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Blow-up time estimates in porous medium equations with nonlinear boundary conditions

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Abstract. In this paper, we consider the blow-up problem of the following porous medium equations with nonlinear boundary conditions

$$\begin{cases} u_t = \Delta u^m + k(t)f(u) & \text{in } \Omega \times (0, t^*), \\ \dfrac{\partial u}{\partial \nu} = g(u) & \text{on } \partial\Omega \times (0, t^*), \\ u(x, 0) = u_0(x) & \text{in } \overline{\Omega}, \end{cases}$$

where $m > 1$, $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded convex domain with smooth boundary. Under appropriate assumptions on the data, a criterion is given to guarantee that solution u blows up at finite time, and an upper bound for blow-up time is derived. Moreover, a lower bound for blow-up time is also obtained.

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1. Introduction

Porous medium equations as representative examples of parabolic equations have been widely studied by many authors [1,5–7,14,23,26]. There are a lot of physical applications where porous medium model appears in a natural way to describe processes involving diffusion or heat transfer. The best known of them is the description of the gas or fluid in porous media [24,28].

The purpose of this paper is to investigate the blow-up phenomena of the following porous medium problems

$$\begin{cases} u_t = \Delta u^m + k(t)f(u) & \text{in } \Omega \times (0, t^*), \\ \dfrac{\partial u}{\partial \nu} = g(u) & \text{on } \partial\Omega \times (0, t^*), \\ u(x, 0) = u_0(x) & \text{in } \overline{\Omega}, \end{cases} \tag{1.1}$$

where $m > 1$, $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded convex domain with smooth boundary, t^* is the blow-up time, $\partial/\partial\nu$ is the outward normal derivative on $\partial\Omega$. Set $\mathbb{R}_+ = (0, +\infty)$. We suppose, in this paper, that f and g are nonnegative $C(\overline{\mathbb{R}_+})$ functions, and k is a positive $C^1(\overline{\mathbb{R}_+})$ function, and $u_0(x)$ is a nonnegative $C^1(\overline{\Omega})$ function satisfying the compatibility condition.

At present, many articles are known for the study of blow-up phenomena for the parabolic equations (for instance, [8,10–13,16–18,25,27]). Recently, some new developments have been made in the study of the blow-up time estimates for parabolic equations under nonlinear boundary conditions. We refer to [9,15,20–22]. In order to investigate the blow-up problems of (1.1), we focus on the papers [15,20,22]. In [22], Payne and Schaefer studied the following problems

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$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times (0, t^*), \\ \dfrac{\partial u}{\partial \nu} = g(u) & \text{on } \partial\Omega \times (0, t^*), \\ u(x, 0) = u_0(x) & \text{in } \overline{\Omega}, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded convex domain with smooth boundary. When $\Omega \subset \mathbb{R}^3$, authors got a lower bound estimate for the blow-up time under the suitable conditions on the nonlinearities. In addition, when $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$, authors derived a sufficient condition to guarantee that solution blows up and obtained an upper bound for t^* . Payne et al. [20] considered the following problems

$$\begin{cases} u_t = \Delta u - f(u) & \text{in } \Omega \times (0, t^*), \\ \dfrac{\partial u}{\partial \nu} = g(u) & \text{on } \partial\Omega \times (0, t^*), \\ u(x, 0) = u_0(x) & \text{in } \overline{\Omega}, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded convex domain with smooth boundary. They established conditions on data sufficient to ensure that solution u exists for all time t as well as conditions on data forcing u to blow up in finite time t^*. Moreover, an upper bound for t^* was obtained. Under more restrictive conditions, when $\Omega \subset \mathbb{R}^3$, a lower bound for t^* was derived. Li and Li [15] studied the following problems

$$\begin{cases} u_t = \displaystyle\sum_{i,j=1}^{n} \left(a^{ij}(x) u_{x_i}\right)_{x_j} - f(u) & \text{in } \Omega \times (0, t^*), \\ \displaystyle\sum_{i,j=1}^{n} a^{ij}(x) u_{x_i} \nu_j = g(u) & \text{on } \partial\Omega \times (0, t^*), \\ u(x, 0) = u_0(x) & \text{in } \overline{\Omega}, \end{cases}$$

where Ω is a bounded convex domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary $\partial\Omega$. Under certain conditions on data, they showed that the solution blows up or remains global for $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$. For $\Omega \subset \mathbb{R}^3$, a lower bound for blow-up time was also derived.

Inspired by the above research, we study the blow-up phenomena of (1.1). The crucial idea in the paper is to make use of the Sobolev inequality in $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ and a modified differential inequality technique. The key to using this method is to construct some suitable auxiliary functions. However, we find that the auxiliary functions defined in [15, 20, 22] seem to be no longer applicable to (1.1). As an innovation of this paper, we need to construct new auxiliary functions to complete our research.

We proceed as follows. In Sect. 2, when $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$, we establish conditions to guarantee that the solution u blows up in finite time and derive an upper bound for t^*. In Sect. 3, when $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$, a lower bound for t^* is obtained when blowup does occur. Section 4 is devoted to presenting an example to illustrate abstract results of this paper.

2. An upper bound for t^*

We seek, in this section, the sufficient conditions on data under which the nonnegative classical solution of (1.1) blows up at finite time t^*. To this end, we assume that functions f and k satisfy

$$f(s) \geq a s^q, \quad s \geq 0, \qquad k(t) \geq r, \quad t \geq 0, \tag{2.1}$$

where a, r, q are positive constants and

$$q > m. \tag{2.2}$$

Let λ_1 be the first eigenvalue and ω_1 be the corresponding eigenfunction of the following fixed membrane problem

$$\begin{cases} \Delta \omega + \lambda \omega = 0, \quad \omega > 0, & \text{in } \Omega, \\ \omega = 0, & \text{on } \partial\Omega \end{cases} \tag{2.3}$$

with

$$\int_\Omega \omega^2 \mathrm{d}x = 1. \tag{2.4}$$

Inspired by [3,4,29], we define the auxiliary function of the form

$$B(t) = \int_\Omega \omega_1^2 u \mathrm{d}x, \quad t \geq 0.$$

Theorem 2.1 is the main result of this section.

Theorem 2.1. *Let u be a nonnegative classical solution of problem (1.1). Suppose (2.1) and (2.2) hold. In addition, we also assume that the initial value u_0 satisfies*

$$arB^{q-m}(0) - 2\lambda_1 > 0. \tag{2.5}$$

Then, the solution u blows up at t^* in measure $B(t)$ with

$$t^* \leq \int_{B(0)}^{+\infty} \frac{\mathrm{d}\eta}{ar\eta^q - 2\lambda_1 \eta^m}.$$

Proof. Differentiating $B(t)$ and using (2.1), (2.3), and the divergence theorem, we have

$$\begin{aligned} B'(t) &= \int_\Omega \omega_1^2 u_t \mathrm{d}x = \int_\Omega \omega_1^2 \left[\Delta u^m + k(t) f(u)\right] \mathrm{d}x \\ &\geq \int_\Omega \omega_1^2 \Delta u^m \mathrm{d}x + ak(t) \int_\Omega \omega_1^2 u^q \mathrm{d}x \\ &= \int_\Omega \Delta \omega_1^2 u^m \mathrm{d}x + ak(t) \int_\Omega \omega_1^2 u^q \mathrm{d}x \\ &= 2\int_\Omega u^m |\nabla \omega_1|^2 \mathrm{d}x + 2\int_\Omega \omega_1 u^m \Delta \omega_1 \mathrm{d}x + ak(t) \int_\Omega \omega_1^2 u^q \mathrm{d}x \\ &= 2\int_\Omega u^m |\nabla \omega_1|^2 \mathrm{d}x - 2\lambda_1 \int_\Omega \omega_1^2 u^m \mathrm{d}x + ak(t) \int_\Omega \omega_1^2 u^q \mathrm{d}x \\ &\geq -2\lambda_1 \int_\Omega \omega_1^2 u^m \mathrm{d}x + ar \int_\Omega \omega_1^2 u^q \mathrm{d}x. \end{aligned} \tag{2.6}$$

It follows from (2.2), (2.4), and the Hölder inequality that

$$\int_\Omega \omega_1^2 u^m \mathrm{d}x \leq \left(\int_\Omega \omega_1^2 u^q \mathrm{d}x\right)^{\frac{m}{q}} \left(\int_\Omega \omega_1^2 \mathrm{d}x\right)^{\frac{q-m}{q}} = \left(\int_\Omega \omega_1^2 u^q \mathrm{d}x\right)^{\frac{m}{q}}. \tag{2.7}$$

Inserting (2.7) into (2.6), we get

$$\begin{aligned} B'(t) &\geq -2\lambda_1 \left(\int_\Omega \omega_1^2 u^q \mathrm{d}x\right)^{\frac{m}{q}} + ar \int_\Omega \omega_1^2 u^q \mathrm{d}x \\ &= \left(\int_\Omega \omega_1^2 u^q \mathrm{d}x\right)^{\frac{m}{q}} \left(ar \left(\int_\Omega \omega_1^2 u^q \mathrm{d}x\right)^{\frac{q-m}{q}} - 2\lambda_1\right). \end{aligned} \tag{2.8}$$

Since

$$\int_\Omega \omega_1^2 u \mathrm{d}x \le \left(\int_\Omega \omega_1^2 u^q \mathrm{d}x\right)^{\frac{1}{q}} \left(\int_\Omega \omega_1^2 \mathrm{d}x\right)^{\frac{q-1}{q}} = \left(\int_\Omega \omega_1^2 u^q \mathrm{d}x\right)^{\frac{1}{q}}, \tag{2.9}$$

we can rewrite (2.8) as follows

$$\begin{aligned} B'(t) &\ge \left(\int_\Omega \omega_1^2 u^q \mathrm{d}x\right)^{\frac{m}{q}} \left(ar\left(\int_\Omega \omega_1^2 u \mathrm{d}x\right)^{q-m} - 2\lambda_1\right) \\ &= \left(\int_\Omega \omega_1^2 u^q \mathrm{d}x\right)^{\frac{m}{q}} \left(arB^{q-m}(t) - 2\lambda_1\right). \end{aligned} \tag{2.10}$$

We note that (2.5) means

$$arB^{q-m}(t) - 2\lambda_1 > 0, \quad t \ge 0. \tag{2.11}$$

In fact, if inequality (2.11) does not hold, then we let

$$t_1 = \min\left\{t > 0 \;\middle|\; arB^{q-m}(t) - 2\lambda_1 \le 0\right\}. \tag{2.12}$$

We deduce

$$arB^{q-m}(t) - 2\lambda_1 > 0, \ 0 \le t < t_1.$$

By (2.10), we have

$$B'(t) > 0, \ 0 \le t < t_1,$$

from which and (2.5), we get

$$B(t_1) > B(0) > \left(\frac{2\lambda_1}{ar}\right)^{\frac{1}{q-m}}.$$

Hence

$$arB^{q-m}(t_1) - 2\lambda_1 > 0,$$

which contradicts with (2.12). This contradiction shows that inequality (2.11) holds.

Now, inserting (2.9) into (2.10) and using (2.11), we have

$$\begin{aligned} B'(t) &\ge \left(\int_\Omega \omega_1^2 u^q\right)^{\frac{m}{q}} \left(arB^{q-m}(t) - 2\lambda_1\right) \ge \left(\int_\Omega \omega_1^2 u \mathrm{d}x\right)^{m} \left(arB^{q-m}(t) - 2\lambda_1\right) \\ &= arB^q(t) - 2\lambda_1 B^m(t) > 0, \quad t \ge 0. \end{aligned} \tag{2.13}$$

We integrate (2.13) over $[0, t]$ to obtain

$$t \le \int_{B(0)}^{B(t)} \frac{\mathrm{d}\eta}{ar\eta^q - 2\lambda_1 \eta^m}. \tag{2.14}$$

Inequality (2.14) implies that solution u blows up at finite time t^* in the measure $B(t)$. In fact, if the solution u remains global in the measure $B(t)$, we have

$$B(t) < +\infty, \quad t \ge 0$$

and

$$t \leq \int_{B(0)}^{B(t)} \frac{\mathrm{d}\eta}{ar\eta^q - 2\lambda_1\eta^m} < \int_{B(0)}^{+\infty} \frac{\mathrm{d}\eta}{ar\eta^q - 2\lambda_1\eta^m}, \quad t \geq 0.$$

Furthermore, letting $t \to +\infty$, we derive

$$\int_{B(0)}^{+\infty} \frac{\mathrm{d}\eta}{ar\eta^q - 2\lambda_1\eta^m} = +\infty.$$

The fact that $q > m > 1$ and $B(0) > 0$ implies

$$\int_{B(0)}^{+\infty} \frac{\mathrm{d}\eta}{ar\eta^q - 2\lambda_1\eta^m} < +\infty,$$

which is a contradiction. Therefore, u blows up at finite time t^* in the measure $B(t)$. Passing to the limit as $t \to t^*$ in (2.14), we get

$$t^* \leq \int_{B(0)}^{+\infty} \frac{\mathrm{d}\eta}{ar\eta^q - 2\lambda_1\eta^m}.$$

□

3. A lower bound for t^*

We look for, in this section, a lower bound for t^* by restricting $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$. To achieve it, we suppose that functions f, g, and k satisfy

$$f(s) \leq as^q, \quad g(s) \leq bs^p, \quad s \geq 0, \quad k(t) \leq M, \quad t \geq 0, \tag{3.1}$$

where a, b, p, q, M are some positive constants and

$$m + 2p > q + 2, \quad p > 1. \tag{3.2}$$

The auxiliary function is defined as follows

$$\Phi(t) = \int_\Omega u^\beta \mathrm{d}x$$

with

$$\beta > \max\{1,\ 3 - m,\ n(p-1)\}. \tag{3.3}$$

In this section, we need to use the following Sobolev inequality (see [2], Corollary 9.14, p.284])

$$\left(\int_\Omega (u^{\frac{m+\beta-1}{2}})^{\frac{2n}{n-2}} \mathrm{d}x\right)^{\frac{n-2}{2n}} \leq C_s \left(\int_\Omega u^{m+\beta-1}\mathrm{d}x + \int_\Omega |\nabla u^{\frac{m+\beta-1}{2}}|^2 \mathrm{d}x\right)^{\frac{1}{2}}, \tag{3.4}$$

where $C_s = C_s(n, \Omega)$ is an embedding constant depending on n $(n \geq 3)$ and Ω. We state our result in Theorem 3.1.

Theorem 3.1. *Let u be a nonnegative classical solution of problem (1.1), which becomes unbounded in the measure $\Phi(t)$ at t^*. Assume that (3.1)–(3.3) hold. Then, there exist computable positive constants C_1, C_2 such that the blow-up time t^* is bounded below by*

$$t^* \geq \int_{\Phi(0)}^{+\infty} \frac{\mathrm{d}\tau}{C_1 + C_2 \tau^{\frac{n(1-p)+m+\beta+2p-3}{n(1-p)+\beta}}}.$$

Proof. Using (3.1), (3.3), and the divergence theorem, we have

$$\begin{aligned}
\Phi'(t) &= \beta \int_\Omega u^{\beta-1} \left[\Delta u^m + k(t) f(u)\right] \mathrm{d}x \\
&= \beta \int_\Omega u^{\beta-1} \nabla \cdot \left(m u^{m-1} \nabla u\right) \mathrm{d}x + \beta k(t) \int_\Omega u^{\beta-1} f(u) \mathrm{d}x \\
&= m\beta \int_\Omega \nabla \cdot \left(u^{m+\beta-2} \nabla u\right) \mathrm{d}x - m\beta(\beta-1) \int_\Omega u^{m+\beta-3} |\nabla u|^2 \mathrm{d}x + \beta k(t) \int_\Omega u^{\beta-1} f(u) \mathrm{d}x \\
&= m\beta \int_{\partial\Omega} u^{m+\beta-2} \frac{\partial u}{\partial \nu} \mathrm{d}S - m\beta(\beta-1) \int_\Omega u^{m+\beta-3} |\nabla u|^2 \mathrm{d}x + \beta k(t) \int_\Omega u^{\beta-1} f(u) \mathrm{d}x \\
&= m\beta \int_{\partial\Omega} u^{m+\beta-2} g(u) \mathrm{d}S - m\beta\left(\beta-1\right) \int_\Omega u^{m+\beta-3} |\nabla u|^2 \mathrm{d}x + \beta k(t) \int_\Omega u^{\beta-1} f(u) \mathrm{d}x \\
&\leq bm\beta \int_{\partial\Omega} u^{m+\beta+p-2} \mathrm{d}S - m\beta(\beta-1) \int_\Omega u^{m+\beta-3} |\nabla u|^2 \mathrm{d}x + a\beta M \int_\Omega u^{\beta+q-1} \mathrm{d}x. \qquad (3.5)
\end{aligned}$$

We note

$$\frac{4}{(m+\beta-1)^2} |\nabla u^{\frac{m+\beta-1}{2}}|^2 = u^{m+\beta-3} |\nabla u|^2. \qquad (3.6)$$

Inserting (3.6) into (3.5), we get

$$\Phi'(t) \leq bm\beta \int_{\partial\Omega} u^{m+\beta+p-2} \mathrm{d}S - \frac{4m\beta(\beta-1)}{(m+\beta-1)^2} \int_\Omega |\nabla u^{\frac{m+\beta-1}{2}}|^2 \mathrm{d}x + a\beta M \int_\Omega u^{\beta+q-1} \mathrm{d}x. \qquad (3.7)$$

To the first term of right side of (3.7), we apply the Lemma in [15] to obtain

$$\int_{\partial\Omega} u^{m+\beta+p-2} \mathrm{d}S \leq \frac{n}{L_0} \int_\Omega u^{m+\beta+p-2} \mathrm{d}x + \frac{(m+\beta+p-2)d}{L_0} \int_\Omega u^{m+\beta+p-3} |\nabla u| \mathrm{d}x, \qquad (3.8)$$

where $L_0 = \min_{\partial\Omega}(x \cdot \nu)$ and $d = \max_{\overline{\Omega}} |x|$. By (3.6), the Hölder inequality, and the Young inequality, we derive

$$\begin{aligned}
\int_\Omega u^{m+\beta+p-3} |\nabla u| \mathrm{d}x &\leq \left(\int_\Omega u^{m+\beta-3} |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega u^{m+\beta+2p-3} \mathrm{d}x\right)^{\frac{1}{2}} \\
&= \left(\varepsilon_1 \int_\Omega u^{m+\beta-3} |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\frac{1}{\varepsilon_1} \int_\Omega u^{m+\beta+2p-3} \mathrm{d}x\right)^{\frac{1}{2}} \\
&\leq \frac{\varepsilon_1}{2} \int_\Omega u^{m+\beta-3} |\nabla u|^2 \mathrm{d}x + \frac{1}{2\varepsilon_1} \int_\Omega u^{m+\beta+2p-3} \mathrm{d}x
\end{aligned}$$

$$= \frac{2\varepsilon_1}{(m+\beta-1)^2}\int\limits_\Omega |\nabla u^{\frac{m+\beta-1}{2}}|^2 \mathrm{d}x + \frac{1}{2\varepsilon_1}\int\limits_\Omega u^{m+\beta+2p-3}\mathrm{d}x, \tag{3.9}$$

where

$$\varepsilon_1 = \frac{L_0(\beta-1)}{bd(m+\beta+p-2)}. \tag{3.10}$$

Substituting (3.8)–(3.10) into (3.7), we deduce

$$\begin{aligned}
\Phi'(t) \le{}& bm\beta\left(\frac{n}{L_0}\int\limits_\Omega u^{m+\beta+p-2}\mathrm{d}x + \frac{(m+\beta+p-2)d}{L_0}\int\limits_\Omega u^{m+\beta+p-3}|\nabla u|\mathrm{d}x\right)\\
&- \frac{4m\beta(\beta-1)}{(m+\beta-1)^2}\int\limits_\Omega |\nabla u^{\frac{m+\beta-1}{2}}|^2\mathrm{d}x + a\beta M\int\limits_\Omega u^{\beta+q-1}\mathrm{d}x\\
\le{}& \frac{bmn\beta}{L_0}\int\limits_\Omega u^{m+\beta+p-2}\mathrm{d}x + \frac{bdm\beta(m+\beta+p-2)}{L_0}\left(\frac{2\varepsilon_1}{(m+\beta-1)^2}\int\limits_\Omega |\nabla u^{\frac{m+\beta-1}{2}}|^2\mathrm{d}x\right.\\
&\left.+\frac{1}{2\varepsilon_1}\int\limits_\Omega u^{m+\beta+2p-3}\mathrm{d}x\right) - \frac{4m\beta(\beta-1)}{(m+\beta-1)^2}\int\limits_\Omega |\nabla u^{\frac{m+\beta-1}{2}}|^2\mathrm{d}x + a\beta M\int\limits_\Omega u^{\beta+q-1}\mathrm{d}x\\
={}& \frac{bmn\beta}{L_0}\int\limits_\Omega u^{m+\beta+p-2}\mathrm{d}x + \frac{bdm\beta(m+\beta+p-2)}{2L_0\varepsilon_1}\int\limits_\Omega u^{m+\beta+2p-3}\mathrm{d}x\\
&+\left(\frac{2bdm\beta(m+\beta+p-2)}{L_0(m+\beta-1)^2}\varepsilon_1 - \frac{4m\beta(\beta-1)}{(m+\beta-1)^2}\right)\int\limits_\Omega |\nabla u^{\frac{m+\beta-1}{2}}|^2\mathrm{d}x + a\beta M\int\limits_\Omega u^{\beta+q-1}\mathrm{d}x\\
={}& \frac{bmn\beta}{L_0}\int\limits_\Omega u^{m+\beta+p-2}\mathrm{d}x + \frac{bdm\beta(m+\beta+p-2)}{2L_0\varepsilon_1}\int\limits_\Omega u^{m+\beta+2p-3}\mathrm{d}x\\
&- \frac{2m\beta(\beta-1)}{(m+\beta-1)^2}\int\limits_\Omega |\nabla u^{\frac{m+\beta-1}{2}}|^2\mathrm{d}x + a\beta M\int\limits_\Omega u^{\beta+q-1}\mathrm{d}x.
\end{aligned} \tag{3.11}$$

We use the Hölder inequality and the Young inequality to the first and fourth terms of right side of (3.11) to obtain

$$\begin{aligned}
\int\limits_\Omega u^{m+\beta+p-2}\mathrm{d}x \le{}& \left(\int\limits_\Omega u^{m+\beta+2p-3}\mathrm{d}x\right)^{\frac{m+\beta+p-2}{m+\beta+2p-3}}|\Omega|^{\frac{p-1}{m+\beta+2p-3}}\\
\le{}& \frac{m+\beta+p-2}{m+\beta+2p-3}\int\limits_\Omega u^{m+\beta+2p-3}\mathrm{d}x + \frac{p-1}{m+\beta+2p-3}|\Omega|
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
\int\limits_\Omega u^{\beta+q-1}\mathrm{d}x \le{}& \left(\int\limits_\Omega u^{m+\beta+2p-3}\mathrm{d}x\right)^{\frac{\beta+q-1}{m+\beta+2p-3}}|\Omega|^{\frac{m+2p-q-2}{m+\beta+2p-3}}\\
\le{}& \frac{\beta+q-1}{m+\beta+2p-3}\int\limits_\Omega u^{m+\beta+2p-3}\mathrm{d}x + \frac{m+2p-q-2}{m+\beta+2p-3}|\Omega|,
\end{aligned} \tag{3.13}$$

where $0<\frac{m+\beta+p-2}{m+\beta+2p-3}<1$ and $0<\frac{m+2p-q-2}{m+\beta+2p-3}<1$ in consideration of (3.2) and (3.3), and $|\Omega|$ is the measure of Ω. Inserting (3.12) and (3.13) into (3.11), we have

$$\Phi'(t) \le A_1 + A_2 \int_\Omega u^{m+\beta+2p-3}\mathrm{d}x - \frac{2m\beta(\beta-1)}{(m+\beta-1)^2}\int_\Omega |\nabla u^{\frac{m+\beta-1}{2}}|^2\mathrm{d}x, \tag{3.14}$$

where

$$A_1 = \frac{bmn\beta(p-1)+a\beta L_0 M(m+2p-q-2)}{L_0(m+\beta+2p-3)}|\Omega| \tag{3.15}$$

and

$$A_2 = \frac{bmn\beta(m+\beta+p-2)+a\beta M L_0(\beta+q-1)}{L_0(m+\beta+2p-3)} + \frac{bdm\beta(m+\beta+p-2)}{2L_0\varepsilon_1}. \tag{3.16}$$

By making use of (3.4) and the Hölder inequality, the second term on the right-hand side of (3.14) can be estimated as follows

$$\begin{aligned}
&\int_\Omega u^{m+\beta+2p-3}\mathrm{d}x\\
&\le \left(\int_\Omega u^\beta \mathrm{d}x\right)^{\frac{2n(1-p)+2(m+\beta+2p-3)}{n(m-1)+2\beta}} \left(\int_\Omega \left(u^{\frac{m+\beta-1}{2}}\right)^{\frac{2n}{n-2}}\mathrm{d}x\right)^{\frac{(m+2p-3)(n-2)}{n(m-1)+2\beta}}\\
&\le \left(\int_\Omega u^\beta \mathrm{d}x\right)^{\frac{2n(1-p)+2(m+\beta+2p-3)}{n(m-1)+2\beta}} \left(C_s^{\frac{2n}{n-2}}\left(\int_\Omega u^{m+\beta-1}\mathrm{d}x + \int_\Omega |\nabla u^{\frac{m+\beta-1}{2}}|^2\mathrm{d}x\right)^{\frac{n}{n-2}}\right)^{\frac{(m+2p-3)(n-2)}{n(m-1)+2\beta}}\\
&= C_s^{\frac{2n(m+2p-3)}{n(m-1)+2\beta}}\left(\int_\Omega u^\beta \mathrm{d}x\right)^{\frac{2n(1-p)+2(m+\beta+2p-3)}{n(m-1)+2\beta}} \left(\int_\Omega u^{m+\beta-1}\mathrm{d}x + \int_\Omega |\nabla u^{\frac{m+\beta-1}{2}}|^2\mathrm{d}x\right)^{\frac{n(m+2p-3)}{n(m-1)+2\beta}},
\end{aligned} \tag{3.17}$$

where $0<\frac{(m+2p-3)(n-2)}{n(m-1)+2\beta}<1$ in view of (3.3). For (3.17), by using the following basic inequality

$$(j_1+j_2)^l \le 2^l(j_1^l+j_2^l), \quad j_1>0,\ j_2>0,\ l>0,$$

we deduce

$$\begin{aligned}
&\int_\Omega u^{m+\beta+2p-3}\mathrm{d}x\\
&\le \left(2C_s^2\right)^{\frac{n(m+2p-3)}{n(m-1)+2\beta}}\left(\int_\Omega u^\beta \mathrm{d}x\right)^{\frac{2n(1-p)+2(m+\beta+2p-3)}{n(m-1)+2\beta}} \left(\int_\Omega u^{m+\beta-1}\mathrm{d}x\right)^{\frac{n(m+2p-3)}{n(m-1)+2\beta}}\\
&\quad + \left(2C_s^2\right)^{\frac{n(m+2p-3)}{n(m-1)+2\beta}}\left(\int_\Omega u^\beta \mathrm{d}x\right)^{\frac{2n(1-p)+2(m+\beta+2p-3)}{n(m-1)+2\beta}} \left(\int_\Omega |\nabla u^{\frac{m+\beta-1}{2}}|^2\mathrm{d}x\right)^{\frac{n(m+2p-3)}{n(m-1)+2\beta}}.
\end{aligned} \tag{3.18}$$

Due to (3.3), we have

$$0<\frac{n(m+2p-3)}{n(m-1)+2\beta}<1. \tag{3.19}$$

By (3.19) and the Young inequality, the first term of (3.18) can be rewritten as

$$
\begin{aligned}
&\left(2C_s^2\right)^{\frac{n(m+2p-3)}{n(m-1)+2\beta}}\left(\int_\Omega u^\beta \mathrm{d}x\right)^{\frac{2n(1-p)+2(m+\beta+2p-3)}{n(m-1)+2\beta}}\left(\int_\Omega u^{m+\beta-1}\mathrm{d}x\right)^{\frac{n(m+2p-3)}{n(m-1)+2\beta}}\\
&=\left(\left(2C_s^2\right)^{\frac{n(m+2p-3)}{2n(1-p)+2\beta}}\left(\frac{n(m-1)+2\beta}{n(m+2p-3)}\right)^{-\frac{n(m+2p-3)}{2n(1-p)+2\beta}}\left(\int_\Omega u^\beta \mathrm{d}x\right)^{\frac{n(1-p)+m+\beta+2p-3}{n(1-p)+\beta}}\right)^{\frac{2n(1-p)+2\beta}{n(m-1)+2\beta}}\\
&\quad\times\left(\frac{n(m-1)+2\beta}{n(m+2p-3)}\int_\Omega u^{m+\beta-1}\mathrm{d}x\right)^{\frac{n(m+2p-3)}{n(m-1)+2\beta}}\\
&\le \frac{2n(1-p)+2\beta}{n(m-1)+2\beta}\left(2C_s^2\right)^{\frac{n(m+2p-3)}{2n(1-p)+2\beta}}\left(\frac{n(m-1)+2\beta}{n(m+2p-3)}\right)^{-\frac{n(m+2p-3)}{2n(1-p)+2\beta}}\left(\int_\Omega u^\beta \mathrm{d}x\right)^{\frac{n(1-p)+m+\beta+2p-3}{n(1-p)+\beta}}\\
&\quad+\int_\Omega u^{m+\beta-1}\mathrm{d}x. \qquad (3.20)
\end{aligned}
$$

It follows from the Hölder inequality and the Young inequality that

$$
\begin{aligned}
&\int_\Omega u^{m+\beta-1}\mathrm{d}x\\
&\le\left(\frac{m+\beta+2p-3}{2(m+\beta-1)}\int_\Omega u^{m+\beta+2p-3}\mathrm{d}x\right)^{\frac{m+\beta-1}{m+\beta+2p-3}}\left(\left(\frac{m+\beta+2p-3}{2(m+\beta-1)}\right)^{-\frac{m+\beta-1}{2(p-1)}}|\Omega|\right)^{\frac{2(p-1)}{m+\beta+2p-3}}\\
&\le\frac{1}{2}\int_\Omega u^{m+\beta+2p-3}\mathrm{d}x+\frac{2(p-1)}{m+\beta+2p-3}\left(\frac{m+\beta+2p-3}{2(m+\beta-1)}\right)^{-\frac{m+\beta-1}{2(p-1)}}|\Omega|, \qquad (3.21)
\end{aligned}
$$

where $0<\frac{m+\beta-1}{m+\beta+2p-3}<1$ in view of (3.2) and (3.3). For the second term of (3.18), we apply (3.19) and the Young inequality to obtain

$$
\begin{aligned}
&\left(2C_s^2\right)^{\frac{n(m+2p-3)}{n(m-1)+2\beta}}\left(\int_\Omega u^\beta \mathrm{d}x\right)^{\frac{2n(1-p)+2(m+\beta+2p-3)}{n(m-1)+2\beta}}\left(\int_\Omega |\nabla u^{\frac{m+\beta-1}{2}}|^2\mathrm{d}x\right)^{\frac{n(m+2p-3)}{n(m-1)+2\beta}}\\
&=\left(\left(2C_s^2\right)^{\frac{n(m+2p-3)}{2n(1-p)+2\beta}}\varepsilon_2^{-\frac{n(m+2p-3)}{2n(1-p)+2\beta}}\left(\int_\Omega u^\beta \mathrm{d}x\right)^{\frac{n(1-p)+m+\beta+2p-3}{n(1-p)+\beta}}\right)^{\frac{2n(1-p)+2\beta}{n(m-1)+2\beta}}\\
&\quad\times\left(\varepsilon_2\int_\Omega |\nabla u^{\frac{m+\beta-1}{2}}|^2\mathrm{d}x\right)^{\frac{n(m+2p-3)}{n(m-1)+2\beta}}\\
&\le\frac{2n(1-p)+2\beta}{n(m-1)+2\beta}\left(2C_s^2\right)^{\frac{n(m+2p-3)}{2n(1-p)+2\beta}}\varepsilon_2^{-\frac{n(m+2p-3)}{2n(1-p)+2\beta}}\left(\int_\Omega u^\beta \mathrm{d}x\right)^{\frac{n(1-p)+m+\beta+2p-3}{n(1-p)+\beta}}
\end{aligned}
$$

$$+\frac{n(m+2p-3)}{n(m-1)+2\beta}\varepsilon_2\int\limits_\Omega|\nabla u^{\frac{m+\beta-1}{2}}|^2\mathrm{d}x, \tag{3.22}$$

where

$$\varepsilon_2=\frac{m\beta(\beta-1)\left[n(m-1)+2\beta\right]}{nA_2(m+2p-3)(m+\beta-1)^2}. \tag{3.23}$$

Now inserting (3.20)–(3.22) into (3.18), we get

$$\begin{aligned}
&\int\limits_\Omega u^{m+\beta+2p-3}\mathrm{d}x\\
&\quad\le\frac{4n(1-p)+4\beta}{n(m-1)+2\beta}\left(2C_s^2\right)^{\frac{n(m+2p-3)}{2n(1-p)+2\beta}}\left(\left(\frac{n(m-1)+2\beta}{n(m+2p-3)}\right)^{-\frac{n(m+2p-3)}{2n(1-p)+2\beta}}+\varepsilon_2^{-\frac{n(m+2p-3)}{2n(1-p)+2\beta}}\right)\\
&\quad\times\left(\int\limits_\Omega u^\beta\mathrm{d}x\right)^{\frac{n(1-p)+m+\beta+2p-3}{n(1-p)+\beta}}+\frac{4(p-1)}{m+\beta+2p-3}\left(\frac{m+\beta+2p-3}{2(m+\beta-1)}\right)^{-\frac{m+\beta-1}{2(p-1)}}|\Omega|\\
&\quad+\frac{2n(m+2p-3)}{n(m-1)+2\beta}\varepsilon_2\int\limits_\Omega|\nabla u^{\frac{m+\beta-1}{2}}|^2\mathrm{d}x.
\end{aligned} \tag{3.24}$$

We substitute (3.24) into (3.14) to derive

$$\begin{aligned}
\Phi'(t)&\le A_1+A_2\frac{4(p-1)}{m+\beta+2p-3}\left(\frac{m+\beta+2p-3}{2(m+\beta-1)}\right)^{-\frac{m+\beta-1}{2(p-1)}}|\Omega|\\
&\quad+A_2\left[\frac{4n(1-p)+4\beta}{n(m-1)+2\beta}\left(2C_s^2\right)^{\frac{n(m+2p-3)}{2n(1-p)+2\beta}}\left(\left(\frac{n(m-1)+2\beta}{n(m+2p-3)}\right)^{-\frac{n(m-1)+2\beta}{2n(1-p)+2\beta}}+\varepsilon_2^{-\frac{n(m+2p-3)}{2n(1-p)+2\beta}}\right)\right]\\
&\quad\times\left(\int\limits_\Omega u^\beta\mathrm{d}x\right)^{\frac{n(1-p)+m+\beta+2p-3}{n(1-p)+\beta}}\\
&=C_1+C_2\Phi^{\frac{n(1-p)+m+\beta+2p-3}{n(1-p)+\beta}}(t),
\end{aligned} \tag{3.25}$$

where

$$C_1=A_1+A_2\frac{4(p-1)}{m+\beta+2p-3}\left(\frac{m+\beta+2p-3}{2(m+\beta-1)}\right)^{-\frac{m+\beta-1}{2(p-1)}}|\Omega| \tag{3.26}$$

and

$$C_2=A_2\frac{4n(1-p)+4\beta}{n(m-1)+2\beta}\left(2C_s^2\right)^{\frac{n(m+2p-3)}{2n(1-p)+2\beta}}\left(\left(\frac{n(m-1)+2\beta}{n(m+2p-3)}\right)^{-\frac{n(m+2p-3)}{2n(1-p)+2\beta}}+\varepsilon_2^{-\frac{n(m+2p-3)}{2n(1-p)+2\beta}}\right). \tag{3.27}$$

Integrating (3.25) from 0 to t, we have

$$t\ge\int\limits_{\Phi(0)}^{\Phi(t)}\frac{\mathrm{d}\tau}{C_1+C_2\tau^{\frac{n(1-p)+m+\beta+2p-3}{n(1-p)+\beta}}}.$$

Since u blows up in measure $\Phi(t)$ at t^*, we pass the limits as $t \to t^{*-}$ to obtain a lower bound

$$t^* \geq \int_{\Phi(0)}^{+\infty} \frac{\mathrm{d}\tau}{C_1 + C_2 \tau^{\frac{n(1-p)+m+\beta+2p-3}{n(1-p)+\beta}}},$$

where $\frac{n(1-p)+m+\beta+2p-3}{n(1-p)+\beta} > 1$ in view of (3.2) and (3.3). □

4. Application

We present an example to demonstrate applications of Theorems 2.1 and 3.1.

Example 4.1. Let u be a nonnegative classical solution of the following problem

$$\begin{cases} u_t = \Delta u^2 + 10\left(3 - \mathrm{e}^{-t}\right) u^3, & \text{in } \Omega \times (0, t^*), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{1}{128} u^2 & \text{on } \partial\Omega \times (0, t^*), \\ u(x, 0) = \dfrac{1}{256}|x|^2 + \dfrac{255}{256} & \text{in } \overline{\Omega}, \end{cases}$$

where $\Omega = \left\{x = (x_1, x_2, x_3) \mid |x|^2 = x_1^2 + x_2^2 + x_3^2 < 1\right\}$ be a unit ball of $\mathbb{R}^3$. Now

$$f(u) = 10u^3, \quad k(t) = 3 - \mathrm{e}^{-t}, \quad g(u) = \frac{1}{128}u^2, \quad u(x,0) = \frac{1}{256}|x|^2 + \frac{255}{256}, \quad m = 2, \quad n = 3.$$

It follows from (2.3) and (2.4) that $\lambda_1 = \pi^2$ and $\omega_1(x) = \frac{\sin \pi |x|}{\sqrt{2\pi}|x|}$. We then have

$$B(t) = \int_\Omega \omega_1^2 u \mathrm{d}x = \int_\Omega \left(\frac{\sin \pi|x|}{\sqrt{2\pi}|x|}\right)^2 u \mathrm{d}x$$

and

$$B(0) = \int_\Omega \omega_1^2 u_0 \mathrm{d}x = \int_\Omega \left(\frac{\sin \pi|x|}{\sqrt{2\pi}|x|}\right)^2 \left(\frac{1}{256}|x|^2 + \frac{255}{256}\right) \mathrm{d}x = 0.9972.$$

By choosing $a = 10$, $r = 2$, and $q = 3$, we easily check that (2.1), (2.2), and (2.5) hold. From Theorem 2.1, it follows that u blows up at finite time t^* in measure $B(t)$ and

$$t^* \leq \int_{B(0)}^{+\infty} \frac{\mathrm{d}\eta}{ar\eta^q - 2\lambda_1 \eta^m} = \int_{0.9972}^{+\infty} \frac{\mathrm{d}\eta}{20\eta^3 - 2\pi^2\eta^2} = 0.1842, \tag{4.1}$$

which is an upper bound for t^*.

In order to obtain a lower bound for t^*, we select $a = 10$, $b = \frac{1}{128}$, $M = 3$, $p = 2$, $q = 3$, and $\beta = 8$. By a simple computation, we have $L_0 = 1$, $d = 1$, and $|\Omega| = \frac{4}{3}\pi$. It is easy to see that (3.1)–(3.3) are valid. It follows from Theorems 2.1 and 3.2 in [19] that the Sobolev embedding constant $C_s = 5.6948$. Putting the above paraments into (3.10), (3.15)–(3.16), (3.23), and (3.26)–(3.27), we get $\varepsilon_1 = 89.6$, $\varepsilon_2 = 1.0633 \times 10^{-2}$, $A_1 = 91.5346$, $A_2 = 274.5227$, $C_1 = 3926.7819$, and $C_2 = 7.4362 \times 10^5$. Now we have

$$\Phi(t) = \int_\Omega u^8 \mathrm{d}x$$

and

$$\Phi(0)=\int\limits_{\Omega}u_0^8\mathrm{d}x=\int\limits_{\Omega}\left(\frac{1}{256}|x|^2+\frac{255}{256}\right)^8\mathrm{d}x=4.1369.$$

Since u blows up in measure $B(t)$ at t^*, u must blow up in measure $\Phi(t)$ at t^*. By Theorem 3.1, we obtain

$$t^*\geq\int\limits_{\Phi(0)}^{+\infty}\frac{\mathrm{d}\tau}{C_1+C_2\tau^{\frac{n(1-p)+m+\beta+2p-3}{n(1-p)+\beta}}}=\int\limits_{4.1369}^{+\infty}\frac{\mathrm{d}\tau}{3926.7819+7.4362\times10^5\tau^{\frac{8}{5}}}=9.5580\times10^{-7}, \tag{4.2}$$

which is a lower bound for t^*. Combining (4.1) and (4.2), we have

$$9.5580\times10^{-7}\leq t^*\leq0.1842.$$

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