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On the formal origins of dark energy

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Abstract. The existence of so-called dark energy and matter in the universe implies that the conventional accounting of mass and energy is incorrect. Here, we use the framework of special relativity and validation through Lorentz invariance to develop an alternative accounting of mass and energy. We assume the usual Einstein relations of special relativity, but we make the distinction between the particle energy $e = mc^2$ and the actual work done by the particle $\mathscr{E}$, and we adopt the perspective that it is not just the momentum vector $\mathbf{p} = m\mathbf{u}$ that contributes to the work done $\mathscr{E}$, but rather the intrinsic particle energy e itself plays an important role through the combined potentials $(\mathbf{p}, e/c)$ as a well-defined four vector within special relativity. The resulting formulation provides a natural extension of Newton's second law, emerges as a fully consistent development of special relativity that is properly invariant under the Lorentz group, and yields an extension of Einstein's famous equation for the work done involving new terms. The new work done expressions can involve the log function and possibly generate extremely large energies that might well represent the first formal indication of the origin of dark energy. Two alternative expressions are both well defined as a limiting case for energy–mass waves travelling at the speed of light and are in complete accord with well-established theory for photons and light for which energy is known to vary linearly with momentum. The present formulation suggests that large energies might be generated even for slowly moving systems, and that dark energy might arise in consequence of conventional mechanical theory neglecting the work done in the direction of time.

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1. Introduction

The purpose of this paper is to show that within the confines of Lorentz invariance and special relativity, there are alternative energy accounting procedures available, for which it is not difficult to envisage the generation of large energies that are well in excess of that predicted by Einstein's expression. Einstein's formulae describing the variation with velocity u for the energy and mass of a particle $e = mc^2$ and $m(u) = m_0[1 - (u/c)^2]^{-1/2}$, where m_0 denotes the rest mass and c is the speed of light, have been overwhelmingly verified in our local environment. On a cosmological scale, our understanding and our accounting for energy and mass are not as successful, and remain improperly understood (see for example [5]). Einstein's formulae are based on the assumption that the particle energy e accrues from and coincides with the work done $\mathscr{E}$ and is derived from an energy rate equation $\mathrm{d}\mathscr{E}/\mathrm{d}t = \mathbf{u}.(\mathrm{d}\mathbf{p}/\mathrm{d}t)$, sometimes referred to as the rate-of-working equation, where $\mathbf{p} = m\mathbf{u}$ is the momentum and $\mathbf{u}$ is the velocity vector. Here, we make a distinction between particle energy e and the work done $\mathscr{E}$, but we recognise the importance of special relativity, and seek to develop a theory in a manner that embraces the essential features of the theory of special relativity.

Assuming all quantities are both position $\mathbf{x}$ and time t dependent and assuming the usual formulae of special relativity, namely $e = mc^2$ and $m(u) = m_0[1 - (u/c)^2]^{-1/2}$, we make the distinction between the particle energy $e = mc^2$ and the actual work done by the particle $\mathscr{E}$. We suggest that it is not just the momentum vector $\mathbf{p} = m\mathbf{u}$ that contributes to the work done $\mathscr{E}$, but also the intrinsic particle energy

e itself plays an important role through the combined potentials $(\mathbf{p}, e/c)$ as a well-defined four vector within special relativity. Specifically, we propose that the force $\mathbf{f}$ and energy–mass production g are such that $(\mathbf{f}, gc)$ constitutes a well-defined four vector and we propose

$$\mathbf{f} = \frac{\partial \mathbf{p}}{\partial t} + \nabla e, \qquad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \nabla \cdot \mathbf{p}, \tag{1.1}$$

as the formal extension of Newton's second law, noting that the second equation of (1.1) is merely the generally accepted mass continuity equation but including an energy–mass production term g. We observe that the second equation might be viewed as Newton's second law in the direction of time, and that clearly this formulation is by no means novel and is well rooted in traditional mechanics. We observe that these relations are left unchanged by the gauge transformation

$$\mathbf{p}' = \mathbf{p} + \nabla \psi, \qquad e' = e - \frac{\partial \psi}{\partial t},$$

where $\psi(\mathbf{x}, t)$ satisfies the wave equation

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \nabla^2 \psi.$$

In the following section, it is proved that both $\mathbf{f}$ and g are properly invariant under the Lorentz group so that $(\mathbf{f}, gc)$ is a well-defined four vector.

In attempting to extend the conventional notion of work done, say $\mathrm{d}W$ from the usual notion of force times distance, we might propose that the incremental work done $\mathrm{d}W$ arises as the scalar product of the two four vectors $(\mathbf{f}, gc)$ and $(\mathbf{dx}, c\mathrm{d}t)$, thus

$$\mathrm{d}W = \mathbf{f} \cdot \mathbf{dx} + gc^2 \mathrm{d}t = \left(\frac{\partial \mathbf{p}}{\partial t} + \nabla e\right) \cdot \mathbf{dx} + \left(\frac{\partial e}{\partial t} + c^2 \nabla \cdot \mathbf{p}\right) \mathrm{d}t, \tag{1.2}$$

which immediately simplifies to yield

$$d(W - e) = \frac{\partial \mathbf{p}}{\partial t} \cdot \mathbf{dx} + c^2 (\nabla \cdot \mathbf{p}) \mathrm{d}t. \tag{1.3}$$

and from which it is clear that the particle energy e appears twice in the work done calculation; once directly on the left-hand side and once through the first term on the right-hand side. Accordingly, in attempting to generalise the conventional work done equation $\mathrm{d}\mathscr{E} = (\partial \mathbf{p}/\partial t).\mathbf{dx}$, we propose

$$d\mathscr{E} = \frac{\partial \mathbf{p}}{\partial t} \cdot \mathbf{dx} + c^2 (\nabla \cdot \mathbf{p}) \mathrm{d}t. \tag{1.4}$$

The resulting formulation (1.4) is clearly different from the usual rate-of-working equation, and embodies an entirely new term $c^2(\nabla \cdot \mathbf{p})\mathrm{d}t$ that does not appear in the conventional approach, and it is this term that might be identified as the formal source of dark energy. It is important to emphasise that even if g is taken to be zero, as is normally the case, equation (1.4) still gives different results to conventional theory. The two terms of (1.4) produce two extreme limits of the proposed theory, corresponding to either purely temporal or purely spatial and yielding, respectively, for one spatial dimension, the two suggestive equations $\mathrm{d}\mathscr{E}/\mathrm{d}p = u$ and $\mathrm{d}\mathscr{E}/\mathrm{d}p = c^2/u$, which are of course, the particle and de Broglie wave velocities, respectively. Subsequently, we show that these two extreme limits arise as special cases of the more general Lorentz-invariant equation

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}p} = c\left(\frac{\lambda + \sin\phi}{1 + \lambda \sin\phi}\right) = c\left(\frac{\lambda + u/c}{1 + \lambda u/c}\right),$$

where $u = c\sin\phi$, and involving the arbitrary constant λ for which the above two special cases correspond, respectively, to the values $\lambda = 0$ and $\lambda = \pm\infty$. Further, we will show that the work done expressions corresponding to these two extreme cases are given, respectively, by

$$\mathscr{E}(\phi) = \frac{e_0}{\cos\phi}, \quad \mathscr{E}(\phi) = e_0 \left\{ \frac{1}{\cos\phi} + \log\left(\tan\frac{\phi}{2}\right)\right\},$$

in terms of the angle ϕ, or directly in terms of the velocity u by

$$\mathscr{E}(u) = \frac{e_0}{(1-(u/c)^2)^{1/2}},$$

$$\mathscr{E}(u) = e_0 \left\{ \frac{1}{(1-(u/c)^2)^{1/2}} + \log\left(\frac{(u/c)}{1+(1-(u/c)^2)^{1/2}}\right)\right\}, \tag{1.5}$$

where $e_0 = m_0c^2$ is the rest mass energy, noting that the first is evidently the Einstein expression, and the second is singular at both $u = 0$ and $u = \pm c$. From the second, it is not difficult to envisage that extremely large negative energies might be generated for relatively slowly moving bodies. For all other values of λ, there are two distinct work done expressions corresponding to $\pm\lambda$ (see Eqs. (4.3) and (4.4)) and it is only at the extremes that the two work done expressions coalesce. In general, the log terms in Eq. (4.4) become singular whenever

$$\tan\phi^* = \frac{f_0}{e_0}(\lambda x^* + ct^*) = \pm\frac{1}{(\lambda^2-1)^{1/2}}, \qquad \frac{u^*}{c} = \frac{\pm 1}{\lambda},$$

where f_0 is the constant appearing in the force equation (3.9), x^*, t^*, ϕ^* and u^* designate the values at the singularity, and the momentum p is related through the expression $pc = e_0 \tan\phi$, while the equation $u^*/c = \pm 1/\lambda$ infers that at the singularity the particle velocity u^* coincides with the phase velocity $-c/\lambda$, which means that at this critical juncture, the particles move with the wave and are not left behind.

Further, simply assuming the existence of either W or $\mathscr{E}$ imposes certain constraints, and it is apparent from (1.2) that $\mathbf{f}$ and g must satisfy the compatibility condition

$$\frac{\partial \mathbf{f}}{\partial t} = c^2 \nabla g, \tag{1.6}$$

in order that either (1.2) or (1.4) represent well-defined differential relations for W and $\mathscr{E}$, respectively. This is a new equation that generates an important constraint on allowable solutions. From (1.6), it is clear that the momentum $\mathbf{p}$ satisfies

$$\frac{\partial^2 \mathbf{p}}{\partial t^2} = c^2\nabla(\nabla\cdot\mathbf{p}) = c^2\nabla^2\mathbf{p} + c^2\nabla_\wedge(\nabla_\wedge\mathbf{p}),$$

which in one spatial dimension simply becomes the wave equation, and further wave-like solutions of this equation are presented in Appendix A. We subsequently show that the operator appearing in (1.4) is Lorentz invariant, and that in consequence the work done transfer rates $d\mathscr{E}/\mathrm{d}p$ or $d\mathscr{E}/d\xi$ are Lorentz invariant, where $\xi = \lambda x + ct$ and these details are presented in Appendix B.

In the following section, we present a brief summary of some of the basic equations of special relativity leading to the Lorentz-invariant energy–momentum relations (2.4) and establishing the invariance of the operator appearing in (1.4). In the subsequent two sections, we show how the above formulation yields an additional term in the famous Einstein equation for the work done $\mathscr{E} = mc^2$. The new term is well-defined as a limiting case for energy–mass waves travelling at the speed of light, yields precisely $\mathscr{E} = pc$ which is in complete accord with the well-established relations for photons and light, namely $p = h\nu/c$ and $\mathscr{E} = h\nu$ where h is Planck's constant and ν denotes the frequency. The new terms appearing in the work done equation for $\mathscr{E}$ can involve a logarithmic singularity giving rise to massive energies, and this characteristic could represent the first formal indication of the origin of dark energy and dark matter in the universe.

We comment that the formula $m(u) = m_0[1-(u/c)^2]^{-1/2}$ is only one of many expressions showing a particular variation of mass with its velocity, and this expression has a long and extensive history involving many eminent scientists such as Abraham, Bücherer, Lorentz, Ehrenfest, Kaufmann and of

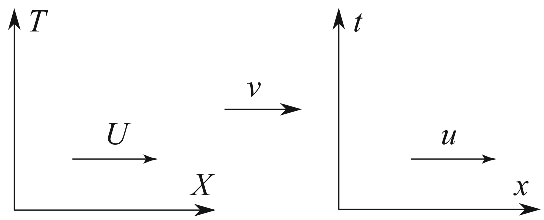

FIG. 1. Inertial frames moving along x-axis with relative velocity v

course Einstein, who first grappled with the notion that the 'transverse and longitudinal' masses may be distinct. The story describing the development of the Einstein expression is fully detailed by Weinstein [6].

2. Special relativity

We consider a rectangular Cartesian frame $\mathbf{X} = (X, Y, Z)$ and another frame $\mathbf{x} = (x, y, z)$ moving with constant velocity v relative to the first frame and the motion is assumed to be in the aligned X and x directions as indicated in Fig. 1. Time is measured from the (X, Y, Z) frame with the variable T and from the (x, y, z) frame with the variable t. Following normal practice, we assume that $y = Y$ and $z = Z$, so that (X, T) and (x, t) are the variables of principal interest. For $0 \leqslant v < c$, the standard Lorentz transformations are

$$x = \frac{X - vT}{[1 - (v/c)^2]^{1/2}}, \quad t = \frac{T - vX/c^2}{[1 - (v/c)^2]^{1/2}}, \tag{2.1}$$

and various derivations of these equations can be found in many standard textbooks, such as Feynmann et al. [1] and Landau and Lifshitz [2], and other novel derivations are given by Lee and Kalotas [3] and Levy-Leblond [4]. With velocities $U = \mathrm{d}X/\mathrm{d}T$ and $u = \mathrm{d}x/\mathrm{d}t$, (2.1) yields the addition of velocity law

$$u = \frac{U - v}{(1 - Uv/c^2)}, \tag{2.2}$$

which is well known and due to Einstein, and an immediate consequence is the identity

$$[1 - (u/c)^2](1 - Uv/c^2)^2 = [1 - (v/c)^2][1 - (U/c)^2]. \tag{2.3}$$

For $v, u, U < c$, assuming the Einstein mass variation in both frames

$$m(u) = \frac{m_0}{[1 - (u/c)^2]^{1/2}}, \quad M(U) = \frac{m_0}{[1 - (U/c)^2]^{1/2}},$$

and with momenta $P = MU$ and $p = mu$, we have on multiplication of (2.2) by $m_0 \left[1 - (u/c)^2\right]^{-1/2}$ and using the square root identity from (2.3), we may readily deduce the Lorentz-invariant energy–momentum relations in consideration of the formulae $e = mc^2$ and $E = Mc^2$, thus

$$p = \frac{P - Ev/c^2}{[1 - (v/c)^2]^{1/2}}, \quad e = \frac{E - Pv}{[1 - (v/c)^2]^{1/2}}, \tag{2.4}$$

and noting that (2.1) and (2.4) give rise to the two Lorentz invariants

$$x^2 - (ct)^2 = X^2 - (cT)^2, \qquad e^2 - (pc)^2 = E^2 - (Pc)^2 = e_0^2, \tag{2.5}$$

where $e_0 = m_0c^2$ denotes the rest mass energy.

In one spatial dimension x, the proposed equations (1.1) become simply

$$f = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x}, \qquad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x}, \tag{2.6}$$

and it is not difficult to show that these equations remain invariant under the Lorentz group (2.1) and (2.4); in other words the following relations hold

$$f = \frac{\partial p}{\partial t} + \frac{\partial e}{\partial x} = \frac{\partial P}{\partial T} + \frac{\partial E}{\partial X}, \qquad g = \frac{1}{c^2}\frac{\partial e}{\partial t} + \frac{\partial p}{\partial x} = \frac{1}{c^2}\frac{\partial E}{\partial T} + \frac{\partial P}{\partial X},$$

which we establish as follows. From Eq. (2.1), we have the differential relations

$$\frac{\partial}{\partial x} = \frac{1}{(1-(v/c)^2)}\left\{\frac{\partial}{\partial X} + \frac{v}{c^2}\frac{\partial}{\partial T}\right\}, \qquad \frac{\partial}{\partial t} = \frac{1}{(1-(v/c)^2)}\left\{\frac{\partial}{\partial T} + v\frac{\partial}{\partial X}\right\},$$

so that on using (2.4) and the subscript notation for partial derivatives we have

$$\begin{aligned} & p_t + e_x \\ = \; & \frac{1}{(1-(v/c)^2)}\left\{(P_T - \frac{v}{c^2}E_T) + v(P_X - \frac{v}{c^2}E_X) + (E_X - vP_X) + \frac{v}{c^2}(E_T - vP_T)\right\} \\ = \; & P_T + E_X, \end{aligned}$$

and similarly

$$\begin{aligned} & \frac{1}{c^2}e_t + p_x \\ = \; & \frac{1}{(1-(v/c)^2)}\left\{\frac{1}{c^2}(E_T - vP_T) + \frac{v}{c^2}(E_X - vP_X) + (P_X - \frac{v}{c^2}E_X) + \frac{v}{c^2}(P_T - \frac{v}{c^2}E_T)\right\} \\ = \; & \frac{1}{c^2}E_T + P_X. \end{aligned}$$

This key outcome indicates at the very least that the two equations (1.1) are well formulated.

We may also confirm that the operator appearing in (1.3) and (1.4) is also Lorentz invariant, namely for any three-dimensional spatial vector $\mathbf{q}$ we have

$$\frac{\partial \mathbf{q}}{\partial t}\cdot \mathbf{dx} + c^2(\nabla\cdot\mathbf{q})\mathrm{d}t = \frac{\partial \mathbf{q}}{\partial T}\cdot \mathbf{dX} + c^2(\nabla^{**}\cdot\mathbf{q})\mathrm{d}T,$$

where ∇^{**} denotes the del operator with respect to the $\mathbf{X}$ variables. This equation follows since for one dimension and relative frame velocity in the x-direction, we have from (2.1) and the above differential relations

$$\begin{aligned} & q_t\mathrm{d}x + c^2 q_x\mathrm{d}t \\ = \; & \frac{1}{(1-(v/c)^2)}\left\{(q_T + vq_X)(\mathrm{d}X - v\mathrm{d}T) + c^2(q_X + \frac{v}{c^2}q_T)(\mathrm{d}T - \frac{v}{c^2}\mathrm{d}X)\right\} \\ = \; & q_T\mathrm{d}X + c^2 q_X\mathrm{d}T, \end{aligned}$$

as required. In the following two sections, we apply the above formulation to identify a new term in the Einstein formula for the work done arising from equation (1.4).

3. Wave-like solutions for (1.1)

In this section, assuming that the momentum $p = mu$ and particle energy $e = mc^2$ are functions of velocity u only, we seek to determine solutions arising from (1.1) that we subsequently exploit to determine an extension of Einstein's formula for the work done $\mathscr{E}$. We assume the two basic equations (2.6), and on introducing the angle $\phi(x,t)$ such that $u = c\sin\phi$ we have from (2.5) the following relations

$$u = c\sin\phi, \quad m = m_0\sec\phi, \quad e = e_0\sec\phi, \quad pc = e_0\tan\phi, \tag{3.1}$$

where $e_0 = m_0c^2$ is the rest mass energy. On substitution of these relations into the two basic equations (2.6), we may readily deduce two equations for the determination of the partial derivatives ϕ_x and ϕ_t; thus

$$\phi_t + c\sin\phi\phi_x = a(\phi)\cos^2\phi, \qquad \sin\phi\phi_t + c\phi_x = b(\phi)\cos^2\phi,$$

where we have introduced $a(\phi) = cf(\phi)/e_0$ and $b(\phi) = c^2g(\phi)/e_0$. On solving these equations as two equations in the two unknowns ϕ_x and ϕ_t, we find

$$c\phi_x = b(\phi) - a(\phi)\sin\phi, \qquad \phi_t = a(\phi) - b(\phi)\sin\phi. \tag{3.2}$$

On cross differentiation of these equations and equating two expressions for ϕ_{xt}, we may deduce the following simple equation

$$\frac{\mathrm{d}(b/a)}{\mathrm{d}\phi} = \left(1 - (b/a)^2\right)\sec\phi,$$

which may be readily integrated, and further simplification yields

$$\frac{b(\phi)}{a(\phi)} = \left(\frac{\sin\phi + \lambda}{1 + \lambda\sin\phi}\right), \tag{3.3}$$

where λ denotes a non-dimensional constant of integration. In terms of the force $f(u)$ and the energy–mass production $g(u)$, we have the implied relation

$$cg(u) = f(u)\left(\frac{(u/c) + \lambda}{1 + \lambda(u/c)}\right),$$

noting that the case $\lambda = 0$ gives $g(u) = f(u)u/c^2$ while the case $\lambda = v/c$, where v denotes the relative frame velocity, produces $g(u) = f(u)U/c^2$ where $U = (u + v)/(1 + uv/c^2)$ arising from (2.2).

On substitution of (3.3) into (3.2) to eliminate $a(\phi)$, we obtain

$$c\phi_x = \frac{b(\phi)\lambda\cos^2\phi}{(\sin\phi + \lambda)}, \qquad \phi_t = \frac{b(\phi)\cos^2\phi}{(\sin\phi + \lambda)}, \tag{3.4}$$

and from these two equations it is apparent that $c\phi_x = \lambda\phi_t$, so that $\phi(x,t) = \phi(\xi)$ where $\xi = \lambda x + ct$, and indicating that with the assumption that the momentum $p = mu$ and the particle energy $e = mc^2$ are functions of velocity u only, then only wave-like solutions are permitted by this formulation. Further substitution of this expression for $\phi(x,t)$ into either of equations (3.4) yields

$$c\frac{d\phi}{d\xi} = \frac{b(\phi)\cos^2\phi}{(\sin\phi + \lambda)}. \tag{3.5}$$

Now from the one-dimensional version of the compatibility condition equation (1.6), namely

$$\frac{\partial f}{\partial t} = c^2\frac{\partial g}{\partial x}, \tag{3.6}$$

we may deduce $a_t = cb_x$, and from $\phi(x,t) = \phi(\xi)$, we find

$$\frac{da(\phi)}{d\phi} = \lambda\frac{db(\phi)}{d\phi}, \tag{3.7}$$

which together with (3.3) constitutes a second equation for the determination of $a(\phi)$ and $b(\phi)$. On elimination of one of $a(\phi)$ and $b(\phi)$ and integration of the resulting first-order ordinary differential equation, or more simply by direct integration of (3.7), we may deduce

$$a(\phi) = a_0(1 + \lambda\sin\phi), \qquad b(\phi) = a_0(\sin\phi + \lambda), \tag{3.8}$$

where a_0 denotes the constant of integration. In terms of the force $f(u)$ and the energy–mass production $g(u)$, these equations become

$$f(u) = f_0(1 + \lambda(u/c)), \qquad cg(u) = f_0(\lambda + (u/c)), \tag{3.9}$$

where f_0 is a re-defined constant given by $f_0 = e_0 a_0/c$, and it is clear from these results that with the assumption that the momentum $p = mu$ and the particle energy $e = mc^2$ are functions of velocity u only, the formulation only permits forces $f(u)$ and energy–mass production $g(u)$ that are linear functions in the particle velocity u. From Eqs. (3.5) and (3.8), we may readily deduce $cd\phi/d\xi = a_0 \cos^2\phi$ which readily integrates to yield $c\tan\phi = a_0(\xi - \xi_0)$, where ξ_0 denotes the constant of integration, reflecting the arbitrary choice of coordinate and time origins. On taking $\xi_0 = 0$, the equation for ϕ simplifies to give the explicit formula for the particle velocity $u(x,t)$, thus

$$u(x,t) = c\left\{\frac{\lambda x + ct}{((e_0/f_0)^2 + (\lambda x + ct)^2)^{1/2}}\right\}, \tag{3.10}$$

where again $e_0 = m_0c^2$ is the rest mass energy and f_0 is the arbitrary constant appearing in equation (3.9). We comment that since $\xi = \lambda x + ct$, particle-like dynamics occurs for $\lambda^2 > 1$, wave-light dynamics for $\lambda^2 < 1$ and light-like dynamics for $\lambda^2 = 1$, and these various cases arise in the following section.

4. Work done equation (1.4)

In this section, we utilise the solution of the previous section to determine an extended Einstein's formula for the work done $\mathscr{E}$. For one-dimensional motion, we have from (1.4) that the incremental work done $d\mathscr{E} = (\partial p/\partial t)\mathrm{d}x + c^2(\partial p/\partial x)\mathrm{d}t$ becomes on using $p = mu$

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = u\frac{\partial p}{\partial t} + c^2\frac{\partial p}{\partial x} = m_0\left\{\frac{u\frac{\partial u}{\partial t} + c^2\frac{\partial u}{\partial x}}{(1-(u/c)^2)^{3/2}}\right\},$$

and on using $u = c\sin\phi$ and $cd\phi/d\xi = a_0\cos^2\phi$ we might deduce

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = e_0\left\{\frac{\lambda + \sin\phi}{\cos^2\phi}\right\}c\frac{d\phi}{d\xi} = cf_0(\lambda + \sin\phi), \tag{4.1}$$

noting the relation $cf_0 = e_0a_0$. Now on using the relation $d\xi/\mathrm{d}t = \lambda u + c$, equation (4.1) becomes

$$\mathrm{d}\mathscr{E} = f_0\left(\frac{\lambda + \sin\phi}{1 + \lambda\sin\phi}\right)d\xi = e_0\sec^2\phi\left(\frac{\lambda + \sin\phi}{1 + \lambda\sin\phi}\right)d\phi, \tag{4.2}$$

and the formal Lorentz invariance of the transfer rates $d\mathscr{E}/\mathrm{d}p$ or $d\mathscr{E}/d\xi$ specifically for $d\mathscr{E}$ arising from (4.2) is outlined in Appendix B. The integral, although elementary, is lengthy and evaluation involves the substitution $z = \tan\phi$ to yield

$$\int\left(\frac{\lambda + \sin\phi}{1 + \lambda\sin\phi}\right)\sec^2\phi d\phi = \int\left(\frac{\lambda + (1-\lambda^2)z(1+z^2)^{1/2}}{1 + (1-\lambda^2)z^2}\right)dz,$$

and there are two cases to consider. If $\lambda^2 < 1$ (wave-like dynamics), we make the substitution $(1-\lambda^2)^{1/2}z = \tan\Phi$, while if $\lambda^2 > 1$ (particle-like dynamics) we make the substitution $(\lambda^2-1)^{1/2}z = \sin\Psi$, and the final results are as follows. For $\lambda^2 < 1$ we find

$$\mathscr{E}(\phi) = e_0\left\{\frac{1}{\cos\phi} + \frac{\lambda}{(1-\lambda^2)^{1/2}}\tan^{-1}\left(\frac{(\lambda + \sin\phi)}{(1-\lambda^2)^{1/2}\cos\phi}\right)\right\} + \mathscr{E}_0, \tag{4.3}$$

while if $\lambda^2 > 1$ we obtain

$$\mathscr{E}(\phi) = e_0\left\{\frac{1}{\cos\phi} + \frac{\lambda}{2(\lambda^2-1)^{1/2}}\log\left(\frac{(\lambda + \sin\phi) - (\lambda^2-1)^{1/2}\cos\phi}{(\lambda + \sin\phi) + (\lambda^2-1)^{1/2}\cos\phi}\right)\right\} + \mathscr{E}_0, \tag{4.4}$$

where again $u = c\sin\phi$ and in both cases $\mathscr{E}_0$ denotes a suitable constant, which solely for purposes of making a consistent numerical comparison in the following section, we chose such that $\mathscr{E}(0) = e_0$. Thus for $\lambda^2 < 1$, we have

$$\mathscr{E}_0 = -\frac{\lambda e_0}{(1-\lambda^2)^{1/2}} \tan^{-1}\left(\frac{\lambda}{(1-\lambda^2)^{1/2}}\right), \tag{4.5}$$

while for $\lambda^2 > 1$ we have

$$\mathscr{E}_0 = -\frac{\lambda e_0}{2(\lambda^2-1)^{1/2}} \log\left(\frac{\lambda-(\lambda^2-1)^{1/2}}{\lambda+(\lambda^2-1)^{1/2}}\right), \tag{4.6}$$

assuming the logarithms are appropriately always well-defined. We observe that the underlying symmetry in both (4.3) and (4.4) is $\mathscr{E}(\phi,\lambda) = \mathscr{E}(-\phi,-\lambda)$; that is the work done $\mathscr{E}(\phi)$ remains invariant under simultaneous changing of both ϕ and λ to $-\phi$ and $-\lambda$, respectively.

We further observe that clearly the first terms in both (4.3) and (4.4) correspond to the Einstein contribution, and formally arise in the limit $\lambda \to 0$ and in this limit we have

$$\mathscr{E}(\phi) \approx e_0\left(\frac{1}{\cos\phi} + \lambda\phi\right) + \mathscr{E}_0.$$

On recalling that the inverse tangent function admits a number of curious features, and in particular satisfies the relation $\tan^{-1}(z) + \tan^{-1}(1/z) = \pi/2$, we may if necessary adopt the reciprocal of the argument of the inverse tangent function, together with a change of sign and a re-definition of the arbitrary constant $\mathscr{E}_0$. By this means, we observe that in both cases, the additional terms are well-defined for $\lambda = 1$, which is the case of energy–mass waves travelling at the speed of light, and both (4.3) and (4.4) simplify to give

$$\mathscr{E}(\phi) = e_0\left(\frac{1}{\cos\phi} - \frac{\cos\phi}{1+\sin\phi}\right) + \mathscr{E}_0 = e_0\tan\phi + \mathscr{E}_0 = e_0 + pc, \tag{4.7}$$

with an appropriate choice for $\mathscr{E}_0$. Similarly, for energy–mass waves travelling at the speed of light in the opposite direction, $\lambda = -1$, and we may deduce

$$\mathscr{E}(\phi) = e_0\left(\frac{1}{\cos\phi} - \frac{\cos\phi}{1-\sin\phi}\right) + \mathscr{E}_0 = -e_0\tan\phi + \mathscr{E}_0 = e_0 - pc, \tag{4.8}$$

and we observe that both formulae are in complete accord with the well-established relations for photons and light, namely $p = h\nu/c$ and $\mathscr{E} = h\nu$ where h is Planck's constant and ν denotes the frequency, which together yield $\mathscr{E} = pc$ and noting that e_0 is generally adopted to be zero for photons.

For $\lambda \to \pm\infty$ we have from (4.4) the following limiting expression:

$$\mathscr{E}(\phi) \approx e_0\left\{\frac{1}{\cos\phi} + \log\left(\frac{\sin\phi}{1+\cos\phi}\right)\right\} + \mathscr{E}_0 = e_0\left\{\frac{1}{\cos\phi} + \log\left(\tan\frac{\phi}{2}\right)\right\} + \mathscr{E}_0, \tag{4.9}$$

which gives rise to Eq. (1.5), and also formally arises directly from (1.4) assuming that all variables are spatially dependent only, so that (1.4) becomes

$$\mathrm{d}\mathscr{E} = c^2\frac{\mathrm{d}p}{\mathrm{d}x}\mathrm{d}t = \frac{e_0 du}{u\left(1-(u/c)^2\right)^{3/2}} = \frac{e_0 d\phi}{\sin\phi\cos^2\phi},$$

after making the substitution $u = c\sin\phi$, and this expression readily integrates to yield (4.9). Further, assuming that $\lambda^2 \approx 1$, expanding both the inverse tangent and the logarithm, and retaining in each case only the leading term in each expansion gives rise to the following simple approximate expression

$$\mathscr{E}(\phi) \approx e_0\frac{(1+\lambda\sin\phi)\tan\phi}{(\lambda+\sin\phi)} + \mathscr{E}_0,$$

valid for both $\lambda^2 < 1$ and $\lambda^2 > 1$, and this expression is exact when $\lambda = 1$ and $\lambda = -1$ giving precisely (4.7) and (4.8), respectively.

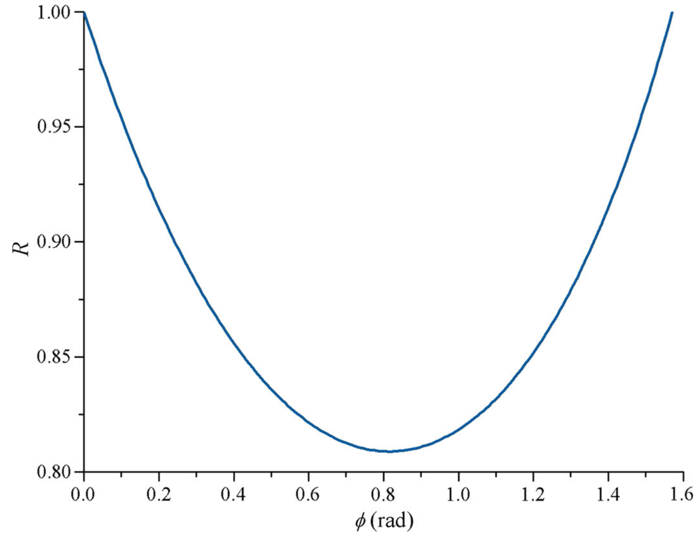

FIG. 2. Typical variation of ratio R defined by (5.1) ($\lambda = 1/2$)

5. Numerical results and conclusions

It is clear from the work done expressions derived in the previous section involving log that large energies might be readily generated for argument values less than unity and tending to zero. For typical values of the arguments, we illustrate the relative magnitude of the new terms in the derived expressions, using the datum energy levels given by (4.5) and (4.6), to evaluate the ratio defined by $R =$ Einstein energy/(Einstein energy + Additional energy). The numerator is simply $\mathscr{E}(\phi) = e_0/\cos\phi$ and the denominator is given by one of the expressions (4.3) and (4.4). With $R = (1+Q)^{-1}$, for $\lambda^2 < 1$ we find

$$Q = \frac{\lambda\cos\phi}{(1-\lambda^2)^{1/2}}\left(\tan^{-1}\left(\frac{(\lambda+\sin\phi)}{(1-\lambda^2)^{1/2}\cos\phi}\right) - \tan^{-1}\left(\frac{\lambda}{(1-\lambda^2)^{1/2}}\right)\right), \tag{5.1}$$

while if $\lambda^2 > 1$ we obtain

$$Q = \frac{\lambda\cos\phi}{2(\lambda^2-1)^{1/2}}\left(\log\left(\frac{(\lambda+\sin\phi)-(\lambda^2-1)^{1/2}\cos\phi}{(\lambda+\sin\phi)+(\lambda^2-1)^{1/2}\cos\phi}\right) - \log\left(\frac{\lambda-(\lambda^2-1)^{1/2}}{\lambda+(\lambda^2-1)^{1/2}}\right)\right), \tag{5.2}$$

noting that for $\lambda = 1$, both cases yield simply $R = 1/(\sin\phi + \cos\phi)$, and consequently the same value is produced for the particular two angles shown in the table. For a consistent numerical comparison, this particular expression for R arises directly from (4.7), namely $\mathscr{E}(\phi) = e_0\,(1+\tan\phi)$, in contrast to the more usual formula for light $\mathscr{E}(\phi) = e_0\tan\phi$.

The typical variation of the ratio R defined by (5.1) for $\lambda = 1/2$ is shown in Figure 2, and similar behaviour is obtained from (5.2) for $\lambda > 1$. If the new terms were to somehow represent dark energy, then conventional wisdom might infer that the numerical values of these ratios might be smaller than about 1/10. From the figure and the actual numerical values shown in the table, it is clear that this is generally not the case. However, it is also clear that there are other scenarios for which the ratio R can be made arbitrarily small simply by making the argument of the logarithm close to zero.

In special relativity, the formulae for the energy and mass of a particle $\mathscr{E} = mc^2$ and $m(u) = m_0[1-(u/c)^2]^{-1/2}$ are based on the assumption that the particle energy e accrues from and coincides with the work done $\mathscr{E}$, and formally arises as a consequence of the rate-of-working equation $\mathrm{d}\mathscr{E}/\mathrm{d}t = \mathbf{u}\cdot(\mathrm{d}\mathbf{p}/\mathrm{d}t)$

Table 1. *Table of numerical values of ratio R defined by (5.1) and (5.2) for various values of λ*

	$\phi = \pi/6$	$\phi = \pi/3$
$\lambda = 1/4$	0.9036	0.8950
$\lambda = 1/2$	0.8321	0.8239
$\lambda = 1$	0.7321	0.7321
$\lambda = 2$	0.6158	0.6328
$\lambda = 10$	0.3851	0.4433

where $\mathbf{p} = m\mathbf{u}$ is the momentum and $\mathbf{u}$ is the velocity vector. Here we have adopted all the usual formulae of special relativity, except that we have made a distinction between particle energy $e = mc^2$ and work done $\mathscr{E}$, and we have proposed (1.1) as a fully Lorentz-invariant alternative of Newton's second law, that is based on the fact that $(\mathbf{p}, e/c)$ is a well-defined special relativistic four vector, and we have formulated an alternative rate-of-working equation $d\mathscr{E}/\mathrm{d}t = \mathbf{u} \cdot \partial\mathbf{p}/\partial t + c^2(\nabla \cdot \mathbf{p})$ which arises by considering the scalar product of two four vectors $(\mathbf{f}, gc)$ and $(\mathbf{dx}, cdt)$. This new rate-of-working equation, Eq. (1.4), is such that the work done transfer rate $d\mathscr{E}/\mathrm{d}p$ or $d\mathscr{E}/d\xi$ is Lorentz invariant, where ξ is the variable $\xi = \lambda x + ct$ and this result is established in Appendix B.

In the new formulation, the compatibility condition (1.6), or in one dimension (3.6), represents a new equation and an important constraint that is not present in conventional theory. In attempting to deduce analogous results to the Einstein formulae, namely that both momentum and energy are functions of u only, then only wave-like solutions are possible, and in one dimension involve the variable $\xi = \lambda x + ct$ and the velocity $u(x, t)$ is given explicitly by (3.10). Based on this solution, there are new terms arising in the work done equation that are identified for the two cases $\lambda^2 < 1$ and $\lambda^2 > 1$ and are given explicitly in (4.3) and (4.4). The cases $\lambda = \pm 1$ correspond to energy–mass waves travelling at the speed of light and arise as well-defined limiting cases from either (4.3) or (4.4), giving the work done by the simple expressions (4.7) and (4.8) that are entirely consistent with well-established results for photons and light. In a single-spatial dimension, the momentum $p = mu$ satisfies the wave equation, and further wave-like solutions of this equation are presented in Appendix A.

The new formulation retains all the major features of special relativity and emerges as a natural extension that might well generate insight into the possible origins of dark energy. In particular, it suggests that dark energy might arise from neglecting the work done in the direction of time, and the consequent logarithmic singularities that can be produced even for slowly moving mechanical structures. Traditional mechanical theory and thinking pre-suppose smooth and sensible physical behaviour, and an absence of singularities. In fact, expectations of decreasing energy for slowing systems, and avoiding singularities lie at the very heart of mechanical thinking of natural systems. We may accommodate a singularity at the speed of light because it is believed to constitute an unattainable physical barrier. The formalism developed here hints at the prospect of physical singularities, supported by large energies, for possibly slowly moving mechanical structures, that perhaps display characteristics of unstoppable mechanical systems.

Note added

The idea formulated here, namely that the combined four potentials $(\mathbf{p}, e/c)$, might play a more fundamental role in mechanics, emanates from electro-magnetism and originally occurred to the author after attending a series of five lectures by Professor Germain Rousseaux presented at the University of Poitiers on five successive Tuesdays, 9th May through to 6th June 2017. Professor Rousseaux as a passionate and life long student of the life and work of James Clerk Maxwell, presented five remarkably insightful lectures, principally dealing with the life of Maxwell and his numerous outstanding contributions to electro-magnetism that included three important themes. Firstly, that much of our present understanding of electro-magnetism developed from mechanical analogues from both solid and fluid mechanics, and that the interplay between mechanics and electro-magnetism continues to this day. Secondly, the recognition as to the importance as to which variables constitute the force and which identify the associated flux, and the assumed linear connection between them, such as in standard notation in electro-magnetic theory $\mathbf{B} = \mu \mathbf{H}$, $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{J} = \sigma \mathbf{E}$. The third and perhaps the most important theme that permeated throughout the five lectures was the more fundamental importance of the vector and scalar potentials $(\mathbf{A}, V)$ as compared to the fields $(\mathbf{E}, \mathbf{B})$ and that are related by the formulae

$$\mathbf{B} = \nabla \wedge \mathbf{A}, \qquad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V.$$

The vector potential $\mathbf{A}$ corresponds to the electro-magnetic momentum and is the analogue of mechanical momentum, and referred to by both Faraday and Maxwell as the electro-tonic intensity, while the potential V is the analogue of the velocity potential in fluid mechanics. After attending these lectures, it occurred to the author that if the four potentials $(\mathbf{A}, V)$ play such an important role in electro-magnetism, might there not also be a corresponding set of potentials that might play a similar role in mechanics. Given that momentum and mass conservation are necessarily partnered in a special relativistic four vector sense, and the requirement of Lorentz-invariant energy–momentum relations, the only available option is the adoption of momentum $\mathbf{p} = m\mathbf{u}$, particle energy $e = mc^2$ and necessarily with the Einstein mass variation.

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Appendices

Appendix A: Further one-dimensional wave-like solutions of (2.6)

In this appendix, we examine further wave-like solutions of (2.6). Assuming the condition (3.6), it is clear that the momentum $p(x,t)$ satisfies the wave equation, namely

$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2}, \tag{A.1}$$

so that the general solution for $p(x,t) = F(\zeta) + G(\eta)$ where $\zeta = x + ct$ and $\eta = x - ct$, and where both F and G denote arbitrary functions of their arguments. Further, from the definition of momentum $p = mu$ and $m(u) = m_0[1 - (u/c)^2]^{-1/2}$ we may deduce

$$u(x,t) = c\left\{\frac{p(x,t)}{((m_0 c)^2 + (p(x,t))^2)^{1/2}}\right\},$$

and therefore the general solution for the velocity profile is $u(x,t)$ is given by

$$u(x,t) = c\left\{\frac{F(\zeta) + G(\eta)}{((m_0 c)^2 + (F(\zeta) + G(\eta))^2)^{1/2}}\right\}, \tag{A.2}$$

where again F and G denote arbitrary functions of $\zeta = x + ct$ and $\eta = x - ct$, respectively. We observe that the solution (3.10) is recovered from (A.2) with $F(\zeta) = \alpha\zeta$ and $G(\eta) = \beta\eta$ where α and β are constants such that $\alpha - \beta = 1$ and the constant λ in (3.10) is given by $\lambda = \alpha + \beta$.

We now seek the most general $f(x,t)$ and $g(x,t)$ satisfying (3.6) such that (2.6) admit non-trivial solutions. Again on introducing ϕ through $u = c\sin\phi$ there follows the relations (3.1) and we might readily deduce

$$c\phi_x = b(x,t) - a(x,t)\sin\phi, \qquad \phi_t = a(x,t) - b(x,t)\sin\phi, \tag{A.3}$$

where $a(x,t)$ and $b(x,t)$ are given by $a(x,t) = cf(x,t)/e_0$ and $b(x,t) = c^2 g(x,t)/e_0$ and $e_0 = m_0c^2$. In terms of $a(x,t)$ and $b(x,t)$ Eq. (3.6) and the equation obtained by equating expressions for ϕ_{xt} become

$$a_t = cb_x, \qquad b_t = ca_x + (a^2 - b^2)\cos\phi,$$

which constitutes two equations for $a(x,t)$ and $b(x,t)$ but also involving the function $\phi(x,t)$. In order to obtain an equation in the single variable $\phi(x,t)$, we use $a_t = cb_x$ to introduce the function $\psi(x,t)$ such that $a = c\psi_x$ and $b = \psi_t$, and Eq. (A.3) become

$$c\phi_x = \psi_t - c\psi_x\sin\phi, \qquad \phi_t = c\psi_x - \psi_t\sin\phi,$$

and from which we may deduce

$$\psi_t = (c\phi_x + \phi_t\sin\phi)\sec^2\phi, \qquad c\psi_x = (\phi_t + c\phi_x\sin\phi)\sec^2\phi, \tag{A.4}$$

and on elimination of $\psi(x,t)$ by equating expressions for the derivative ψ_{xt}, we obtain

$$\phi_{tt} + 2\tan\phi{\phi_t}^2 = c^2(\phi_{xx} + 2\tan\phi{\phi_x}^2),$$

which simplifies to give

$$\frac{\partial^2(\tan\phi)}{\partial t^2} = c^2\frac{\partial^2(\tan\phi)}{\partial x^2}.$$

This equation is of course merely (A.1) with $pc = e_0\tan\phi$. This means that the most general $f(x,t)$ and $g(x,t)$ satisfying (3.6) are determined from (A.4) through the relations

$$a(x,t) = (\tan\phi)_t + c\sin\phi(\tan\phi)_x, \qquad b(x,t) = c(\tan\phi)_x + \sin\phi(\tan\phi)_t,$$

along with $\tan\phi = c(F(\zeta) + G(\eta))/e_0$. The final relations for $f(x,t)$ and $g(x,t)$ become

$$f(x,t) = c\left\{(F'(\zeta) - G'(\eta)) + \sin\phi(F'(\zeta) + G'(\eta))\right\},$$

$$g(x,t) = \left\{(F'(\zeta) + G'(\eta)) + \sin\phi(F'(\zeta) - G'(\eta))\right\},$$

where the primes denote differentiation with respect to the argument indicated, and with $\sin\phi = u(x,t)/c$ given explicitly by

$$\sin\phi = \frac{u(x,t)}{c} = \left\{\frac{(F(\zeta) + G(\eta))}{((e_0/c)^2 + (F(\zeta) + G(\eta))^2)^{1/2}}\right\}.$$

Appendix B: Formal Lorentz invariance of $d\mathscr{E}/dp$ or $d\mathscr{E}/d\xi$ for $d\mathscr{E}$ arising from (4.2)

In this appendix, we show the formal Lorentz invariance of $d\mathscr{E}/dp$ or $d\mathscr{E}/d\xi$ for the specific $d\mathscr{E}$ given by (4.2). From $\xi = \lambda x + ct$ and the Lorentz transformations (2.1) we have

$$\xi = \frac{\lambda(X - vT) + c(T - vX/c^2)}{[1 - (v/c)^2]^{1/2}} = \frac{(\lambda - v/c)\,X + (1 - \lambda v/c)\,cT}{[1 - (v/c)^2]^{1/2}},$$

and therefore with $\eta = \mu X + cT$, we have $\xi = \sigma\eta$ where μ is the essential constant in the (X,T) variables that corresponds to λ in the (x,t) variable and μ and σ are given by

$$\mu = \left(\frac{\lambda - v/c}{1 - \lambda v/c}\right), \qquad \sigma = \frac{(1 - \lambda v/c)}{[1 - (v/c)^2]^{1/2}}. \tag{B.1}$$

It proves convenient to introduce the new variables $\alpha = \lambda c$ and $\beta = \mu c$ so that from equation (B.1) we have the reciprocal relations

$$\beta = \left(\frac{\alpha - v}{1 - \alpha v/c^2} \right), \quad \alpha = \left(\frac{\beta + v}{1 + \beta v/c^2} \right). \tag{B.2}$$

From Eq. (4.2), we find that

$$\frac{\mathrm{d}\mathscr{E}}{d\xi} = f_0 \left(\frac{\lambda + \sin \phi}{1 + \lambda \sin \phi} \right) = f_0 \left(\frac{\alpha + u}{1 + \alpha u/c^2} \right),$$

and from (2.2) and (B.2) we may by direct substitution establish the formal identity

$$\frac{\alpha + u}{1 + \alpha u/c^2} = \frac{\beta + U}{1 + \beta U/c^2}.$$

The Lorentz invariance $\mathrm{d}\mathscr{E}/\mathrm{d}\xi = \mathrm{d}\mathscr{E}^{**}/\mathrm{d}\eta$ or $\mathrm{d}\mathscr{E}/\mathrm{d}p = \mathrm{d}\mathscr{E}^{**}/\mathrm{d}P$ now follows on noting the relation $pc = f_0 \xi$, where $\mathscr{E}^{**}$ denotes the work done with respect to the (X, T) variables, and f_0 and the corresponding quantity in the (X, T) variables F_0 are related by the equation $F_0 = \sigma f_0$ where σ is defined by the latter relation of (B.1).

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