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Exponential stability in type III thermoelasticity with microtemperatures

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Abstract. In this paper we consider the type III thermoelastic theory with microtemperatures. We study the time decay of the solutions and we prove that under suitable conditions for the constitutive tensors, the solutions decay exponentially. This fact is in somehow shocking because it differs from the behavior of the solutions in the classical model of thermoelasticity with microtemperatures.

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1. Introduction and basic equations

Experimental observation shows that the classical heat continuum theory cannot be used to describe satisfactorily some thermal phenomena. At the same time, the behavior of the thermal waves obtained from the combination of the Fourier law with the equation

$$c\dot{\theta} = -\nabla \mathbf{q}$$

violates the principle of causality (in the above equation θ denotes the temperature, $\mathbf{q}$ the heat flux and c is the thermal capacity). To overcome these drawbacks, new mathematical models have been introduced. As a matter of illustration, let us recall the Green and Lindsay [13] or the Lord and Shulman [37] theories. Both of them are proposed from the Cattaneo–Maxwell heat conduction equation [5]. We can also cite the two temperatures model proposed by Chen and Gurtin [6] and Chen, Gurtin and Williams [7,8] or the time reversal thermoelasticity [18]. Green and Nagdhi proposed three other thermoelastic theories that they named of type I, II and III, respectively [14–16]. The first one coincides with the classical theory in the linear case. The second one is known as thermoelasticity without energy dissipation because there is no dissipation and the energy is conserved. The third one is the most general, because it contains the former two as limit cases.

On the other side, recently there is an increasing interest concerning models with microstructure [12,22–26,28]. An important case appears when the microstructure is given by thermal effects as microtemperatures [4,9,10,41,42]. Applications of them have been proposed in the literature [27,43]. Grot [17] developed a theory of thermodynamics for elastic materials with microstructure whose microelements, in addition to microdeformations, possess microtemperatures in the context of the classical theory. Theories with microtemperatures are currently under deep investigation.

From now on, we are going to work in a three-dimensional bounded domain Ω with boundary smooth enough to allow the application of the divergence theorem. We wil use the standard notation where "$,i$" means the partial derivative with respect to the variable x_i, a superposed dot means time derivative and summation on repeated indices is assumed.

In this paper we are interested in the thermoelastic theory of type III with microtemperatures, and the system of equations that we want to study (proposed by Aouadi, Ciarletta and Passarella [3]) is the following:

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$$\begin{cases} \rho \ddot{u}_i = (A_{ijkl}u_{k,l} - a_{ij}\theta + B_{ijkl}R_{k,l})_{,j} \\ c\ddot{\tau} = -a_{ij}\dot{u}_{i,j} - (d_{ij}\dot{R}_i)_{,j} + (K_{ij}\tau_{,j} + K^*_{ij}\dot{\tau}_{,j})_{,i} - b_{ij}\dot{R}_{i,j} \\ c_{ij}\ddot{R}_j = (B_{klij}u_{k,l} - b_{ij}\dot{\tau} + C_{ijkl}R_{k,l})_{,j} - d_{ij}\dot{\tau}_{,j} + (C^*_{ijkl}\dot{R}_{k,l})_{,j} \end{cases} \tag{1.1}$$

Here, u_i is the displacement vector, θ is the temperature and M_i are the microtemperatures. Moreover, τ is the *thermal displacement* introduced by Green and Naghdi and R_i are the *microthermal displacements*, defined, respectively, by:

$$\tau(\mathbf{x},t) = \tau_0(\mathbf{x}) + \int_0^t \theta(\mathbf{x},s)\,ds \text{ and } R_i(\mathbf{x},t) = R_i^0(\mathbf{x}) + \int_0^t M_i(\mathbf{x},s)\,ds.$$

As usual, ρ denotes the mass density and c the thermal capacity. A_{ijkl} is the elastic tensor, a_{ij} is the coupling tensor between the displacement and the temperature, B_{ijkl} is the coupling tensor between the displacement and the microtemperatures, d_{ij} and b_{ij} are the coupling tensors between the temperature and the microtemperatures, K_{ij} is the tensor introduced by Green and Naghdi, K^*_{ij} is the thermal conductivity tensor, c_{ij} is a typical tensor of the theories with microtemperatures, and, finally, C_{ijkl} and C^*_{ijkl} are the specific type III tensors with microtemperatures.

To have a well posed problem we need to impose initial and boundary conditions. As initial conditions we assume

$$\begin{aligned} &u_i(\mathbf{x},0) = u_i^0(\mathbf{x}), \dot{u}_i(\mathbf{x},0) = v_i^0(\mathbf{x}), \tau(\mathbf{x},0) = \tau^0(\mathbf{x}), \dot{\tau}(\mathbf{x},0) = \theta^0(\mathbf{x}) \\ &R_i(\mathbf{x},0) = R_i^0(\mathbf{x}), \dot{R}_i(\mathbf{x},0) = M_i^0(\mathbf{x}) \text{ for } \mathbf{x} \in \Omega. \end{aligned} \tag{1.2}$$

And we impose homogeneous boundary conditions:

$$u_i(\mathbf{x},t) = \tau(\mathbf{x},t) = R_i(\mathbf{x},t) = 0 \text{ for } \mathbf{x} \in \partial\Omega, t \geq 0. \tag{1.3}$$

The following symmetries are satified:

$$A_{ijkl} = A_{klij}, K_{ij} = K_{ji}, K^*_{ij} = K^*_{ji}, C_{ijkl} = C_{klij}, c_{ij} = c_{ji}. \tag{1.4}$$

It is well known that the axioms of thermomechanics imply that the thermal conductivity tensor, K^*_{ij}, cannot have negative sign. However, these axioms do not imply any other condition over none of the remaining tensors, including the elasticity tensor [29]. It is also known that for elastic materials initially prestressed, the elasticity tensor does not have necessarily positive sign [20,21,29]. Therefore, it is relevant to analyze the problem determined by thermoelastic systems when the elasticity tensor is not positive definite (see, for example, [2,31,34,35,44]). It is important to notice that this problem can be ill posed in the sense of Hadamard. Hence it is difficult to deal with. For this reason it is difficult to obtain results about the qualitative properties of the solutions. Nevertheless, results on uniqueness, instability of solutions, continuous dependence in the sense of Hölder, structural stability, etc., have been found for different situations [1,32,33,40]. In fact, this kind of results can also be obtained for the type III thermoelasticity with microtemperatures, the situation we analyze. Nevertheless, the techniques used to prove them are quite standard and we will not write the developments here. We will focus on the exponential stability of the solutions.

On the other hand, it is worth noting that the existence of solutions is, in general, a very difficult property to prove. Knops [30] showed the existence of solutions in a weak sense for the isothermal case, but, up to now, this result has not been extended to any other thermoelastic situation. From a mathematical point of view, the existence of solutions is strongly easier when the elasticity tensor is positive definite. Usually, this fact implies the stability of the system in the sense of Lyapunov. With this hypothesis, the existence and uniqueness of solutions were established [3] for our problem, but the exponential stability remained unsolved. In this work we prove that under suitable conditions for the constitutive tensors, the system is exponentially stable. This fact differs from the result known for the classical heat conduction theory.

We want to highlight that, in their work, Aouadi, Ciarletta and Passarella [3] set the system of equations, gave conditions to have a well posed problem and obtained asymptotic stability of the solutions under appropriate hypotheses. In this paper, using the semigroups method, we obtain exponential stability of the solutions under suitable conditions for the constitutive tensors (our conditions differ from the ones used in [3] to prove the asymptotic stability, although related to them). The exponential stability is a remarkable fact because it does not happen for the three-dimensional case in the context of the classical theory with microtemperatures. In our case, a strong coupling between the displacement and the microtemperatures appears. This coupling is not present in the classical theory.

2. Exponential decay of solutions

In this section we will prove the exponential decay of the solutions of system (1.1). We need to impose some assumptions over the constitutive coefficients. For each vector ξ_i and each pair of tensors ξ_{ij} and η_{ij} the following inequalities are assumed:

$$\begin{aligned} &A_{ijkl}\xi_{ij}\xi_{kl} + 2B_{ijkl}\xi_{ij}\eta_{kl} + C_{ijkl}\eta_{ij}\eta_{kl} \geq C_0(\xi_{ij}\xi_{ij} + \eta_{ij}\eta_{ij}),\\ &K_{ij}\xi_i\xi_j \geq C_1\xi_i\xi_i, \quad K^*_{ij}\xi_i\xi_j \geq C_2\xi_i\xi_i\\ &c_{ij}\eta_i\eta_j \geq C_3\eta_i\eta_i, \quad C^*_{ijkl}\eta_{ij}\eta_{kl} \geq C_4\eta_{ij}\eta_{ij},\\ &\rho \geq \rho_0 > 0, \quad c \geq c_0 > 0, \end{aligned} \tag{2.1}$$

for positive constants $c_0, C_0, C_1, C_2, C_3, C_4$ and ρ_0.

Besides the assumptions for the coefficients given by (2.1), we also assume that

$$B_{klij}\xi_{kl}\xi_{ij} \geq C\xi_{ij}\xi_{ij} \quad \text{or} \quad B_{klij}\xi_{kl}\xi_{ij} \leq -C\xi_{ij}\xi_{ij} \tag{2.2}$$

for a positive constant C.

Our assumptions agree with the thermomechanical axioms and the empirical experience. We want to emphasize that the first condition in (2.1) can be interpreted with the help of the elastic stability as well as the condition on the tensor K_{ij}. We have also hardened a little bit the condition on the thermal conductivity. The assumption concerning the thermal capacity is also obvious. The condition (2.2) says that the coupling between microtemperatures and the displacement is very strong, which is a different property with respect what happens in the classical theory.

We will use the semigroup arguments, and we will follow several ideas of the work of Aouadi, Ciarletta and Passarella [3]. First of all, we transform the initial boundary problem defined by system (1.1), initial conditions (1.2) and boundary conditions (1.3) to an abstract problem on a suitable Hilbert space. To this end, we introduce the following notation:

$$v_i = \dot{u}_i, \theta = \dot{\tau}, M_i = \dot{R}_i.$$

Let $\mathcal{H}$ be the Hilbert space defined by

$$\mathcal{H} = \{(u_i, v_i, \tau, \theta, R_i, M_i) : u_i, \tau, R_i \in W_0^{1,2}(\Omega), v_i, \theta, M_i \in L^2(\Omega)\},$$

where $W_0^{1,2}(\Omega)$ and $L^2(\Omega)$ are the usual Sobolev spaces. As we consider that our functions take values in the complex field[1], the inner product that we define in $\mathcal{H}$ is

$$\langle \mathcal{U}, \mathcal{U}^* \rangle = \frac{1}{2}\int_\Omega \left(\rho v_i \overline{v}_i^* + c\theta\overline{\theta}^* + c_{ij}M_i\overline{M}_j^* + 2\mathcal{W}[(u_i,\tau,R_i),(u_i^*,\tau^*,R_i^*)]\right) dV, \tag{2.3}$$

where $\mathcal{U} = (u_i, v_i, \tau, \theta, R_i, M_i)$, $\mathcal{U}^* = (u_i^*, v_i^*, \tau^*, \theta^*, R_i^*, M_i^*)$ and

$$2\mathcal{W}[(u_i,\tau,R_i),(u_i^*,\tau^*,R_i^*)] = A_{ijkl}u_{i,j}\overline{u}_{k,l} + B_{ijkl}(u_{i,j}\overline{R}^*_{k,l} + \overline{u}^*_{i,j}R_{k,l}) + K_{ij}\tau_{,i}\overline{\tau}^*_{,j} + C_{ijkl}R_{i,j}\overline{R}^*_{k,l}.$$

[1] It is worth noting that this consideration is a difference with respect the approach followed in [3]. However, it does not represent a relevant difference in the arguments.

The inner product defined at (2.3) is equivalent to the usual one in $\mathcal{H}$.

In order to obtain a synthetic expression for our problem, and following [3] we consider the operators defined below:

$$\begin{aligned}
&A_i(\mathbf{u}) = \tfrac{1}{\rho}(A_{ijkl}u_{k,l})_{,j}\\
&B_i\theta = -\tfrac{1}{\rho}(a_{ji}\theta)_{,j}\\
&C_i\mathbf{R} = \tfrac{1}{\rho}(B_{jikl}R_{l,k})_{,j}\\
&D\mathbf{v} = -\tfrac{1}{c}a_{ij}v_{i,j}\\
&E\tau = \tfrac{1}{c}(K_{ij}\tau_{,j})_{,i}\\
&G\theta = \tfrac{1}{c}(K^*_{ij}\theta_{,j})_{,i}\\
&J\mathbf{M} = -\tfrac{1}{c}\left((d_{ij}M_j)_{,i} + b_{ji}M_{i,j}\right)\\
&L_s\mathbf{u} = l_{si}\left(B_{klij}u_{k,lj}\right)_{,i}\\
&Z_s\theta = -l_{si}\left((b_{ji}\theta)_{,j} + d_{ij}\theta_{,j}\right)\\
&N_s\mathbf{R} = l_{si}\left(C_{jikl}R_{k,l}\right)_{,j}\\
&P_s\mathbf{M} = l_{si}\left(C^*_{jikl}M_{k,l}\right)_{,j}
\end{aligned}$$

where l_{si} is defined by $l_{si}c_{ij} = \delta_{sj}$, being δ_{sj} the Kronecker delta. Notice that that means that l_{si} is the inverse of the matrix c_{ij}.

Therefore, system (1.1) with initial conditions (1.2) and boundary conditions (1.3) can be written as

$$\frac{d}{dt}\mathcal{U}(t) = \mathcal{A}\mathcal{U}(t), \quad \mathcal{U}(0) = \mathcal{U}_0, \tag{2.4}$$

where $\mathcal{U}_0 = \left({u_i}^0, {v_i}^0, \tau^0, \theta^0, R_i^0, M_i^0\right)$, and $\mathcal{A}$ is the following matrix operator

$$\mathcal{A} = \begin{pmatrix} 0 & \mathbf{Id} & 0 & 0 & 0 & 0\\ \mathbf{A} & 0 & 0 & \mathbf{B} & \mathbf{C} & 0\\ 0 & 0 & 0 & Id & 0 & 0\\ 0 & D & E & G & 0 & J\\ 0 & 0 & 0 & 0 & 0 & \mathbf{Id}\\ \mathbf{L} & 0 & 0 & \mathbf{Z} & \mathbf{N} & \mathbf{P} \end{pmatrix}, \tag{2.5}$$

where $\mathbf{A} = (A_i)$, $\mathbf{B} = (B_i)$, $\mathbf{C} = (C_i)$, $\mathbf{L} = (L_s)$ and $\mathbf{Z} = (Z_s)$.

The domain of the operator $\mathcal{A}$ is $D(\mathcal{A}) = \{\mathcal{U} \in \mathcal{H} : \quad \mathcal{A}\mathcal{U} \in \mathcal{H}\}$. It is clear that it contains a dense subspace of $\mathcal{H}$ and, therefore, $D(\mathcal{A})$ is dense in the Hilbert space $\mathcal{H}$.

Aouadi et al. (Lemma 1 in [3]) proved that $\mathcal{A}$ is a dissipative operator because

$$\mathrm{Re}\langle \mathcal{A}\mathcal{U}, \mathcal{U}\rangle_{\mathcal{H}} \le 0.$$

In fact, they found that

$$\mathrm{Re}\langle \mathcal{A}\mathcal{U}, \mathcal{U}\rangle_{\mathcal{H}} = -\frac{1}{2}\int_{\Omega} (K^*_{ij}\theta_{,i}\overline{\theta}_{,j} + C^*_{ijkl}M_{i,j}\overline{M}_{l,k})\, dV.$$

They also proved (Lemma 2 in [3]) that the operator $\mathcal{A}$ satisfies that $Range(I - \mathcal{A}) = \mathcal{H}$. Following an analogous argument, it can be proved that 0 belongs to the resolvent of $\mathcal{A}$ (in short, $0 \in \rho(\mathcal{A})$).

Therefore, using the Lumer–Phillips theorem (see, e.g., [38]), we get the following result.

Theorem 2.1. *The operator given by matrix $\mathcal{A}$ generates a contraction C_0-semigroup $S(t) = \{e^{\mathcal{A}t}\}_{t\ge 0}$ in $\mathcal{H}$.*

We have now the basic tools to prove the main result of this section. But before that, we recall the caractherization stated in the book of Liu and Zheng that ensures the exponential decay (see [19], [36] or [39]).

Theorem 2.2. *Let $S(t)=\{e^{\mathcal{A}t}\}_{t\geq 0}$ be a C_0-semigroup of contractions on a Hilbert space. Then $S(t)$ is exponentially stable if and only if the following two conditions are satisfied:*

(i) $i\mathbb{R}\subset\rho(\mathcal{A})$,
(ii) $\overline{\lim}_{|\lambda|\to\infty}\|(i\lambda\mathcal{I}-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}<\infty$.

Lemma 2.3. *The operator $\mathcal{A}$ defined at (2.5) satisfies that $i\mathbb{R}\subset\rho(\mathcal{A})$.*

Proof. Following the arguments given by Liu and Zheng ([36], page 25), the proof consists of the following three steps:

(i) Since 0 is in the resolvent of $\mathcal{A}$, using the contraction mapping theorem, we have that for any real λ such that $|\lambda|<||\mathcal{A}^{-1}||^{-1}$, the operator $i\lambda\mathcal{I}-\mathcal{A}=\mathcal{A}(i\lambda\mathcal{A}^{-1}-\mathcal{I})$ is invertible. Moreover, $||(i\lambda\mathcal{I}-\mathcal{A})^{-1}||$ is a continuous function of λ in the interval $(-||\mathcal{A}^{-1}||^{-1},||\mathcal{A}^{-1}||^{-1})$.

(ii) If $\sup\{||(i\lambda\mathcal{I}-\mathcal{A})^{-1}||,|\lambda|<||\mathcal{A}^{-1}||^{-1}\}=M<\infty$, then by the contraction theorem, the operator

$$i\lambda\mathcal{I}-\mathcal{A}=(i\lambda_0\mathcal{I}-\mathcal{A})\Big(\mathcal{I}+i(\lambda-\lambda_0)(i\lambda_0\mathcal{I}-\mathcal{A})^{-1}\Big),$$

is invertible for $|\lambda-\lambda_0|<M^{-1}$. It turns out that, by choosing λ_0 as close to $||\mathcal{A}^{-1}||^{-1}$ as we can, the set $\{\lambda,|\lambda|<||\mathcal{A}^{-1}||^{-1}+M^{-1}\}$ is contained in the resolvent of $\mathcal{A}$ and $||(i\lambda\mathcal{I}-\mathcal{A})^{-1}||$ is a continuous function of λ in the interval $(-||\mathcal{A}^{-1}||^{-1}-M^{-1},||\mathcal{A}^{-1}||^{-1}+M^{-1})$.

(iii) Let us assume that the intersection of the imaginary axis and the spectrum is not empty, then there exists a real number ϖ with $||\mathcal{A}^{-1}||^{-1}\leq|\varpi|<\infty$ such that the set $\{i\lambda,|\lambda|<|\varpi|\}$ is in the resolvent of $\mathcal{A}$ and $\sup\{||(i\lambda\mathcal{I}-\mathcal{A})^{-1}||,|\lambda|<|\varpi|\}=\infty$. Therefore, there exist a sequence of real numbers λ_n with $\lambda_n\to\varpi$, $|\lambda_n|<|\varpi|$ and a sequence of vectors $\mathcal{U}_n=(\mathbf{u}_n,\mathbf{v}_n,\tau_n,\theta_n,\mathbf{R}_n,\mathbf{M}_n)$ in the domain of the operator $\mathcal{A}$ and with unit norm such that

$$\|(i\lambda_n\mathcal{I}-\mathcal{A})\mathcal{U}_n\|\to 0. \tag{2.6}$$

If we write (2.6) in components, we obtain the following conditions:

$$i\lambda_n\mathbf{u}_n-\mathbf{v}_n\to\mathbf{0},\quad \text{in } \mathbf{W}^{1,2} \tag{2.7}$$

$$i\lambda_n\mathbf{v}_n-\mathbf{A}_n\mathbf{u}_n-\mathbf{B}\theta_n-\mathbf{C}\mathbf{R}_n\to\mathbf{0},\quad \text{in } \mathbf{L}^2 \tag{2.8}$$

$$i\lambda_n\tau_n-\theta_n\to 0,\quad \text{in } W^{1,2} \tag{2.9}$$

$$i\lambda_n\theta_n-\mathbf{D}\mathbf{v}_n-E\tau_n-G\theta_n-J\mathbf{M}_n\to\mathbf{0},\quad \text{in } L^2 \tag{2.10}$$

$$i\lambda_n\mathbf{R}_n-\mathbf{M}_n\to\mathbf{0},\quad \text{in } \mathbf{W}^{1,2} \tag{2.11}$$

$$i\lambda_n\mathbf{M}_n-\mathbf{L}\mathbf{u}_n-\mathbf{Z}\theta_n-\mathbf{N}\mathbf{R}_n-\mathbf{P}\mathbf{M}_n\to 0,\quad \text{in } \mathbf{L}^2. \tag{2.12}$$

In view of the dissipative term for the operator, we see that

$$\theta_n,\ \mathbf{M}_n\to 0 \text{ in } \mathbf{W}^{1,2}. \tag{2.13}$$

From (2.9) we also have that $\tau_n\to 0$ in $W^{1,2}$. And from (2.11), $\mathbf{R}_n\to 0$ in $\mathbf{W}^{1,2}$.

If we multiply (2.12) by $\mathbf{u}_n$ we obtain that

$$\langle i\lambda_n\mathbf{M}_n,\mathbf{u}_n\rangle-\langle\mathbf{L}\mathbf{u}_n,\mathbf{u}_n\rangle-\langle\mathbf{Z}\theta_n,\mathbf{u}_n\rangle-\langle\mathbf{N}\mathbf{R}_n,\mathbf{u}_n\rangle-\langle\mathbf{P}\mathbf{M}_n,\mathbf{u}_n\rangle\to 0. \tag{2.14}$$

Or, equivalently,

$$\langle\mathbf{M}_n,-i\lambda_n\mathbf{u}_n\rangle-\langle\mathbf{L}\mathbf{u}_n,\mathbf{u}_n\rangle-\langle\mathbf{Z}\theta_n,\mathbf{u}_n\rangle-\langle\mathbf{N}\mathbf{R}_n,\mathbf{u}_n\rangle-\langle\mathbf{P}\mathbf{M}_n,\mathbf{u}_n\rangle\to 0. \tag{2.15}$$

As $i\lambda_n \mathbf{u}_n$ is bounded in $\mathbf{L}^2$ and $\mathbf{u}_n$ is bounded in $\mathbf{W}^{1,2}$, we conclude that

$$\langle \mathbf{M}_n, -i\lambda_n \mathbf{u}_n \rangle \to 0, \langle \mathbf{Z}\theta_n, \mathbf{u}_n \rangle \to 0, \langle \mathbf{N}\mathbf{R}_n, \mathbf{u}_n \rangle \to 0 \text{ and } \langle \mathbf{P}\mathbf{M}_n, \mathbf{u}_n \rangle \to 0.$$

Therefore,

$$\langle \mathbf{L}\mathbf{u}_n, \mathbf{u}_n \rangle \to 0.$$

Taking into account the definition of the operators and also assumptions (2.2), we obtain that $\mathbf{u}_n \to 0$ in $\mathbf{W}^{1,2}$ and, in consequence, from (2.7)–(2.8), $\mathbf{v}_n \to 0$ in $\mathbf{L}^2$. These facts show that it is not possible the existence of such a unit sequence, and, therefore, the imaginary axis in contained in the resolvent of $\mathcal{A}$. □

Lemma 2.4. *The operator $\mathcal{A}$ defined at (2.5) satisfies that*

$$\overline{\lim_{|\lambda|\to\infty}} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Proof. The proof is similar to the one proposed for Lemma 2.3. □

Theorem 2.5. *The C_0-semigroup $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$ is exponentially stable. That is, there exist two positive constants M and α such that $\|S(t)\| \leq M\|S(0)\|e^{-\alpha t}$.*

Proof. The proof is a direct consequence of Lemma 2.3, Lemma 2.4 and Theorem 2.2. □

It is worth noting that the behavior of the solutions using this model completely differs from the behavior in the three-dimensional classical thermoelasticity, where slow decay or even undamped solutions are observed. The exponential stability obtained in this case is a consequence of the strong coupling between the displacement and the microtemperatures, coupling that is not present in the classical model. This behavior is a shocking effect of the type III thermoelasticity theory with microtemperatures.

3. Conclusions

In this paper we have proved that under hypotheses of positivity for different tensors, the solutions of the system of equations that models the type III thermoelasticity with microtemperatures decay exponentially. This behavior differs extremely from the one obtained for the classical theory. Even more, the exponential stability holds for symmetric domains. This fact is also very different from what happens in the classical case, where undamped solutions could appear [11].

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