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Infinite speed behavior of two-temperature Green–Lindsay thermoelasticity theory under temperature-dependent thermal conductivity

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Abstract. The present work attempts to analyze the effects of temperature-dependent thermal conductivity on thermoelastic interactions in a medium with a spherical cavity under two-temperature Green–Lindsay thermoelasticity theory. An attempt is made to compare the results with the corresponding results under other three thermoelastic models. The thermal conductivity of the material is assumed to be depending affinely on the conductive temperature. It is assumed that the conductive temperature is prescribed at the stress-free boundary of the spherical cavity. Assuming spherical symmetry motion, the resulting thermoelastic system in one space dimension is solved by using the Kirchhoff transformation, Laplace transform technique and expansion in modified Bessel functions. The paper concludes with numerical results on the solution of the problem for specific parameter choices. Various graphs depict the behavior of the conductive and thermodynamic temperature, the displacement and two nonzero components of stress. A detailed analysis of the results is given by showing the effects of the assumed temperature dependence of the material property. The effect of employing the two-temperature model is discussed in detail. We observe an infinite domain of influence under the two-temperature model as compared to the classical Green–Lindsay model, which we hope will be a useful insight.

Mathematics Subject Classification. 74A15.

Keywords. Non-classical heat conduction, Green–Lindsay thermoelasticity, Two-temperature thermoelasticity, Temperature dependency of material parameters, Kirchhoff transformation.

1. Introduction

The coupled thermoelasticity theory based on Fourier law was introduced by Biot [1] and it is well understood that it suffers from the drawback of an infinite speed of propagation of thermal wave due to parabolic type heat conduction equation involved in this theory. Consequently, the classical coupled theory of thermoelasticity [1] has been extended during the last few decades to remove this apparent drawback in the theory. We recall the generalized thermoelasticity theories developed by Lord and Shulman [2], Green and Lindsay [3], Green and Naghdi [4–6], Chandrasekharaiah [7] and Roychoudhuri [8] in this context. We also refer to the book by Ignaczak and Starzewski [9] for detailed discussion of these generalized thermoelastic models.

In the mechanics of continuous media, a material is said to have memory effect or hereditary characteristics if the behavior of the material at time t is specified in terms of the experience of the body up to the time t. Coleman [10] formulated a theory of materials with memory. In 1966, Gurtin and Williams [11] proposed a modified form of the Classius inequality involving two temperatures: the conductive temperature and the thermoelastic temperature. Subsequently, an alternative thermoelasticity theory called two-temperature thermoelasticity theory has been proposed by Chen and Gurtin [12] and Chen et al. [13,14]. This two-temperature thermoelasticity theory proposes that the heat conduction on a deformable body depends on two different temperatures—the conductive temperature and the thermodynamic temperature [11,13,14]. According to the two-temperature thermoelasticity theory, the entropy contribution due to heat conduction is governed by thermodynamic temperature and that of the heat supply by the conductive temperature. The stress, energy, entropy, heat-flux and the thermodynamic temperature at a given time depend on the histories up to that time of the deformation gradient, the conductive temperature and the gradient of this temperature. Chen et al. [13] suggested that the difference between the two temperatures is proportional to heat supply and in case of the absence of heat supply, the two temperatures are equal for a time-independent situation. However, for time-dependent cases, the two temperatures are in general different, regardless of heat supply. By employing the two-temperature theory, thermoelastic wave propagation from the cylindrical and spherical cavity had been studied by Warren [15]. Uniqueness and reciprocity theorems for the two-temperature thermoelasticity theory in case of a homogeneous and isotropic solid medium have been reported by Iesan [16]. Subsequently, wave propagation in the two-temperature theory is investigated by Warren and Chen [17]. This model has also been investigated by Amos [18] and Chakrabarti [19]. The two-temperature thermoelasticity theory is being revisited once again in recent years. Some interesting results on this theory are reported by Puri and Jordan [20] and also by Quintanilla [21]. Youssef [22] extended this theory in the context of the generalized theory of heat conduction and formulated two generalized two-temperature theories of thermoelasticity by providing the uniqueness theorem. Later on, Magana and Quintanilla [23] studied the uniqueness and growth of solutions of this theory. Subsequently, Youssef [24], Youssef and Al-Lehaibi [25], Kumar et al. [26,27], Youssef and Bassiouny [28], Abbas and Youssef [29], Ezzat et al. [30], Mukhopadhyay and Kumar [31], Kumar and Mukhopadhyay [32], Banik and Kanoria [33] and Mukhopadhyay et al. [34] carried out some investigations on two-temperature generalized thermoelasticity and indicated some significant features of the theories given by Youssef [22].

In 1991, Noda [35] reported that the thermoelastic parameters are assumed to be constant in general, but these parameters remain no longer constants for thermoelastic materials at very high temperature. It has been shown with practical results (see Noda [35]) that thermal conductivity of the materials decreases linearly with temperature. Thermoelastic materials at high temperature provide much different practical and theoretical results from the expectations. Therefore, it is quite necessary to consider the dependency of these parameters on temperature in the analysis of the behavior of materials when it is kept at a very high temperature. In recent years, a lot of work has been carried out with generalized theory of thermoelasticity by taking into account the dependency of thermoelastic parameters on temperature. It is worth to recall that Suhara [36] solved a thermoelastic model by considering the shear modulus depending on temperature and discussed the effect in detail. Youssef et al. [37] discussed the results by solving a thermoelastic problem for an unbounded medium with a spherical cavity by assuming that thermal conductivity and modulus of elasticity depend on temperature. A characteristic feature has been discussed for a two-dimensional thermoelastic problem with temperature-dependent elastic moduli by Othman [38–41]. Zenkour and Abbas [42] discussed the effects of temperature -dependent properties of the materials assuming the density and other thermoelastic properties depending on temperature. A study on temperature-dependent thermal conductivity for generalized thermoelasticity theory is reported by Kumar and Mukhopadhyay [43].

The present work analyzed the effects of temperature-dependent thermal conductivity on thermoelastic interactions inside a medium with a spherical cavity under a two-temperature generalized thermoelastic theory that involves two thermal relaxation parameters. The thermal conductivity of the material is assumed to vary with temperature linearly. Initially, the temperature at the boundary of the spherical cavity is assumed to be subjected to a thermal shock, and it is assumed that there is no stress on the surface of the cavity. We solve the problem by using Kirchhoff transformation along with Laplace transform technique. Various graphs are plotted to display the distributions of different field variables like conductive temperature, thermodynamic temperature, displacement and two nonzero components of stress. An attempt is also made to compare the results in the present context with the corresponding results predicted by other thermoelasticity theories. A detailed analysis of the results due to temperature-dependent material properties and effects of employing two-temperature thermoelastic model is presented. We highlight some important features of the present two-temperature model in the context of temperature-dependent thermal conductivity.

2. Governing equations and problem formulation

A thermoelastic process is a coupled dynamical process of an exchange of mechanical energy into thermal energy and vice-versa under the action of externally applied thermo-mechanical loading [9]. Such a process is accompanied by strain and temperature changes inside the body all of which vanish upon the removal of the applied loading. The process can be described in terms of the physical field variables like temperature, displacement vector and strain tensor. The two-temperature thermoelasticity theory is developed on basis of a modified form of the Classius inequality proposed by Gurtin and Williams [11] that involves two different temperatures: the conductive temperature and the thermoelastic temperature. Youssef [22] extended this two-temperature theory in the context of the generalized theory of heat conduction and formulated two generalized two-temperature models of thermoelasticity, namely two-temperature LS (TLS) model and two-temperature GL (TGL) model. In the present study, we consider the TGL model. By following Youssef [22] and Ignaczak and Starzewski [9], the basic equations of two-temperature thermoelasticity with two relaxation parameters (TGL model) for an isotropic elastic medium in absence of heat sources and body forces can be written as follows:

Strain-displacement relation:

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{1}$$

The stress–strain-temperature relation:

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e_{kk}\delta_{ij} - \gamma(\theta + \tau_1\dot{\theta})\delta_{ij} \tag{2}$$

The equation of motion:

$$\sigma_{ji,j} = \rho\ddot{u}_i \tag{3}$$

The energy balance equation:

$$q_{i,i} = -\rho T_0\dot{\eta} \tag{4}$$

The entropy equation:

$$\rho T_0\eta = \rho c_E(\theta + \tau_0\dot{\theta}) + \gamma T_0 e_{kk} \tag{5}$$

The heat conduction law:

$$q_i = -K\phi_{,i} \tag{6}$$

The relation between two temperatures [12]:

$$\phi - \theta = \alpha\phi_{,ii} \tag{7}$$

where u_i are the components of the displacement vector, t is the time, e_{ij} are the components of elastic strain tensor, σ_{ij} are the components of stress tensor, and e_{kk} is the dilatation. θ is the thermodynamic temperature above the reference temperature T_0, and ϕ is the conductive temperature above reference temperature T_0, here $\left|\frac{\phi}{T_0}\right| \ll 1$. λ and μ are the Lamé's constants, ρ is the mass density, $\gamma = (3\lambda + 2\mu)\beta_\tau$, where β_τ is the coefficient of linear thermal expansion, c_E is the specific heat at constant strain, τ_0 and τ_1 $(\tau_1 \geq \tau_0)$ are the thermal relaxation parameters [3,9]. α is the two-temperature parameter. K is the thermal conductivity of the material of the medium.

Now, in the present study, we consider a material whose thermal conductivity K is varying with the conductive temperature, ϕ. Hence, from Eqs. (4–6), the derived heat conduction equation for the TGL theory in the present context is obtained as

$$\left(K\phi_{,i}\right)_{,i} = \rho c_E \left(1 + \tau_0 \frac{\partial}{\partial t}\right) \frac{\partial \theta}{\partial t} + T_0 \gamma \dot{e}_{kk} \tag{8}$$

By combining Eqs. (2) and (3), we have

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ij} - \gamma(\theta_{,i} + \tau_1 \dot{\theta}_{,i}) = \rho \ddot{u}_i \tag{9}$$

Further, we consider the corresponding heat conduction equation and equation of motion for two-temperature thermoelasticity with one relaxation parameter (TLS model [22]) as:

$$\left(K\phi_{,i}\right)_{,i} = \rho c_E \left(1 + \tau_0 \frac{\partial}{\partial t}\right) \frac{\partial \theta}{\partial t} + T_0 \gamma \left(1 + \tau_0 \frac{\partial}{\partial t}\right) \frac{\partial e_{kk}}{\partial t} \tag{10}$$

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ij} - \gamma \theta_{,i} = \rho \ddot{u}_i \tag{11}$$

In order to study the present problem in the contexts of four models, namely LS, TLS, GL and TGL models, we can write the unified form of Eqns. (9–10) as

$$\left(K\phi_{,i}\right)_{,i} = \rho c_E \left(1 + \tau_0 \frac{\partial}{\partial t}\right) \frac{\partial \theta}{\partial t} + T_0 \gamma \left(1 + \xi \tau_0 \frac{\partial}{\partial t}\right) \frac{\partial e_{kk}}{\partial t} \tag{12}$$

$$\mu u_{i,jj} + (\lambda + \mu)\, u_{j,ij} - \gamma \left(\theta_{,i} + \tau_1 \frac{\partial \theta_{,i}}{\partial t}\right) = \rho \ddot{u}_i \tag{13}$$

where ξ is a constant parameter used to write the above equations under four models in a unified way. Hence, Eqs. (2), (7) and Eqs. (12–13) are considered as governing equations for the present study, and these Eqs. (2), (7),(12–13) reduce to the particular set of equations of TGL, GL, TLS and LS models by providing the particular values to the parameters α, τ_0, τ_1 and ξ as follows:

- **TGL model:** $\alpha \neq 0$, $\tau_0 \neq 0$, $\tau_1 \neq 0$, $\xi = 0$.
- **GL model:** $\alpha = 0$, $\tau_0 \neq 0$, $\tau_1 \neq 0$, $\xi = 0$.
- **TLS model:** $\alpha \neq 0$, $\tau_0 \neq 0$, $\tau_1 = 0$, $\xi = 1$.
- **LS model:** $\alpha = 0$, $\tau_0 \neq 0$, $\tau_1 = 0$, $\xi = 1$.

The thermal conductivity K of metals generally decreases exponentially with temperature as mentioned by Noda [36]. Therefore, considering only the first two terms of the exponential function, the variation in thermal conductivity with conductive temperature is taken in linear form as

$$K(\phi) = K_0(1 + K_1 \phi), \tag{14}$$

where K_0 is the thermal conductivity at reference temperature T_0. $K_1 (K_1 \neq 0)$ is a constant to the present study, and its value is zero for those materials whose thermal conductivity is invariant with temperature.

Now, we consider the exterior of a ball of radius, $a(a > 0)$, centered at origin and occupied by a thermoelastic medium at uniform reference temperature T_0. Introducing spherical polar coordinates (r, ϑ, φ) and assuming spherical symmetry, Eqs. (2 , 12) and (13) reduce to

$$\sigma_{rr} = 2\mu \frac{\partial u}{\partial r} + \lambda e - \gamma \left(\theta + \tau_1 \frac{\partial \theta}{\partial t}\right) \tag{15}$$

$$\sigma_{\varphi\varphi} = \sigma_{\vartheta\vartheta} = 2\mu \frac{u}{r} + \lambda e - \gamma \left(\theta + \tau_1 \frac{\partial \theta}{\partial t}\right) \tag{16}$$

$$\left[K \nabla^2 \phi + \frac{\partial K}{\partial \phi} \left(\frac{\partial \phi}{\partial r}\right)^2\right] = \rho c_E \left(1 + \tau_0 \frac{\partial}{\partial t}\right) \frac{\partial \theta}{\partial t} + T_0 \gamma \left(1 + \xi \tau_0 \frac{\partial}{\partial t}\right) \frac{\partial e}{\partial t} \tag{17}$$

$$(\lambda + 2\mu) \frac{\partial e}{\partial r} - \gamma \left(1 + \tau_1 \frac{\partial}{\partial t}\right) \frac{\partial \theta}{\partial r} = \rho \frac{\partial^2 u}{\partial t^2} \tag{18}$$

where u is the single nonzero component of displacement. σ_{rr}, $\sigma_{\varphi\varphi}$ and $\sigma_{\vartheta\vartheta}$ are nonzero stress components, and we use the notations $e = \frac{\partial u}{\partial r} + \frac{2}{r}u$ and $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$.

In view of Eqs. (7, 14), we find that Eq. (17) is nonlinear, and therefore, to tackle the nonlinearity, we consider a new function Φ expressing the temperature with Kirchhoff's transformation as

$$\Phi = \frac{1}{K_0}\int_0^{\phi} K(p)\mathrm{d}p = \phi + \frac{1}{2}K_1\phi^2. \tag{19}$$

Hence, by using Eqs. (14) and (19), we have

$$K_0\frac{\partial \Phi}{\partial r} = K\frac{\partial \phi}{\partial r} \text{ and } K_0\frac{\partial^2 \Phi}{\partial r^2} = K\frac{\partial^2 \phi}{\partial r^2} + \frac{\partial K}{\partial \phi}\left(\frac{\partial \phi}{\partial r}\right)^2.$$

Adding above equations, we get

$$\left[K\nabla^2\phi + \frac{\partial K}{\partial \phi}\left(\frac{\partial \phi}{\partial r}\right)^2\right] = K_0\nabla^2\Phi. \tag{20}$$

Substituting (20) into (17), we obtain

$$K_0\nabla^2\Phi = \rho c_E\left(1 + \tau_0\frac{\partial}{\partial t}\right)\frac{\partial \theta}{\partial t} + T_0\gamma\left(1 + \xi\tau_0\frac{\partial}{\partial t}\right)\frac{\partial e}{\partial t}. \tag{21}$$

Now, for convenience, we use the following symbols and notations in order to make Eqs. (7), (15–16),(18) and (21) dimensionless:

$(r', u') = c_0\eta\,(r, u)$, $\left(t', \tau_0', \tau_1'\right) = c_0^2\eta\,(t, \tau_0, \tau_1)$, $\left(\theta', \phi', \Phi'\right) = \frac{1}{T_0}\,(\theta, \phi, \Phi)$, $e' = e$,

$\sigma_{ij}' = \frac{\sigma_{ij}}{(\lambda+2\mu)}$, $c_0^2 = \frac{(\lambda+2\mu)}{\rho}$, $a_1 = \frac{\gamma T_0}{(\lambda+2\mu)}$, $a_2 = \frac{\gamma}{K_0\eta}$, $\alpha' = \alpha c_0^2\eta^2$, $\beta^2 = \frac{\lambda}{(\lambda+2\mu)}$, where $\eta = \frac{\rho c_E}{K_0}$.

Therefore, Eqs. (7), (15–16), (18) and (21) change to their dimensionless forms as follows:

$$\phi - \theta = \alpha\nabla^2\phi \tag{22}$$

$$\sigma_{rr} = \left(1-\beta^2\right)\frac{\partial u}{\partial r} + \beta^2 e - a_1\left(\theta + \tau_1\frac{\partial \theta}{\partial t}\right) \tag{23}$$

$$\sigma_{\varphi\varphi} = \sigma_{\vartheta\vartheta} = \left(1-\beta^2\right)\frac{u}{r} + \beta^2 e - a_1\left(\theta + \tau_1\frac{\partial \theta}{\partial t}\right) \tag{24}$$

$$\nabla^2\Phi = \left(1 + \tau_0\frac{\partial}{\partial t}\right)\frac{\partial \theta}{\partial t} + a_2\left(1 + \xi\tau_0\frac{\partial}{\partial t}\right)\frac{\partial e}{\partial t} \tag{25}$$

$$\frac{\partial e}{\partial r} - a_1\left(1 + \tau_1\frac{\partial}{\partial t}\right)\frac{\partial \theta}{\partial r} = \frac{\partial^2 u}{\partial t^2} \tag{26}$$

Here, for sake of simplicity of notations, we dropped the primes in Eqs. (22–26).

3. Solution of the problem

We apply Laplace transform to Eqs. (22)–(26) with homogeneous initial conditions and obtain

$$\bar{\phi} - \bar{\theta} = \alpha\nabla^2\bar{\phi} \tag{27}$$

$$\bar{\sigma}_{rr} = \left(1-\beta^2\right)\frac{\partial \bar{u}}{\partial r} + \beta^2\bar{e} - b_0(s)\bar{\theta} \tag{28}$$

$$\bar{\sigma}_{\varphi\varphi} = \bar{\sigma}_{\nu\nu} = \left(1-\beta^2\right)\frac{\bar{u}}{r} + \beta^2\bar{e} - b_0(s)\bar{\theta} \tag{29}$$

$$\nabla^2\bar{\Phi} = b_{11}(s)\bar{\theta} + a_2 b_{12}(s)\bar{e} \tag{30}$$

$$\frac{\partial \bar{e}}{\partial r} - b_0(s)\frac{\partial \bar{\theta}}{\partial r} = s^2\bar{u} \tag{31}$$

where s is the Laplace transform parameter, and we used the notations $b_0(s) = a_1(1+\tau_1 s)$, $b_{11}(s) = s(1+\tau_0 s)$ and $b_{12}(s) = s(1+\xi\tau_0 s)$.

Now, operating $(\frac{\partial}{\partial r} + \frac{2}{r})$ to both sides of Eq. (31), we get

$$\nabla^2\bar{e} - b_0(s)\nabla^2\bar{\theta} = s^2\bar{e} \tag{32}$$

From Eqs. (30) and (27) and using the fact $\left|\frac{\phi}{T_0}\right| << 1$, we find

$$b_2(s)\nabla^2\bar{\Phi} = b_{11}(s)\bar{\Phi} + a_2 b_{12}(s)\bar{e}, \tag{33}$$

where $b_2(s) = 1 + \alpha b_{11}(s)$.

Similarly, by using Eq. (27) into Eq. (32), we get the following equation:

$$\alpha b_0(s)\nabla^4\bar{\Phi} - b_0(s)\nabla^2\bar{\Phi} + \nabla^2\bar{e} = s^2\bar{e}. \tag{34}$$

Applying Eq. (33) into (34), we find

$$m_0(s)\nabla^4\bar{\Phi} - m_1(s)\nabla^2\bar{\Phi} + m_2(s)\bar{\Phi} = 0, \tag{35}$$

where $m_0(s) = a_2\alpha b_0(s)b_{12}(s) + b_2(s)$, $m_1(s) = a_2 b_0(s)b_{12}(s) + b_{11}(s) + s^2 b_2(s)$, $m_2(s) = s^2 b_{11}(s)$.

Further, using Eqs. (33) into (34), we find

$$p_0(s)\bar{\Phi} + p_1(s)\nabla^2\bar{e} + p_2(s)\bar{e} = 0, \tag{36}$$

where $p_0(s) = b_0(s)b_{11}(s)\left[\alpha b_{11}(s) - b_2(s)\right]$, $p_1(s) = b_2(s)\left[a_2\alpha b_0(s)b_{12}(s) + b_2(s)\right]$,
$p_2(s) = a_2\alpha b_0(s)b_{11}(s)b_{12}(s) - a_2 b_0(s)b_{12}(s)b_2(s) - s^2 b_2^2(s)$.

Applying Eq. (36) into Eq. (33), we have

$$n_0(s)\nabla^4\bar{e} - n_1(s)\nabla^2\bar{e} + n_2(s)\bar{e} = 0, \tag{37}$$

where $n_0(s) = b_2(s)p_1(s)$, $n_1(s) = p_1(s)b_{11}(s) - p_2(s)b_2(s)$, $n_2(s) = a_2 b_{12}(s)p_0(s) - b_{11}(s)p_2(s)$.

Equations (35) and (37) can be put in the forms

$$\left[\nabla^2 - c_1(s)\right]\left[\nabla^2 - c_2(s)\right]\bar{\Phi} = 0 \tag{38}$$

$$\left[\nabla^2 - d_1(s)\right]\left[\nabla^2 - d_2(s)\right]\bar{e} = 0 \tag{39}$$

where $c_j = \pm c_j(s)$ and $d_j = \pm d_j(s)$, $(j = 1, 2)$ are the roots of the equation $m_0(s)x^4 - m_1(s)x^2 + m_2(s) = 0$ and $n_0(s)x^4 - n_1(s)x^2 + n_2(s) = 0$, respectively.

The factors of Eqs. (38) and (39) are in the form of modified spherical Bessel differential equations. Therefore, its solution bounded at infinity can be taken in the following forms:

$$\bar{\Phi}(r, s) = \frac{1}{\sqrt{r}}\sum_{j=1}^{2} A_j(s)K_{1/2}[c_j(s)r] \tag{40}$$

$$\bar{e}(r, s) = \frac{1}{\sqrt{r}}\sum_{j=1}^{2} B_j(s)K_{1/2}[d_j(s)r] \tag{41}$$

where $A_j = A_j(s)$ and $B_j = B_j(s)$ are arbitrary constants independent from r, but dependent on Laplace transform parameter s. $K_\nu(r)$ is the representation of modified Bessel function of order ν of the second kind.

Using Eqs. (40) , (41) and (33), we have

$$K_{1/2}[d_j(s)r]B_j(s) = b_3^{(j)}(s)K_{1/2}[c_j(s)r]A_j(s), \quad j = 1, 2, \tag{42}$$

where $b_3^{(j)}(s) = \frac{b_2(s)c_j^2 - b_{11}(s)}{a_2 b_{12}(s)}$.

Using Eqs. (27), (40) and (41) into (31), we have

$$\bar{u}(r,s)=\frac{1}{\sqrt{r}}\left[-\sum_{j=1}^{2}\frac{d_j(s)}{s^2}K_{3/2}[d_j(s)r]B_j(s)+\sum_{j=1}^{2}b_4^{(j)}(s)K_{3/2}[c_j(s)r]A_j(s)\right], \tag{43}$$

where $b_4^{(j)}(s)=\frac{b_0(s)c_j\left[1-\alpha c_j^2(s)\right]}{s^2}$, $j=1,2$.

From Eqs. (28) and (27), we get

$$\begin{aligned}\bar{\sigma}_{rr}=&\frac{1}{\sqrt{r}}\sum_{j=1}^{2}\left[\frac{\left(1-\beta^2\right)\left(d_j(s)\right)^2}{s^2}+\beta^2\right]K_{1/2}[d_j(s)r]B_j(s)\\&+\frac{1}{r^{3/2}}\sum_{j=1}^{2}\frac{2d_j(s)\left(1-\beta^2\right)}{s^2}K_{3/2}[d_j(s)r]B_j(s)\\&-\frac{1}{\sqrt{r}}\sum_{j=1}^{2}\left[\left(1-\beta^2\right)c_j(s)+\frac{s^2}{c_j(s)}\right]b_4^{(j)}(s)K_{1/2}[c_j(s)r]A_j(s)\\&-\frac{1}{r^{3/2}}\sum_{j=1}^{2}2\left(1-\beta^2\right)b_4^{(j)}(s)K_{3/2}[c_j(s)r]A_j(s)\end{aligned} \tag{44}$$

Similarly, from Eqs. (29) and (27) , we have

$$\begin{aligned}\bar{\sigma}_{\varphi\varphi}=\bar{\sigma}_{\vartheta\vartheta}=&\frac{1}{\sqrt{r}}\sum_{j=1}^{2}\beta^2K_{1/2}[d_j(s)r]B_j(s)-\frac{1}{r^{3/2}}\sum_{j=1}^{2}\frac{d_j(s)\left(1-\beta^2\right)}{s^2}K_{3/2}[d_j(s)r]B_j(s)\\&-\frac{1}{\sqrt{r}}\sum_{j=1}^{2}\frac{s^2b_4^{(j)}}{c_j(s)}K_{1/2}[c_j(s)r]A_j(s)+\frac{1}{r^{3/2}}\sum_{j=1}^{2}\left(1-\beta^2\right)b_4^{(j)}(s)K_{3/2}[c_j(s)r]A_j(s)\end{aligned} \tag{45}$$

Boundary conditions

We consider the boundary $r=a$ of the spherical cavity is traction free and is subjected to a unit step increase in temperature. Therefore, the boundary conditions in the dimensionless forms can be written as:

$$\phi(r,t)=\phi_0^*H(t)\text{ and }\quad\sigma_{rr}(r,t)=0\text{ at }r=a \tag{46}$$

where ϕ_0^* is a constant temperature, and $H(t)$ is the Heaviside unit step function.

Therefore, using Eq. (19) and applying Laplace transform to the boundary conditions given by (46), we find

$$\bar{\Phi}(a,s)=\frac{\phi_0^*}{s}\left(1+\frac{1}{2}K_1\phi_0^*\right),\quad\bar{\sigma}_{rr}(a,s)=0. \tag{47}$$

From Eqs. (40), (42) and (47) , we obtain a linear system of two equations as given by

$$X_1A_1+X_2A_2=\frac{\phi_0^*}{s}\left(1+\frac{1}{2}K_1\phi_0^*\right) \tag{48}$$

$$Y_1A_1+Y_2A_2=0 \tag{49}$$

where

$$X_j = \frac{1}{\sqrt{a}} K_{1/2}(c_j a), \quad j = 1, 2.$$

$$Y_j = \frac{1}{\sqrt{a}} \left[\frac{\left(1-\beta^2\right)\left(d_j(s)\right)^2}{s^2} + \beta^2 \right] b_3^{(j)} K_{1/2}[c_j(s)a] + \frac{2d_j(s)\left(1-\beta^2\right) b_3^{(j)}}{s^2 a^{3/2}} \frac{K_{1/2}[c_j(s)a] K_{3/2}[d_j(s)a]}{K_{1/2}[d_j(s)a]}$$
$$- \frac{1}{\sqrt{a}} \left[\left(1-\beta^2\right) c_j(s) + \frac{s^2}{c_j(s)} \right] b_4^{(j)}(s) K_{1/2}[c_j(s)a] - \frac{2\left(1-\beta^2\right) b_4^{(j)}(s)}{a^{3/2}} K_{3/2}[c_j(s)a], \quad j = 1, 2.$$

After solving Eqs. (48)–(49), we can find the unknowns $A_j(s)$ $(j = 1, 2)$ and hence the constants $B_j(s)$, $j = 1, 2$, are derived from Eq. (42). Displacement, radial stress and circumferential stress can be obtained with the help of Eqs. (43)–(45), respectively and conductive temperature $\bar{\phi}$ can be obtained by combining Eqs. (40) with (19). Finally, we obtain the solution for thermodynamic temperature θ in the Laplace transform domain by using Eq. (30). This completes the solution of the present problem in the Laplace transform domain.

4. Numerical results and discussion

The solution in the physical space-time domain can be recovered by taking the inverse Laplace transforms of the solutions in Laplace transform domain obtained in the previous section. However, solutions of $\bar{\theta}$, $\bar{u}$, $\bar{\sigma}_{rr}$, $\bar{\sigma}_{\phi\phi}$ involve $K_{1/2}(s)$ and complicated expressions of $A_j(s)$, $c_j(s)$, $d_j(s)$. Hence, to obtain inverse Laplace transforms of these functions analytically is highly complicated and closed form analytical solution of field variables in the space-time domain is a formidable task. Therefore, we make an attempt to obtain the inversion numerically. We employ the numerical method of Laplace inversion given by Graver–Stehfest [44] and find the solution of physical variables like conductive temperature, thermodynamic temperature, displacement, radial stress and circumferential stress by using computer programming in MATLAB software. According to Graver–Stehfest [44,45] method, if $\overline{f}(s)$ is the Laplace inverse of the function $f(t)$, then

$$f(t) = \frac{\ln(2)}{2} \sum_{k=1}^{N} V_k \overline{f} \left(k \frac{\ln(2)}{t} \right), \tag{50}$$

where N is a suitable positive integer and V_k is given by

$$V_k = (-1)^{(k+N/2)} \sum_{j=[(k+1)/2]}^{\min(k,N/2)} \frac{j^{\frac{N}{2}} (2j)!}{\left(\frac{N}{2} - j\right)!\, j!\, (j-1)!\, (k-j)!\, (2j-k)!}. \tag{51}$$

We consider copper material and the physical data as follows [46]:
$\lambda = 7.76 \times 10^{10}$ N m^{-2}, $\mu = 3.86 \times 10^{10}$ N m^{-2}, $\beta_\tau = 1.78 \times 10^{-5}$ K^{-1}, $\eta = 8886.73$ s m^{-2}, $c_E = 383.1$ J Kg^{-1} K^{-1}, $\rho = 8954$ Kg m^{-3}, $T_0 = 293$ K.
We assume the following dimensionless values of the constants:
$\alpha = 0.071301$, $\tau_0 = 0.01$, $\tau_1 = 0.02$, $\phi_0^* = 1$, $a = 1$.

We make an attempt to discuss the effects of temperature-dependent thermal conductivity on the behavior of physical field variables in the context of TGL model. We further aim to compare the results under all four models, namely LS model, GL model, TGL model and TLS model. Hence, we compute the numerical values of physical variables, u, ϕ, σ_{rr}, $\sigma_{\varphi\varphi}$ and θ, at different time and show the results in different graphs. Figures 1a, 2a, 3a, 4a and 5a show the variations of displacement, conductive temperature, thermodynamic temperature, radial stress and hoop stress, respectively, under TGL theory for

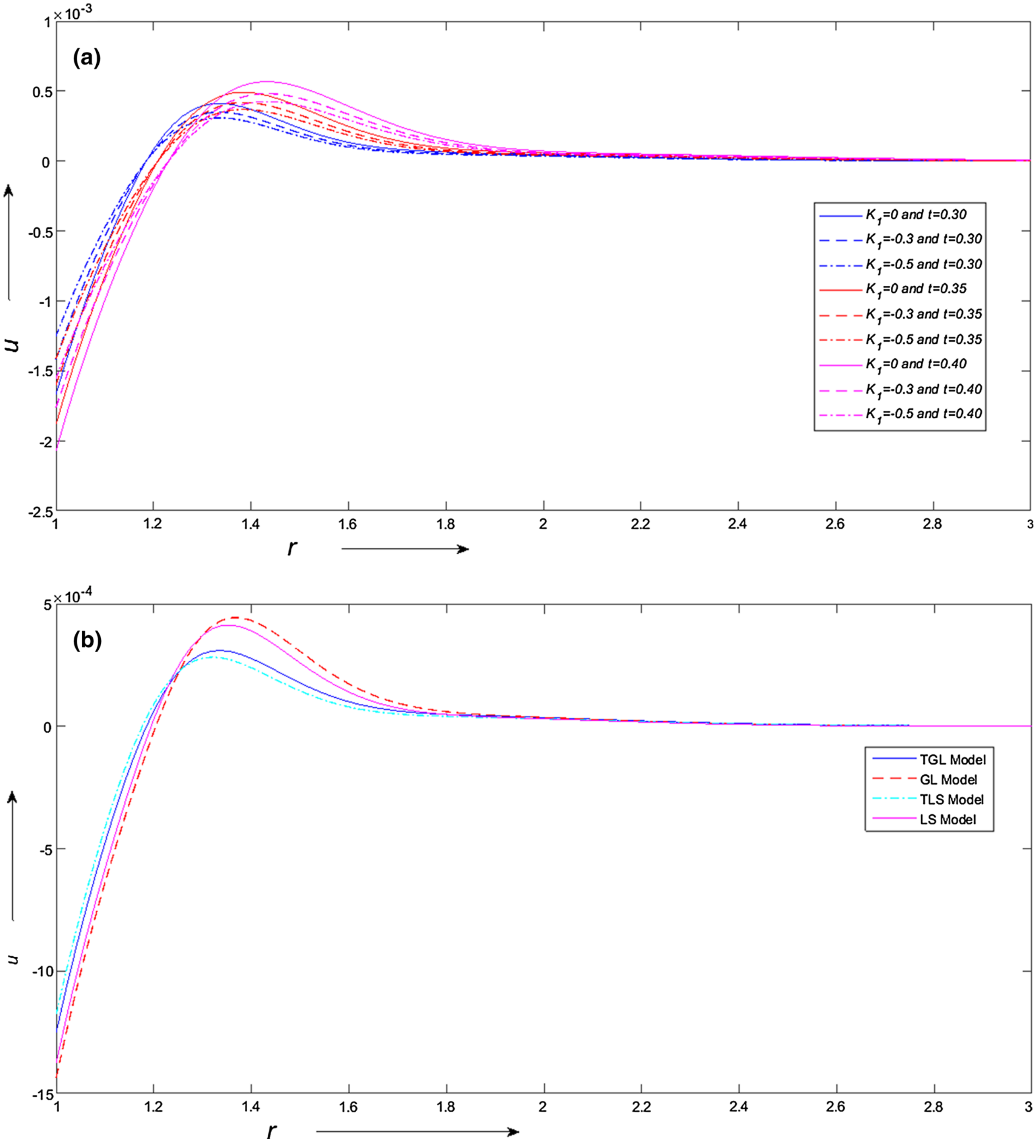

FIG. 1. **a** Variation of u vs. r under TGL model for different values of t and K_1, **b** Variation of u vs. r under different models for $K_1 = -0.5$ and $t = 0.3$

different values of the parameter K_1 and at different non-dimensional time ($t = 0.30$, $t = 0.35$, $t = 0.40$). Figures 1b, 2b, 3b, 4b and 5b reveal the variations of all the field variables under TGL, GL, TLS and LS thermoelasticity models for $K_1 = -0.5$ and at $t = 0.3$. We find the following observations:

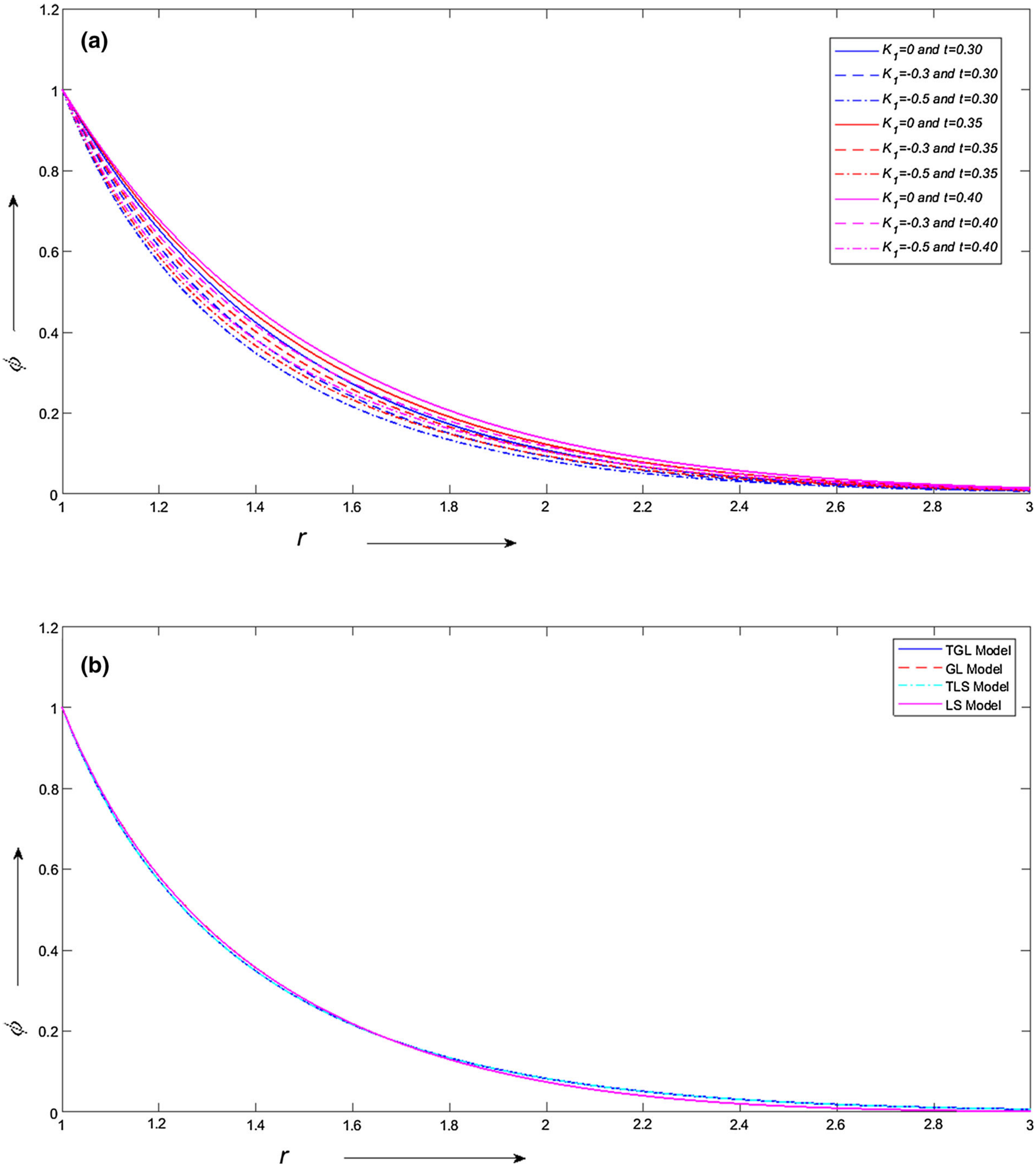

Fig. 2. **a** Variation of ϕ vs. r under TGL model for different values of t and K_1, **b** Variation of ϕ vs. r under different models for $K_1 = -0.5$ and $t = 0.3$

Displacement (u)

Figure 1a displays the effects of K_1 on displacement for TGL model and indicates that the influence of the temperature-dependent effect parameter (K_1) on displacement at any time is significant. The influence

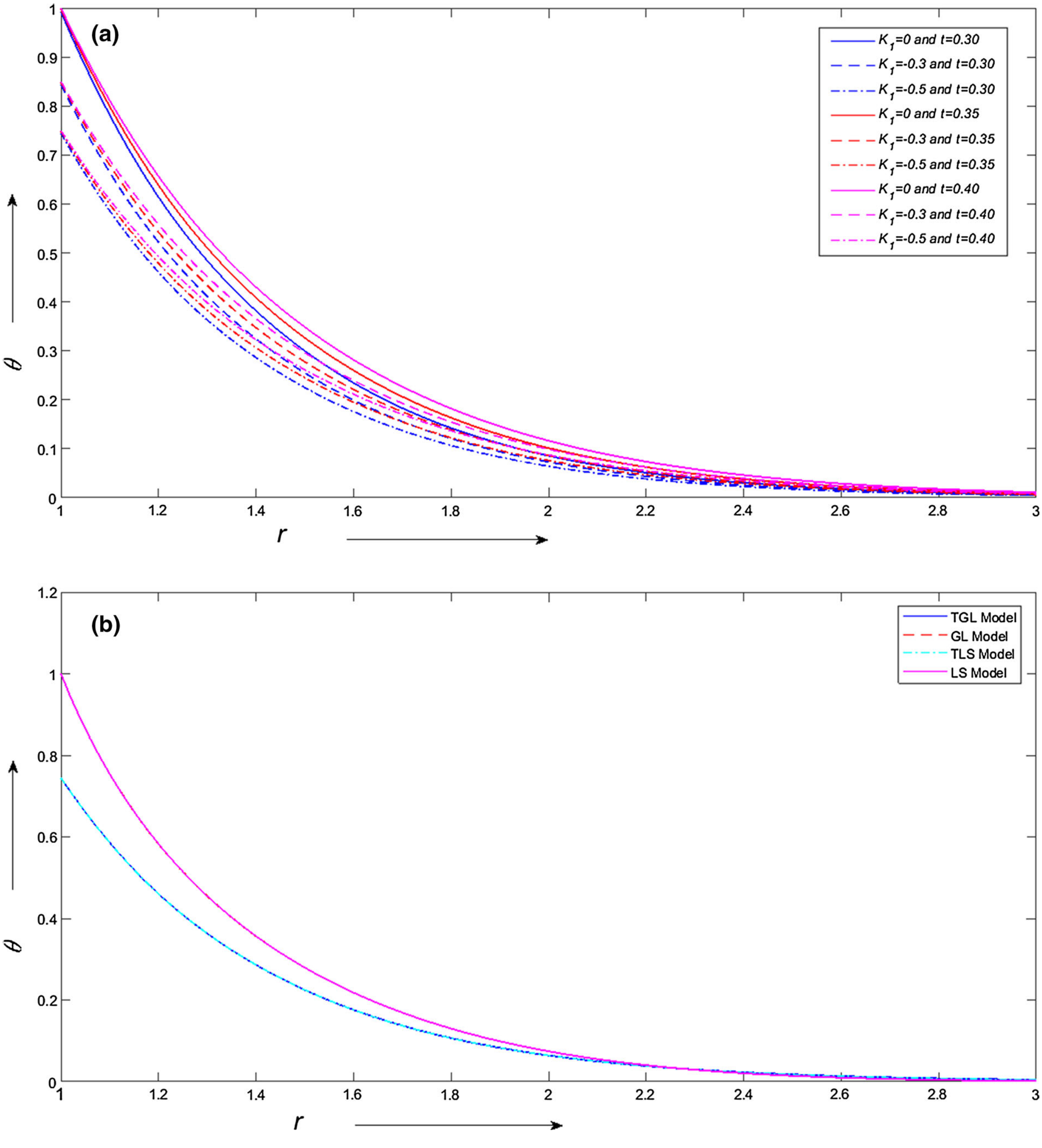

Fig. 3. **a** Variation of θ vs. r under TGL model for different values of t and K_1, **b** Variation of θ vs. r under different models for $K_1 = -0.5$ and $t = 0.3$

region is observed to be dominant near the boundary of the spherical cavity, and the influence becomes insignificant when we move away from the boundary of the cavity. It is observed that at any time, the displacement increases as the value of K_1 goes to higher negative values and then starts decreasing for

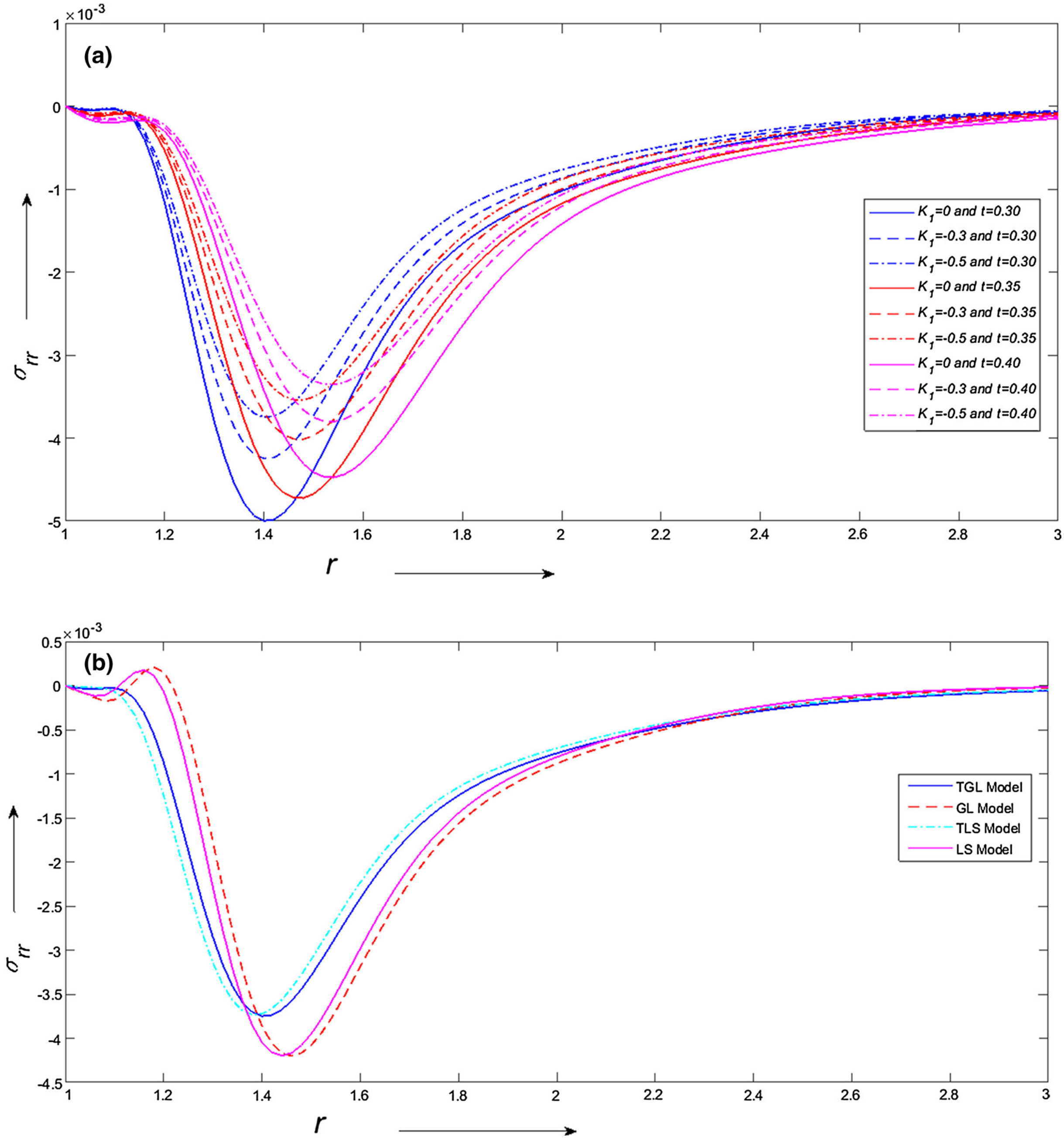

FIG. 4. **a** Variation of σ_{rr} vs. r under TGL model for different values of t and K_1, **b** Variation of σ_{rr} vs. r under different models for $K_1 = -0.5$ and $t = 0.3$

higher negative values of K_1. It is further observed that displacement is inversely proportional to t for a fixed value of K_1, i.e., u decreases with increase in time, t. However, the effect of temperature dependence is more significant at higher time.

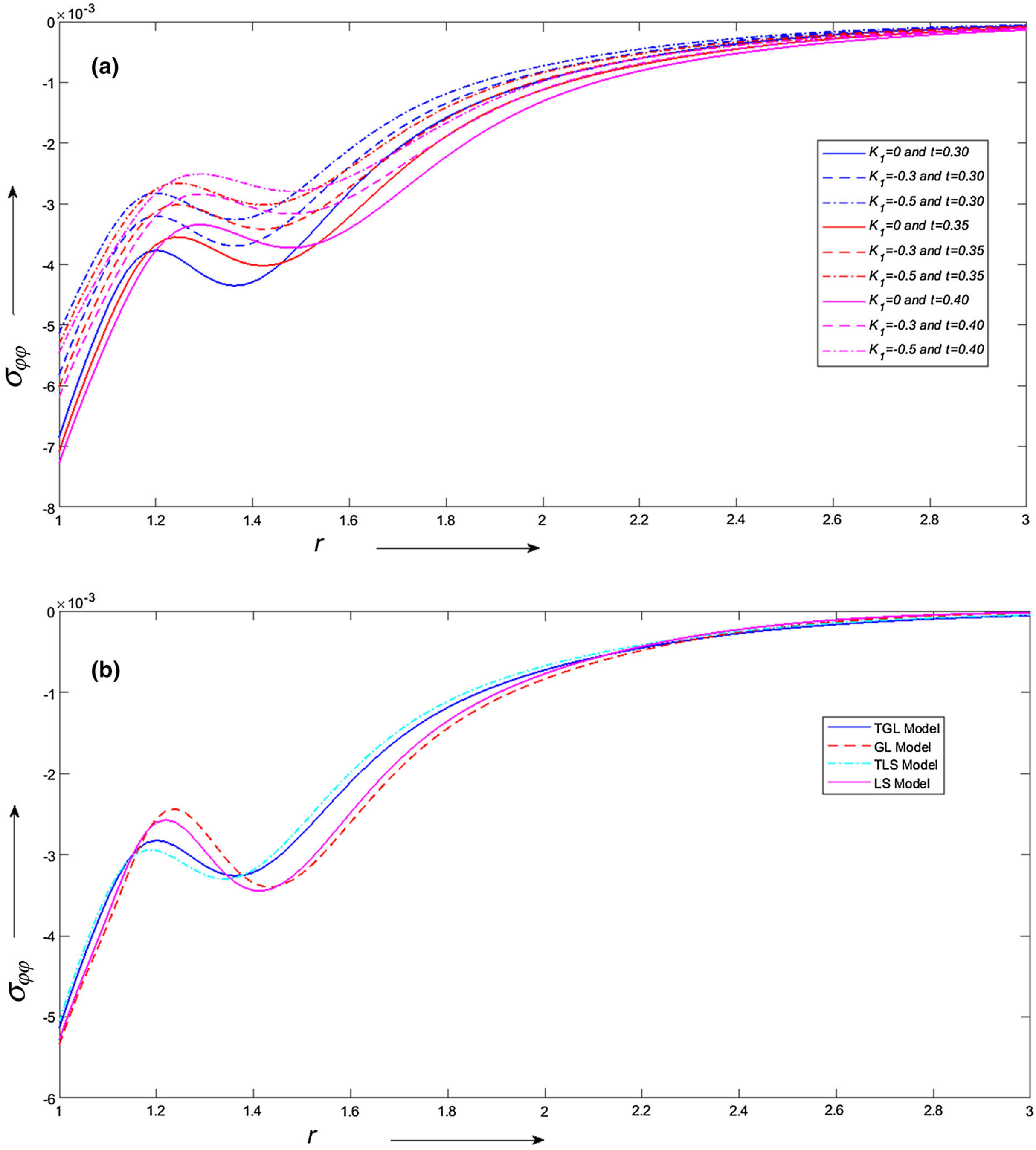

FIG. 5. **a** Variation of $\sigma_{\varphi\varphi}$ vs. r under TGL model for different values of t and K_1, **b** Variation of $\sigma_{\varphi\varphi}$ vs. r under different models for $K_1 = -0.5$ and $t = 0.3$

We can observe the difference in results for displacement field predicted by different models from Fig. 1b. It is noted that there is no prominent difference in displacement for GL and LS models or between TGL and TLS thermoelasticity theories. However, the difference is notable when we compare the results under the models with and without a second temperature. Near the boundary of the cavity, displacement has lesser numerical value for two-temperature thermoelasticity theory, and at a distance from the boundary, it shows the reverse behavior. Further, the domain of influence is larger for two-temperature theory.

Conductive temperature (ϕ)

Figure 2a, b displays the variation of conductive temperature. It is indicated from Fig. 2a that distribution of ϕ is in agreement with the given boundary condition. It further shows that for TGL thermoelasticity theory, the effect of the temperature-dependent parameter K_1 is much prominent at any time. It is observed that the conductive temperature decreases as the value of K_1 goes to higher negative values and it is directly proportional to t. From the figure, it is clear that the conductive temperature is bounded at infinity (i.e., $\phi \to 0$ as $r \to \infty$) as we were interested in such solution.

Similar to displacement, Fig. 2b shows that the variation in ϕ is not prominent between the results under TGL and TLS models. However, there is a notable difference between two-temperature thermoelasticity theory and generalized thermoelasticity theory in the absence of the second temperature.

Thermodynamic temperature, θ

The variation of thermodynamic temperature (θ) for TGL model for different values of parameter K_1 at different time is displayed in Fig. 3a. It is observed that θ is largely affected by temperature-dependent parameter, K_1.

However, a similar variation in thermodynamic temperature is observed for a particular value of K_1. The thermodynamic temperature decreases for higher negative values of K_1. However, θ decreases as time increases for a fixed value of the parameter, K_1 under TGL model.

Figure 3b shows the variation of thermodynamic temperature under four different models for a fixed value of the parameter, K_1 at a fixed time. Here, it is observed that like the case of displacement, there is no prominent difference in variation of θ for TGL and TLS models or between GL and LS models, but the difference in the variation of θ is significant for a two-temperature theory and a theory without second temperature. The thermodynamic temperature has higher values under GL and LS models as compared to the two-temperature models. The thermodynamic temperature converges to zero for all four models with the same convergence rate. It is clearly notable from Figs. 2b and 3b that there is a significant difference in results for conductive temperature and thermodynamic temperature under two -temperature theories. However, they are the same under LS theory and GL theory.

Discussion on stress components ($\sigma_{rr}, \sigma_{\varphi\varphi}$)

The variation of radial stress (σ_{rr}) is displayed in Fig. 4a, b. Both of these figures reveal that the radial stress is in agreement with provided boundary condition and it is compressive in nature under TGL model. It is evident from Fig. 4a that with respect to TGL model, the variation in radial stress is greatly affected by the temperature-dependent effect parameter at any time. Furthermore, σ_{rr} increases when K_1 gets higher negative values. It is also observed that near the boundary of the cavity, σ_{rr} increases when t increases and it starts decreasing for higher value of r. Figure 4b shows that the variation in σ_{rr} is prominent only with respect to a two-temperature theory and a theory without second temperature as in the case of displacement distribution. However, the difference between TGL model and TLS model or between GL model and LS model is not very prominent.

The variation of circumferential stress ($\sigma_{\varphi\varphi}$) is shown in Fig. 5a, b. Figure 5(a) shows that the circumferential stress is compressive in nature and the variation in $\sigma_{\varphi\varphi}$ is prominently different with respect to different values of the parameter K_1. Like the case of other field variables, $\sigma_{\varphi\varphi}$ decreases with higher negative values of K_1 for a fixed time. Here, we again note that like the cases of other field variables, the difference in the nature of circumferential stress distribution under TGL model and GL model or between TLS and LS model is very much significant.

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