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Large time behavior of strong solutions to the 1D non-resistive full compressible MHD system with large initial data

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Abstract. The Cauchy and initial-boundary value problems for one-dimensional compressible magnetohydrodynamics (MHD) system with non-resistive are studied in this article. Global-in-time, strong solutions to this system are shown to exist uniquely and be asymptotically stable as the time tends to infinity for large initial data. The main difficulties lie in the uniform-in-time estimate of first-order derivative of magnetic and the estimates of positive lower and upper uniform-in-time bounds of the density and temperature. This is a development of Zhang and Zhao (J Math Phys 58:031504, 2017).

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1. Introduction

In this article, we consider the following 1D non-resistive magnetohydrodynamics (MHD) system,

$$\begin{cases} \rho_t + (\rho u)_y = 0, \\ (\rho u)_t + (\rho u^2)_y + \left(R\rho\theta + \frac{b^2}{2}\right)_y = (\lambda u_y)_y, \\ b_t + (ub)_y = 0, \\ (\rho e)_t + (\rho u e)_y + R\rho\theta u_y = (\kappa\theta_y)_y + \lambda u_y^2. \end{cases} \tag{1.1}$$

As it is well known, the motion of a conducting fluid (plasma) in an electromagnetic field is governed by the equations of MHD, which is a coupled system of the induction equation of the magnetic field and the Navier–Stokes equations of fluid dynamics (see also [1,4–7,12,26]):

$$\begin{cases} \rho_t + \operatorname{div}(\rho \boldsymbol{u}) = 0, \qquad \boldsymbol{x} \in \mathbb{R}^3, \quad t \in \mathbb{R}^+, \\ (\rho \boldsymbol{u})_t + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla P = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \operatorname{div}\mathbb{S}, \\ \boldsymbol{B} - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = -\nabla \times (\sigma \nabla \times \boldsymbol{B}), \quad \operatorname{div}\boldsymbol{B} = 0, \\ \left(\rho e + \frac{1}{2}\rho|\boldsymbol{u}|^2 + \frac{1}{2}|\boldsymbol{B}|^2\right)_t + \operatorname{div}\left(\left(\rho e + \frac{1}{2}\rho|\boldsymbol{u}|^2 + P\right)\boldsymbol{u} - \kappa\nabla\theta\right) \\ \qquad\qquad = \operatorname{div}((\boldsymbol{u} \times \boldsymbol{B}) \times \boldsymbol{B} + \sigma \boldsymbol{B} \times (\nabla \times \boldsymbol{B}) + \mathbb{S}\boldsymbol{u}), \end{cases} \tag{1.2}$$

where ρ is the density, $\boldsymbol{u} \in \mathbb{R}^3$ the velocity, $\boldsymbol{B} \in \mathbb{R}^3$ the magnetic field, and θ the temperature; the pressure P and the internal energy e are related with the density and temperature of the flow by the equations of state:

$$P = P(\rho, \theta) = R\rho\theta, \quad \text{and} \quad e = e(\rho, \theta) = c_v\theta, \tag{1.3}$$

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where the specific gas constant R and the specific heat at constant volume c_v are positive constants, respectively; the symbol $\mathbb{S}$ denotes the viscous stress tensor

$$\mathbb{S} = \lambda'(\mathrm{div}\boldsymbol{u})\mathbb{I} + \mu(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top), \tag{1.4}$$

where $\mathbb{I}$ is the 3×3 identity matrix, $(\nabla \boldsymbol{u})^\top$ is the transpose of the matrix $\nabla \boldsymbol{u}$, and λ', μ are the constants viscosity coefficients of the flow which satisfy

$$\mu > 0, 2\mu + 3\lambda' > 0.$$

κ is the heat conductivity coefficients, while σ is the magnetic diffusivity acting as the magnetic diffusion coefficient of the magnetic field.

When $\sigma = 0$, Eq. (1.2) implies that in a highly conducting fluid, the magnetic field lines move along exactly with the fluid, rather than simply diffusing out. This type of behavior is physically described as that *the magnetic field lines are frozen into the fluid*. In effect, the fluid can flow freely along the magnetic field lines, but any motion of the conducting fluid, perpendicular to the field lines, carries them with the fluid. The "frozen-in" nature of magnetic fields plays very important roles and has a very wide range of applications in both astrophysics and nuclear fusion theory, where the magnetic Reynolds number $R_m \sim 1/\nu$ is usually very high. A typical illustration of the "frozen-in" behavior is the phenomenon of sunspots. For more details of its physical background and applications, we refer to [4–7,12,26].

However, similarly to those for the Navier–Stokes equations, many physically important and mathematically fundamental problems of MHD are still open. For example, to the author's knowledge, the global well-posedness of the multi-dimensional compressible non-resistive MHD equations remains unknown, even that the data are sufficiently close to the non-vacuum equilibrium state in a similar sense as that in [30] for the compressible Navier–Stokes equations. Here, we would like to refer to the recent works [3,8,10,11,22,24,25,28,31,37,38]. The articles [8,10,11] established the local solution of incompressible non-resistive MHD system in Besev and Sobolev space, respectively. The global well-posedness of the Cauchy problem of two/three-dimensional incompressible non-resistive MHD (MHD-type) equations obtained in [22,24,25,28,31,38] where the small initial data are announced. For the isotropic compressible MHD system, Bian and Yuan [3] and Wu and Wu [37] obtained the local solution and global small solution, respectively.

Due to the complex structure of multi-dimensional equations, similarly to that in [2,13,14,19,21], we consider the simpler compressible, viscous, heat-conducting, non-resistive MHD equations for ideal polytropic fluids in dimension one (1.1), which based on the specific choice of dependent variables with $y \in \mathcal{I} \subset \mathbb{R}$ and $t \in \mathbb{R}^+$:

$$\rho = \rho(y,t), \quad \boldsymbol{u} = (u(y,t),0,0)^\top, \quad \theta = \theta(y,t), \quad \boldsymbol{B} = (0,0,b(y,t))^\top, \quad \lambda = 2\mu + \lambda',$$

where $\mathcal{I} := \mathbb{R}$, or $\mathcal{I} := (0,\infty)$, or $\mathcal{I} := (0,1)$. Clearly, the magnetic field obeys the divergence constraint $\mathrm{div}\boldsymbol{B} = 0$ due to the special dependent variables. The system (1.1) is supplemented with the initial data and boundary condition:

$$\begin{cases} (\rho,u,b,\theta)|_{t=0} = (\rho_0,u_0,b_0,\theta_0)(y), \quad y \in \mathcal{I}, \\ (u,\partial_y\theta)|_{\partial\mathcal{I}} = 0, \qquad t > 0, \text{ if } \mathcal{I} := (0,\infty) \text{ or } \mathcal{I} := (0,1). \end{cases} \tag{1.5}$$

For the systems (1.1) and (1.5) in $\mathcal{I} = (0,1)$, we [40] obtained the strong solution with large initial data, and obtained the resistive limit. But the large time behavior of strong solution to (1.1) and (1.5) cannot be arrived. For the Navier–Stokes equations with temperature-dependent viscosity coefficients, Liu et al. [29] and Wang and Zhao [36] got the strong solutions and the large time behavior of solutions. Recently, Wan and Wang [35] developed the result of [29]. For the Navier–Stokes equations with constant viscosity coefficients, Li and Liang [23] established the large time behavior of solutions in $(0,\infty)$ and $\mathbb{R}$. Borrow the method of [23], Wan and Wang [34] obtained the large time behavior for cylindrically symmetric flows. Inspired by the above results, we [33] obtained the large time behavior of the 1D solution of (1.2) with some assumption when the viscosity coefficients are dependent on temperature and $\sigma \neq 0$. Recently, we

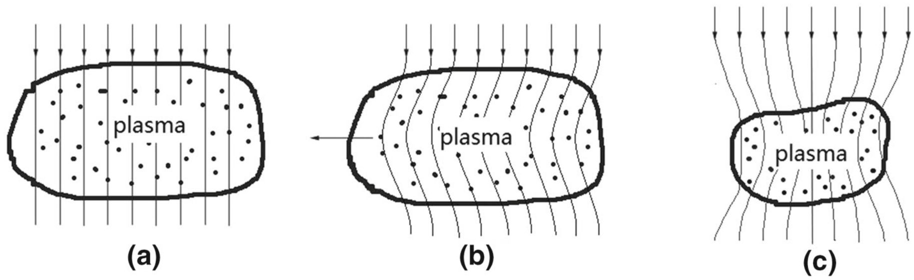

FIG. 1. The frozen-in behavior of magnetic fields in plasma

[32] develop the result of [33] to the screw pinches for the plasma physics. Inspired of Jiang and Zhang [18], Li and Sun [27] obtain the large time behavior of weak solution for 1D isentropic of the non-resistive compressible MHD system. However, the methods of [27] cannot be used to obtain the uniform-in-time higher-order estimates of strong solutions. Hence, we have to find new ways to establish the large time behavior of global strong solutions to the Cauchy and initial-boundary value problems for (1.1) with large initial data.

For this purpose, let x be the Lagrangian space variable, t be the time variable, and $v = \frac{1}{\rho}$ the specific volume. Then, the system (1.1) becomes

$$\begin{cases} v_t = u_x, \\ u_t + \left(\frac{R\theta}{v} + \frac{b^2}{2}\right)_x = \left(\frac{\lambda u_x}{v}\right)_x, \\ (vb)_t = 0, \\ c_v\theta_t + \frac{R\theta u_x}{v} = \left(\frac{\kappa\theta_x}{v}\right)_x + \frac{\lambda u_x^2}{v}, \end{cases} \qquad (x,t) \in \Omega \times \mathbb{R}_+, \tag{1.6}$$

where $\Omega = \mathbb{R}$, or $\Omega = (0,1)$, or $\Omega = (0,\infty)$.

Due to the frozen-in behavior of magnetic fields, the magnetic line in the body moves along with the plasma (as shown in Fig. 1b, Fig. 1a is a situation where a plasma is stationary in a magnetic field), when the plasma is moving in a magnetic field. Moreover, with the plasma compressed, the magnetic sense lines are also compressed (Fig. 1c).

In view of the above, we can reasonably assume that there exists a constant $\bar{b} \neq 0$ such that

$$v_0 b_0(x) = \bar{b}. \tag{1.7}$$

Remark 1.1. Since we are interested in showing that $(v,u,b,\theta) \to (1,0,\bar{b},1)$ as $t \to \infty$ in a strong sense, due to the conservation $(1.6)_3$, we may conclude that

$$vb = \bar{b}, \tag{1.8}$$

which means that the assumption (1.7) is reasonably. On the other hand, the condition (1.7) implies that there exists a magnetic background $\bar{b}$ in it.

The system (1.6) is supplemented with the initial condition

$$(v,u,b,\theta)(0,x) = (v_0,u_0,b_0,\theta_0)(x), \quad x \in \overline{\Omega}, \tag{1.9}$$

and three types of far-field and boundary conditions:

(1) Cauchy problem

$$\Omega = \mathbb{R}, \quad \lim_{|x|\to\infty}(v(x,t),u(x,t),b(x,t),\theta(x,t)) = (1,0,\bar{b},1), \quad t > 0; \tag{1.10}$$

(2) boundary and far-field conditions for $\Omega = (0, \infty)$,

$$u(0,t) = 0,\ \theta_x(0,t) = 0,\ \lim_{x\to\infty}(v(x,t), u(x,t), b(x,t), \theta(x,t)) = (1, 0, \bar{b}, 1), \quad t > 0; \tag{1.11}$$

(3) boundary conditions for $\Omega = (0, 1)$,

$$u(0,t) = u(1,t) = 0, \quad \theta_x(0,t) = \theta_x(1,t) = 0, \qquad t > 0. \tag{1.12}$$

These boundary conditions are supposed to be compatible with the initial data.

Notations. For the convenience, we define

$$\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)}, \|\cdot\|_{H^k} := \|\cdot\|_{H^k(\Omega)}, \int \cdot := \int_\Omega \cdot \mathrm{d}x$$

where $1 \le p \le \infty$ and $k \in \mathbb{N}$. The symbol $A \lesssim B$ ($A \gtrsim B$) means that $A \le CB$ ($A \ge CB$) holds uniformly for some t-independent constant C which depends only on the initial data. The positive constants c and C are different on line to line. C_ε denotes positive constant depending on ε. Without loss of generality, we assume that $R = 1$.

Under the condition (1.7), we have the following main results.

Theorem 1.1. *Assume that the initial data $(v_0 - 1, u_0, b_0 - \bar{b}, \theta_0 - 1)(x) \in H^1(\Omega)$ are compatible with (1.11), (1.12) and satisfy*

$$\inf_{x\in\Omega}\{v_0(x), \theta_0(x)\} > 0, \qquad v_0 b_0(x) = \bar{b}. \tag{1.13}$$

Then the problem (1.6)–(1.10), or (1.6)–(1.9), (1.11), or (1.6)–(1.9), (1.12) has a unique global solution $(v, u, b, \theta)(t, x)$ satisfying

$$\begin{cases} C^{-1} \le v(t,x), \theta(t,x) \le C, \quad \forall (t,x) \in [0, +\infty) \times \Omega, \\ (v - 1, u, b - \bar{b}, \theta - 1) \in L^\infty(0, T; H^1(\Omega)), \\ (v_x, b_x, v_t, u_t, b_t, \theta_t) \in L^2([0,T] \times \Omega), \\ (u_x, \theta_x) \in L^2(0, T; H^1(\Omega)), \end{cases} \tag{1.14}$$

and

$$\lim_{t\to\infty}\Big(\|(v - 1, u, b - \bar{b}, \theta - 1)(t)\|_p + \|(v_x, u_x, b_x, \theta_x)(t)\|_2\Big) = 0, \qquad \text{for } \Omega = \mathbb{R} \text{ and } (0, \infty), \tag{1.15}$$

$$\|(v - \bar{v}, u, b - \bar{b}/\bar{v}, \theta - \bar{\theta})(t)\|_{H^1} \lesssim e^{-ct}, \qquad \text{for } \Omega = (0,1), \tag{1.16}$$

where

$$2 < p \le \infty, \qquad \bar{v} := \frac{1}{|\Omega|}\int v_0, \qquad \bar{\theta} := \frac{1}{|\Omega|}\int \left[\theta_0 + \frac{1}{2c_v}(u_0^2 + v_0 b_0^2)\right]. \tag{1.17}$$

In the following, we comment on the analysis of the paper. First, this paper is inspired by [15,17, 20,23,34]. However, taking into account of the hydrodynamic and electrodynamic effects, the problem becomes considerably complicated. In view of the additional nonlinear terms induced by the magnetic field, our first main difficulties in the proof of the uniform-in-time pointwise upper and lower bounds for $v(t, x)$. To overcome these difficulties, we, firstly, obtain the lower bound of v which is different from Wan et. al. [34]. Secondly, based on the lower bound of v and the particularity of Eq. $(1.6)_3$ we arrive the upper bound of v (see details in Lemma 2.2 and 3.1). Yet, it is also hard to establish the estimate of the first-order derivative of magnetic. Thanks to the phenomenon that the magnetic field lines are frozen into the fluid, we can reasonably suppose that $v_0 b_0 = \bar{b}$ to solve this problem. Moreover, employing the Poincaré's inequality, we arrive that the solutions converge exponentially to the constant state as the time tends to infinity in the bounded domain $\Omega = (0, 1)$. To the similar result of Theorem 1.1 studied in screw pinches arisen from plasma physics [32,39] with zero and nonzero resistive term, we cannot overcome the above difficulties, which is also our research in the future.

The rest of this paper is organized as follows. First of all, we derive a number of desired *a priori* estimates independent of time in Sect. 2 for $\Omega = \mathbb{R}$ and $(0, \infty)$ and in Sect. 3 for $\Omega = (0, 1)$. Combining the *a priori* estimates in Sects. 2 and 3, the existence, uniqueness, and large time behavior of the solutions are proved in Sect. 4.

2. Global estimates of (1.6)–(1.10) and (1.6)–(1.9), (1.11)

The global well-posedness of strong solutions to the problems (1.6)–(1.10) and (1.6)–(1.9), (1.11) can be shown in the same way as that in [15,23,34]. So, the main purpose of this section is to derive the global t-independent estimates of the solutions in $\Omega = \mathbb{R}$ and $\Omega = (0, \infty)$, which will be used to justify the existence, uniqueness, and large time behavior of global-in-time solutions. We start with the following basic energy estimate.

Lemma 2.1. (basic energy estimate) *Assume that the conditions listed in Theorem 1.1 hold. Then for* $t \geq 0$,

$$\int \eta(v, u, b, h, \theta)(t, x) + \int_0^t \int \left[\frac{u_x^2}{v\theta} + \frac{\theta_x^2}{v\theta^2}\right] \mathrm{d}s \lesssim 1, \tag{2.1}$$

where

$$\eta(v, u, b, h, \theta) := \hat{\theta}\phi\left(\frac{v}{\hat{v}}\right) + \frac{u^2 + v(b - v_0 b_0/\hat{v})^2}{2} + c_v\hat{\theta}\phi\left(\frac{\theta}{\hat{\theta}}\right), \quad \phi(z) = z - \log z - 1. \tag{2.2}$$

Proof. Let $\hat{v}, \hat{\theta} > 0$ be some arbitrary but fixed constants. Multiplying $(1.6)_1$–$(1.6)_4$ by $\hat{\theta}(\hat{v}^{-1} - v^{-1})$, u, $b - \frac{v_0 b_0}{\hat{v}}$, and $(1 - \hat{\theta}\theta^{-1})$, respectively, and adding them together, one obtains

$$\begin{aligned} &\left[\hat{\theta}\phi\left(\frac{v}{\hat{v}}\right) + c_v\hat{\theta}\phi\left(\frac{\theta}{\hat{\theta}}\right) + \frac{1}{2}u^2 + \frac{1}{2}v\left(b - \frac{v_0 b_0}{\hat{v}}\right)^2\right]_t + \frac{\lambda u_x^2\hat{\theta}}{v\theta} + \frac{\kappa\theta_x^2\hat{\theta}}{v\theta^2} \\ &= \left[\frac{\kappa\theta_x}{v} - \frac{\kappa\theta_x\hat{\theta}}{v\theta} + \frac{\lambda u_x u}{v} - \left(\frac{\theta}{v} + \frac{b^2 - (v_0 b_0/\hat{v})^2}{2} - \frac{\hat{\theta}}{\hat{v}}\right)u\right]_x. \end{aligned} \tag{2.3}$$

Integrating (2.3) over $\Omega \times (0, t)$, (2.1) is immediately arrived. □

For the simplicity, we take $\hat{\theta} = 1$ and $\hat{v} = 1$ in the following estimates. Employing Jensen's inequality to the convex function ϕ, we can derive the following corollary from the estimate Lemma 2.1. Because the proof is standard, we omit it for simplicity (see [20]).

Corollary 2.1. *Assume that the conditions listed in Theorem 1.1 hold. Then for all* $(t, x, i) \in [0, T) \times \Omega \times Z$, *there are* $a_i(t), b_i(t) \in U_i := [i, i+1]$ *such that*

$$C^{-1} \leq \int_{U_i} v(t, y)\mathrm{d}y,\ v(t, a_i(t)),\ \int_{U_i} \theta(t, y)\mathrm{d}y,\ \theta(t, b_i(t)) \leq C. \tag{2.4}$$

Next, by means of Lemma 2.1 and Corollary 2.1, we derive the upper and lower bounds of v.

Lemma 2.2. *Assume that the conditions listed in Theorem 1.1 hold. Then for* $(x, t) \in \Omega \times [0, T)$,

$$v(x, t) \lesssim 1, \qquad v(x, t) \geq \begin{cases} C(t), & \text{if } T < \infty, \\ C, & \text{if } T = \infty. \end{cases} \tag{2.5}$$

Proof. The proof is divided into three steps.

Step 1 (Representation formula for v) We define that

$$\psi(x)=\begin{cases}1, & x<[z]+1,\\ [z]+2-x, & [z]+1\le x<[z]+2,\\ 0, & x\ge [z]+2,\end{cases} \tag{2.6}$$

where z is arbitrary but fixed point of $\overline{\Omega}$ and $[z]$ denotes the largest integer that is less or equal to z. Multiplying $(1.6)_2$ by ψ and integrating over $(y,\infty)\times[0,t]$, one has

$$\begin{aligned}\log v(t,y)-\log v(0,y)&=\frac{1}{\lambda}\int_0^t\left(\frac{\theta}{v}+\frac{b^2}{2}\right)(s,y)\mathrm{d}s+\frac{1}{\lambda}\int_y^\infty(u_0-u)\psi\mathrm{d}x\\&\quad+\frac{1}{\lambda}\int_0^t\int_{[z]+1}^{[z]+2}\left(\frac{\lambda u_x}{v}-\frac{\theta}{v}-\frac{b^2}{2}\right)\mathrm{d}x\mathrm{d}s,\end{aligned} \tag{2.7}$$

where $y\in([z]-1,[z]+1)\cap\Omega:=U$. By virtue of (2.7), one has

$$vD^{-1}B^{-1}=\exp\left\{\frac{1}{\lambda}\int_0^t\left(\frac{\theta}{v}+\frac{b^2}{2}\right)\mathrm{d}s\right\}, \tag{2.8}$$

where

$$\begin{aligned}B&:=\exp\left\{\frac{1}{\lambda}\int_0^t\int_{[z]+1}^{[z]+2}\left(\frac{\lambda u_x}{v}-\frac{\theta}{v}-\frac{b^2}{2}\right)\mathrm{d}x\mathrm{d}s\right\},\\D&:=v_0\exp\left\{\frac{1}{\lambda}\int_y^\infty(u_0-u)\psi\mathrm{d}x\right\}.\end{aligned}$$

Multiplying (2.8) by $\frac{1}{\lambda}\left[\frac{\theta}{v}+\frac{b^2}{2}\right]$, one obtains

$$v(y,t)=D(t)B(t)+\frac{1}{\lambda}\int_0^t\frac{D(t)B(t)}{D(s)B(s)}\left(\theta+\frac{vb^2}{2}\right)(y,s)\mathrm{d}s. \tag{2.9}$$

Step 2 (Lower bound for v) First of all, we need some estimate of D, B, and θ. By means of Lemma 2.1 and Cauchy–Schwarz's inequality, it follows

$$\int_y^\infty(u_0-u)\psi\mathrm{d}x\lesssim 1+\int_y^{[z]+2}[u^2+u_0^2]\mathrm{d}x\lesssim 1, \tag{2.10}$$

which yields

$$C^{-1}\le D\le C. \tag{2.11}$$

By Cauchy–Schwarz's inequality, Lemma 2.1, and Hölder's inequality, we have

$$\begin{aligned}\int_s^t\int_{[z]+1}^{[z]+2}\left(\frac{\lambda u_x}{v}-\frac{\theta}{v}-\frac{b^2}{2}\right)\mathrm{d}x\mathrm{d}\tau &\le C\int_s^t\int_{[z]+1}^{[z]+2}\frac{\lambda u_x^2}{v\theta}\mathrm{d}x\mathrm{d}\tau-\frac{1}{2}\int_s^t\int_{[z]+1}^{[z]+2}\frac{\theta}{v}\mathrm{d}x\mathrm{d}\tau\\ &\le C-\frac{1}{2}\int_s^t\inf_{([z]+1,[z]+2)}\theta(\cdot,\tau)\left[\int_{[z]+1}^{[z]+2}v\mathrm{d}x\right]^{-1}\mathrm{d}\tau\le C-C^{-1}\int_s^t\inf_{([z]+1,[z]+2)}\theta(\cdot,\tau)\mathrm{d}\tau.\end{aligned}\tag{2.12}$$

Similar as [16], one has

$$-C^{-1}\int_s^t\inf_{([z]+1,[z]+2)}\theta(\cdot,\tau)\mathrm{d}\tau\le C-C^{-1}(t-s).\tag{2.13}$$

In fact, by Cauchy–Schwarz's inequality, Lemma 2.1, and Corollary 2.1, there exist $b_j(\tau)\in([z]+1,[z]+2)$ such that

$$\int_s^t\int_{b_j(\tau)}^{\xi}\frac{\theta_x(x,\tau)}{\theta(x,\tau)}\mathrm{d}x\mathrm{d}\tau\le\int_s^t\left(\int_{[z]+1}^{[z]+2}\frac{\theta_x^2}{v\theta^2}\right)^{1/2}\left(\int_{[z]+1}^{[z]+2}v\right)^{1/2}\mathrm{d}\tau\le C+C(t-s),\tag{2.14}$$

where $\xi\in([z]+1,[z]+2)$. By Jensen's inequality to convex function e^x, Corollary 2.1, and (2.14), it follows

$$\begin{aligned}\int_s^t\theta(\xi,\tau)\mathrm{d}\tau &=\int_s^t\exp\{\log\theta(\xi,\tau)\}\mathrm{d}\tau\ge(t-s)\exp\left\{\frac{1}{t-s}\int_s^t\log\theta\mathrm{d}\tau\right\}\\ &=(t-s)\exp\left\{\frac{1}{t-s}\int_s^t\left(\log\frac{\theta(\xi,\tau)}{\theta(b_j(\tau),\tau)}+\log\theta(b_j(\tau),\tau)\right)\mathrm{d}\tau\right\}\\ &=(t-s)\exp\left\{\frac{1}{t-s}\int_s^t\left(\int_{b_j(\tau)}^{\xi}\frac{\theta_x}{\theta}\mathrm{d}x+\log\theta(b_j(\tau),\tau)\right)\mathrm{d}\tau\right\}\\ &\ge(t-s)\exp\left\{\log C-\frac{1}{t-s}\left|\int_s^t\int_{b_j(\tau)}^{\xi}\frac{\theta_x}{\theta}\mathrm{d}x\right|\right\}\ge C(t-s)\exp\left\{\frac{-C}{t-s}\right\},\end{aligned}\tag{2.15}$$

which means that (2.13) is valid. Inserting (2.13) into (2.12), one obtains

$$0\le B(t)\le C\mathrm{e}^{-t/c},\qquad\frac{B(t)}{B(s)}\le C\mathrm{e}^{-(t-s)/c}.\tag{2.16}$$

Integrating (2.9) over U, by Corollary 2.1 and Lemma 2.1, we can derive

$$B^{-1}(t)\lesssim B^{-1}(t)\int_U v\mathrm{d}y\lesssim 1+\int_0^t B^{-1}(s)\int_U\left(\theta+\frac{vb^2}{2}\right)\mathrm{d}y\mathrm{d}s\lesssim 1+\int_0^t B^{-1}(s)\mathrm{d}s,\tag{2.17}$$

which combined with Gronwall's inequality yields

$$B^{-1}\le C\exp\left\{c\int_0^t 1\mathrm{d}s\right\}\le C\mathrm{e}^{ct}.\tag{2.18}$$

Hence, for $t \le t_0 < \infty$, one has

$$v \ge DB \ge C\mathrm{e}^{-ct} \ge C\mathrm{e}^{-ct_0}. \tag{2.19}$$

For the enough large t, one deduces

$$v \ge C \int_0^t \frac{B(s)}{B(t)} \theta \mathrm{d}s. \tag{2.20}$$

So, we need the estimates of θ. By means of Corollary 2.1, there exits $b_j(t) \in U$, such that

$$C^{-1} \le \theta(b_j(t), t) \le C. \tag{2.21}$$

Hence, by Hölder's inequality and Corollary 2.1, for $x \in U$ there exist $b_j(t) \in U$ such that

$$\begin{aligned} |\log(\theta(x,t)+1) - \log\Big(\theta\big(b_j(t),t\big)+1\Big)| &\lesssim \int_U \frac{\theta_x}{\theta} \mathrm{d}x \\ &\lesssim \left(\int \frac{\theta_x^2}{v\theta^2}\right)^{1/2} \left(\int_U v \mathrm{d}x\right)^{1/2} \lesssim \left(\int \frac{\theta_x^2}{v\theta^2}\right)^{1/2}, \end{aligned} \tag{2.22}$$

which yields

$$\theta \ge C - C\int \frac{\theta_x^2}{v\theta^2}. \tag{2.23}$$

Integrating (2.9) over U again, by Lemma 2.1, one has

$$1 \lesssim B(t) + \int_0^t \frac{B(t)}{B(s)} \int_U (\theta + \frac{vb^2}{2}) \mathrm{d}x\mathrm{d}s \lesssim \mathrm{e}^{-ct} + \int_0^t \frac{B(t)}{B(s)} \mathrm{d}s, \tag{2.24}$$

that is

$$\int_0^t \frac{B(t)}{B(s)} \mathrm{d}s \ge C - C\mathrm{e}^{-ct}. \tag{2.25}$$

Putting (2.23) into (2.20), by (2.16), (2.25), and Lemma 2.1, for the enough large t, it follows

$$\begin{aligned} v &\ge C \int_0^t \frac{B(s)}{B(t)} \theta \mathrm{d}s \ge C \int_0^t \frac{B(s)}{B(t)} \left(1 - \int \frac{\theta_x^2}{v\theta^2}\right) \mathrm{d}s \\ &\ge C - C\mathrm{e}^{-ct} - C\left(\int_0^{t/2} + \int_{t/2}^t\right) \frac{B(s)}{B(t)} \int \frac{\theta_x^2}{v\theta^2} \mathrm{d}s \\ &\ge C - C\mathrm{e}^{-ct} - C\int_0^{t/2} \mathrm{e}^{-(t-s)c} \int \frac{\theta_x^2}{v\theta^2} \mathrm{d}s - C\int_{t/2}^t \int \frac{\theta_x^2}{v\theta^2} \mathrm{d}s \\ &\ge C - C\mathrm{e}^{-ct} - C\mathrm{e}^{-ct/2} - C\int_{t/2}^t \int \frac{\theta_x^2}{v\theta^2} \mathrm{d}s \ge C. \end{aligned} \tag{2.26}$$

Step 3 (Upper bound for v) Integrating $(1.6)_3$ over $[0,t]$, one has

$$vb = v_0 b_0, \tag{2.27}$$

which yields

$$vb^2 = v_0^2 b_0^2 v^{-1}. \tag{2.28}$$

Hence, by (2.28) and the lower bound of v, we obtain

$$vb^2 \le C. \tag{2.29}$$

By the boundedness of B, Lemma 2.1, (2.11), (2.16), and (2.29), one obtains

$$\|v\|_{L^\infty(U)} \le C + C\int_0^t \left(\mathrm{e}^{-c(t-s)}\|v\|_{L^\infty(U)}\int \frac{\theta_x^2}{v\theta^2}\right)\mathrm{d}s, \tag{2.30}$$

where we have used the result

$$\|\theta\|_{L^\infty(U)} \lesssim 1 + \|v\|_{L^\infty(U)}\int \frac{\theta_x^2}{v\theta^2}. \tag{2.31}$$

In fact, by Hölder's inequality and Corollary 2.1, there exits $b_j(t) \in U$ such that

$$\begin{aligned}|\theta^{1/2}(y,t) - \theta^{1/2}(b_j(t),t)| &\lesssim \int_U \theta^{1/2}\theta_x \mathrm{d}x \lesssim \|v\|_{L^\infty(U)}^{1/2}\left(\int \frac{\theta_x^2}{v\theta^2}\right)^{1/2}\left(\int_U \theta \mathrm{d}x\right)^{1/2}\\ &\lesssim \|v\|_{L^\infty(U)}^{1/2}\left(\int \frac{\theta_x^2}{v\theta^2}\right)^{1/2},\end{aligned} \tag{2.32}$$

which means that (2.31) is valid. By Gronwall's inequality, (2.30), and Lemma 2.1, it follows

$$\sup_{y\in U} v(t,y) \le C, \qquad \forall t \in [0,\infty). \tag{2.33}$$

□

According to Lemma 2.1, 2.2 and Corollary 2.1, we have the following results.

Corollary 2.2. *Assume that the conditions listed in Theorem 1.1 hold. Then for* $t \ge 0$,

$$C^{-1} \le |b| \le C, \tag{2.34}$$

$$\int_0^t \|u\|_\infty^2 \mathrm{d}s \lesssim 1. \tag{2.35}$$

Proof. By Lemma 2.2, (2.27), and (1.13), (2.34) can be immediately obtained. Next, we prove (2.35) in $\Omega = \mathbb{R}$. By $u|_{x=\infty} = 0$, Hölder's inequality, Corollary 2.1, and Lemma 2.2, we have

$$\|u\|_\infty^2 \lesssim \sum_{i=-\infty}^{\infty}\left(\int_{U_i}|u_x|\right)^2 \lesssim \|v\|_\infty \sup_i \int_{U_i}\theta \sum_{i=-\infty}^{\infty}\int_{U_i}\frac{u_x^2}{v\theta} \lesssim \int \frac{u_x^2}{v\theta}, \tag{2.36}$$

where U_i is defined in Corollary 2.1. By (2.36) and Lemma 2.1, one has (2.35). The proof of (2.35) in $\Omega = (0,\infty)$ is similar as the case $\Omega = \mathbb{R}$, we omit it for simplicity. □

Lemma 2.3. *Assume that the conditions listed in Theorem 1.1 hold. Then for* $t \ge 0$,

$$\|\big((\theta-1), u^2, v_x, b_x\big)(t)\|_2^2 + \int_0^t \Big(\|(\theta_x, uu_x, (1,\sqrt{\theta})(v_x,u_x), b_x)(s)\|_2^2\Big)\mathrm{d}s \lesssim 1. \tag{2.37}$$

Proof. For each $t \geq 0$ and $a > 1$, we define

$$\Omega_a := \{x \in \Omega : \theta > a\}, \qquad (\theta - a)_+ := \max\{\theta - a, 0\}.$$

Taking inner product $(1.6)_2$ and $(1.6)_4$ with $2u(\theta-2)_+$ and $(\theta-2)_+$ over Ω, respectively, and adding them together, by integration by parts, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\left[\frac{c_v}{2}(\theta-2)_+^2+(\theta-2)_+u^2\right]+\int_{\Omega_2}\frac{\kappa\theta_x^2}{v}\mathrm{d}x+\int(\theta-2)_+\frac{\lambda u_x^2}{v}\\
&=\int\frac{\theta}{v}(\theta-2)_+u_x+2\int\frac{\theta}{v}\partial_x(\theta-2)_+u-2\int bb_x(\theta-2)_+u\\
&\quad-2\lambda\int_{\Omega_2}\theta_x\frac{u_xu}{v}\mathrm{d}x+\int_{\Omega_2}\theta_tu^2\mathrm{d}x:=\sum_{i=1}^{5}I_i.
\end{aligned} \tag{2.38}$$

By Cauchy–Schwarz's inequality, Lemma 2.1, 2.2, and Corollary 2.2, one deduces

$$\begin{aligned}
\sum_{i=1}^{4}I_i&\leq\varepsilon\int_{\Omega_2}\theta u_x^2\mathrm{d}x+C_\varepsilon\int_{\Omega_2}\theta(\theta-2)_+^2\mathrm{d}x+\varepsilon\int_{\Omega_2}\theta_x^2\mathrm{d}x+C_\varepsilon\int\theta^2u^2\\
&\quad+\varepsilon\|b_x\|_2^2+C_\varepsilon\int(\theta-2)_+^2u^2+\varepsilon\|\theta_x\|_2^2+C_\varepsilon\|uu_x\|_2^2\\
&\leq\varepsilon\|\sqrt{\theta}u_x,u_x,b_x,\theta_x\|_2^2+C_\varepsilon\sup_{\Omega}\left(\theta-\frac{3}{2}\right)_+^2+C_\varepsilon\|uu_x\|_2^2,
\end{aligned} \tag{2.39}$$

where we have used the fact

$$\int_{\Omega_2}\theta\mathrm{d}x\lesssim\int\phi(\theta)\lesssim1.$$

By means of $(1.6)_4$, one has

$$I_5=\int_{\Omega_2}\left(\frac{\lambda u_x^2}{c_vv}-\frac{\theta u_x}{c_vv}\right)u^2\mathrm{d}x+c_v^{-1}\int_{\Omega_2}\left(\frac{\kappa\theta_x}{v}\right)_xu^2\mathrm{d}x:=I_{51}+I_{52}. \tag{2.40}$$

By Cauchy–Schwarz's inequality and Lemma 2.1, we have

$$I_{51}\lesssim\int_{\Omega_2}(u_x^2+\theta^2)u^2\mathrm{d}x\lesssim\int u_x^2u^2+\sup_{\Omega}\left(\theta-\frac{3}{2}\right)_+^2. \tag{2.41}$$

To tackle the term I_{52}, we define

$$\varphi_\xi(\theta):=\begin{cases}1, & \theta-2\geq\xi,\\ (\theta-2)/\xi, & 0\leq\theta-2<\xi,\\ 0, & \theta-2<0.\end{cases}$$

Thanks to Lebesgue's dominated convergence theorem, integration by parts, and Lemma 2.2, it follows

$$\begin{aligned}
I_{52}&=\frac{\kappa}{c_v}\lim_{\xi\to0^+}\int_{\Omega}\varphi_\xi(\theta)\left(\frac{\theta_x}{v}\right)_xu^2\mathrm{d}x\leq-\frac{2\kappa}{c_v}\lim_{\xi\to0^+}\int_{\Omega}\varphi_\xi(\theta)\frac{\theta_x}{v}uu_x\mathrm{d}x\\
&\leq\varepsilon\|\theta_x\|_2^2+C_\varepsilon\int_{\Omega_2}u^2u_x^2\mathrm{d}x.
\end{aligned} \tag{2.42}$$

Putting (2.39)–(2.42) into (2.38), by integration over $(0,t)$ and Lemma 2.1, 2.2, one can deduce

$$\begin{aligned}&\int(\theta-2)_+^2+\int_0^t\int\left[\theta_x^2+(1+\theta)u_x^2\right]\\&\quad\lesssim 1+\int_0^t\sup_\Omega\left(\theta-\frac{3}{2}\right)_+^2(s,\cdot)\mathrm{d}s+\varepsilon\int_0^t\|b_x\|_2^2\mathrm{d}s+\int_0^t\|uu_x\|_2^2\mathrm{d}s,\end{aligned}\tag{2.43}$$

where we have used the facts

$$\begin{aligned}&\int_0^t\int\theta_x^2\mathrm{d}s\le\int_0^t\left(\int_{\Omega_2}+\int_{\Omega\backslash\Omega_2}\right)\theta_x^2\mathrm{d}x\mathrm{d}s\le\int_0^t\int_{\Omega_2}\theta_x^2\mathrm{d}x\mathrm{d}s+C\int_0^t\int_{\Omega\backslash\Omega_2}\frac{\theta_x^2}{v\theta^2}\mathrm{d}x\mathrm{d}s\\&\le\int_0^t\int_{\Omega_2}\theta_x^2\mathrm{d}x\mathrm{d}s+C,\end{aligned}\tag{2.44}$$

$$\begin{aligned}&\int_0^t\int(1+\theta)u_x^2\mathrm{d}s\lesssim\int_0^t\int\left(\frac{1}{\theta}+\theta\right)u_x^2\mathrm{d}s\lesssim\int_0^t\int\frac{u_x^2}{v\theta}\mathrm{d}s+\int_0^t\left(\int_{\Omega_3}+\int_{\Omega\backslash\Omega_3}\right)\theta u_x^2\mathrm{d}x\mathrm{d}s\\&\lesssim\int_0^t\int\frac{u_x^2}{v\theta}\mathrm{d}s+\int_0^t\int_{\Omega_3}(\theta-2)_+u_x^2\mathrm{d}x\mathrm{d}s\lesssim 1+\int_0^t\int_{\Omega_3}(\theta-2)_+u_x^2\mathrm{d}x\mathrm{d}s.\end{aligned}\tag{2.45}$$

By means of Lemma 2.1, 2.2 and Hölder's inequality, one also has

$$\int(\theta-1)^2=\left(\int_{\Omega_3}+\int_{\Omega\backslash\Omega_3}\right)(\theta-1)^2\mathrm{d}x\lesssim\int_{\Omega_3}(\theta-2)_+^2\mathrm{d}x+1,\tag{2.46}$$

$$\begin{aligned}&\int_0^t\sup_\Omega\left(\theta-\frac{3}{2}\right)_+^2\mathrm{d}s=\int_0^t\sup_\Omega\left(\int_x^\infty\partial_x\left(\theta-\frac{3}{2}\right)_+\mathrm{d}x\right)^2\mathrm{d}s\\&\lesssim\int_0^t\left(\int_{\Omega_{3/2}}|\theta_x|\mathrm{d}x\right)^2\mathrm{d}s\lesssim\int_0^t\int_{\Omega_{3/2}}\frac{\theta_x^2}{\theta}\mathrm{d}x\int_{\Omega_{3/2}}\theta\mathrm{d}x\mathrm{d}s\\&\lesssim\int_0^t\int_{\Omega_{3/2}}\frac{\theta_x^2}{\theta}\mathrm{d}x\mathrm{d}s\lesssim\varepsilon\int_0^t\int\theta_x^2\mathrm{d}s+C_\varepsilon\int_0^t\int\frac{\theta_x^2}{\theta^2}\mathrm{d}s\\&\lesssim\varepsilon\int_0^t\int\theta_x^2\mathrm{d}s+C_\varepsilon.\end{aligned}\tag{2.47}$$

Substituting (2.46), (2.47) into (2.43), one obtains

$$\int(\theta-1)^2+\int_0^t\int\left[\theta_x^2+(1+\theta)u_x^2\right]\le C_\varepsilon+\varepsilon\int_0^t\|b_x\|_2^2\mathrm{d}s+C\int_0^t\|uu_x\|_2^2\mathrm{d}s.\tag{2.48}$$

Taking inner product $(1.6)_2$ with $4u^3$ over Ω, by integration by parts, Cauchy–Schwarz's and Hölder's inequalities, Lemma 2.1, 2.2, and Corollary 2.2, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int u^4+12\lambda\int\frac{u^2u_x^2}{v}&=12\int\left(\frac{\theta-1}{v}+\frac{1-v}{v}\right)u^2u_x-4\int bb_xu^3\\&\lesssim C_\varepsilon\|u\|_\infty^2\int(\theta-1)^2+\varepsilon\|uu_x\|_2^2+C_\varepsilon\|u\|_\infty^2\int(1-v)^2+\varepsilon\|b_x\|_2^2+C_\varepsilon\|u\|_\infty^2\int u^4\\&\lesssim\varepsilon(\|uu_x\|_2^2+\|b_x\|_2^2)+C_\varepsilon\|u\|_\infty^2(\|\theta-1\|_2^2+\|u\|_4^4+1).\end{aligned}\tag{2.49}$$

Integrating (2.49) over $[0,t]$, by Corollary 2.2, one has

$$\|u\|_4^4+\int_0^t\|uu_x\|_2^2\lesssim 1+\varepsilon\int_0^t\|b_x\|_2^2+C_\varepsilon\int_0^t\|u\|_\infty^2(\|\theta-1\|_2^2+\|u\|_4^4).\tag{2.50}$$

Before the estimate of $\|b_x\|_2$, we have the following calculation. Acting ∂_x to (2.27), thanks to the initial condition (1.13), we can deduce

$$\frac{v_x}{v}=-\frac{b_x}{b}.\tag{2.51}$$

By $(1.6)_1$ and $(1.6)_2$, we have

$$\left(\frac{\lambda v_x}{v}-u\right)_t=\left(\frac{\theta}{v}+\frac{b^2}{2}\right)_x.\tag{2.52}$$

Multiplying (2.52) by $\frac{v_x}{v}$, by $(1.6)_1$, (2.51), Lemma 2.2, and integration by parts, one deduces

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\left[\frac{\lambda}{2}\left(\frac{v_x}{v}\right)^2-\frac{uv_x}{v}\right]+c\int(\theta v_x^2+b_x^2)&\le C\left|\int\theta_xv_x\right|-\int u\left(\frac{u_x}{v}\right)_x\\&\lesssim\left|\int\theta_xv_x\right|+\int\frac{u_x^2}{v}\lesssim\varepsilon\int\theta v_x^2+\varepsilon\|v_x\|_2^2\int\frac{\theta_x^2}{\theta^2}+C_\varepsilon\int\theta_x^2+C\|u_x\|_2^2,\end{aligned}\tag{2.53}$$

where we have used the fact:

$$\int v_x^2\le C\|v_x\|_2^2\int\frac{\theta_x^2}{\theta^2}+C\int\theta v_x^2.\tag{2.54}$$

In fact, by (2.23), one has

$$\theta+\int\frac{\theta_x^2}{\theta^2}\gtrsim 1,\tag{2.55}$$

which means that (2.54) is valid. Integrating (2.53) over $[0,t]$, one has

$$\|v_x\|_2^2+\int_0^t\left(\int\theta v_x^2+\int b_x^2\right)\mathrm{d}s\lesssim 1+\int_0^t\|v_x\|_2^2\int\frac{\theta_x^2}{\theta^2}\mathrm{d}s+\int_0^t\|\theta_x,u_x\|_2^2\mathrm{d}s.\tag{2.56}$$

Choosing suitable small $\delta>0$ and suitable large $\eta>0$, multiplying (2.56) and (2.50) by δ and η, respectively, and adding them together with (2.48), by Gronwall's inequality, Lemma 2.1, 2.2, Corollary 2.2, and (2.51), one can derive that

$$\|\theta-1,u^2,v_x,b_x\|_2^2+\int_0^t\|\theta_x,u_x,b_x,uu_x\|_2^2\mathrm{d}s+\int_0^t\int\theta(u_x^2+v_x^2)\mathrm{d}s\lesssim 1.\tag{2.57}$$

Thanks to (2.54), (2.57), and Lemma 2.1, we have

$$\int_0^t \int v_x^2 \mathrm{d}s \le \|v_x\|_2^2 \int_0^t \int \frac{\theta_x^2}{\theta^2} \mathrm{d}s + \int_0^t \int \theta v_x^2 \mathrm{d}s \lesssim 1, \tag{2.58}$$

which combined with (2.57) completes the proof of this lemma. □

Based on the above results, we now estimate the first-order derivative of velocity.

Lemma 2.4. *Assume that the conditions listed in Theorem 1.1 hold. Then for $t \ge 0$,*

$$\|u_x(t)\|_2^2 + \int_0^t \|(u_{xx}, u_t)(s)\|_2^2 \mathrm{d}s \lesssim 1. \tag{2.59}$$

Proof. Taking inner product $(1.6)_2$ with $2u_{xx}$ on Ω, by integration by parts, Cauchy–Schwarz's inequality, and Lemma 2.2, 2.3, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u_x\|_2^2 + c\int u_{xx}^2 &\le C\left|\int u_{xx}\left(\frac{b^2}{2} + \frac{\theta-1}{v} + \frac{1-v}{v}\right)_x\right| + C\int |u_x v_x u_{xx}| \\ &\le \varepsilon\|u_{xx}\|_2^2 + C_\varepsilon\|u_x\|_\infty^2\|v_x\|_2^2 + C_\varepsilon\|\theta_x, (\theta-1)v_x, v_x, b_x\|_2^2 \le \varepsilon\|u_{xx}\|_2^2 + C_\varepsilon\left\|u_x, \theta_x, v_x, b_x\right\|_2^2,\end{aligned} \tag{2.60}$$

where we have used the fact:

$$\|\theta-1\|_\infty^2 \lesssim \|\theta-1\|_2^2 + \int |\theta-1||\theta_x| \lesssim \|\theta-1\|_2^2 + \|\theta_x\|_2^2, \tag{2.61}$$

$$\int (\theta-1)^2 v_x^2 \le \|\theta-1\|_\infty^2\|v_x\|_2^2 \le \|\theta-1\|_2^2\|v_x\|_2^2 + \|\theta_x\|_2^2\|v_x\|_2^2 \le \|v_x\|_2^2 + \|\theta_x\|_2^2, \tag{2.62}$$

$$\|u_x\|_\infty^2 \le C\|u_x\|_2^2 + C\|u_x\|_2\|u_{xx}\|_2 \le \varepsilon\|u_{xx}\|_2^2 + C_\varepsilon\|u_x\|_2^2. \tag{2.63}$$

By (2.60), Gronwall's inequality, and Lemma 2.1, 2.3, we deduce

$$\|u_x(t)\|_2^2 + \int_0^t \|u_{xx}\|_2^2 \mathrm{d}s \lesssim 1. \tag{2.64}$$

By means of $(1.6)_2$ and Lemma 2.2, one has

$$|u_t| \lesssim |u_{xx}, u_x v_x| + |\theta_x, (\theta-1)v_x, v_x, b_x|, \tag{2.65}$$

which combined with Lemma 2.3 and (2.61)–(2.64) yields

$$\int_0^t \|u_t\|_2^2 \mathrm{d}s \lesssim \int_0^t \left\|u_x, u_{xx}, v_x, \theta_x, b_x\right\|_2^2 \mathrm{d}s \lesssim 1, \tag{2.66}$$

where we have used the fact:

$$\|u_x v_x\|_2 \lesssim \|u_x\|_\infty\|v_x\|_2 \lesssim \|u_x\|_{H^1}. \tag{2.67}$$

□

Next, we estimate the temperature.

Lemma 2.5. *Assume that the conditions listed in Theorem 1.1 hold. Then for $t \ge 0$,*

$$\|\theta_x(t)\|_2^2 + \int_0^t \|\theta_t, v_t, \theta_{xx}\|_2^2 \mathrm{d}s \lesssim 1. \tag{2.68}$$

Proof. Multiplying $(1.6)_5$ by θ_{xx}, integrating over Ω on x, by integration by parts, Cauchy–Schwarz's inequality, and Lemma 2.2–2.4, we have

$$\begin{aligned}&\frac{c_v}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\theta_x^2+\int\frac{\kappa\theta_{xx}^2}{v}=\int\theta_{xx}\left[\left(\frac{\theta-1}{v}+\frac{1-v}{v}\right)u_x+\frac{\kappa\theta_x v_x}{v^2}-\frac{\lambda u_x^2}{v}\right]\\&\le\varepsilon\int\theta_{xx}^2+C_\varepsilon\int\left[(\theta-1)^2u_x^2+(1-v)^2u_x^2+\frac{\theta_x^2v_x^2}{v^4}+\frac{\lambda^2u_x^4}{v^2}\right]\\&\lesssim\varepsilon\|\theta_{xx}\|_2^2+\|u_x\|_\infty^2\|\theta-1,v-1,u_x\|_2^2+\|\theta_x\|_\infty^2\|v_x\|_2^2\\&\lesssim\varepsilon\|\theta_{xx}\|_2^2+\|u_x,u_{xx}\|_2^2+\|\theta_x\|_2\|\theta_{xx}\|_2\lesssim\varepsilon\|\theta_{xx}\|_2^2+\|u_x,u_{xx}\|_2^2+C_\varepsilon\|\theta_x\|_2^2,\end{aligned}\tag{2.69}$$

which combined with Gronwall's inequality and Lemma 2.3–2.4, one has

$$\|\theta_x(t)\|_2^2+\int_0^t\|\theta_{xx}\|_2^2\mathrm{d}s\lesssim 1.\tag{2.70}$$

Hence, similar as (2.66), by means of $(1.6)_1$, $(1.6)_5$, Lemma 2.1–2.4, and (2.70), one can deduce

$$\int_0^t\|v_t,\theta_t\|_2^2\mathrm{d}s\lesssim 1.\tag{2.71}$$

□

Lemma 2.6. *Assume that the conditions listed in Theorem 1.1 hold. Then for $(x,t)\in[0,T)\times\Omega$,*

$$\theta(x,t)\lesssim 1,\qquad \theta(x,t)\ge\begin{cases}C(t), & \text{if } T<\infty,\\ C, & \text{if } T=\infty.\end{cases}\tag{2.72}$$

Proof. By (2.61), Lemma 2.3, and Lemma 2.5, one obtains

$$\|\theta\|_\infty\lesssim 1+\|\theta-1\|_\infty\lesssim 1+\|\theta-1\|_2\|\theta_x\|_2\lesssim 1.\tag{2.73}$$

Next, we prove the lower bound of θ. Multiplying $(1.6)_5$ by θ^{-2}, we have

$$c_v\left(\frac{1}{\theta}\right)_t-\left[\frac{\kappa r^2}{v}\left(\frac{1}{\theta}\right)_x\right]_x=-\frac{2\theta\kappa}{v}\left(\frac{1}{\theta}\right)_x^2-\frac{\lambda}{v\theta^2}\left(u_x-\frac{\theta}{2\lambda}\right)^2+\frac{1}{4\lambda v}\le\frac{1}{4\lambda v}\le C_1(t).\tag{2.74}$$

Let $G(t,x):=\theta^{-1}-c_v^{-1}\int_0^t C_1(s)\mathrm{d}s$, then

$$\begin{cases}c_vG_t\le\left[\frac{\kappa}{v}G_x\right]_x, & (t,x)\in[0,T]\times\Omega,\\ H(0,x)=\frac{1}{\theta(0,x)}\le\frac{1}{\inf_\Omega\theta_0} & \text{for } x\in\Omega.\end{cases}\tag{2.75}$$

In view of the maximum principle (see Evans [9]), we infer that

$$G(t,x)\le\frac{1}{\inf_\Omega\theta_0}\qquad\text{for all } (t,x)\in[0,T]\times\Omega,\tag{2.76}$$

which implies

$$\theta\ge C(t),\tag{2.77}$$

If the time tends to infinity, then by (2.69) and Lemma 2.3–2.5, one has

$$\int_0^\infty\left|\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_x\|_2^2\right|\mathrm{d}t\le C,\tag{2.78}$$

which yields

$$\lim_{t\to\infty}\|\theta_x\|_2=0.\tag{2.79}$$

Hence, by Sobolev's inequality, Lemma 2.3, and (2.79), we obtain

$$\lim_{t\to\infty} \|\theta - 1\|_\infty = 0, \tag{2.80}$$

which yields

$$\theta(x,t) \geq C, \qquad x \in \mathbb{R} \text{ or } (0,\infty) \tag{2.81}$$

for the large enough time t. □

3. Global estimates of (1.6)–(1.9), (1.12)

In the bounded domain $\Omega = (0,1)$, the estimates of the solutions are similar as Sect. 2 besides the proofs of Lemma 2.2, 2.3, Corollary 2.2, and the lower bound of θ when time is enough large. For simplicity, we just give out these different estimates. First of all, the following estimates of lower and upper bounds of v are different from Lemma 2.2.

Lemma 3.1. *Assume that the conditions listed in Theorem 1.1 hold. Then for $(x,t) \in [0,T) \times \Omega$,*

$$v(x,t) \lesssim 1, \qquad v(x,t) \geq \begin{cases} C(t), & \text{if } T < \infty, \\ C, & \text{if } T = \infty. \end{cases} \tag{3.1}$$

Proof. The proof is divided into three steps.

Step 1 (Representation formula for v) By means of $(1.6)_1$, integrating $(1.6)_2$ over $[0,t] \times [x_1(t), x]$, we have

$$\Big[\log v(x,t) - \log v(x_1(t),t)\Big] - \Big[\log v_0(x) - \log(x_1(t),0)\Big] = \frac{1}{\lambda}\int_{x_1(t)}^{x} (u - u_0)(\xi)\mathrm{d}\xi$$
$$+\frac{1}{\lambda}\int_0^t \left[\frac{\theta}{v} + \frac{b^2}{2}\right](x,s)\mathrm{d}s - \frac{1}{\lambda}\int_0^t \left[\frac{\theta}{v} + \frac{b^2}{2}\right](x_1(t),s)\mathrm{d}s, \tag{3.2}$$

where $x_1(t) \in [0,1]$ is determined by the following progresses. Next, for the convenience, we define

$$F =: \frac{\lambda u_x}{v} - \frac{\theta}{v} - \frac{b^2}{2}, \qquad \varphi =: \int_0^t F(x,s)\mathrm{d}s + \int_0^x u(\xi,0)\mathrm{d}\xi,$$

which means that

$$\varphi_x = u, \qquad \varphi_t = F, \tag{3.3}$$
$$(v\varphi)_t - (ru\varphi)_x = vF - u^2 = \lambda u_x - \theta - \frac{vb^2}{2} - u^2. \tag{3.4}$$

By $(1.6)_1$, one has

$$\frac{1}{\lambda}\int_0^t \left[\frac{\theta}{v} + \frac{b^2}{2}\right](x_1(t),s)\mathrm{d}s = \frac{1}{\lambda}\int_0^t \left(\frac{\lambda u_x}{v} - F\right)(x_1(t),s)\mathrm{d}s$$
$$= (\log v(x_1(t),t) - \log v(x_1(t),0)) - \frac{1}{\lambda}\int_0^t F(x_1(t),s)\mathrm{d}s. \tag{3.5}$$

Integrating (3.4) over $[0,t]\times\Omega$, by virtue of mean value theorem, there exits $x_1(t)\in[0,1]$ such that

$$\varphi(x_1(t),t)=\int\varphi v=\int v_0\int_0^x u_0(\xi)\mathrm{d}\xi-\int_0^t\int\left[\theta+\frac{vb^2}{2}+u^2\right]\mathrm{d}s. \tag{3.6}$$

By the definition of φ, (3.3), and (3.6), one obtains

$$\begin{aligned}\int_0^t F(x_1(t),s)\mathrm{d}s&=\varphi(x_1(t),t)-\int_0^{x_1(t)}u_0(\xi,t)\mathrm{d}\xi=\int v_0\int_0^x u_0(\xi)\mathrm{d}\xi\\&\quad-\int_0^t\int\left[\theta+\frac{vb^2}{2}+u^2\right]\mathrm{d}s-\int_0^{x_1(t)}u_0(\xi,t)\mathrm{d}\xi.\end{aligned} \tag{3.7}$$

Putting (3.5) and (3.7) into (3.2), we derive

$$vD^{-1}B=\exp\left\{\frac{1}{\lambda}\int_0^t\left(\frac{\theta}{v}+\frac{b^2}{2}\right)(x,s)\mathrm{d}s\right\}, \tag{3.8}$$

where

$$\begin{aligned}B&:=\exp\left\{\frac{1}{\lambda}\int_0^t\int\left(u^2+\theta+\frac{vb^2}{2}\right)\mathrm{d}s\right\},\\D&:=v_0\exp\left\{\frac{1}{\lambda}\left(\int v_0\int_0^x u_0(\xi)\mathrm{d}\xi-\int_0^{x_1(t)}u_0(\xi,t)\mathrm{d}\xi-\int_{x_1(t)}^x(u-u_0)(\xi)\mathrm{d}\xi\right)\right\}.\end{aligned}$$

Multiplying (3.8) by $\frac{1}{\lambda}\left[\frac{\theta}{v}+\frac{b^2}{2}\right]$, one obtains

$$v(x,t)=D(t)B^{-1}(t)+\frac{1}{\lambda}\int_0^t\frac{B(s)}{B(t)}\frac{D(t)}{D(s)}\left(\theta+\frac{vb^2}{2}\right)(x,s)\mathrm{d}s. \tag{3.9}$$

Step 2 (Lower bound for v)

First of all, we need some estimate of D, θ, and B. Employing Jensen's inequality to the convex function ϕ, one has

$$\int z-\log\int z-1\le\int\phi(z). \tag{3.10}$$

By (3.10) and Lemma 2.1, one obtains

$$C^{-1}\le\int v,\int\theta\le C. \tag{3.11}$$

By the definition of D, Lemma 2.1, and (3.11), one can deduce

$$C^{-1}\le D\le C. \tag{3.12}$$

Hence, for $t\le t_0<\infty$, one has

$$v\ge DB^{-1}\ge C\mathrm{e}^{-ct}\ge C\mathrm{e}^{-ct_0}. \tag{3.13}$$

For the enough large t, one deduces

$$v \geq C \int_0^t \frac{B(s)}{B(t)} \theta \mathrm{d}s. \tag{3.14}$$

So, we need the estimates of θ and $\frac{B(s)}{B(t)}$. By mean value theorem and (3.11), there exits $x_2(t) \in [0,1]$, such that

$$C^{-1} \leq \theta(x_2(t), t) \leq C. \tag{3.15}$$

By Cauchy–Schwarz's inequality and (3.11), one has

$$|\log(\theta+1) - \log(\theta(x_2(t),t)+1)| \lesssim \int \frac{\theta_x}{\theta} \lesssim \left(\int \frac{\theta_x^2}{v\theta^2}\right)^{1/2} \left(\int v\right)^{1/2} \lesssim \left(\int \frac{\theta_x^2}{v\theta^2}\right)^{1/2}, \tag{3.16}$$

which means that

$$\theta \geq C - C \int \frac{\theta_x^2}{v\theta^2}. \tag{3.17}$$

By Lemma 2.1, the definition of B, and (3.9), we obtain

$$\mathrm{e}^{-c_1(t-s)} \leq \frac{B(s)}{B(t)} \leq \mathrm{e}^{-c_2(t-s)}, \tag{3.18}$$

$$\int v \lesssim \mathrm{e}^{-ct} + \int_0^t \frac{B(s)}{B(t)} \mathrm{d}s, \tag{3.19}$$

that is

$$\int_0^t \frac{B(s)}{B(t)} \mathrm{d}s \geq C - C\mathrm{e}^{-ct}. \tag{3.20}$$

Putting (3.17) into (3.14), by (3.18), (3.20), and Lemma 2.1, for the enough large t, it follows

$$\begin{aligned} v &\geq C \int_0^t \frac{B(s)}{B(t)} \theta \mathrm{d}s \geq C \int_0^t \frac{B(s)}{B(t)} \left(1 - \int \frac{\theta_x^2}{v\theta^2}\right) \mathrm{d}s \\ &\geq C - C\mathrm{e}^{-ct} - C \left(\int_0^{t/2} + \int_{t/2}^t\right) \frac{B(s)}{B(t)} \int \frac{\theta_x^2}{v\theta^2} \mathrm{d}s \\ &\geq C - C\mathrm{e}^{-ct} - C \int_0^{t/2} \mathrm{e}^{-(t-s)c} \int \frac{\theta_x^2}{v\theta^2} \mathrm{d}s - C \int_{t/2}^t \int \frac{\theta_x^2}{v\theta^2} \mathrm{d}s \\ &\geq C - C\mathrm{e}^{-ct} - C\mathrm{e}^{-ct/2} - C \int_{t/2}^t \int \frac{\theta_x^2}{v\theta^2} \mathrm{d}s \geq C. \end{aligned} \tag{3.21}$$

Step 3 (Upper bound for v) Integrating $(1.6)_3$ over $[0,t]$, one has

$$vb = v_0 b_0, \tag{3.22}$$

which yields

$$vb^2 = v_0^2 b_0^2 v^{-1}. \tag{3.23}$$

Hence, by (3.23) and the lower bound of v, we obtain

$$vb^2 \leq C. \tag{3.24}$$

By the boundedness of B^{-1}, Lemma 2.1, (3.12), (3.18), and (3.24), one obtains

$$\|v\|_\infty \leq C + C\int_0^t \left(\mathrm{e}^{-c_2(t-s)}\|v\|_\infty \int \frac{\theta_x^2}{v\theta^2}\right) \mathrm{d}s, \tag{3.25}$$

where we have used the result

$$\|\theta\|_\infty \lesssim 1 + \|v\|_\infty \int \frac{\theta_x^2}{v\theta^2}. \tag{3.26}$$

In fact, by Hölder's inequality

$$\begin{aligned}|\theta^{1/2}(x,t) - \theta^{1/2}(x_2(t),t)| &\lesssim \int \theta^{1/2}\theta_x \lesssim \|v\|_\infty^{1/2} \left(\int \frac{\theta_x^2}{v\theta^2}\right)^{1/2} \left(\int \theta\right)^{1/2} \\ &\lesssim \|v\|_\infty^{1/2} \left(\int \frac{\theta_x^2}{v\theta^2}\right)^{1/2}.\end{aligned} \tag{3.27}$$

which means that (3.26) is valid. By Gronwall's inequality, (3.25), and Lemma 2.1, it follows

$$\sup_{0\leq t\leq T} \|v\|_\infty \leq C. \tag{3.28}$$

□

According to Lemma 2.1 and Lemma 3.1, we have the following results.

Corollary 3.1. *Assume that the conditions listed in Theorem 1.1 hold. Then for $t \geq 0$,*

$$C^{-1} \leq \int v, \int \theta \leq C, \tag{3.29}$$

$$\|b\|_\infty \lesssim 1, \tag{3.30}$$

$$\int_0^t \|u\|_\infty^2 \mathrm{d}s \lesssim 1. \tag{3.31}$$

Proof. By (3.11), (3.22), and Lemma 3.1, (3.29)–(3.30) can be immediately obtained. By $u|_{\partial\Omega} = 0$, Hölder's inequality, and Lemma 3.1, we have

$$\|u\|_\infty^2 \lesssim \left(\int |u_x|\right)^2 \lesssim \|v\|_\infty \int \frac{u_x^2}{v\theta} \int \theta \leq \int \frac{u_x^2}{v\theta}. \tag{3.32}$$

By (3.32) and Lemma 2.1, one has (3.31). □

In the bounded domain, Lemma 2.3 cannot be established employing the method of unbounded domain. Next, we give out the estimate similar as Lemma 2.3 in domain $(0, 1)$.

Lemma 3.2. *Assume that the conditions listed in Theorem 1.1 hold. Then for $t \geq 0$,*

$$\int \left(\theta^2 + u^4 + u^2 + v_x^2 + b_x^2\right)(t) + \int_0^t \int \left(\theta_x^2 + u^2u_x^2 + u_x^2 + \theta v_x^2 + v_x^2 + b_x^2\right)(s)\mathrm{d}s \lesssim 1. \tag{3.33}$$

Proof. Taking inner product $(1.6)_2$ with $4u^3$ over Ω, by Hölder's and Cauchy–Schwarz's inequalities and Lemma 3.1, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int u^4+12\lambda\int\frac{u^2u_x^2}{v}&=12\int\frac{\theta}{v}u^2u_x+12\int\frac{b^2}{2}u_xu^2\\&\lesssim C_\varepsilon\|u\|_\infty^2\left(\int\theta^2+1\right)+\varepsilon\|uu_x\|_2^2.\end{aligned}\tag{3.34}$$

Multiplying $(1.6)_2$ by u, adding together with $(1.6)_4$, we have

$$\left(c_v\theta+\frac{u^2}{2}\right)_t+\left[\left(P+\frac{b^2}{2}\right)u\right]_x-\frac{b^2}{2}u_x=\left(\frac{\kappa\theta_x}{v}\right)_x+\left(\frac{\lambda uu_x}{v}\right)_x.\tag{3.35}$$

Taking inner product (3.35) with $c_v\theta+\frac{u^2}{2}$ over Ω, by integration by parts, Hölder's, Cauchy–Schwarz's and Poincaré's inequalities and Corollary 3.1, one can deduce

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\left[\left(c_v\theta+\frac{u^2}{2}\right)^2\right]+c_v\kappa\int\frac{\theta_x^2}{v}+\lambda\int\frac{u^2u_x^2}{v}\\&=-(\kappa+c_v\lambda)\int\frac{\theta_xu_xu}{v}+\int(P+\frac{b^2}{2})u(c_v\theta_x+uu_x)+\int\frac{b^2}{2}u_x(c_v\theta+\frac{u^2}{2})\\&\lesssim\int|\theta_xuu_x|+\int|\theta u+ub^2||\theta_x+uu_x|+\int|u_x\theta|\\&\lesssim\varepsilon\|\theta_x\|_2^2+C_\varepsilon\|uu_x\|_2^2+C_\varepsilon\|u\|_\infty^2\left(\int\theta^2+1\right)+C_\varepsilon\|u_x\|_2^2.\end{aligned}\tag{3.36}$$

By virtue of (3.34), (3.36), and Gronwall's inequality, one can deduce

$$\int\left(\theta^2+u^4\right)(t)+\int_0^t\int\left(\theta_x^2+u^2u_x^2\right)(s)\mathrm{d}s\lesssim\int_0^t\|u_x\|_2^2\mathrm{d}s.\tag{3.37}$$

Before the estimate of $\|v_x\|_2$, we have the following calculations. By $(1.6)_3$, one has

$$(\log v)_t=-(\log b)_t.\tag{3.38}$$

Acting ∂_x to (3.38), thanks to the initial condition (1.13), it follows

$$\frac{v_x}{v}=-\frac{b_x}{b}.\tag{3.39}$$

By $(1.6)_1$-$(1.6)_2$, we have

$$\left(\frac{\lambda v_x}{v}-u\right)_t=\left(\frac{\theta}{v}+\frac{b^2}{2}\right)_x.\tag{3.40}$$

Multiplying (3.40) by $\frac{v_x}{v}$, by $(1.6)_1$, (3.39), Lemma 3.1, and integration by parts, one deduces

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\left[\frac{\lambda}{2}\left(\frac{v_x}{v}\right)^2-\frac{uv_x}{v}\right]+c\int(\theta v_x^2+b_x^2)&\le C\left|\int\theta_xv_x\right|-\int u\left(\frac{u_x}{v}\right)_x\\&\lesssim\left|\int\theta_xv_x\right|+\int\frac{u_x^2}{v}\lesssim\varepsilon\int\theta v_x^2+\varepsilon\|v_x\|_2^2\int\frac{\theta_x^2}{\theta^2}+C_\varepsilon\int\theta_x^2+C\|u_x\|_2^2,\end{aligned}\tag{3.41}$$

where we have used the fact:

$$\int v_x^2\le C\|v_x\|_2^2\int\frac{\theta_x^2}{\theta^2}+C\int\theta v_x^2.\tag{3.42}$$

In fact, by (3.27), there exits a constant a such that $(\theta^{1/2}-a^{1/2})\lesssim\int\frac{\theta_x^2}{\theta^2}$. Hence,

$$a^{1/2}\int v_x^2=-\int_0^T\Big(\theta^{1/2}-a^{1/2}\Big)v_x^2+\int\theta^{1/2}v_x^2\le\varepsilon\int v_x^2+C_\varepsilon\int\Big(\theta^{1/2}-a^{1/2}\Big)^2v_x^2+C_\varepsilon\int\theta v_x^2$$
$$\le\varepsilon\int v_x^2+C_\varepsilon\int\frac{\theta_x^2}{\theta^2}\int v_x^2+C_\varepsilon\int\theta v_x^2, \tag{3.43}$$

which means that (3.42) is valid. In (3.41), only the term $\|u_x\|_2$ needs to be tackled. Taking inner product $(1.6)_2$ with u over Ω, by Cauchy–Schwarz's inequality, it yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2+c\|u_x\|_2^2\le C\left|\int(\frac{\theta}{v}+b^2)_xu\right|\le\varepsilon\|\theta_x,v_x,b_x\|_2^2+C_\varepsilon\|u\|_2^2$$
$$\le\varepsilon\int\theta v_x^2+\varepsilon\int\frac{\theta_x^2}{\theta^2}\int v_x^2+\varepsilon\|\theta_x,b_x\|_2^2+C_\varepsilon\|u\|_\infty^2. \tag{3.44}$$

By means of Corollary 3.1, (3.41), (3.44), Gronwall's inequality, and (3.39), one can derive that

$$\int\Big(u^2+v_x^2+b_x^2\Big)(t)+\int_0^t\int\Big(b_x^2+\theta v_x^2+u_x^2\Big)(s)\mathrm{d}s\lesssim 1. \tag{3.45}$$

Thanks to (3.42) and (3.45), we have

$$\int_0^t\int v_x^2\mathrm{d}s\le\int v_x^2\int_0^t\int\frac{\theta_x^2}{\theta^2}\mathrm{d}s+\int_0^t\int\theta v_x^2\mathrm{d}s\lesssim 1, \tag{3.46}$$

which combined with (3.37) and (3.45) completes the proof of this lemma. □

Next, we give out the proof of lower bound of θ for the enough large time t. Similar as (2.79), one has

$$\lim_{t\to\infty}\|\theta_x\|_2=0. \tag{3.47}$$

Hence, in the domain $\Omega=(0,1)$, by (4.11), Lemma 2.4, Lemma 3.1, Lemma 2.5, and (3.47), one can deduce

$$\lim_{t\to\infty}\|\theta-\bar{\theta}\|_\infty\le\lim_{t\to\infty}\|\theta-\bar{\theta}\|_2^{1/2}\|\theta_x\|_2^{1/2}=0, \tag{3.48}$$

which yields

$$\theta(x,t)\ge C,\qquad x\in(0,1) \tag{3.49}$$

for the large enough time t.

4. Proof of Theorem 1.1

In order to show the global existence of the solutions to the problems (1.6)–(1.10), or (1.6)–(1.9), (1.11), or (1.6)–(1.9), (1.12), one can use the standard argument in [15,23,34], that is, first to construct an approximate problem in the bounded interval $(0,k)$ and to prove the *a priori* estimates independent of k similar to those obtained in Sects. 2 and 3, then to let k tend to infinity for the purpose of getting the global strong solutions as the limit. We omit the details for proving (1.14), while we deduce the assertions (1.15) and (1.16) of Theorem 1.1 as follows.

4.1. Proof of (1.15)

Applying ∂_x to $(1.6)_1$ and taking inner product with v_x over Ω, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v_x\|_2^2 \le \int (ru)_{xx}v_x \le \|u, v_x, u_x, u_{xx}\|_2^2. \tag{4.1}$$

By (4.1), (2.60), (2.69), and Lemma 2.2-2.5, one can deduce

$$\int_0^\infty \left|\frac{\mathrm{d}}{\mathrm{d}t}\|v_x, u_x, \theta_x\|_2^2\right| \mathrm{d}t \le C, \tag{4.2}$$

which combined with (2.51), Lemma 2.2, and Corollary 2.2 yields

$$\lim_{t\to\infty}\|v_x, b_x, u_x, \theta_x\|_2^2 = 0. \tag{4.3}$$

By means of Lemma 2.1, 2.2 and Lemma 2.6, one has

$$\|v-1\|_2^2 + \|\theta - 1\|_2^2 \lesssim \int \phi(v) + \phi(\theta) \lesssim 1, \tag{4.4}$$

By virtue of $(1.6)_3$, one has

$$b - b_0v_0 = b_0v_0(v-1)/v, \tag{4.5}$$

which combined with (4.4) and Lemma 2.2 yields

$$\|b - b_0v_0\|_2 \lesssim 1. \tag{4.6}$$

By means of (4.4), (4.6), Lemma 2.1, and (4.3), it follows

$$\lim_{t\to\infty}\|v-1, u, b-b_0v_0, \theta-1\|_p = 0, \qquad 2 < p \le \infty. \tag{4.7}$$

The proof of (1.15) is thus completed by (4.3) and (4.7). □

4.2. Proof of (1.16)

By mean value theorem and (3.22), there exists $\alpha(t) \in (0,1)$ such that

$$v(\alpha(t)) = \bar{v}, \qquad b(\alpha(t)) = v_0b_0/\bar{v}. \tag{4.8}$$

By Poincaré's inequality, we have

$$\|v-\bar{v}, u, b - v_0b_0/\bar{v}\|_2 \lesssim \|v_x, u_x, b_x\|_2. \tag{4.9}$$

Multiplying $(1.6)_2$-$(1.6)_3$ by u and $b - v_0b_0/\bar{v}$, respectively, adding them with $(1.6)_4$, and integrating the resulting identity over $[0,t]\times\Omega$, one can deduce

$$\int \left[c_v\theta + \frac{1}{2}(u^2 + v(b - v_0b_0/\bar{v})^2)\right](t,x)\mathrm{d}x = c_v\bar{\theta} \qquad \forall t \in [0,\infty), \tag{4.10}$$

where $\bar{\theta}$ is given by (1.17). We use Poincaré's inequality, Lemma 3.1, and (4.9) to obtain

$$\begin{aligned}\|(\theta-\bar{\theta})(t)\|_2^2 &\le \int |\theta(t,x) - \int \theta(t,y)dy|^2\mathrm{d}x + \|(u, \sqrt{v}(b - v_0b_0/\bar{v})\|_2^2 \\ &\lesssim \|(u_x, b_x, \theta_x)(t)\|_2^2. \end{aligned} \tag{4.11}$$

Taking $\hat{v} = \bar{v}$ and $\hat{\theta} = \bar{\theta}$, by means of (2.3), (3.41), (2.60), (2.69), Lemma 3.1, Lemma 2.6, and (3.49), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\bar{\theta}\phi(\frac{v}{\bar{v}}) + c_v\bar{\theta}\phi\left(\frac{\theta}{\bar{\theta}}\right) + \frac{1}{2}u^2 + \frac{1}{2}v\left(b - \frac{v_0b_0}{\bar{v}}\right)^2\right) + c\int(u_x^2 + \theta_x^2) \le 0, \tag{4.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left[\frac{\lambda}{2}\left(\frac{v_x}{v}\right)^2-\frac{uv_x}{v}\right]+c\int(v_x^2+b_x^2)\le C\|\theta_x,u_x\|_2^2, \tag{4.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_x\|_2^2+c\|u_{xx}\|_2^2\le C\left\|u_x,\theta_x,v_x,b_x\right\|_2^2, \tag{4.14}$$

$$\frac{c_v}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_x\|_2^2+c\|\theta_{xx}\|_2^2\le C\|u_x,u_{xx},\theta_x\|_2^2, \tag{4.15}$$

Choosing suitable small $\eta_1<\eta_2<\eta_3<1$, multiplying (4.13)-(4.15) by η_3, η_2, and η_1, respectively, adding them together with (4.12), we can deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t)+c\|u_x,\theta_x,v_x,b_x,u_{xx},\theta_{xx}\|_2^2\le 0. \tag{4.16}$$

where

$$\begin{aligned}A(t):=&\int\left(\bar\theta\phi\left(\frac{v}{\bar v}\right)+c_v\bar\theta\phi\left(\frac{\theta}{\bar\theta}\right)+\frac12u^2+\frac12v\left(b-\frac{v_0b_0}{\bar v}\right)^2\right)\\&+\eta_3\int\left[\frac{\lambda}{2}\left(\frac{v_x}{v}\right)^2-\frac{uv_x}{v}\right]+\eta_2\|u_x\|_2^2+\frac{\eta_1c_v}{2}\|\theta_x\|_2^2.\end{aligned} \tag{4.17}$$

By means of Cauchy–Schwarz's inequality, Lemma 3.1, Lemma 2.6, (3.49), (4.9), and (4.11), one has

$$A(t)\lesssim\|v-\bar v,\theta-\bar\theta,u,b-v_0b_0/\bar v,u_x,\theta_x\|_2^2\lesssim\|v_x,\theta_x,u_x,b_x\|_2^2. \tag{4.18}$$

By virtue of Cauchy–Schwarz's inequality, one has

$$-\int\frac{uv_x}{v}\ge-\frac{\lambda}{4}\int\frac{v_x^2}{v^2}-C\|u\|_2^2. \tag{4.19}$$

By (4.19), one can choose suitable small $\eta_3>0$ such that

$$\eta_3\int\left[\frac{\lambda}{2}\left(\frac{v_x}{v}\right)^2-\frac{uv_x}{v}\right]\ge\frac{\eta_3\lambda}{4}\int\frac{v_x^2}{v^2}-\frac14\|u\|_2^2. \tag{4.20}$$

Inserting (4.20) into (4.17), by (3.39), Lemma 3.1, and Corollary 3.1, it follows

$$A(t)\gtrsim\|v_x,u_x,\theta_x\|_2^2\gtrsim\|v_x,u_x,\theta_x,b_x\|_2^2. \tag{4.21}$$

By virtue of (4.16), (4.18), and (4.21), one can derive

$$\|v_x,u_x,\theta_x,b_x\|_2^2\lesssim A(t)\lesssim \mathrm{e}^{-ct}, \tag{4.22}$$

which combined with (4.9) and (4.11) completes the proof of (1.16). □

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