solutions via Schauder fixed point theorem. The following lemma is an application of the Schauder fixed point theorem via the aid of Lemmas 2.8 and 2.12. Note that we attach the parameters (c,n) to the solution of problem (2.18) and (2.19) to emphasize its dependence on (c,n).

Lemma 2.13. *For any given $c\geq c_*$ and $n>z^c_4$, problem* (2.18) *and* (2.19) *admits a nonnegative solution (S^c_n,I^c_n,R^c_n) on $[-n,n]$. Furthermore, (S^c_n,I^c_n,R^c_n) satisfies*

$$S^c_-\leq S^c_n\leq S_\infty,\ \ I^c_-\leq I^c_n\leq I^c_+,\ \ R^c_-\leq R^c_n\leq R^c_+\ \ \text{on } [-n,n],\text{ and } (S^c_n)'>0\ \ \text{in } (-n,n).\tag{2.30}$$

2.3. Proof of Theorem 1.2(II)

Now we are in a position to prove Theorem 1.2(II). The proof is divided into two parts. First, we show that for each $c\geq c_*$ and $S_\infty>0$, there exists a solution (S^c,I^c,R^c) of system (1.3) satisfying the conditions $(S^c,I^c,R^c)(+\infty)=(S_\infty,0,0)$ (see Lemma 2.14). Then we prove that such a solution (S^c,I^c,R^c) satisfies the boundary conditions at $-\infty$ specified in Theorem 1.2(II) (see Lemma 2.15).

2.3.1. A solution $(S^c,I^c,R^c)(z)$ of system (1.3) with the conditions $(S^c,I^c,R^c)(+\infty)=(S_\infty,0,0)$.

Lemma 2.14. *Assume that $\mathcal{R}_0>1$. Then for any given $S_\infty>0$ and $c\geq c_*$, system (1.3) admits a nonnegative solution (S^c,I^c,R^c) such that $(S^c,I^c,R^c)(+\infty)=(S_\infty,0,0)$. Moreover, $((S^c)',(I^c)',(R^c)')(+\infty)=(0,0,0)$, and the following asymptotical behavior holds:*

$$I^c(z)=\mathcal{O}(\mathrm{e}^{-\lambda_1 z})\ (\textit{resp. }\mathcal{O}(z\mathrm{e}^{-\lambda_* z})),\ \textit{as } z\to+\infty,\ \textit{for } c>c_*\ (\textit{resp. for } c=c_*).\tag{2.31}$$

Proof. The proof consists of two parts.

Step 1: Existence of solutions. Given $c\geq c_*$, let $(S^c_n(z),I^c_n(z),R^c_n(z))$ be the solution of the problem (2.18) and (2.19) for each $n\in\mathbb{N}$ and $n>z^c_4$. For any fixed $K\in\mathbb{N}$ and $K>z^c_4$, from Lemma 2.12 it follows that the sequences $\{S^c_n(z)\}_{n\geq K}$, $\{I^c_n(z)\}_{n\geq K}$ and $\{R^c_n(z)\}_{n\geq K}$ are uniformly bounded on $[-K,K]$, which, together with comparison lemma in [10, Lemma A.2] and system (2.18), implies that the sequences $\{\big((S^c_n)',(I^c_n)',(R^c_n)'\big)\}_{n\geq K}$, $\{\big((S^c_n)'',(I^c_n)'',(R^c_n)''\big)\}_{n\geq K}$, and $\{\big((S^c_n)''',(I^c_n)''',(R^c_n)'''\big)\}_{n\geq K}$

are uniformly bounded on $[-K, K]$. Taken together, using the Arzela–Ascoli theorem and diagonal arguments, we can choose a subsequence, still denoted by $\{(S_n^c, I_n^c, R_n^c)\}_{n\in\mathbb{N}}$ for simplicity, such that $(S_n^c, I_n^c, R_n^c) \to (S^c, I^c, R^c)$ as $n \to +\infty$ in $C^2_{loc}(\mathbb{R})$ for some functions S^c, I^c and R^c in $C^2(\mathbb{R})$. Thus, (S^c, I^c, R^c) is a solution of (1.3) on $\mathbb{R}$. Now, using the asymptotical behavior of $(S^c_\pm, I^c_\pm, R^c_\pm)(z)$ as $z \to +\infty$, it follows from (2.30) that $(S^c, I^c, R^c)(+\infty) = (S_\infty, 0, 0)$ and the relation (2.31) holds. Moreover, we have $(S^c)'(z) \geq 0$ on $\mathbb{R}$.

Step 2: Show that $((S^c)', (I^c)', (R^c)')(+\infty) = (0, 0, 0)$. To begin with, integrating the S-equation of (1.3) from 0 to z, we see that

$$d_1[(S^c)'(z) - (S^c)'(0)] + c(S^c(z) - S^c(0)) = \int_0^z \frac{\beta I^c(y + c\tau)S^c(y)}{S^c(y) + I^c(y) + R^c(y)} \mathrm{d}y, \tag{2.32}$$

which infers that $(S^c)'(+\infty)$ exists if and only if the integral

$$\int_0^{+\infty} \frac{\beta I^c(y + c\tau)S^c(y)}{S^c(y) + I^c(y) + R^c(y)} \mathrm{d}y \tag{2.33}$$

converges. If the integral (2.33) diverges, then it must be $+\infty$. Together with (2.32) and the fact that $S^c(+\infty) = S_\infty < +\infty$, it follows $(S^c)'(+\infty) = +\infty$. This in turn implies $S^c(+\infty) = +\infty$, a contradiction. Hence, the integral (2.33) converges. Together with (2.32), this suggests that $(S^c)'(+\infty)$ exists, and so $(S^c)'(+\infty) = 0$. Summing the equations in (1.3), it follows that

$$d_1(S^c)'' + d_2(I^c)'' + c[(I^c)' + (S^c)'] = (\gamma + \delta)I^c. \tag{2.34}$$

Next, we integrate (2.34) from 0 to z to deduce that the following equality holds for $z \in \mathbb{R}$,

$$d_1[(S^c)'(z) - (S^c)'(0)] + d_2[(I^c)'(z) - (I^c)'(0)] + c[I^c(z) - I^c(0) + S^c(z) - S^c(0)] = (\gamma + \delta) \int_0^z I^c(y)\mathrm{d}y.$$

Since $I^c(z) \to 0$ exponentially fast as $z \to +\infty$, the integral $\int_0^{+\infty} I^c(y)\mathrm{d}y$ converges. Then taking the limit in the above equation as $z \to +\infty$, the limit $(I^c)'(+\infty) := \lim_{z\to+\infty}(I^c)'(z)$ exists, and $(I^c)'(+\infty) = 0$ due to $I^c(+\infty) = 0$. Finally, integrating the R-equation of (1.3) from 0 to z and using similar arguments as above, one can deduce $(R^c)'(+\infty) = 0$. The proof is thus completed. □

2.3.2. The asymptotical behavior of $(S^c, I^c, R^c)(z)$ as $z \to -\infty$. In the following lemma, we discuss the asymptotical behavior of $(S^c, I^c, R^c)(z)$ as $z \to -\infty$.

Lemma 2.15. *For each $c \geq c_*$, the solution (S^c, I^c, R^c) satisfies the following:*

(i) *The limit $(S^c, I^c, R^c)(-\infty)$ exists and $I^c(-\infty) = 0$.*
(ii) $((S^c)', (I^c)', (R^c)')(-\infty) = (0, 0, 0)$.
(iii) $(S^c)' > 0$ *and* $(R^c)' < 0$ *in* $\mathbb{R}$.
(iv) *The relation* (1.5) *holds.*

Proof. We divide the proof into four steps.

Step 1: Prove that the limits $S_{-\infty} := S^c(-\infty) \geq 0$ *and* $(S^c)'(-\infty) = 0$, *and* $(S^c)' > 0$ *in* $\mathbb{R}$. To see this, we first recall that $(S^c)' \geq 0$ and $0 \leq S^c \leq S_\infty$ on $\mathbb{R}$. Thus, the limit $S_{-\infty} := S^c(-\infty)$ exists and is nonnegative. Integrating the S-equation of system (1.3) from z to $+\infty$ and then rearranging the resulting identity, we have

$$0 \leq d_1(S^c)'(z) = c(S_\infty - S^c(z)) - \int_z^{+\infty} \frac{\beta I(y + c\tau)S^c(y)}{S^c(y) + I^c(y) + R^c(y)} \mathrm{d}y, \ z \in \mathbb{R}, \tag{2.35}$$

which implies that

$$cS_\infty \geq cS^c(z) + \int_z^{+\infty} \frac{\beta I(y+c\tau)S^c(y)}{S^c(y)+I^c(y)+R^c(y)}\mathrm{d}y,\ z\in\mathbb{R}.$$

Thus, by taking the limit in Eq. (2.35) as $z\to-\infty$ and using the fact that $S^c(-\infty)$ exists, we have the limit $(S^c)'(-\infty)=0$. Next, we show that $(S^c)'(z)>0$ in $\mathbb{R}$. In fact, multiplying the S-equation of (1.3) by $\mathrm{e}^{(c/d_1)z}$, and then integrating the resulting equation from $-\infty$ to z, we obtain

$$(S^c)'(z) = \frac{\beta}{d_1}\cdot \int_{-\infty}^{z} \frac{I(y+c\tau)S(y)}{S(y)+I(y)+R(y)}\mathrm{e}^{\frac{c}{d_1}(y-z)}\,\mathrm{d}y,\ z\in\mathbb{R},$$

which implies $(S^c)'(z)>0$ for $z\in\mathbb{R}$. This completes the proof of this step.

Step 2: Show that the limits $I^c(-\infty)=(I^c)'(-\infty)=0$. Integrating (2.34) from z to $+\infty$ and using Step 1 and Step 2 in Lemma 2.14, we have

$$d_2(I^c)'(z)+cI^c(z) = -(\gamma+\delta)\int_z^{+\infty} I^c(y)\mathrm{d}y - d_1(S^c)'(z)+c[S_\infty - S^c(z)],\quad z\in\mathbb{R}. \tag{2.36}$$

Since $I^c(z)\to 0$ exponentially fast as $z\to+\infty$, the integral $\int_z^{+\infty} I^c(y)\mathrm{d}y$ exists. By (2.36), the limits $S^c(-\infty)$ and $(S^c)'(-\infty)$, and the relation $0\leq I^c\leq L_0$ on $\mathbb{R}$, we have that the integral $\int_{\mathbb{R}} I^c(y)\mathrm{d}y$ converges, which implies $\liminf_{z\to-\infty} I^c(z)=0$. Now by taking the limit of (2.36) along a sequence $\{z_n\}_{n\in\mathbb{N}}$ with $z_n\to-\infty$ as $n\to\infty$, we can exclude $\limsup_{z\to-\infty} I^c(z)>0$. Hence, we have $I^c(-\infty)=0$. Finally, from Eq. (2.36) and the convergence of the integral $\int_{\mathbb{R}} I^c(y)\mathrm{d}y$, the limit $\lim_{z\to-\infty}(I^c(z)+(I^c)'(z))$ exists, and so $(I^c)'(-\infty)=0$. This proves the assertion of this step.

Step 3: Prove that the limit $R_{-\infty}:=R^c(-\infty)$ *exists,* $(R^c)'(-\infty)=0$, *and* $(R^c)'<0$ *in* $\mathbb{R}$. We first claim that

$$\lim_{z\to-\infty} \mathrm{e}^{\frac{c}{d_3}z}(R^c)'(z)=0. \tag{2.37}$$

Indeed, integrating the R-equation of (1.3) from z to $+\infty$ and using the fact that $R^c(+\infty)=(R^c)'(+\infty)=0$, it follows that

$$(R^c)'(z)+\frac{c}{d_3}R^c(z) = \frac{\gamma}{d_3}\int_z^{+\infty} I^c(y)\mathrm{d}y>0\ \text{ for } z\in\mathbb{R}. \tag{2.38}$$

Recall from Sect. 2.1.3 that for each $c\geq c_*$, $R_+^c(z)=\mathcal{O}(\mathrm{e}^{-(\lambda_1^R)z})$ as $z\to-\infty$, and $\lambda_1^R<c/d_3$. Then we have $\mathrm{e}^{(c/d_3)z}R^c(z)\leq \mathrm{e}^{(c/d_3)z}R_+^c(z)\to 0$ as $z\to-\infty$. Recall from *Step 2* that the integral $\int_{\mathbb{R}} I^c(y)\mathrm{d}y$ converges for $c\geq c_*$. Taken together, (2.38) implies that $\lim_{z\to-\infty}\mathrm{e}^{(c/d_3)z}(R^c)'(z)$ exists and equals 0. Thus, the assertion of this claim is proved.

Next, multiplying the R-equation of system (1.3) by $\mathrm{e}^{(c/d_3)z}$ and then integrating the resulting equation from $-\infty$ to z, we have

$$(R^c)'(z) = -\frac{\gamma}{d_3}\mathrm{e}^{-\frac{c}{d_3}z}\int_{-\infty}^{z} \mathrm{e}^{\frac{c}{d_3}y}I^c(y)\mathrm{d}y<0\ \text{ for } z\in\mathbb{R}, \tag{2.39}$$

where we have used (2.37). Recall from (2.3) that for each $c \geq c_*$, $0 < I_+^c(z) \leq L_0$ for $z \in \mathbb{R}$. Hence, the integral on the right-hand side of (2.39) converges. Moreover, from (2.39) it follows that

$$|(R^c)'(z)| \leq \frac{\gamma}{d_3} \int_{-\infty}^{z} I^c(y)\mathrm{d}y \leq \frac{\gamma}{d_3} \int_{\mathbb{R}} I^c(y)\mathrm{d}y, \quad z \in \mathbb{R} \tag{2.40}$$

and so the limit $R^c(-\infty)$ either exists as a positive number or equals $+\infty$. The second case is excluded by the fact that the integral $\int_{\mathbb{R}} I^c(y)\mathrm{d}y$ exists, and (2.40) and (2.38). Then using (2.38) again it follows that $(R^c)'(-\infty)$ exists, and so $(R^c)'(-\infty) = 0$. This step is thus proved.

Step 4: Verification of the relation (1.5). First, taking the limit in (2.35) as $z \to -\infty$ and using $(S^c)'(-\infty) = 0$, the rightmost equality in (1.5b) holds. Second, taking the limit in (2.36) as $z \to -\infty$ and using $I^c(-\infty) = (I^c)'(-\infty) = 0$, the leftmost equality in (1.5b) holds. Finally, taking the limit in (2.38) as $z \to -\infty$ and using $(R^c)'(-\infty) = 0$, we obtain $R^c(-\infty) = (\gamma/c)\int_{\mathbb{R}} I^c(y)\mathrm{d}y$. This, together with (1.5b), yields (1.5a). The proof of this lemma is thus completed. □

3. Nonexistence of traveling wave

In this section, we will establish Theorems 1.1 and 1.2(I). The proof is divided into two cases: (i) $\mathcal{R}_0 > 1$ and $c \in (0, c_*)$; and (ii) $\mathcal{R}_0 \leq 1$ and $c \in \mathbb{R}$.

We first observe that if (S, I, R) is a nonnegative solution of (1.3) and (1.4), then I satisfies

$$I'(\pm\infty) = I''(\pm\infty) = 0. \tag{3.1}$$

This follows from the limits $I(\pm\infty) = 0$ and the following integral representation of the species I

$$I(z) = \frac{1}{\rho} \int_{-\infty}^{z} \mathrm{e}^{\lambda_-(z-y)} \frac{\beta S(y)I(y+c\tau)}{S(y)+I(y)+R(y)}\mathrm{d}y + \frac{1}{\rho} \int_{z}^{+\infty} \mathrm{e}^{\lambda_+(z-y)} \frac{\beta S(y)I(y+c\tau)}{S(y)+I(y)+R(y)}\mathrm{d}y,$$

where $\lambda_- < 0 < \lambda_+$ are the solutions of $d_2\lambda^2 + c\lambda - (\gamma+\delta) = 0$ and $\rho = d_2(\lambda_+ - \lambda_-)$.

3.1. The case $\mathcal{R}_0 > 1$ and $c \in (0, c_*)$

Lemma 3.1. *Assume that $\mathcal{R}_0 > 1$ and $c \in (0, c_*)$. Then system* (1.3) *with boundary conditions* (1.4) *has no nonnegative non-trivial solutions.*

Proof. For contradiction, we assume that there exists such a nonnegative non-trivial solution (S, I, R). The strategy of the remaining proof consists of two major parts: First, we modify the idea in [29] to give the decaying rate of $(I(z), R(z))$ for large z, and then use the two-sided Laplace transform suggested in [7] to arrive at a contradiction. The proof consists of four steps.

Step 1: Prove that the integral

$$\mathcal{I}(z) := \int_{z}^{+\infty} I(y)\mathrm{d}y$$

converges for all $z \in \mathbb{R}$. Indeed, by (1.4), we have $\frac{\beta S(z)}{S(z)+I(z)+R(z)} \to \beta$ as $z \to +\infty$. Together with $\beta > \gamma + \delta$, we can find a large $Z_0 > 0$ such that $\frac{\beta S(z)}{S(z)+I(z)+R(z)} > \frac{\beta+\gamma+\delta}{2}$ for all $z \geq Z_0$. By applying this to the I-equation of (1.3) and rearranging the resulting inequality, we have

$$-d_2 I''(z) - cI'(z) \geq \frac{\beta+\gamma+\delta}{2}[I(z+c\tau) - I(z)] + \frac{\beta-(\gamma+\delta)}{2}I(z), \; z \geq Z_0.$$

An integration of this inequality from z to M (with $M > z + c\tau$) gives

$$\begin{aligned} & - d_2[I'(M) - I'(z)] - c[I(M) - I(z)] \\ & \geq \frac{\beta+\gamma+\delta}{2} \int_z^M [I(y+c\tau) - I(y)]\mathrm{d}y + \frac{\beta-(\gamma+\delta)}{2} \int_z^M I(y)\mathrm{d}y \\ & \geq -\frac{\beta+\gamma+\delta}{2} \int_z^{z+c\tau} I(y)\mathrm{d}y + \frac{\beta-(\gamma+\delta)}{2} \int_z^M I(y)\mathrm{d}y \end{aligned}$$

for all $z \geq Z_0$ and $M > z + c\tau$.

Since $I(+\infty) = I'(+\infty) = 0$, the above inequality implies that the integral $\mathcal{I}(z)$ converges for $z \in \mathbb{R}$, and $\mathcal{I}(z)$ satisfies

$$\frac{\beta-(\gamma+\delta)}{2}\mathcal{I}(z) \leq d_2 I'(z) + cI(z) + \frac{\beta+\gamma+\delta}{2} \int_z^{z+c\tau} I(y)\mathrm{d}y, \ z \geq Z_0. \tag{3.2}$$

Step 2: Show that

$$I(z) \leq C_0 \mathrm{e}^{-\lambda_0 z} \quad \text{for } z \geq 0 \text{ and for some } \lambda_0, C_0 > 0. \tag{3.3}$$

Indeed, with the use of Fubini's theorem, we have

$$\begin{aligned} \int_z^{+\infty} \int_x^{x+c\tau} I(y)\mathrm{d}y\mathrm{d}x &= \int_{z+c\tau}^{+\infty} \int_{y-c\tau}^{y} I(y)\mathrm{d}x\mathrm{d}y + \int_z^{z+c\tau} \int_z^{y} I(y)\mathrm{d}x\mathrm{d}y \\ &\leq c\tau \int_{z+c\tau}^{+\infty} I(y)\mathrm{d}y + c\tau \int_z^{z+c\tau} I(y)\mathrm{d}y = c\tau\mathcal{I}(z). \end{aligned}$$

Together with an integration of both sides of (3.2) from $z \geq Z_0$ to $+\infty$, it holds that

$$d_2 I(z) + \frac{\beta-(\gamma+\delta)}{2} \int_z^{+\infty} \mathcal{I}(y)\mathrm{d}y \leq \ c\left[1 + \tau(\frac{\beta+\gamma+\delta}{2})\right]\mathcal{I}(z), \ z \geq Z_0, \tag{3.4}$$

which implies that $\int_z^{+\infty} \mathcal{I}(y)\mathrm{d}y \leq k\mathcal{I}(z)$ for $z \geq Z_0$ with $k = c[2 + \tau(\beta+\gamma+\delta)]/(\beta-\gamma-\delta)$. Since $\mathcal{I}(z)$ is non-increasing in z, we have $\eta\mathcal{I}(z+\eta) \leq \int_z^{z+\eta} \mathcal{I}(y)\mathrm{d}y \leq k\mathcal{I}(z)$ for $z \geq Z_0$ and $\eta > 0$. Choose a large η such that $\eta > 2k$. Then the above inequality implies $\mathcal{I}(z+\eta) \leq \frac{1}{2}\mathcal{I}(z)$ for $z \geq Z_0$. Set $\hat{\mathcal{I}}(z) = \mathrm{e}^{\lambda_0 z}\mathcal{I}(z)$ with $\lambda_0 = \ln 2/\eta > 0$. Then we have $\hat{\mathcal{I}}(z+\eta) \leq \hat{\mathcal{I}}(z)$ for $z \geq Z_0$, and hence, $\hat{\mathcal{I}}(z)$ is bounded as $z \to +\infty$, which, together with (3.4), infers that the inequality (3.3) holds.

Step 3: Prove that for $\hat{\lambda}_0 \in (0, \min\{\lambda_0, d_3/c\})$, *there exists a* $C_1 > 0$ *such that*

$$R(z) \leq C_1 \mathrm{e}^{-\hat{\lambda}_0 z} \quad \text{for } z \in \mathbb{R}. \tag{3.5}$$

Recall that $(S+I+R)(-\infty) = S_{-\infty} + R_{-\infty} > 0$ and $(S+I+R)(+\infty) = S_\infty > 0$. Hence, in view of (3.3), we can deduce that the functions $\mathrm{e}^{\lambda_0 z}I(z)$ and $\mathrm{e}^{\lambda_0 z} \cdot \frac{I(z+c\tau)}{S(z)+I(z)+R(z)}$ are bounded on $\mathbb{R}$. Using $R(+\infty) = 0$ and the variation in constant method, $R(z)$ can be represented by

$$R(z) = \frac{\gamma}{c} \int_z^{+\infty} I(y)\mathrm{d}y + \frac{\gamma}{c} \int_0^z \mathrm{e}^{-\frac{c}{d_3}(z-y)} I(y)\mathrm{d}y + C\mathrm{e}^{-\frac{c}{d_3}z},$$

where C is a positive constant. Choose $\hat{\lambda}_0 \in (0, \min\{\lambda_0, c/d_3\})$. Then using (3.3), we have

$$\begin{aligned} e^{\hat{\lambda}_0 z} R(z) &= \frac{\gamma}{c} e^{\hat{\lambda}_0 z} \int_z^{+\infty} I(y) dy + \frac{\gamma}{c} \int_0^z e^{\hat{\lambda}_0 z} e^{-\frac{c}{d_3}(z-y)} I(y) dy + C e^{-(\frac{c}{d_3} - \hat{\lambda}_0) z} \\ &\le \frac{\gamma}{c} e^{\lambda_0 z} \int_z^{+\infty} I(y) dy + \frac{\gamma}{c} \int_0^z e^{\hat{\lambda}_0 z} e^{-\hat{\lambda}_0 (z-y)} I(y) dy + C e^{-(\frac{c}{d_3} - \hat{\lambda}_0) z} \\ &\le \frac{\gamma}{c} \big[\hat{\mathcal{I}}(z) + C_0 \int_0^z e^{(\hat{\lambda}_0 - \lambda_0) y} dy \big] + C, \ z \ge Z_0, \end{aligned}$$

where C_0 is a constant. This, with the fact that $\hat{\mathcal{I}}(z)$ is bounded for $z \ge 0$, implies that $e^{\hat{\lambda}_0 z} R(z)$ is bounded for $z \ge 0$, which, together with $R(-\infty) = R_{-\infty} > 0$, in turn suggests that (3.5) holds.

Step 4: Final proof. We introduce the two-sided Laplace transform of I defined by $\mathcal{L}(\lambda) = \int_{-\infty}^{+\infty} e^{\lambda \xi} I(\xi) d\xi$. Due to (3.3), $\mathcal{L}(\lambda)$ is well defined for $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \in (0, \lambda_0)$. Now rewrite the I-equation of (1.3) as follows:

$$d_2 I''(z) + c I'(z) + \beta I(z + c\tau) - (\gamma + \delta) I(z) = \ I(z + c\tau) \cdot \frac{\beta \big[I(z) + R(z) \big]}{S(z) + I(z) + R(z)}. \tag{3.6}$$

Multiplying both sides of (3.6) by $e^{\lambda \xi}$ with $\lambda \in (0, \lambda_0)$, and integrating the resulting equation by parts over $\mathbb{R}$, we then have

$$f(\lambda, c) \mathcal{L}(\lambda) = \int_{-\infty}^{+\infty} e^{\lambda \xi} I(\xi + c\tau) \cdot \frac{\beta \big[I(\xi) + R(\xi) \big]}{S(\xi) + I(\xi) + R(\xi)} d\xi. \tag{3.7}$$

Now fix a $\hat{\lambda}_0 \in (0, \min\{\lambda_0, d_3/c\})$ such that (3.5) holds. By the properties of Laplace transform (see [33, p. 58]), it follows that either (i) both sides of (3.7) can be analytically continued to the right half plane of $\mathbb{C}$, or (ii) there exists $\hat{\lambda}_1 > 0$ such that the function $\mathcal{L}$ is analytic for all λ with $\operatorname{Re}\lambda \in (0, \hat{\lambda}_1)$ and is singular at $\lambda = \hat{\lambda}_1$. We shall show that (i) is true. For this, suppose, for contradiction, that (ii) holds. Then, by (3.3), we have $\hat{\lambda}_1 \ge \lambda_0 > \hat{\lambda}_0$. From (3.3) and (3.5), and the fact that $S + I + R$ has a positive lower bound on $\mathbb{R}$, it follows that for all $\lambda \in [0, \hat{\lambda}_0)$, $I(z) + R(z) \sim o(e^{-\lambda z})$ as $z \to +\infty$, and hence, $e^{\lambda z}(I(z) + R(z))/(S(z) + I(z) + R(z))$ is bounded on $\mathbb{R}$. Note that $\int_{-\infty}^{+\infty} e^{\lambda \xi} I(\xi + c\tau) d\xi = e^{-\lambda c\tau} \mathcal{L}(\lambda)$, provided that $\mathcal{L}(\lambda)$ exists. Now fix an $\epsilon \in (0, \hat{\lambda}_0/2)$. Then we find that the integral

$$\int_{-\infty}^{+\infty} e^{(\hat{\lambda}_1 + \epsilon) \xi} I(\xi + c\tau) \cdot \frac{\beta \big(I(\xi) + R(\xi) \big)}{S(\xi) + I(\xi) + R(\xi)} d\xi = \int_{-\infty}^{+\infty} \Big[e^{(\hat{\lambda}_1 - \epsilon) \xi} I(\xi + c\tau) \Big] \cdot \left[e^{2\epsilon \xi} \frac{\beta \big(I(\xi) + R(\xi) \big)}{S(\xi) + I(\xi) + R(\xi)} \right] d\xi$$

converges. Together with (3.7), this in turn implies that $\mathcal{L}(\hat{\lambda}_1 + \epsilon)$ exists, which is a contradiction with the definition of $\hat{\lambda}_1$. Hence, (i) is true.

On the other hand, due to the assumption that $\beta > (\gamma+\delta)$ it follows that $f(\lambda, c) > 0$ for all $\lambda > 0$ and $c \in (0, c_*)$. Then with the use of (3.7), we have

$$0 < \mathcal{L}(\lambda) f(\lambda, c) = \int_{-\infty}^{+\infty} \mathrm{e}^{\lambda\xi} I(\xi + c\tau) \cdot \frac{\beta\big(I(\xi) + R(\xi)\big)}{S(\xi) + I(\xi) + R(\xi)} \mathrm{d}\xi$$
$$\leq \mathcal{L}(\lambda) \cdot \left\{ \beta \mathrm{e}^{-\lambda c\tau} \Big\| \frac{I(\cdot) + R(\cdot)}{S(\cdot) + I(\cdot) + R(\cdot)} \Big\|_{L^\infty(\mathbb{R})} \right\} \leq \beta \mathcal{L}(\lambda),$$

which in turn implies $f(\lambda, c) \leq \beta$ for all $\lambda > 0$ and $c \in (0, c_*)$. However, since $f(\lambda, c) \to +\infty$ as $\lambda \to +\infty$ for any fixed $c \in (0, c_*)$, we thus arrive at a contradiction. Therefore, there are no nonnegative non-trivial solutions of (1.3) and (1.4) if $\mathcal{R}_0 > 1$ and $0 < c < c_*$. This completes the proof of this lemma. □

3.2. The case $\mathcal{R}_0 \in (0, 1]$ and $c \in \mathbb{R}$

Lemma 3.2. *Assume that $\mathcal{R}_0 \leq 1$. Then (1.3) and (1.4) admit no nonnegative non-trivial solutions.*

Proof. For contradiction, suppose that (S, I, R) is such a nonnegative non-trivial solution of (1.3) and (1.4). By (3.1), we integrate the I-equation of (1.3) over $\mathbb{R}$ to get

$$(\gamma + \delta) \int_{\mathbb{R}} I(y) dy = \int_{\mathbb{R}} \frac{\beta I(y + c\tau) S(y)}{S(y) + I(y) + R(y)} \mathrm{d}y.$$

From the assumption $\mathcal{R}_0 = \beta/(\gamma + \delta) \leq 1$ and the above equality, it follows that

$$0 \leq (\gamma + \delta - \beta) \int_{\mathbb{R}} I(y) \mathrm{d}y = (\gamma + \delta) \int_{\mathbb{R}} I(y) \mathrm{d}y - \beta \int_{\mathbb{R}} I(y + c\tau) \mathrm{d}y$$
$$= -\beta \int_{\mathbb{R}} \frac{I(y + c\tau)(I(y) + R(y))}{S(y) + I(y) + R(y)} \mathrm{d}y < 0.$$

This is a contradiction and thus competes the proof of this lemma. □

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