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The existence of steady states for a bimolecular model with autocatalysis and saturation law

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Abstract. In this paper, a reaction–diffusion system known as a bimolecular model with autocatalysis and saturation law is considered. Firstly, we briefly obtain some characterizations for the positive solutions, including the a priori estimate of the positive solutions and the nonexistence of non-constant positive solution. Secondly, we emphatically discusses the bifurcation from the unique positive constant solution with both simple eigenvalues and double eigenvalues in one-dimensional case. Meanwhile, some other existence results are shown to supplement the analytical conclusions with the degree theory in N dimensional case.

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1. Introduction

In this work, we deal with a bimolecular autocatalytic reaction–diffusion model with autocatalysis and saturation law and attempt to present some existence analysis for the corresponding stationary problem. First of all, let us give a brief description regarding the derivation of the system. The reaction process [1,2] of the model is given by

$$\begin{aligned}
&A \xrightarrow{k_1} Y,\\
&X + pY \xrightarrow{k_2} X \ (autocatalysis),\\
&X \xrightarrow{S(k_3,k_4)} P \quad (saturation\ law),
\end{aligned}$$

in which A, X, Y and P are chemical reactants and products, and the system is considered open to in-and-out-flow of A and P. In addition, p denotes the order of the reaction with respect to the autocatalytic species; k_1, k_2, k_3 and k_4 represent the reaction rates, and $S(k_3, k_4)$ accounts for the Langmuir–Hinshelwood law in heterogeneous catalysis and adsorption, the Michaelis–Menten law in enzyme-controlled processes and the Holling law in ecology. It is assumed that all three steps of the reaction process are irreversible and the concentrations of A and P are independent of time and spatial variables, that is, the concentration of these two chemicals is kept uniform throughout the reactor. Disregarding convective phenomena and considering isothermal processes only, the above scheme can be described by the nonlinear partial differential equations

$$\begin{aligned}
&\tfrac{\partial[X]}{\partial t} - D_{[X]}\Delta[X] = k_2[X][Y]^p - \tfrac{k_3[X]}{1+k_4[X]},\\
&\tfrac{\partial[Y]}{\partial t} - D_{[Y]}\Delta[Y] = k_1[A] - k_2[X][Y]^p,
\end{aligned} \tag{1.1}$$

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where Δ is the Laplace operator, showing the spatial dependence of the reaction, $[A], [X]$ and $[Y]$ are the concentrations of A, X and Y, respectively; $D_{[X]}$ and $D_{[Y]}$ denote the diffusion coefficients which are assumed to be positive constants, and the nonlinear term $\frac{k_3[X]}{1+k_4[X]}$ represents the Holling II functional response or the Monod equation.

To simplify the reaction–diffusion system (1.1), Peng et al. [3] introduce the following quantities with $p = 1$:

$$\begin{aligned} &U = \tfrac{k_2}{k_3}[X], \quad V = \tfrac{k_3[Y]}{k_1[A]}, \quad \bar{t} = k_3 t, \\ &\lambda' = \tfrac{k_1 k_2}{k_3}[A], \quad k = \tfrac{k_3 k_4}{k_2}, \quad d_1 = \tfrac{D_{[X]}}{k_3}, \quad d_2 = \tfrac{D_{[Y]}}{k_3}. \end{aligned}$$

By dropping the upper bar on $\bar{t}$, system (1.1) becomes

$$\begin{aligned} &\tfrac{\partial U}{\partial t} - d_1 \Delta U = \lambda' UV - \tfrac{U}{1+kU} && x \in \Omega, t > 0, \\ &\tfrac{\partial V}{\partial t} - d_2 \Delta V = 1 - UV && x \in \Omega, t > 0, \end{aligned} \tag{1.2}$$

where Ω is a bounded domain in the Euclidean space $\mathbb{R}^N$ with smooth boundary, denoted as $\partial\Omega$. Peng et al. [3] considered the existence and nonexistence of non-constant stationary solutions of system (1.2) when the diffusion rate of a certain reactant is large or small, which showed that the diffusion rate of this reactant and the size of the reactor play decisive roles in leading to the formation of stationary patterns. Soon afterward, Yi et al. [4] discussed the Hopf bifurcations and steady-state bifurcations which bifurcate from the unique constant positive equilibrium solution of system (1.2) in the one-dimensional space both theoretically and numerically. They obtained the existence of spatially non-homogeneous periodic orbits and non-constant positive stationary solutions, which implied the possibility of rich spatiotemporal patterns in this diffusive bimolecular system. To further explore this model, Peng and Yi [5] continued with the analysis on the steady-state bifurcation and the effect of various parameters on spatiotemporal patterns was discussed.

By introducing the following quantities:

$$\begin{aligned} &u = (k_1[A])^{-1}[X], \quad V = k_2^{1/p}[Y], \\ &d_1 = D_{[X]}, \quad d_2 = D_{[Y]}, \quad c = k_3, \quad b = k_1 k_4 [A], \quad \lambda = k_1 k_2^{1/p}[A], \quad a = c - b, \end{aligned}$$

Zhou [6] simplified system (1.1) as follows:

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + uv^p - \frac{(a+b)u}{1+bu} & x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} = d_2 \Delta v + \lambda(1 - uv^p) & x \in \Omega, t > 0, \\ \frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \geq\not\equiv 0, v(x,0) = v_0(x) \geq\not\equiv 0, & x \in \Omega, \end{cases} \tag{1.3}$$

where u and v represent the concentrations of the two reactants, respectively; $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator in $\mathbb{R}^N$; Ω is a bounded domain in the Euclidean space $\mathbb{R}^N$ with smooth boundary, denoted as $\partial\Omega$, ν is the unit outer normal vector on $\partial\Omega$; d_1 and d_2 are the diffusion coefficients; p denotes the order of the reaction with respect to the autocatalytic species. The constants $a, b, d_1, d_2, p, \lambda$ are positive constants, but $p > 1$. By analyzing the eigenvalues of the linearized system and using the bifurcation theory, Zhou [6] derived the conditions to occurrence Turing instability and Hopf bifurcation from the unique constant solution of system (1.3).

Our main aim in this paper is to investigate the steady-state solutions of system (1.3) mathematically. This leads us to investigate the associated steady-state problem, i.e., the following coupled elliptic system:

$$\begin{cases} d_1\Delta u + uv^p - \frac{(a+b)u}{1+bu} = 0 & x \in \Omega, \\ d_2\Delta v + \lambda(1-uv^p) = 0 & x \in \Omega, \\ \frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & x \in \partial\Omega. \end{cases} \tag{1.4}$$

It has been emphasized that the order p is a positive integer with $p > 1$, throughout this paper. It is easy to check that (1.2) has a unique positive constant solution $U^* = (u^*, v^*) = (1/a, \sqrt[p]{a})$.

The present paper is organized as follows: In Sect. 2, we briefly obtain some characterizations for the positive solutions of (1.4), including the a priori estimate of the positive solutions and the nonexistence of non-constant positive solution of (1.4). In Sect. 3, by taking λ as the parameter, we emphatically analyze the bifurcation solution which emanates from the constant solution $U^* = (u^*, v^*) = (1/a, \sqrt[p]{a})$, with both simple eigenvalues and double eigenvalues, respectively, in Sects. 3.1 and 3.2. Combined with the a priori estimates, some other existence results are shown to supplement the analytical conclusions with the degree theory in Sect. 4. Finally, the results obtained in this paper are summarized in Sect. 5.

2. Some characterizations of positive solutions

This section is devoted to some basic properties of non-homogeneous steady-state solutions of (1.3), namely the non-constant positive solutions of (1.4). We first set out to seek for the a priori estimate of positive solutions of (1.4).

The following three lemmas derived from [7, Lemma 2.1-2.3] are the main tools in obtaining our estimate. The first lemma is based on the result which is well known as a local result for weak super-solutions of linear elliptic equations (see, for example, [8, Theorem 8.18] and [9, Theorem 6.40])

Lemma 2.1. (See also [10, Lemma 2.1]) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, and let Λ be a nonnegative constant. Suppose that $z \in W^{1,2}(\Omega)$ is a nonnegative weak solution of the inequalities*

$$\Delta z - \Lambda z \leq 0 \ \text{in } \Omega, \qquad \frac{\partial z}{\partial \nu} \leq 0 \ \text{on } \partial\Omega.$$

Then, for any $q \in [1, \frac{N}{N-2})$, there exists a positive constant C_0, depending only on q, Λ and Ω, such that

$$||z||_q \leq C_0 \inf_{\Omega} z.$$

The next lemma is a simple but useful result, which was first derived by virtue of the Maximum Principle in [11, Proposition 2.2].

Lemma 2.2. (See also [12, Lemma 2.1], [13, Lemma 2.1] and [14, Lemma 2.1]) *Assume that $g \in C(\overline{\Omega} \times \mathbb{R}^1)$.*

1) Assume that $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and satisfies: $\Delta w(x) + g(x, w(x)) \geq 0$ in Ω, $\frac{\partial w}{\partial n} \leq 0$ on $\partial\Omega$. If $w(x_0) = \max_{\overline{\Omega}} w$, then $g(x_0, w(x_0)) \geq 0$.

2) Assume that $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and satisfies: $\Delta w(x) + g(x, w(x)) \leq 0$ in Ω, $\frac{\partial w}{\partial n} \geq 0$ on $\partial\Omega$. If $w(x_0) = \min_{\overline{\Omega}} w$, then $g(x_0, w(x_0)) \leq 0$.

Lastly, we have the following Harnack inequality for weak solutions, which is an analog of [8, Theorem 8.16].

Lemma 2.3. (See also [15, Lemma 2.2]) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, and let $c(x) \in L^q(\Omega)$ for some $q > N/2$. Suppose that $z \in W^{1,2}(\Omega)$ is a nonnegative weak solution of the boundary value problem*

$$\Delta z + c(x)z = 0 \ \text{in } \Omega, \qquad \frac{\partial z}{\partial \nu} = 0 \ \text{on } \partial\Omega.$$

Then, there exists a positive constant C_1, depending only on $q, ||c||_q$ and Ω, such that

$$\sup_{\Omega} z \leq C_1 \inf_{\Omega} z.$$

Theorem 2.1. *There exists two positive constant C_0 and C_2, depending possibly on a, b, d_1, and Ω, such that any positive solution $(u(x), v(x))$ of (1.4) satisfies the following inequalities:*

$$\frac{|\Omega|}{C_0(a+b)} \leq u \leq C_2, \qquad \frac{1}{\sqrt[p]{C_2}} \leq v \leq \sqrt[p]{\frac{C_0(a+b)}{|\Omega|}}.$$

Proof. Assume that $(u(x), v(x))$ is a positive solution of (1.4). By integrating the two equations in (1.4) over Ω, we first get

$$\int_{\Omega} uv^p \mathrm{d}x = |\Omega| = \int_{\Omega} \frac{(a+b)u}{1+bu} \mathrm{d}x \leq (a+b) \int_{\Omega} u \mathrm{d}x. \tag{2.1}$$

Since

$$\begin{aligned} & d_1 \Delta u + uv^p - \frac{(a+b)u}{1+bu} = 0 \\ \Leftrightarrow \quad & \Delta u + \frac{uv^p}{d_1} - \frac{(a+b)u}{d_1(1+bu)} = 0 \\ \Leftrightarrow \quad & \Delta u + \frac{uv^p}{d_1} - \frac{(a+b)u}{d_1(1+bu)} = 0 \\ \Rightarrow \quad & \Delta u - \frac{(a+b)u}{d_1} \leq 0, \end{aligned}$$

it follows from Lemma 2.1 (when $q = 1$) and (2.1) that there exists a $C_0 := C_0(a, b, d_1, \Omega)$, such that

$$\frac{|\Omega|}{a+b} \leq \int_{\Omega} u \mathrm{d}x \leq C_0 \inf_{\Omega} u.$$

Thus, we have

$$\inf_{\Omega} u \geq \frac{|\Omega|}{C_0(a+b)}. \tag{2.2}$$

Since

$$\begin{aligned} & d_2 \Delta v + \lambda(1 - uv^p) = 0 \\ \Leftrightarrow \quad & \Delta v + \frac{\lambda(1 - uv^p)}{d_2} = 0, \end{aligned}$$

it follows from Lemma 2.2 that

$$\frac{\lambda[1 - u(x_0)v(x_0)^p]}{d_2} \geq 0,$$

where $v(x_0) = \sup_{\Omega} v(x)$ for some $x_0 \in \overline{\Omega}$. Thus, by (2.2) we have

$$v(x_0) \leq \frac{1}{\sqrt[p]{u(x_0)}} \leq \frac{1}{\sqrt[p]{\frac{|\Omega|}{C_0(a+b)}}} = \sqrt[p]{\frac{C_0(a+b)}{|\Omega|}}. \tag{2.3}$$

Let $c(x) = \frac{v^p}{d_1} - \frac{a+b}{d_1(1+bu)}$. Since

$$\begin{aligned} & d_1 \Delta u + uv^p - \frac{(a+b)u}{1+bu} = 0 \\ \Leftrightarrow \quad & \Delta u + u \left[\frac{v^p}{d_1} - \frac{a+b}{d_1(1+bu)} \right] = 0 \end{aligned}$$

and $||c(x)||_\infty = ||\frac{v^p}{d_1} - \frac{a+b}{d_1(1+bu)}||_\infty \le \frac{C_0(a+b)}{|\Omega|d_1} + \frac{a+b}{d_1}$. It follows from Lemma 2.3 (when $q=\infty$) that there exists a $C_1 := C_1(a,b,d_1,\Omega)$, such that

$$\sup_\Omega u \le C_1 \inf_\Omega u. \tag{2.4}$$

Now we claim that there is a positive constant C_2 such that the following inequation holds:

$$\sup_\Omega u \le C_2. \tag{2.5}$$

Suppose that (2.5) does not hold. Then by (2.4) above, we have $u(x) = \infty$ almost everywhere (a.e.) in Ω. Let $w = \lambda d_1 u + d_2 v$. It follows from (1.4) that w satisfies

$$\Delta w + \lambda\left[1 - \frac{(a+b)u}{1+bu}\right] = 0. \tag{2.6}$$

By integrating two sides of (2.6) over Ω, we get

$$\int_\Omega \left[1 - \frac{(a+b)u}{1+bu}\right] \mathrm{d}x = 0. \tag{2.7}$$

Since

$$\left[1 - \frac{(a+b)u}{1+bu}\right] \overset{a.e.}{=\!=} 1 - \frac{a+b}{b} = -\frac{a}{b} < 0, \text{ in } \Omega,$$

which is a contradiction with (2.7), this verifies that (2.5) holds, where $C_2 := C_2(a,b,d_1,\Omega)$.

Similarly as the proof of the upper boundedness (2.3) of v, it follows from Lemma 2.2 again that

$$v(x_1) = \inf_\Omega v(x) \ge \frac{1}{\sqrt[p]{u(x_0)}} \ge \frac{1}{\sqrt[p]{C_2}}. \tag{2.8}$$

for some $x_1 \in \overline{\Omega}$. Summarizing (2.2), (2.3), (2.5) and (2.8), the proof is completed. □

By Theorem 2.1, we can obtain the following result easily, so we omit the proof here.

Corollary 2.1. *Assume that $(u(x), v(x))$ is a positive solution of (1.4) and denote $\overline{v} = \frac{1}{|\Omega|}\int_\Omega v(x)\mathrm{d}x$. Then, there exists a positive constant C_3 such that the following inequality holds:*

$$|v^p - \overline{v}^p| = |v-\overline{v}| \cdot |v^{p-1} + v^{p-2}\overline{v} + \cdots + v\overline{v}^{p-2} + \overline{v}^{p-1}| \le C_3 \cdot |v - \overline{v}|, \tag{2.9}$$

where C_3 is depending possibly on a, b, d_1, p and Ω.

By Theorem 2.1 and Corollary 2.1, we will give the sufficient conditions for the nonexistence of non-constant positive solutions to (1.4).

Theorem 2.2. *Assume that the following inequality holds:*

$$\mu_1 > \max\left\{\frac{(a+b)(2C_0 + 2|\Omega| + C_0\lambda) + C_2C_3|\Omega|}{2d_1|\Omega|}, \frac{C_0\lambda(a+b) + C_2C_3|\Omega|(1+2\lambda)}{2d_2|\Omega|}\right\}, \tag{2.10}$$

where C_0, C_2 are defined in Theorem 2.1, and C_3 in Corollary 2.1. Then, (1.4) has no non-constant positive solutions.

Proof. Denote $\overline{w} = \frac{1}{|\Omega|}\int_\Omega w(x)\mathrm{d}x$. It is obviously that $\int_\Omega (u-\overline{u})\mathrm{d}x = \int_\Omega (v-\overline{v})\mathrm{d}x = 0$. Multiplying the both sides of the first two equations in (1.4) by $(u-\overline{u})$ and $(v-\overline{v})$, respectively, it follows that

$$
\begin{aligned}
d_1 \int_\Omega |\nabla(u-\overline{u})|^2 \mathrm{d}x &= \int_\Omega \left[uv^p - \frac{(a+b)u}{1+bu}\right](u-\overline{u})\mathrm{d}x \\
&= \int_\Omega \left[uv^p - \overline{uv}^p + \frac{(a+b)\overline{u}}{1+b\overline{u}} - \frac{(a+b)u}{1+bu}\right](u-\overline{u})\mathrm{d}x \\
&= \int_\Omega \left\{(uv^p - \overline{u}v^p) + (\overline{u}v^p - \overline{uv}^p) + \left[\frac{(a+b)\overline{u}}{1+b\overline{u}} - \frac{(a+b)u}{1+bu}\right]\right\}(u-\overline{u})\mathrm{d}x \\
&= \int_\Omega \left[v^p(u-\overline{u}) + \overline{u}(v^p - \overline{v}^p) - \frac{(a+b)(u-\overline{u})}{(1+b\overline{u})(1+b\overline{u})}\right](u-\overline{u})\mathrm{d}x \\
&= \int_\Omega \left\{\left[v^p - \frac{a+b}{(1+b\overline{u})(1+b\overline{u})}\right](u-\overline{u}) + \overline{u}(v^p-\overline{v}^p)\right\}(u-\overline{u})\mathrm{d}x \\
&\le \int_\Omega \left[\frac{C_0(a+b)}{|\Omega|} + (a+b)\right](u-\overline{u})^2\mathrm{d}x + \int_\Omega C_2C_3|u-\overline{u}||v-\overline{v}|\mathrm{d}x \\
&\le \left[\frac{C_0(a+b)}{|\Omega|} + (a+b) + \frac{C_2C_3}{2}\right]\int_\Omega (u-\overline{u})^2\mathrm{d}x + \frac{C_2C_3}{2}\int_\Omega (v-\overline{v})^2\mathrm{d}x,
\end{aligned}
\tag{2.11}
$$

and

$$
\begin{aligned}
d_2 \int_\Omega |\nabla(v-\overline{v})|^2\mathrm{d}x &= \lambda\int_\Omega (1-uv^p)(v-\overline{v})\mathrm{d}x = \lambda\int_\Omega (\overline{uv}^p - uv^p)(v-\overline{v})\mathrm{d}x \\
&= \lambda\int_\Omega [\overline{uv}^p - u\overline{v}^p + u\overline{v}^p - uv^p](v-\overline{v})\mathrm{d}x \\
&= \lambda\int_\Omega [(\overline{u}-u)\overline{v}^p + u(\overline{v}^p - v^p)](v-\overline{v})\mathrm{d}x \\
&\le \lambda\int_\Omega [(\overline{u}-u)\overline{v}^p + u(\overline{v}^p - v^p)](v-\overline{v})\mathrm{d}x \\
&\le \lambda\int_\Omega \frac{C_0(a+b)}{|\Omega|}|u-\overline{u}||v-\overline{v}|\mathrm{d}x + \lambda\int_\Omega C_2C_3(v-\overline{v})^2\mathrm{d}x \\
&\le \frac{C_0(a+b)\lambda}{2|\Omega|}\int_\Omega (u-\overline{u})^2\mathrm{d}x + \left[\frac{C_0(a+b)\lambda}{2|\Omega|} + C_2C_3\lambda\right]\int_\Omega (v-\overline{v})^2\mathrm{d}x.
\end{aligned}
\tag{2.12}
$$

Thanks to the well-known Poincaré inequality

$$
\mu_1 \int_\Omega (w-\overline{w})^2\mathrm{d}x \le \int_\Omega |\nabla(w-\overline{w})|^2\mathrm{d}x, \quad \forall\, w \in H^1(\Omega),
$$

where μ_1 is the first positive eigenvalue of $-\Delta$ with Neumann homogeneous boundary condition, we have

$$\begin{aligned}\mu_1 d_1 \int_\Omega (u-\overline{u})^2 \mathrm{d}x &\le \left[\tfrac{C_0(a+b)}{|\Omega|} + (a+b) + \tfrac{C_2C_3}{2}\right] \int_\Omega (u-\overline{u})^2 \mathrm{d}x + \tfrac{C_2C_3}{2} \int_\Omega (v-\overline{v})^2 \mathrm{d}x,\\ \mu_1 d_2 \int_\Omega (v-\overline{v})^2 \mathrm{d}x &\le \tfrac{C_0(a+b)\lambda}{2|\Omega|} \int_\Omega (u-\overline{u})^2 \mathrm{d}x + \left[\tfrac{C_0(a+b)\lambda}{2|\Omega|} + C_2C_3\lambda\right] \int_\Omega (v-\overline{v})^2 \mathrm{d}x.\end{aligned} \tag{2.13}$$

Adding the two inequalities of (2.13), we get

$$\begin{aligned}\mu_1 d_1 \int_\Omega (u-\overline{u})^2 + \mu_1 d_2 \int_\Omega (v-\overline{v})^2 \mathrm{d}x &\le \frac{(a+b)(2C_0 + 2|\Omega| + C_0\lambda) + C_2C_3|\Omega|}{2|\Omega|} \int_\Omega (u-\overline{u})^2 \mathrm{d}x\\ &\quad + \frac{C_0\lambda(a+b) + C_2C_3|\Omega|(1+2\lambda)}{2|\Omega|} \int_\Omega (v-\overline{v})^2 \mathrm{d}x,\end{aligned} \tag{2.14}$$

which yields that $u=\overline{u}$ and $v=\overline{v}$ by the condition (2.10). That is to say, (1.4) has no non-constant positive solutions. The proof is completed. □

Corollary 2.2. *If λ is small enough or d_2 is large enough, then (1.4) has no non-constant positive solutions.*

Proof. Assume that (u, v) is a positive solution of (1.4). Note that $\overline{v} = \frac{1}{|\Omega|}\int_\Omega v(x)\mathrm{d}x$. It follows from the proof of Theorem 2.2 that $(v-\overline{v})$ satisfies the second inequality of (2.13). So we must have $v=\overline{v}$ if λ is small enough or d_2 is large enough, and then, $u = 1/\sqrt[p]{\overline{v}}$ by the second equation of (1.4). The proof is completed. □

Remark 1. Theorem 2.1 will be still valid, if $p>1$ is a positive constant, but not an integer.

3. Existence of bifurcation solutions: one-dimensional space

3.1. The bifurcation from simple eigenvalues

In this and next section, we will consider the steady-state bifurcations from both simple eigenvalues and double eigenvalues, which imply the existence of non-constant positive solution of (1.4). The idea in constructing this paper is partly due to the techniques developed in [13,16].

By using the Crandall–Rabinowitz bifurcation theorem [17, Theorem 13.5], we first take λ as the parameter to discuss the bifurcation solutions of (1.4) with $\Omega=(0,\pi)$, i.e., the local bifurcation of the following system:

$$\begin{cases} d_1\Delta u + uv^p - \frac{(a+b)u}{1+bu} = 0 & x\in(0,\pi),\\ d_2\Delta v + \lambda(1-uv^p) = 0 & x\in(0,\pi),\\ u_x = v_x = 0, & x = 0,\pi, \end{cases} \tag{3.1}$$

where the Laplace operator Δ can be understood as $\Delta = \frac{\partial^2}{\partial x^2}$ in $\mathbb{R}$.

It is common knowledge that all the eigenvalues of problem (3.2) (See Theorems 2.44 and 2.55 in [18])

$$\begin{cases} \phi'' + \mu u = 0, & x\in(0,\pi),\\ \phi_x = 0, & x=0,\pi \end{cases} \tag{3.2}$$

are $\mu_n = n^2, n = 0,1,2,\ldots$, and the corresponding eigenfunction are

$$\phi_n(x) = \begin{cases} \sqrt{\frac{1}{\pi}}, & n=0,\\ \sqrt{\frac{2}{\pi}}\cos nx, & n>0, \end{cases}$$

which construct the normal orthogonal basis of $L^2(0,\pi)$.

Let $X = \{(u,v) : u, v \in W^{2,p}(0,\pi), u_x = v_x = 0, x = 0,\pi\}$ and $Y = L^p(0,\pi) \times L^p(0,\pi)$. Then, X is a Banach space, and Y is a Hilbert space with the inner product $(U_1, U_2)_Y = (u_1, u_2)_{L^2(0,\pi)} + (v_1, v_2)_{L^2(0,\pi)}$. Define the mapping $F : (0,\infty) \times X \to Y$ by

$$F(\lambda, U) = \begin{pmatrix} d_1\Delta u + g(u,v) - h(u) \\ d_2\Delta v + \lambda[1 - g(u,v)] \end{pmatrix},$$

where $U = (u,v)$, $g(u,v) = uv^p$ and $h(u) = \frac{(a+b)u}{1+bu}$. Then for any $(u,v) \in X$, that $U = (u,v)$ is a zero of (3.1) is equivalent to $F(\lambda, U) = 0$.

The Fréchet derivative of $F(\gamma, U)$, with respect to U at U^*, could be characterized by

$$L(\lambda) = \begin{pmatrix} d_1\Delta + \frac{ab}{a+b} & \frac{p}{\sqrt[p]{a}} \\ -\lambda a & d_2\Delta - \frac{\lambda p}{\sqrt[p]{a}} \end{pmatrix}. \tag{3.3}$$

The characteristic equation of $L(\lambda)$ is $L(\lambda)(\xi,\eta)^\top = \Lambda(\xi,\eta)^\top$, where $(\xi,\eta) \in X$. That is to say, (ξ,η) satisfies

$$\begin{cases} d_1\Delta\xi + \frac{ab}{a+b}\xi + \frac{p}{\sqrt[p]{a}}\eta = \Lambda\xi, & x \in (0,\pi), \\ d_2\Delta\eta - \frac{\lambda p}{\sqrt[p]{a}}\eta - \lambda a\xi = \Lambda\eta, & x \in (0,\pi), \\ u_x = v_x = 0, & x = 0,\pi. \end{cases}$$

Let $\xi = \sum\limits_{n=0}^{\infty} a_n\phi_n, \eta = \sum\limits_{n=0}^{\infty} b_n\phi_n$. Then, the above characteristic equation translates into

$$\sum_{n=0}^{\infty} M_n(\lambda)\begin{pmatrix} a_n \\ b_n \end{pmatrix}\phi_n = 0,$$

where

$$M_n(\lambda) = \begin{pmatrix} -d_1\mu_n + \frac{ab}{a+b} & \frac{p}{\sqrt[p]{a}} \\ -\lambda a & -d_2\mu_n - \frac{\lambda p}{\sqrt[p]{a}} \end{pmatrix}. \tag{3.4}$$

Letting $|M_n(\lambda)| = 0, n = 0,1,2,\ldots$, we have

$$\Lambda^2 - T_n\Lambda + D_n = 0, \quad n = 0,1,2,\ldots, \tag{3.5}$$

where

$$\begin{aligned} T_n(\lambda) &= -(d_1 + d_2)\mu_n + \tfrac{ab}{a+b} - \tfrac{\lambda p}{\sqrt[p]{a}}, \\ D_n(\lambda) &= d_2\mu_n\left(d_1\mu_n - \tfrac{ab}{a+b}\right) + \tfrac{p}{\sqrt[p]{a}}\left(\tfrac{a^2}{a+b} + d_1\mu_n\right)\lambda. \end{aligned}$$

Letting $\Lambda = 0$ in (3.5), which yields $D_n(\lambda) = 0$, we have

$$\lambda_n := \frac{\sqrt[p]{a}d_2(ab - d_1(a+b)\mu_n)\mu_n}{p(a^2 + d_1(a+b)\mu_n)}. \tag{3.6}$$

Hereafter in this paper, we will impose the condition

$$\textbf{(H)}: \quad ab > d_1(a+b).$$

Accordingly, there exists a positive integer n_λ such that $-d_1\mu_n + \frac{ab}{a+b} > 0$ for $1 \le n \le n_\lambda$. Now, we will prove the existence of non-constant positive solutions of $F(\lambda, U) = 0$ near (λ_n, U^*) with $1 \le n \le n_\lambda$.

By the Crandall and Rabinowitz bifurcation theorem (See Theorem 13.5 in [17]), (λ_0, U^*) is a bifurcation point of $F(\lambda, U) = 0$ if the following conditions hold:

(1) F_λ, F_U and $F_{\lambda,U}$ exist and are continuous;
(2) $\dim\ker F_U(\lambda_0, U^*) = \operatorname{codim} R(F_U(\lambda_0, U^*)) = 1$;

(3) $\ker F_U(\lambda_0, U^*) = \operatorname{span}\{\Phi\}$, $F_{\lambda,U}(\lambda_0, U^*)\Phi \notin R(F_U(\lambda_0, U^*))$.

Theorem 3.1. *Suppose that* (**H**) *holds and takes*

$$\lambda_n = \frac{\sqrt[p]{a}d_2(ab - d_1(a+b)\mu_n)\mu_n}{p(a^2 + d_1(a+b)\mu_n)}, \quad \text{where } 1 \le n \le n_\lambda \text{ and } \mu_n = n^2.$$

If $\lambda_i \neq \lambda_j$ when $i \neq j$ $(1 \le i, j \le n_\lambda)$, then (λ_i, U^*) with $1 \le i \le n_\lambda$ is a bifurcation point of $F(\lambda, U) = 0$ with respect to the curve $(\lambda, U^*), \lambda > 0$.

Moreover, there is a curve of non-constant positive solutions

$$(\lambda(s), (u(s), v(s))) = (\lambda(s), (u^* + s(\phi_i + o(s)), v^* + s(b_i\phi_i + o(s))))$$

with $|s| \ll 1$, satisfying $\lambda(0) = \lambda_i, u(0) = u^*, v(0) = v^*$ and $\lambda(s), u(s), v(s)$ are continuous functions with respect to s. Here, $b_i = \frac{-\lambda_i a}{d_2\mu_i + \frac{\lambda_i p}{\sqrt[p]{a}}}$.

Proof. Note that

$$L(\lambda_i) := F_U(\lambda_i, U^*) = \begin{pmatrix} d_1\Delta + \frac{ab}{a+b} & \frac{p}{\sqrt[p]{a}} \\ -\lambda_i a & d_2\Delta - \frac{\lambda_i p}{\sqrt[p]{a}} \end{pmatrix},$$

and from (3.4), we have

$$|M_n(0)| = 0 \quad \Leftrightarrow \quad \lambda = \lambda_n = \frac{d_2\mu_n\left(-d_1\mu_n + \frac{ab}{a+b}\right)}{\frac{p}{\sqrt[p]{a}}\left(\frac{a^2}{a+b} + d_1\mu_n\right)}.$$

By some simple calculations, we obtain that $\ker L(\lambda_i) = \operatorname{span}\{\Phi\}, \Phi = (1, b_i)^\perp\phi_i$, where

$$b_i = \frac{d_1\mu_i - \frac{ab}{a+b}}{\frac{p}{\sqrt[p]{a}}} = \frac{-\lambda_i a}{d_2\mu_i + \frac{\lambda_i p}{\sqrt[p]{a}}} < 0.$$

The adjoint operator of $L(\lambda_i)$ can be expressed as

$$L^*(\lambda_i) = \begin{pmatrix} d_1\Delta + \frac{ab}{a+b} & -\lambda_i a \\ \frac{p}{\sqrt[p]{a}} & d_2\Delta - \frac{\lambda_i p}{\sqrt[p]{a}} \end{pmatrix}.$$

Similarly, we get $\ker L^*(\lambda_i) = \operatorname{span}\{\Phi^*\}, \Phi^* = (1, b_i^*)^\perp\phi_i$, where

$$b_i^* = \frac{-d_1\mu_i + \frac{ab}{a+b}}{\lambda_i a} = \frac{\frac{p}{\sqrt[p]{a}}}{d_2\mu_i + \frac{\lambda_i p}{\sqrt[p]{a}}} > 0.$$

Since $R(L) = (\ker L^*)^\perp$, we obtain

$$\operatorname{codim} R(L(\lambda_i)) = \dim \ker L^*(\lambda_i) = 1.$$

Besides,

$$F_{\lambda,U}(\lambda_i, U^*)\Phi = \begin{pmatrix} 0 & 0 \\ -a & -\frac{p}{\sqrt[p]{a}} \end{pmatrix}\Phi = \begin{bmatrix} 0 \\ -\left(d_1\mu_i + \frac{a^2}{a+b}\right) \end{bmatrix}\phi_i$$

and

$$(F_{\lambda,U}(\lambda_i, U^*)\Phi, \Phi^*)_Y = -\left(d_1\mu_i + \frac{a^2}{a+b}\right)\frac{\frac{p}{\sqrt[p]{a}}}{d_2\mu_i + \frac{\lambda_i p}{\sqrt[p]{a}}} < 0, \neq 0. \tag{3.7}$$

Inequation (3.7) implies that $F_{\lambda,U}(\lambda_i, U^*)\Phi \notin R(L(\lambda_i))$, which concludes the proof. □

Remark 2. Theorem 3.1 will be still valid, if $p > 1$ is a positive constant, but not an integer.

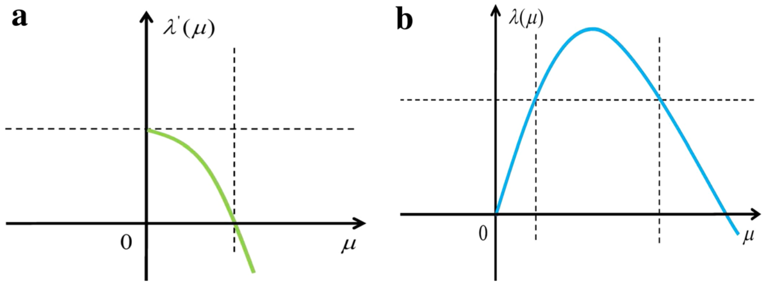

Fig. 1. **a** The relation between $\lambda'(\mu)$ (the vertical axis) and μ (the horizontal axis), **b** the relation between $\lambda(\mu)$ (the vertical axis) and μ (the horizontal axis)

3.2. The bifurcation from double eigenvalues

Recall the definition of λ_n in (3.6) and take $\lambda(\mu) = \frac{d_2\mu\left(-d_1\mu+\frac{ab}{a+b}\right)}{\frac{p}{\sqrt[p]{a}}\left(\frac{a^2}{a+b}+d_1\mu\right)}$ with $\mu \geq 0$. It can be easily verified from the continuity and monotonicity for $\lambda(\mu)$ (see Fig. 1), that λ_i in (3.6) may be or not be equal to λ_j when $i \neq j$, $1 \leq i, j \leq n_\lambda$. So there are two cases:

(1) For any $i, j \in [1, n_\lambda]$, $\lambda_i \neq \lambda_j$ when $i \neq j$;
(2) For any $i, j \in [1, n_\lambda]$, there will be at most two positive integers $i \neq j$ with $\lambda_i = \lambda_j$.

Case (1) was discussed by Theorem 3.1, which implies the bifurcation from simple eigenvalues. In this section, we will discuss the bifurcation solutions of (3.1) from double eigenvalues, i.e., Case (2), by using the implicit function theorem.

Theorem 3.2. *Suppose that* **(H)** *holds and there exist two positive integers $i \neq j$, but $\lambda_i = \lambda_j = \overline{\lambda}$ and $j = 2i$, $1 \leq i, j \leq n_\gamma$. Take $\omega_0 \in \mathbb{R}$ with $\cos\omega_0 \neq 0$ and $(a + \frac{p}{\sqrt[p]{a}} b_i)\cos^2\omega_0 c_1 c_3 \neq (a + \frac{p}{\sqrt[p]{a}} b_j)\sin^2\omega_0 c_2 c_4$. If $1 + b_i b_i^* \neq 0$, $1 + b_j b_j^* \neq 0$, then $(\widehat{\gamma}, U^*)$ is a bifurcation point of $F(\gamma, U) = 0$ with respect to the curve $(\lambda, U^*), \lambda > 0$.*

Moreover, there is a curve of non-constant positive solutions $(\gamma(\omega), U^* + s(\cos\omega\ \Phi_i + \sin\omega\ \Phi_j + W(\omega)))$ with $|\omega - \omega_0| \ll 1$, satisfying $\lambda(\omega_0) = \overline{\lambda}, s(\omega_0) = 0, W(\omega_0) = 0$ and $\lambda(\omega), s(\omega), W(\omega)$ are continuously differentiable functions with respect to ω. Here, λ_i and λ_j are defined by (3.6) and

$$\Phi_i = (1, b_i)^\perp \phi_i, \quad \Phi_j = (1, b_j)^\perp \phi_j,$$
$$b_i = \frac{-\overline{\lambda}a}{d_2\mu_i + \frac{\overline{\lambda}p}{\sqrt[p]{a}}}, \quad b_j = \frac{-\overline{\lambda}a}{d_2\mu_j + \frac{\overline{\lambda}p}{\sqrt[p]{a}}}, \quad b_i^* = \frac{\frac{p}{\sqrt[p]{a}}}{d_2\mu_i + \frac{\overline{\lambda}p}{\sqrt[p]{a}}}, \quad b_j^* = \frac{\frac{p}{\sqrt[p]{a}}}{d_2\mu_j + \frac{\overline{\lambda}p}{\sqrt[p]{a}}}.$$

Proof. Take the translation $U = U^* + (y, z)$, then $F(\gamma, U) = 0$ yields a new map $\mathcal{F} : \mathbb{R} \times X \to Y$:

$$\begin{aligned}\mathcal{F}(\lambda, (y, z)) &= \begin{pmatrix} d_1\Delta y + g(u^* + y, v^* + z) - h(u^* + y) \\ d_2\Delta z + \lambda - \lambda g(u^* + y, v^* + z) \end{pmatrix} \\ &= L(\lambda)(y, z)^\top + (f_1, f_2)^\top,\end{aligned} \tag{3.8}$$

where $L(\lambda)$ is defined by (3.3), $g(u, v) = uv^p$, $h(u) = \frac{(a+b)u}{1+bu}$ and

$$f_1 \doteq f_1(y, z) = g(u^* + y, v^* + z) - h(u^* + y) - \frac{ab}{a+b}y - \frac{p}{\sqrt[p]{a}}z,$$
$$f_2 \doteq f_2(y, z) = \lambda - \lambda g(u^* + y, v^* + z) + \lambda a y + \frac{\lambda p}{\sqrt[p]{a}}z.$$

By the most basic version of Taylor's theorem, the third-order expansion of f_1 and f_2 at $(0,0)$ can be expressed as follows:

$$\begin{aligned} f_1 &= \tfrac{1}{2}Ay^2 + Byz + \tfrac{1}{2}Cz^2 + \tfrac{1}{6}Dy^3 + \tfrac{1}{2}Eyz^2 + \tfrac{1}{6}Hz^3 + \mathcal{O}(|y|^4, |y||z|^3), \\ f_2 &= \widehat{B}yz + \tfrac{1}{2}\widehat{C}z^2 + \tfrac{1}{2}\widehat{E}yz^2 + \tfrac{1}{6}\widehat{H}z^3 + \mathcal{O}(|y||z|^3, |z|^4), \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} &A = \tfrac{2a^3b}{(a+b)^2} > 0, && B = pa^{\frac{p-1}{p}} > 0, \\ &C = p(p-1)a^{-\frac{2}{p}} > 0, && D = -\tfrac{6a^4b^2}{(a+b)^3} < 0, \\ &E = p(p-1)a^{\frac{p-2}{p}} > 0, && H = p(p-1)(p-2)a^{-\frac{3}{p}}, \\ &\widehat{B} = -\lambda pa^{\frac{p-1}{p}} < 0, && \widehat{C} = -\lambda p(p-1)a^{-\frac{2}{p}} < 0. \\ &\widehat{E} = \lambda p(p-1)a^{\frac{p-2}{p}} < 0 && \widehat{H} = -\lambda p(p-1)(p-2)a^{-\frac{3}{p}}. \end{aligned} \tag{3.10}$$

Recall $L(\overline{\lambda})$ in (3.3) and $M_n(\lambda)$ in (3.4) and let $L(\overline{\lambda})(\xi,\eta)^\top = 0$. It's easy to check that $|M_n(0)| = 0$ is equivalent to $n = i, j$. Accordingly, we have

$$\ker\, L(\overline{\lambda}) = \text{span}\{\Phi_i, \Phi_j\}, \quad \Phi_i = (1, b_i)^\top \phi_i, \quad \Phi_j = (1, b_j)^\top \phi_j,$$

where

$$b_i = \frac{-\overline{\lambda}a}{d_2\mu_i + \frac{\overline{\lambda}p}{\sqrt[p]{a}}} < 0, \quad b_j = \frac{-\overline{\lambda}a}{d_2\mu_j + \frac{\overline{\lambda}p}{\sqrt[p]{a}}} < 0, \quad b_i \neq b_j.$$

Similarly, for the adjoint operator $L^*(\overline{\lambda})$ we have

$$\ker\, L^*(\overline{\lambda}) = \text{span}\{\Phi_i^*, \Phi_j^*\}, \quad \Phi_i = (1, b_i^*)^\top \phi_i, \quad \Phi_j = (1, b_j^*)^\top \phi_j,$$

where

$$b_i^* = \frac{\frac{p}{\sqrt[p]{a}}}{d_2\mu_i + \frac{\overline{\lambda}p}{\sqrt[p]{a}}} > 0, \quad b_j^* = \frac{\frac{p}{\sqrt[p]{a}}}{d_2\mu_j + \frac{\overline{\lambda}p}{\sqrt[p]{a}}} > 0, \quad b_i^* \neq b_j^*.$$

Therefore, the image space of $L(\overline{\lambda})$ can be expressed by

$$R(L(\overline{\lambda})) = \left\{ (\varphi,\psi) \in Y : \int_0^\pi (\varphi + b_i^*\psi)\phi_i dx = \int_0^\pi (\varphi + b_j^*\psi)\phi_j dx = 0 \right\},$$

which yields

$$\text{codim}\, R(L(\overline{\lambda})) = \dim \ker\, L^*(\overline{\lambda}) = 2.$$

Set $X_1 = \text{span}\{\Phi_i, \Phi_j\}$. Decomposes $X = X_1 \oplus X_2$ with

$$X_2 = \left\{ (\varphi,\psi) \in Y : \int_0^\pi (\varphi + b_i\psi)\phi_i dx = \int_0^\pi (\varphi + b_j\psi)\phi_j dx = 0 \right\}.$$

And again, we define a projection from Y to $X_1 \subset Y$ by

$$P\begin{pmatrix} y \\ z \end{pmatrix} = \frac{1}{1 + b_i b_i^*}\left[\int_0^\pi (\varphi + b_i^*\psi)\phi_i \mathrm{d}x\right]\Phi_i + \frac{1}{1 + b_j b_j^*}\left[\int_0^\pi (\varphi + b_j^*\psi)\phi_j \mathrm{d}x\right]\Phi_j.$$

Thus, we can decomposes $Y = Y_1 \oplus Y_2$ with $Y_1 = R(P) = X_1$ and $Y_2 = ker(P) = R(L(\overline{\lambda}))$. One can see [19] for the detailed decomposition.

Now, we will find solutions of $\mathcal{F}(\lambda,(y,z))=0$ in the following form

$$(y,z)^\top = s(\cos\omega\ \Phi_i + \sin\omega\ \Phi_j + W), \quad W=(w_1,w_2)^\top \in X_2, \tag{3.11}$$

where $s,\omega\in\mathbb{R}$ are parameters. Take a fixed $\omega_0\in\mathbb{R}$ such that $\cos\omega_0\neq 0$ and $(a+\frac{p}{\sqrt[q]{a}}b_i)\cos^2\omega_0 c_1c_3 \neq (a+\frac{p}{\sqrt[q]{a}}b_j)\sin^2\omega_0 c_2c_4$. Define a new mapping $\mathcal{G}(\gamma,s,W;\omega):\mathbb{R}\times\mathbb{R}\times X_2\times(\omega_0-\delta,\omega_0+\delta)\to Y$ by

$$\begin{aligned}\mathcal{G}(\lambda,s,W;\omega) &= s^{-1}\mathcal{F}(\gamma, s(\cos\omega\ \Phi_i+\sin\omega\ \Phi_j+W))\\ &= L(\lambda)(\cos\omega\ \Phi_i+\sin\omega\ \Phi_j+W)+s^{-1}(f_1,f_2)^\top,\\ &= L(\lambda)(\cos\omega\ \Phi_i+\sin\omega\ \Phi_j+W)+s(g_1,g_2)^\top,\end{aligned}$$

where

$$\begin{aligned} g_1 &= \tfrac{1}{2}A\overline{y}^2 + B\overline{yz} + \tfrac{1}{2}C\overline{z}^2 + s(\tfrac{1}{6}D\overline{y}^3 + \tfrac{1}{2}E\overline{yz}^2 + \tfrac{1}{6}H\overline{z}^3) + \mathcal{O}(|s|^2),\\ g_2 &= \widehat{B}\overline{yz} + \tfrac{1}{2}\widehat{C}\overline{z}^2 + s(\tfrac{1}{2}\widehat{E}\overline{yz}^2 + \tfrac{1}{6}\widehat{H}\overline{z}^3) + \mathcal{O}(|s|^2),\end{aligned}$$

and $(\overline{y},\overline{z}) = (\cos\omega\ \phi_i + \sin\omega\ \phi_j + w_1, b_i\cos\omega\ \phi_i + b_j\sin\omega\ \phi_j + w_2)$.

It is obvious that $G(\overline{\lambda},0,(0,0);\omega_0)=0$. At $(\lambda,s,W;\omega)=(\overline{\lambda},0,(0,0);\omega_0)$, the Fréchet derivative of $G(\lambda,s,W;\omega)$ with respect to (λ,s,W) can be characterized by the mapping

$$\begin{aligned} &\mathcal{G}_{(\lambda,s,W)}(\overline{\lambda},0,(0,0);\omega_0)(\lambda,s,W)\\ &= L(\overline{\lambda})W - \lambda\left(a+\tfrac{p}{\sqrt[q]{a}}b_i\right)\cos\omega_0\begin{pmatrix}0\\ \phi_i\end{pmatrix} - \lambda\left(a+\tfrac{p}{\sqrt[q]{a}}b_j\right)\sin\omega_0\begin{pmatrix}0\\ \phi_j\end{pmatrix}\\ &\quad + s\cos\omega_0^2\begin{pmatrix}\frac{1}{2}A+Bb_j+\frac{1}{2}Cb_j^2\\ \widehat{B}b_j+\frac{1}{2}\widehat{C}b_j^2\end{pmatrix}\phi_i^2 + s\sin\omega_0\cos\omega_0\begin{pmatrix}A+Bb_j+Bb_i+Cb_ib_j\\ \widehat{B}b_j+\widehat{B}b_i+\widehat{C}b_ib_j\end{pmatrix}\phi_i\phi_j\\ &\quad + s\sin\omega_0^2\begin{pmatrix}\frac{1}{2}A+Bb_j+\frac{1}{2}Cb_j^2\\ \widehat{B}b_j+\frac{1}{2}\widehat{C}b_j^2\end{pmatrix}\phi_j^2.\end{aligned} \tag{3.12}$$

By the projection P from Y to Y_1, we have

$$\begin{pmatrix}0\\ \phi_i\end{pmatrix} = c_1\Phi_i + \begin{pmatrix}y_1\\ z_1\end{pmatrix}, \quad \begin{pmatrix}0\\ \phi_j\end{pmatrix} = c_2\Phi_j + \begin{pmatrix}y_2\\ z_2\end{pmatrix},$$

where

$$\begin{aligned} c_1 &= \tfrac{b_i^*}{1+b_ib_i^*}\neq 0,\ \begin{pmatrix}y_1\\ z_1\end{pmatrix} = \begin{bmatrix}-c_1\phi_i\\ (1-c_1b_i)\phi_i\end{bmatrix}\in Y_2,\\ c_2 &= \tfrac{b_j^*}{1+b_jb_j^*}\neq 0,\ \begin{pmatrix}y_2\\ z_2\end{pmatrix} = \begin{bmatrix}-c_2\phi_j\\ (1-c_2b_j)\phi_i\end{bmatrix}\in Y_2.\end{aligned}$$

Note that $j=2i$. Thus, we have $\int_0^\pi \phi_i^2\phi_j\mathrm{d}x = \sqrt{\frac{1}{2\pi}}\neq 0$ and $\int_0^\pi \phi_j^2\phi_i\mathrm{d}x = 0$. In addition, $\int_0^\pi \phi_i^3\mathrm{d}x = \int_0^\pi \phi_j^3\mathrm{d}x = 0$ is valid. Accordingly,

$$\begin{pmatrix}y_5\\ z_5\end{pmatrix} \doteq \begin{pmatrix}\frac{1}{2}A+Bb_j+\frac{1}{2}Cb_j^2\\ \widehat{B}b_j+\frac{1}{2}\widehat{C}b_j^2\end{pmatrix}\phi_j^2 \in Y_2.$$

Then, we decompose

$$\begin{pmatrix} \frac{1}{2}A + Bb_j + \frac{1}{2}Cb_j^2 \\ \widehat{B}b_j + \frac{1}{2}\widehat{C}b_j^2 \end{pmatrix} \phi_i^2 = c_3\Phi_j + \begin{pmatrix} y_3 \\ z_3 \end{pmatrix},$$
$$\begin{pmatrix} A + Bb_j + Bb_i + Cb_ib_j \\ \widehat{B}b_j + \widehat{B}b_i + \widehat{C}b_ib_j \end{pmatrix} \phi_i\phi_j = c_4\Phi_i + \begin{pmatrix} y_4 \\ z_4 \end{pmatrix},$$

where

$$c_3 = \sqrt{\frac{1}{2\pi}} \frac{(\frac{1}{2}A + Bb_j + \frac{1}{2}C) + b_j^*(\frac{1}{2}\widehat{A}_0 + \widehat{B}_0 b_j)}{1 + b_j b_j^*},$$
$$c_4 = \sqrt{\frac{1}{2\pi}} \frac{(A + Bb_j + Bb_i + Cb_ib_j) + b_i^*(\widehat{B}b_j + \widehat{B}b_i + \widehat{C}b_ib_j)}{1 + b_i b_i^*},$$

and

$$\begin{pmatrix} y_3 \\ z_3 \end{pmatrix} = \begin{bmatrix} (\frac{1}{2}A + Bb_j + \frac{1}{2}Cb_j^2)\phi_i^2 - c_3\phi_j \\ (\widehat{B}b_j + \frac{1}{2}\widehat{C}b_j^2)\phi_i^2 - c_3 b_j \phi_j \end{bmatrix} \in Y_2,$$
$$\begin{pmatrix} y_4 \\ z_4 \end{pmatrix} = \begin{bmatrix} (A + Bb_j + Bb_i + Cb_ib_j)\phi_i\phi_j - c_4\phi_i \\ (\widehat{B}b_j + \widehat{B}b_i + \widehat{C}b_ib_j)\phi_i\phi_j - c_4 b_i \phi_i \end{bmatrix} \in Y_2.$$

In order to use the implicit function theorem, it suffices to verify that $\mathcal{G}_{(\lambda,s,W)}(\overline{\lambda}, 0, (0,0); \omega_0) : \mathbb{R} \times \mathbb{R} \times X_2 \to Y$ is an isomorphism, in other words, $\mathcal{G}_{(\lambda,s,W)}(\overline{\lambda}, 0, (0,0); \omega_0)$ is injective and surjective. Let

$$\mathcal{G}_{(\lambda,s,W)}(\overline{\lambda}, 0, (0,0); \omega_0)(\lambda, s, W) = 0. \tag{3.13}$$

Note that $L(\overline{\lambda})$ is an isomorphism from X_2 to Y_2, $\Phi_i, \Phi_j \in Y_1$ and $(y_k, z_k)^\top \in Y_2$, $k = 1,2,3,4,5$. By (3.12), (3.13) translates into

$$\begin{cases} \left[-\lambda\left(a + \frac{p}{\sqrt[p]{a}}b_i\right)\cos\omega_0 c_1 + s\sin\omega_0\cos\omega_0 c_4\right]\Phi_i \\ \quad + \left[-\lambda\left(a + \frac{p}{\sqrt[p]{a}}b_j\right)\sin\omega_0 c_2 + s\cos\omega_0^2 c_3\right]\Phi_j = 0, \\ L(\overline{\lambda})W - \lambda\left(a + \frac{p}{\sqrt[p]{a}}b_i\right)\cos\omega_0\begin{pmatrix} y_1 \\ z_1 \end{pmatrix} - \lambda\left(a + \frac{p}{\sqrt[p]{a}}b_j\right)\cos\omega_0\begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \\ \quad + s\cos\omega_0^2\begin{pmatrix} y_3 \\ z_3 \end{pmatrix} + s\sin\omega_0\cos\omega_0\begin{pmatrix} y_4 \\ z_4 \end{pmatrix} + s\sin\omega_0^2\begin{pmatrix} y_5 \\ z_5 \end{pmatrix} = 0, \end{cases} \tag{3.14}$$

Since $(a + \frac{p}{\sqrt[p]{a}}b_i)\cos^2\omega_0 c_1 c_3 \neq (a + \frac{p}{\sqrt[p]{a}}b_j)\sin^2\omega_0 c_2 c_4$, we obtain $\lambda = 0, s = 0$ from the first equation of (3.14). Since $L(\overline{\lambda})$ is an isomorphism from X_2 to Y_2, we obtain $W = 0$ from the second equation of (3.14). This yields that $G_{(\lambda,s,W)}(\overline{\lambda}, 0, (0,0); \omega_0)$ is injective.

Next, for any $(y, z) \in Y$, we will prove that there exists $(\lambda, s, W) \in \mathbb{R} \times \mathbb{R} \times X_2$ such that

$$G_{(\lambda,s,W)}(\overline{\lambda}, 0, (0,0); \omega_0)(\lambda, s, W) = (y, z)^\top. \tag{3.15}$$

By the decomposition of Y, there exist $\alpha, \beta \in \mathbb{R}$ and $(y_0, z_0) \in Y_2$ such that $(y, z)^\top = \alpha\Phi_i + \beta\Phi_j + (y_0, z_0)^\top$. Substituting (y, z) in (3.15), we have

$$\begin{cases} \left[-\lambda\left(a + \frac{p}{\sqrt[p]{a}}b_i\right)\cos\omega_0 c_1 + s\sin\omega_0\cos\omega_0 c_4\right]\Phi_i \\ \quad + \left[-\lambda\left(a + \frac{p}{\sqrt[p]{a}}b_j\right)\cos\omega_0 c_2 + s\cos\omega_0^2 c_3\right]\Phi_j = \alpha\Phi_i + \beta\Phi_j, \\ L(\overline{\lambda})W - \lambda\left(a + \frac{p}{\sqrt[p]{a}}b_i\right)\cos\omega_0\begin{pmatrix} y_1 \\ z_1 \end{pmatrix} - \lambda\left(a + \frac{p}{\sqrt[p]{a}}b_j\right)\cos\omega_0\begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \\ \quad + s\cos\omega_0^2\begin{pmatrix} y_3 \\ z_3 \end{pmatrix} + s\sin\omega_0\cos\omega_0\begin{pmatrix} y_4 \\ z_4 \end{pmatrix} + s\sin\omega_0^2\begin{pmatrix} y_5 \\ z_5 \end{pmatrix} = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}, \end{cases} \tag{3.16}$$

Since $(a + \frac{p}{\sqrt[p]{a}}b_i)\cos^2\omega_0 c_1 c_3 \neq (a + \frac{p}{\sqrt[p]{a}}b_j)\sin^2\omega_0 c_2 c_4$ again and $\cos\omega_0 \neq 0$, we obtain

$$\lambda = \frac{\alpha\cos\omega_0 c_3 - \beta\sin\omega_0 c_4}{(a + \frac{p}{\sqrt[p]{a}}b_j)\sin^2\omega_0 c_2 c_4 - (a + \frac{p}{\sqrt[p]{a}}b_i)\cos^2\omega_0 c_1 c_3},$$

$$s = \frac{\alpha(a + \frac{p}{\sqrt[p]{a}}b_j)\sin\omega_0 c_2 - \beta(a + \frac{p}{\sqrt[p]{a}}b_i)\cos\omega_0 c_1}{(a + \frac{p}{\sqrt[p]{a}}b_j)\sin^2\omega_0 c_2 c_4 - (a + \frac{p}{\sqrt[p]{a}}b_i)\cos^2\omega_0 c_1 c_3}$$

from the first equation of (3.16). Since $L(\overline{\lambda})$ is an isomorphism from X_2 to Y_2 again, we obtain $W = L^{-1}(\lambda)(\widetilde{y}, \widetilde{z})^\top \in X_2$ from the second equation of (3.16), where

$$\begin{pmatrix} \widetilde{y} \\ \widetilde{z} \end{pmatrix} = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} + \lambda\left(a + \frac{p}{\sqrt[p]{a}}b_i\right)\cos\omega_0\begin{pmatrix} y_1 \\ z_1 \end{pmatrix} + \lambda\left(a + \frac{p}{\sqrt[p]{a}}b_j\right)\cos\omega_0\begin{pmatrix} y_2 \\ z_2 \end{pmatrix}$$
$$- s\cos\omega_0^2\begin{pmatrix} y_3 \\ z_3 \end{pmatrix} - s\sin\omega_0\cos\omega_0\begin{pmatrix} y_4 \\ z_4 \end{pmatrix} - s\sin\omega_0^2\begin{pmatrix} y_5 \\ z_5 \end{pmatrix},$$

This yields that $\mathcal{G}_{(\lambda,s,W)}(\overline{\lambda}, 0, (0,0); \omega_0)$ is surjective. Consequently, $\mathcal{G}_{(\lambda,s,W)}(\overline{\lambda}, 0, (0,0); \omega_0) : \mathbb{R} \times \mathbb{R} \times X_2 \to Y$ is an isomorphism.

By using the implicit function theorem to $\mathcal{G}(\lambda, s, W; \omega) = 0$, there is a curve of non-constant solutions $(\lambda(\omega), s(\omega), W(\omega))$ with $|\omega - \omega_0| \ll 1$, satisfying $\lambda(\omega_0) = \overline{\lambda}, s(\omega_0) = 0, W(\omega_0) = 0$, and $\lambda(\omega), s(\omega), W(\omega)$ are continuously differentiable functions with respect to ω. Hence, $(\lambda(\omega), U^* + s(\cos\omega\ \Phi_i + \sin\omega\ \Phi_j + W(\omega)))$ are non-constant positive solutions of $F(\lambda, u) = 0$. The proof is completed. □

Remark 3. Note that if $i = 2j$, then $\int_0^\pi \phi_i\phi_j^2 \mathrm{d}x = \sqrt{\frac{1}{2\pi}} \neq 0$, $\int_0^\pi \phi_i^2\phi_j \mathrm{d}x = 0$. By using the different space decomposition to $\begin{pmatrix} 0 \\ \phi_i \end{pmatrix}, \begin{pmatrix} 0 \\ \phi_j \end{pmatrix}, \begin{pmatrix} \phi_i^2 \\ \phi_i^2 \end{pmatrix}, \begin{pmatrix} \phi_i\phi_j \\ \phi_i^2\phi_j \end{pmatrix}$ and $\begin{pmatrix} \phi_j^2 \\ \phi_j^2 \end{pmatrix}$, ones will come up with similar bifurcation conclusions for (3.1) by repeating the above procedure.

Remark 4. Theorem 3.2 will be still valid, if $p > 1$ is a positive constant, but not an integer.

Remark 5. Note that Sect. 3 mainly discusses the existence of bifurcation solutions of system (1.4) with one-dimensional space, especially the bifurcations from double eigenvalues. Next section will consider the higher dimension case, for the completeness of our results.

4. Existence of non-constant positive steady states: the degree theory

In view of the nonexistence results in Sect. 2, it is particularly interesting to know if (1.4) admit non-constant solutions when the system parameters take on small or larger values. We shall give a positive answer this section, by using a standard approach based on the Leray–Schauder degree theory for compact operators in Banach space.

We first reformulate the system (1.4) in the framework that the degree theory can be easily applied. Let $(u, v) = (u^*v^*) + (y, z)$. It follows from (3.8), (3.9) and (3.10), that (1.4) is shifted to

$$\begin{cases} -d_1\Delta y = a_1 y + a_2 z + f_1(y,z) & x \in \Omega, \\ -d_2\Delta z = b_1 y + b_2 z + f_2(y,z) & x \in \Omega, \\ \frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & x \in \partial\Omega. \end{cases} \tag{4.1}$$

where

$$a_1 = \frac{ab}{a+b}, \quad a_2 = \frac{p}{\sqrt[p]{a}}, \quad b_1 = -\lambda a, \quad b_2 = -\frac{\lambda p}{\sqrt[p]{a}}$$

and f_1, f_2 consist of higher order terms of y and z, which are defined by (3.9). The constant positive solution (u^*v^*) is thus shifted to $(0, 0)$, and if (u, v) is a positive solution of (1.4), then (y, z) must satisfy

$$\mathfrak{R} = \left\{ (y,z) : \frac{|\Omega|}{C_0(a+b)} - \frac{1+\epsilon_0}{a} < u < C_2 - \frac{1-\epsilon_0}{a}, \frac{1}{\sqrt[p]{C_2}} - \sqrt[p]{a} - \epsilon_0 < v < \sqrt[p]{\frac{C_0(a+b)}{|\Omega|}} - \sqrt[p]{a} + \epsilon_0 \right\},$$

where ϵ_0 is a positive constant with $\epsilon_0 \ll 1$ and C_0, C_2 are defined in Theorem 2.1.

Denote a space E by

$$E = \left\{ (u,v) : u, v \in C^{1,\alpha}(\overline{\Omega}), \ \frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}$$

and $W = (y, z)$. Then, (1.4) can be interpreted as the following equation

$$W = K(\lambda)W + H(W) \tag{4.2}$$

in E, where

$$K(\lambda)W = (2a_1K_1y + a_2K_1z, b_1K_2y), \qquad H(W) = (K_1f_1, K_2f_2) = o(W),$$

are two compact linear operators on E, and $K_1 = (-d_1\Delta + a_1)^{-1}, K_2 = (-d_2\Delta - b_2)^{-1}$.

Let $0 = \lambda_0 < \mu_1 \le \mu_2 \le \mu_3 \le \cdots$ be the sequence of eigenvalues for the elliptic operator $-\Delta$ subject to the Neumann boundary condition on Ω, where each μ_i has multiplicity $m_i \ge 1$. Let ϕ_{ij}, $1 \le j \le m_i$, be the normalized eigenfunctions corresponding to μ_i. Then, the set $\{\phi_{ij}\}, i \ge 0, 1 \le j \le m_i$, forms a complete orthonormal basis in $L^2(\Omega)$.

Take

$$\lambda_0 = \frac{\sqrt[p]{a}ab}{p(a+b)}, \qquad \lambda_S(\mu_i) = \frac{\sqrt[p]{a}d_2(ab - d_1(a+b)\mu_i)\mu_i}{p(a^2 + d_1(a+b)\mu_i)}. \tag{4.3}$$

and assume that

$$\textbf{(H')}: \quad ab > d_1\mu_1(a+b), \qquad \text{i.e.,} \quad a_1 > d_1\mu_i.$$

Accordingly, there exists a positive integer $\widetilde{n_\lambda}$ such that $ab > d_1\mu_i(a + b)$, i.e., $a_1 > d_1\mu_i$, for $1 \le i \le \widetilde{n_\lambda}$. Then we can define $\widetilde{\lambda}$ and $\widehat{\lambda}$ by

$$\widetilde{\lambda} := \max_{1 \le i \le \widetilde{n_\lambda}} \{\lambda_S(\mu_i)\}, \qquad \widehat{\lambda} := \max_{i > 0} \{\lambda_S(\mu_i)\}. \tag{4.4}$$

We first give the asymptotic stability of the unique positive constant solution (u^*v^*) for (1.3).

Lemma 4.1. (See [6, Theorem 3.1]) *If* $\lambda > \max\{\lambda_0, \widehat{\lambda}\}$, *then* (u^*v^*) *is linear stable (and then local asymptotic stable) with respect to* (1.3)*; if* $\lambda < \max\{\lambda_0, \widetilde{\lambda}\}$, *then* (u^*v^*) *is unstable.*

Theorem 4.1. *Assume* **(H')** *holds. Assume that* a, b, p, d_1, λ *are fixed positive constant. If* $\mu_1 < \frac{ab}{d_1(a+b)} < \mu_2$ *and* μ_1 *has an odd multiplicity* m_1, *then* (1.4) *possesses at least one non-constant positive solution if* $\lambda < \lambda_S(\mu_1)$ *and* $d_1 \geq 1$.

Proof. By Theorem 2.1, (1.4) has no solution on the boundary of $\mathfrak{R}$. Thus, the Leray–Schauder degree $deg(I - K(\lambda) - H, E \bigcap \mathfrak{R}, 0)$ is well defined, and by the homotopy invariance it is a constant for all $d_2 > 0$. By [20, Theorem 6.3.9], we note that if $(0, 0)$ is an isolated zero of $I - K(\lambda) - H$ and the linear operator $I - K(\lambda)$ is a bijection, then

$$deg(I - K(\lambda) - H, B(0), 0) = i(I - K(\lambda), (0, 0)) = (-1)^{\sigma},$$

where σ is the sum of the algebraic multiplicities of the positive eigenvalues of $K(\lambda) - I$ and $B(0)$ is a small open ball with center at $(0, 0)$. Note that if ζ is an eigenvalue of $K(\lambda) - I$ with a corresponding eigenfunction (ξ, η), then

$$\begin{aligned} -d_1(\zeta + 1)\Delta\xi &= (1 - \zeta)a_1\xi + a_2\eta, \\ -d_2(\zeta + 1)\Delta\eta &= b_1\xi + (\zeta + 1)b_2\eta. \end{aligned}$$

Let

$$\xi = \sum_{0 \leq i < \infty, 1 \leq j \leq m_j} a_{ij}\phi_{ij}, \qquad \eta = \sum_{0 \leq i < \infty, 1 \leq j \leq m_j} b_{ij}\phi_{ij}.$$

Then

$$\sum_{0 \leq i < \infty, 1 \leq j \leq m_j} M_i \begin{pmatrix} a_{ij} \\ b_{ij} \end{pmatrix} \phi_{ij} = 0,$$

where

$$M_i(\lambda) = \begin{pmatrix} (1 - \zeta)a_1 - d_1(\zeta + 1)\mu_i & a_2 \\ b_1 & (\zeta + 1)b_2 - d_2(\zeta + 1)\mu_i \end{pmatrix}.$$

Hence the set of eigenvalues of $K(\lambda) - I$ consists exactly of all roots of the characteristic equation

$$(a_1 + d_1\mu_i)\zeta^2 + 2d_1\mu_i\zeta + d_1\mu_i - a_1 + \frac{a_2b_1}{b_2 - d_2\mu_i} = 0. \tag{4.5}$$

Note $a_1 = \frac{ab}{a+b}, a_2 = \frac{p}{\sqrt[p]{a}}, b_1 = -\lambda a, b_2 = -\frac{\lambda p}{\sqrt[p]{a}}$ and let $P_i := 2d_1\mu_i, Q_i := d_1\mu_i - a_1 + \frac{a_2b_1}{b_2 - d_2\mu_i}$. Then we obtain that

(1) If $i = 0$, then $P_i = 0, Q_i = \frac{a^2}{a+b} > 0$ for any $\lambda > 0$;
(2) If $1 \leq i \leq \widetilde{n_\lambda}$, then $P_i > 0$ and $Q_i > 0$ since $\lambda > \lambda_S(\mu_1) \geq \widehat{\lambda}$;
(3) If $i > \widetilde{n_\lambda}$, it is obviously that $P_i > 0$ and $Q_i > 0$ for any $\lambda > 0$.

In summary, (4.5) has no positive root for any $i \geq 0$, and thus, $K(\lambda) - I$ has no positive eigenvalue, which yields that $i(I - K(\lambda), (0, 0)) = (-1)^{\sigma} = (-1)^0 = 1$.

It follows from Corollary 2.2 that, if we chose a constant λ_0 small enough, then (4.1) with $\lambda = \lambda_0$ has no solution other than $(0, 0)$ in $\mathfrak{R}$. Thus,

$$deg\left(I - K(\lambda) - H, E \bigcap \mathfrak{R}, 0\right) = i(I - K(\lambda_0), (0, 0)) = 1. \tag{4.6}$$

Now suppose for contradiction that there exists $d > D$ such that (1.4) does not admit any non-constant solution under our assumption. Then $(0, 0)$ is an isolated zero of $I - K(\lambda) - H$ in $\mathfrak{R}$. Recall that

$$\lambda_S(\mu_1) = \frac{\sqrt[p]{a}d_2(ab - d_1(a + b)\mu_1)\mu_1}{p(a^2 + d_1(a + b)\mu_1)}.$$

Since $\mu_1 < \frac{ab}{d_1(a+b)} < \mu_2$, we have

(1) For $i = 1$, (4.5) has one positive root ζ_1, and a negative root;
(2) For $i \neq 1$, (4.5) has no roots with positive real parts.

In summary, we conclude that $K(d) - I$ has exactly one positive eigenvalue $\mu_1 > 0$, and all other eigenvalues have non-positive real parts. Thus,

$$deg\left(I - K(\lambda) - H, E\bigcap\mathfrak{R}, 0\right) = i(I - K(\lambda), (0, 0)) = (-1)^{\sigma} \tag{4.7}$$

for all $\lambda < \lambda_S(\mu_1)$.

Next, we prove that $\sigma = m_1$. By the definition of the algebraic multiplicities,

$$\sigma = \dim \bigcup_{i=1}^{\infty} \ker \mathcal{L}^i, \qquad \mathcal{L} = K(\lambda) - (\zeta_1 + 1)I.$$

Note that $m_1 = \dim \ker \mathcal{L}$. It is obviously that $\sigma \geq m_1$. It is now sufficient to prove

$$\ker \mathcal{L}^2 = \ker \mathcal{L}, \tag{4.8}$$

which implies $\ker \mathcal{L}^i = \ker \mathcal{L}$ for all $i \geq 1$. It is known that (4.8) holds if and only if $\ker \mathcal{L} \bigcap R(\mathcal{L}) = \{0\}$. Let $\mathcal{L}^*$ be the adjoint operator of $\mathcal{L}$, then $R(\mathcal{L}) = (\ker \mathcal{L}^*)^{\perp}$. Note that $\mathcal{L}^* = K(\lambda)^* - (\zeta_1 + 1)I$, where $K(\lambda)^*$ is the adjoint of $K(\lambda)$.

By the above statement, we have

$$\ker \mathcal{L} = \left\{ \begin{pmatrix} a_2 \\ (\zeta_1 - 1)a_1 + d_1(\zeta_1 + 1)\mu_1 \end{pmatrix} \phi_{1j}, 1 \leq j \leq m_i \right\}.$$

By some similarly standard argument, we also have that

$$\ker \mathcal{L}^* = \left\{ \begin{pmatrix} d_1(\zeta_1 + 1)(a_1 + \mu_1) \\ a_2 \end{pmatrix} \phi_{1j}, 1 \leq j \leq m_i \right\}.$$

Since $d_1 \geq 1$, we have

$$\begin{aligned}(\zeta_1 - 1)a_1 + d_1(\zeta_1 + 1)\mu_1 + &d_1(\zeta_1 + 1)(a_1 + \mu_1) = \\ &2d_1\mu_1(\zeta_1 + 1) + a_1[\zeta_1(1 + d_1) + (d_1 - 1)] > 0,\end{aligned} \tag{4.9}$$

so, we have $\ker \mathcal{L} \bigcap R(\mathcal{L}) = \{0\}$, which implies that $\sigma = m_1$. Thus, we have

$$deg(I - K(\lambda) - H, E\bigcap\mathfrak{R}, 0) = i(I - K(\lambda), (0, 0)) = (-1)^{\sigma} = (-1)^{m_1} = -1 \tag{4.10}$$

for all $\lambda < \lambda_S(\mu_1)$, which is a contradiction to (4.6). This completes the proof. □

Remark 6. The condition "$d_1 \geq 1$" in Theorem 4.1 is a technical condition in the proof procedure. In fact, we just need to seek out some appropriate parameters such that the inequality of (4.9) is established.

Remark 7. Note that Theorem 4.1 discusses the existence of non-constant positive solutions of system (1.4) with higher dimension space. Unfortunately, it does not give us the definite expressions of the solutionis, as Theorems 3.1 and 3.2.

5. Conclusions

In this paper, we investigate a bimolecular autocatalytic model, which appears in a diffusion–reaction process with autocatalysis and the Langmuir–Hinshelwood (Michaelis–Menten, Holling) saturation law. The main motivation is to propose the effects of the reaction rate λ on the bimolecular reaction kinetics, which is characterized by the parameter k_1 in Sect. 1.

In the introduction, by some changes in variables the original system (1.1) essentially has the same dynamical behavior as the reaction–diffusion system corresponding to (1.3). For the bimolecular stationary model (1.4) and its corresponding reaction–diffusion system (3.1) in one-dimensional case, we have obtained the existence results of spatially non-homogeneous steady states by the bifurcation theory and the Leray–Schauder degree theory. In addition, some boundedness and nonexistence results steady state is investigated in terms of parameters.

On the other hand, the steady-state bifurcations from simple and double eigenvalues are intensively studied. The techniques of space decomposition and implicit function theorem are adopted to deal with the case of double eigenvalues, which is an extension of the classical Crandall and Rabinowitz bifurcation theorem (Bifurcation from simple eigenvalues).

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