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Stability result for viscoelastic wave equation with dynamic boundary conditions

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Abstract. In this paper, we consider wave viscoelastic equation with dynamic boundary condition in a bounded domain, and we establish a general decay result of energy by exploiting the frequency domain method which consists in combining a contradiction argument and a special analysis for the resolvent of the operator of interest with assumptions on past history relaxation function.

Mathematics Subject Classification. 35L05, 35L15, 35L70, 93D15.

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Contents

1. Introduction

We omit the space variable x of $u(x,t)$, $u_t(x,t)$ and for simplicity reason denote $u(x,t) = u$, $u_t(x,t) = u_t$, and when no confusion arises also, the functions considered are all real valued; here, $u_t = \partial u(t)/\partial t$, $u_{tt} = \partial^2 u(t)/\partial t^2$. Our main interest lies in the following system of viscoelastic equation:

$$\begin{cases} u_{tt} - \Delta u + \int\limits_0^\infty g(s)\Delta u(x,t-s)\mathrm{d}s = 0, & x \in \Omega,\ t > 0 \\ u_{tt} = -\left(\dfrac{\partial u}{\partial \nu}(x,t) - \int\limits_0^\infty g(s)\dfrac{\partial u}{\partial \nu}(x,t-s)\mathrm{d}s\right), & x \in \Gamma_1,\ t > 0 \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0 \\ u(x,-t) = u_0(x,t), & x \in \Omega,\ t > 0 \\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

The main difficulty of the problem considered is related to the nonordinary boundary conditions defined on Γ_1. Very little attention has been paid to this type of boundary conditions [2]. From the mathematical point of view, these problems do not neglect acceleration terms on the boundary. Such types of boundary conditions are usually called dynamic boundary conditions. They are not only important from the theoretical point of view but also arise in several physical applications [3,4]. For instance in one space dimension, problem (1.1) can modelize the dynamic evolution of a viscoelastic rod that is fixed at one end and has a tip mass attached to its free end. The dynamic boundary conditions represent the Newton's law for the attached mass (see [2,6] for more details), which arise when we consider the transverse motion of a flexible membrane whose boundary may be affected by the vibrations only in a region. Also some of them as in problem (1.1) appear when we assume that is an exterior domain of $\mathbb{R}^3$ in which homogeneous fluid is at rest except for sound waves. Each point of the boundary is subjected to small normal displacements into the obstacle (see [6] for more details). Among the early results dealing with the dynamic boundary conditions are those of Grobbelaar-van Dalsen [13,14] in which the authors have made contributions to this field, and in [15], the authors have studied the following problem:

$$\begin{cases} u_{tt} - \Delta u + \delta \Delta u_t = |u|^{p-1}u, & x \in \Omega,\ t > 0 \\ u_{tt} = -\left(\dfrac{\partial u(x,t)}{\partial \nu}(x,t) + \delta \dfrac{\partial u(x,t)}{\partial \nu}(x,t) + \alpha |u_t|^{m-1} u(x,t) \right), & x \in \Gamma_1,\ t > 0 \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0 \\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

and they have obtained several results concerning local existence which extended to the global existence by using the concept of stable sets. The authors have also obtained the energy decay and the blow up of the solutions for positive initial energy.

The same problem was treated by [11], they showed the existence and uniqueness of a local in time solution and under some restrictions on the initial data, and the solution continues to exist globally in time. On the other hand, if the interior source dominates the boundary damping, they proved that the solution is unbounded and grows as an exponential function. In addition, in the absence of the strong damping, they proved also the solution ceases to exist and blows up in finite time. Related problem as [9], Cavalcanti et al. [5] studied the following system:

$$\begin{cases} u_{tt} + \mathcal{A}u + a(x) g_1(u_t) = 0, & x \in \Omega,\ t > 0, \\ u_{tt} + \dfrac{\partial u(x,t)}{\partial \nu_{\mathcal{A}}} + \mathcal{A}_T \upsilon + g_2(\upsilon_t) = 0, & x \in \Gamma_1,\ t > 0, \\ u(x,t) = 0 & x \in \Gamma_0,\ t > 0, \\ u(x,t) = \upsilon, & x \in \Gamma_1,\ t > 0, \\ (u_t(x,0), \upsilon_t(x,0)) = (u_1, \upsilon^1), & x \in (\Omega, \Gamma_1), \\ (u(x,0), \upsilon(0)) = (u_0(x), \upsilon^1), & x \in (\Omega, \Gamma_1). \end{cases}$$

They supposed that the second-order differential operators $\mathcal{A}$ and $\mathcal{A}_T$ satisfy certain uniform ellipticity conditions, and they obtained uniform stabilization by using Riemannian geometry methods.
Motivated by the previous works, it is interesting to show more general decay result to that in [9] and [10], we analyze the influence of the viscoelastic, on the solutions to (1.1). Under suitable assumption on function $g(.)$, the initial data and the parameters in the equations.

The content of this paper is organized as follows: In Sect. 2, we provide assumptions that will be used later. In Sect. 3, we state and prove the local existence result. In Sect. 4, by exploiting the frequency domain method used also in [1] we prove the stability result.

2. Preliminaries

In this section, we present some materials and assumptions for the proof of our results. Denote

$$H^1_{\Gamma_0}(\Omega) = \left\{u \in H^1(\Omega) : u_{\Gamma_0} = 0\right\},$$
$$H^1_{\Gamma_0}(\Gamma) = \left\{u \in H^1(\Gamma) : u_{\Gamma_0} = 0\right\},$$

we set γ_1 the trace operator from $H^1_{\Gamma_0}(\Omega)$ on $L^2(\Gamma_1)$ and $H^{\frac{1}{2}}(\Gamma_1) = \gamma_1(H^1_{\Gamma_0}(\Omega))$. We denote by B the norm of γ_1, namely

$$\forall u \in H^1_{\Gamma_0}(\Omega), \quad \|u\|_{2.\Gamma_1} \leq B\|\nabla u\|_2.$$

We will use the following embeddings

$$H^1_{\Gamma_0}(\Omega) \hookrightarrow L^q(\Omega) \text{ for } 2 \leq q \leq \frac{2n}{n-2}, \text{ if } n \geq 3 \text{ and } q \geq 2, \text{ if } n = 1, 2$$
$$L^r(\Omega) \hookrightarrow L^q(\Omega), \quad \text{for } q < r.$$

Then for some $c_s > 0$,

$$\|\nu\|_q \leq c_s\|\nabla\nu\|_2, \quad \|\nu\|_q \leq c_s\|\nu\|_r \quad \text{for} \quad \nu \in H^1_{\Gamma_0}(\Omega).$$

We recall that $H^{\frac{1}{2}}(\Gamma_1)$ is dense in $L^2(\Gamma_1)$. We denote

$$E(\Delta, L^2(\Omega)) = \left\{u \in H^1(\Omega) \ \text{ such that } \Delta u \in L^2(\Omega)\right\}$$

and recall that for a function $u \in E(\Delta, L^2(\Omega))$, $\frac{\partial u}{\partial \nu} \in H^{-\frac{1}{2}}(\Gamma_1)$. We will usually use the following Green's formula

$$\int_\Omega \nabla u(x)\nabla\omega(x)\mathrm{d}x = -\int_\Omega \Delta u(x)\omega(x)\mathrm{d}x + \int_{\Gamma_1} \frac{\partial u}{\partial\nu}(x)\omega(x)\mathrm{d}\Gamma_1, \ \forall\omega \in H^1_{\Gamma_0}(\Omega). \tag{2.3}$$

For studying problem (1.1), we will need the following assumptions (A1).

- The relaxation function g is differentiable function such that, for $s \geq 0$

$$g(s) \geq 0, \quad 1 - \int_0^\infty g(s)\mathrm{d}s = \ell > 0, \tag{2.4}$$

- $$\exists\, \zeta_0, \zeta_1 > 0 : \quad -\zeta_1 g(t) \leq g'(t) \leq -\zeta_0 g(t), \quad \forall\, t \in \mathbb{R}. \tag{2.5}$$

3. Well-posedness of the problem

In order to prove the existence of solutions of problem (1.1), we follow the approach of Dafermos [8], by considering a new auxiliary variable the relative history of u as follows:

$$\eta := \eta^t(x, s) = u(x, t) - u(x, t - s) \quad \text{in} \quad \Omega \times (0, \infty) \times (0, \infty),$$

and the weighted L^2- spaces

$$\begin{aligned}\mathcal{M} &= L^2_g(\mathbb{R}_+; H^1_{\Gamma_0}(\Omega)) \\ &= \left\{\xi : \mathbb{R}_+ \to H^1_{\Gamma_0}(\Omega)) : \int_0^\infty g(s)\|\nabla\xi(s)\|_2^2\mathrm{d}s < \infty\right\},\end{aligned}$$

which is a Hilbert space endowed with inner product and norm consecutively

$$\langle \xi, \zeta \rangle_{\mathcal{M}} = \int_0^\infty g(s) \left(\int_\Omega \nabla \xi(s) \nabla \zeta(s) \mathrm{d}x \right) \mathrm{d}s,$$

and

$$\|\xi\|_{\mathcal{M}}^2 = \int_0^\infty g(s) \|\nabla \xi(s)\|_2^2 \mathrm{d}s.$$

Our analysis is given on the phase space

$$\mathcal{H} = H^1_{\Gamma_0}(\Omega) \times \times L^2(\Omega) \times L^2(\Gamma_1) \times \mathcal{M}. \tag{3.6}$$

If we denote $V := (u, u_t, \gamma_1(u_t), \eta)$, then it is clear that $\mathcal{H}$ is a Hilbert space with respect to the following inner product

$$\begin{aligned} \langle V_1, V_2 \rangle_{\mathcal{H}} = (1 - g_0) \int_\Omega \nabla u_1 \cdot \nabla u_2 \mathrm{d}x + \int_\Omega \upsilon_1 \cdot \upsilon_2 \mathrm{d}x + \int_{\Gamma_1} w_1 \cdot w_2 \mathrm{d}\sigma \\ + \int_0^\infty g(s) \left(\int_\Omega \nabla \eta_1(s) \cdot \nabla \eta_2(s) \mathrm{d}x \right) \mathrm{d}s, \end{aligned} \tag{3.7}$$

for $V_1 = (u_1, \upsilon_1, w_1, \eta_1)^\mathsf{T}$ and $V_2 = (u_2, \upsilon_2, w_2, \eta_2)^\mathsf{T}$. Therefore, problem (1.1) is equivalent to

$$\begin{cases} u_{tt} - \ell \Delta u - \int_0^\infty g(s) \Delta \eta^t(x, s) \mathrm{d}s = 0, & x \in \Omega,\ t > 0, \\ u_{tt} = -\left(\dfrac{\partial u}{\partial \nu}(x, t) + \int_0^\infty g(s) \dfrac{\partial u}{\partial \nu}(x, t - s) \mathrm{d}s \right), & x \in \Gamma_1,\ t > 0, \\ \eta_t^t(x, t) + \eta_s^t(x, s) = u_t(x, t), & x \in \Omega,\ t > 0,\ s > 0, \\ u(x, t) = \eta^t(x, 0) = 0, & x \in \Gamma_0,\ t > 0, \\ u(x, -t) = u_0(x, t), & x \in \Omega,\ t > 0, \\ u_t(x, 0) = u_1(x), & x \in \Omega, \\ u(x, 0) = u_0(x), & x \in \Omega. \end{cases} \tag{3.8}$$

If $V_0 \in \mathcal{H}$ and $V \in \mathcal{H}$, problem (3.8) is formally equivalent to the following abstract evolution equation in the Hilbert space $\mathcal{H}$

$$\begin{cases} V'(t) = \mathcal{A} V(t), \quad t > 0 \\ V(0 = V_0, . \end{cases} \tag{3.9}$$

such that $V_0 = (u_0, u_1, \gamma_1(u_1), \eta^0)^\mathsf{T}$ and the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ \upsilon \\ \omega \\ \eta \end{pmatrix} = \begin{pmatrix} \upsilon \\ (1-g_0)\Delta u + \int_0^\infty g(s)\Delta\eta(s)\mathrm{d}s \\ -\frac{\partial u}{\partial \nu} - \int_0^\infty g(s)\frac{\partial \omega}{\partial \nu}(x,t-s)\mathrm{d}s \\ -\frac{\partial \eta}{\partial s} + \upsilon \end{pmatrix} \tag{3.10}$$

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (u,\upsilon,\omega,\eta) \in \left(H^2(\Omega)\cap H^1_{\Gamma_0}(\Omega)\right) \times H^1_{\Gamma_0}(\Omega) \times L^2(\Gamma_1) \times \mathcal{M}, \\ (1-g_0)u + \int_0^\infty g(s)\eta(s)\mathrm{d}s \in L^2(\Omega),\ \omega = \gamma_1(u) = u_0(.,0),\ \eta(0) = 0 \text{ on } \Gamma_1 \end{array} \right\}$$

Now, our main result concerning this section is stated as follows:

Theorem 3.1. *Let $V_0 \in \mathcal{H}$. Then, system (3.8) has a unique weak solution*

$$V \in \mathcal{C}(\mathbb{R}^+;\mathcal{H})$$

Moreover, if $V_0 \in D(\mathcal{A})$, then the solution of (3.9) satisfies

$$V \in \mathcal{C}^1(\mathbb{R}^+;\mathcal{H}) \cap \mathcal{C}(\mathbb{R}^+;D(\mathcal{A}))$$

Proof. By Lumer–Phillips' theorem[17], it suffices to show that $\mathcal{A}$ is m-dissipative.

We first prove that $\mathcal{A}$ is dissipative. Indeed, for any $V = (u,\upsilon,\omega,\eta)^\mathsf{T} \in D(\mathcal{A})$, we have

$$\begin{aligned} \langle \mathcal{A}V, V\rangle_{\mathcal{H}} &= \left\langle \begin{pmatrix} \upsilon \\ (1-g_0)\Delta u + \int_0^\infty g(s)\Delta\eta(s)\mathrm{d}s \\ -\frac{\partial u}{\partial \nu} - \int_0^\infty g(s)\frac{\partial u}{\partial \nu}(x,t-s)\mathrm{d}s \\ -\frac{\partial \eta}{\partial s} + \upsilon \end{pmatrix}, \begin{pmatrix} u \\ \upsilon \\ \omega \\ \eta \end{pmatrix} \right\rangle \\ &= (1-g_0)\int_\Omega \nabla\upsilon\cdot\nabla u\mathrm{d}x + (1-g_0)\left\{\int_\Omega \Delta u\cdot\upsilon\mathrm{d}x + \int_\Omega\int_0^\infty g(s)\Delta\eta(s)\upsilon(s)\mathrm{d}s\right\} \\ &\quad + \int_{\Gamma_1}\left(\frac{-\partial u}{\partial\nu} - \int_0^\infty g(s)\frac{\partial u}{\partial \nu}(x,t-s)\mathrm{d}s\right)\omega\mathrm{d}\sigma + \left\langle \frac{-\partial\eta}{\partial s} + \upsilon, \eta\right\rangle_{L^2_g}. \end{aligned} \tag{3.11}$$

Noting that

$$\int_{\Gamma_1}\left(\frac{-\partial u}{\partial\nu} - \int_0^\infty g(s)\frac{\partial u}{\partial \nu}(x,t-s)\mathrm{d}s\right)\omega\mathrm{d}\sigma = 0 \tag{3.12}$$

By exploiting Green's formula, integrating by parts and using the fact that $\eta(0) = 0$ (from the definition of $D(\mathcal{A})$), we obtain

$$\left\langle \frac{-\partial \eta}{\partial s}, \eta \right\rangle_{L^2_g} = \frac{1}{2} \int_0^\infty g'(s) \|\nabla \eta(s)\|^2 \mathrm{d}s.$$

Inserting the previous identity into (3.11) , we get

$$\langle \mathcal{A}V, V \rangle_{\mathcal{H}} = \frac{1}{2} \int_0^\infty g'(s) \|\nabla \eta(s)\|^2 \mathrm{d}s,$$

which implies that

$$\langle \mathcal{A}V, V \rangle_{\mathcal{H}} \leq 0,$$

since g is nonincreasing. This means that $\mathcal{A}$ is dissipative. Note that, thanks to (A1) and the fact that $\eta \in L^2_g(\mathbb{R}; H^1_{\Gamma_0}(\Omega))$,

$$\begin{aligned} \left| \int_0^\infty g'(s) \|\nabla \eta(s)\|^2 \mathrm{d}s \right| &= - \int_0^\infty g'(s) \|\nabla \eta(s)\|^2 \mathrm{d}s \\ &\leq \zeta_1 \int_0^\infty g(s) \|\nabla \eta(s)\|^2 \mathrm{d}s \\ &< +\infty. \end{aligned} \tag{3.13}$$

Next, we shall prove that $\lambda I - \mathcal{A}$ is surjective for $\lambda > 0$. Indeed, let $F = (f_1, f_2, f_3, f_4)^\mathsf{T} \in \mathcal{H}$, and we look for $W = (\omega_1, \omega_2, \omega_3, \omega_4)^\mathsf{T} \in D(\mathcal{A})$ satisfying

$$(I\lambda - \mathcal{A})W = F \tag{3.14}$$

As previously, we have

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 \\ (1-g_0)\Delta & 0 & 0 & \int_0^\infty g(s)\Delta \mathrm{d}s \\ -\frac{\partial}{\partial \nu} & 0 & -\int_0^\infty g(s) \frac{\partial}{\partial \nu} \mathrm{d}s & 0 \\ 0 & I & 0 & -\frac{\partial}{\partial s} \end{pmatrix}$$

and (3.14) gives us

$$\begin{cases} \lambda \omega_1 - \omega_2 = f_1 \\ -(1-g_0)\Delta \omega_1 + \lambda \omega_2 - \int_0^\infty g(s) \Delta \omega_4(s) \mathrm{d}s = f_2 \\ \lambda \omega_3 + \frac{\partial \omega_1}{\partial \nu} + \int_0^\infty g(s) \frac{\partial \omega_3(s)}{\partial \nu} \mathrm{d}s = f_3 \\ -\omega_2 + \lambda \omega_4 + \frac{\partial}{\partial s} \omega_4 = f_4. \end{cases} \tag{3.15}$$

We note that the first equation in (3.15) gives

$$\omega_2 = \lambda \omega_1 - f_1 \tag{3.16}$$

and the last equation in (3.15) with $\eta(0)=0$ has unique solution

$$\omega_4(s) = \left(\int_0^s e^y (f_4(y) + \omega_2(y))dy\right) e^{-s}. \tag{3.17}$$

From the first and the second equation in (3.15), we can deduce the following

$$\lambda^2\omega_1 - (1-g_0)\Delta\omega_1 = (f_2 + \lambda f_1) + \int_0^\infty g(s)\omega_4(s)ds. \tag{3.18}$$

Putting $\bar{u} = \omega_1 + \int_0^\infty g(s)\omega_3(s)ds$. Then from equation (3.18), $\bar{u}$ must satisfy

$$\begin{aligned} \lambda^2\bar{u} - (1-g_0)\Delta\bar{u} = \lambda^2 \int_0^\infty g(s)\omega_3(s)ds - (1-g_0)\int_0^\infty g(s)\Delta\omega_3(s)ds \\ + (f_2 + \lambda f_1) + \int_0^\infty g(s)\omega_4(s)ds \end{aligned} \tag{3.19}$$

with the boundary conditions

$$\bar{u} = 0 \quad \text{on} \quad \Gamma_0 \tag{3.20}$$

$$\frac{\partial \bar{u}}{\partial \nu} = f_3 - \lambda\bar{u} + \lambda u_0(x)(1-\ell) \quad \text{on} \quad \Gamma_1. \tag{3.21}$$

It is sufficient to prove that (3.19) has a solution $\bar{u}$ in $H^2(\Omega)\cap H^1_{\Gamma_0}(\Omega)$ and replacing it in (3.17) and (3.16) to conclude that (3.8) has a solution $V \in D(\mathcal{A})$. So we multiply (3.19) by a test function $\varphi \in H^1_{\Gamma_0}(\Omega)$ and we integrate by parts, obtaining the following variational formulation of (3.19):

$$a(\bar{u},\varphi) = l(\varphi) \quad \forall\, \varphi \in H^1_{\Gamma_0}(\Omega) \tag{3.22}$$

where

$$a(\bar{u},\varphi) = \int_\Omega \left[\lambda^2\bar{u}\cdot\varphi + (1-g_0)\,\nabla\bar{u}\cdot\nabla\varphi\right] dx + \lambda\int_{\Gamma_1} \bar{u}(\sigma)\varphi(\sigma)d\sigma \tag{3.23}$$

and

$$\begin{aligned} l(\varphi) = \int_\Omega \left[\lambda^2 \int_0^\infty g(s)\omega_3(s)ds\varphi dx + (1-g_0)\int_0^\infty g(s)\nabla\omega_3(s)ds\nabla\varphi dx + (f_2+\lambda f_1)\varphi dx\right] \\ + \int_\Omega\int_0^\infty g(s)\omega_4(s)ds\varphi dx + \lambda\int_{\Gamma_1} u_0(\sigma)\varphi(\sigma)d\sigma \end{aligned} \tag{3.24}$$

It is clear that a is a bilinear and continuous form on $H^1_{\Gamma_0}(\Omega)$ and l is linear and continuous form on $H^1_{\Gamma_0}(\Omega)$. On the other hand, (3.23) implies that there exists a positive constant a_0 such that

$$\begin{aligned} a(\bar{u},\bar{u}) &= \int_\Omega \lambda^2|\bar{u}|^2dx + (1-g_0)\int_\Omega |\nabla\bar{u}|^2dx + \lambda\int_{\Gamma_1}|\bar{u}(\sigma)|^2d\sigma \\ &\geq a_0\|\bar{u}\|_2^2 \quad \forall\bar{u} \in H^1_{\Gamma_0}(\Omega), \end{aligned} \tag{3.25}$$

which implies that a is coercive. Therefore, using the Lax–Milgram theorem, we conclude that (3.19) has a unique solution $\bar{u}$ in $H^1_{\Gamma_0}(\Omega)$. By classical regularity arguments, we conclude that the solution $\bar{u}$ of

(3.19) belongs into $H^2(\Omega)\cap H^1_{\Gamma_0}(\Omega)$ and satisfies (3.19). Consequently, using (3.16) and (3.17), we deduce that (3.8) has a unique solution $V \in D(\mathcal{A})$. This proves that $(\lambda I - \mathcal{A})$ is surjective, and hence, $\mathcal{A}$ is an infinitesimal generator of a linear $\mathcal{C}_0$ semigroup of contractions on $\mathcal{H}$. □

The energy associated with (3.8) is defined by

$$E(t) = \frac{1}{2}\left\{\|u_t(t)\|^2_{\Gamma_1} + \|\nabla u(t)\|_2^2 + \|\eta\|^2_{L^2_g}\right\}, \tag{3.26}$$

Lemma 3.2. *The functional defined in (3.26) satisfies the following inequality*

$$E'(t) \le \frac{1}{2}\int_0^\infty g'(s)\|\nabla\eta(s)\|_2^2 ds,\ \forall\ t \ge 0, \tag{3.27}$$

Proof. By multiplying the first equation in (3.8) by $u_t(t)$, and integrating over Ω, we get

$$0 = \frac{1}{2}\frac{d}{dt}\left\{\|u_t(t)\|^2_{\Gamma_1} + \|\nabla u(t)\|_2^2\right\} + \int_0^\infty g(s)\int_{\mathbb{R}^n} \nabla\eta(s)\nabla u_t(t) ds dx. \tag{3.28}$$

Since

$$u_t(x,t) = \eta_t(x,s) + \eta_s(x,s), \quad (x,s) \in \Omega\times\mathbb{R}^+,\ t \ge 0,$$

we have

$$\begin{aligned}\int_0^\infty g(s)\int_\Omega \nabla\eta(s)\nabla u_t(t)\mathrm{d}x\mathrm{d}s &= \int_0^\infty g(s)\int_\Omega \nabla\eta(s)\nabla\eta_t(t)\mathrm{d}x\mathrm{d}s \\ &\quad + \int_0^\infty g(s)\int_\Omega \nabla\eta(s)\nabla\eta_s(t)\mathrm{d}x\mathrm{d}s \\ &= \frac{1}{2}\int_0^\infty g(s)\frac{d}{dt}\|\nabla\eta(s)\|_2^2\mathrm{d}s \\ &\quad - \frac{1}{2}\int_0^\infty g'(s)\|\nabla\eta(s)\|_2^2\mathrm{d}s \\ &\quad + \int_0^\infty g(s)\int_\Omega \nabla\eta(s)\nabla\eta_t(t)\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{3.29}$$

Due to Young's inequality, we have for any $\delta > 0$

$$\begin{aligned}\int_0^\infty g(s)\int_\Omega \nabla\eta(s)\nabla\eta_t(t)\mathrm{d}x\mathrm{d}s &\le \int_0^\infty g(s)\left(\frac{1}{4\delta}\|\nabla\eta(s)\|_2^2 + \delta\|\nabla\eta_t\|_2^2\right)\mathrm{d}s \\ &\le \delta\left(\int_0^\infty g(s)\mathrm{d}s\right)\|\nabla\eta_t\|_2^2 + \frac{1}{4\delta}\int_0^\infty g(s)\|\nabla\eta(s)\|_2^2\mathrm{d}s \\ &= \delta g_0\|\nabla\eta_t\|_2^2 + \frac{1}{4\delta}\|\eta\|^2_{L^2_g},\end{aligned} \tag{3.30}$$

by replacing (3.29) and (3.30) into (3.28), we get the desired result. □

4. Stability result

The necessary and sufficient conditions for the exponential stability of the C_0-semigroup of contractions on a Hilbert space were obtained by Gearhart [12] and Huang [16] independently (see also Prüss [18]). We will use the following result due to Gearhart.

Lemma 4.1. *A semigroup* $\{e^{t\mathcal{A}}\}_{t\geq 0}$ *of contractions on a Hilbert space* $\mathcal{X}$ *is exponentially stable if and only if*

$$i\mathbb{R} \equiv \{i\beta; \quad \beta \in \mathbb{R}\} \subset \rho(\mathcal{A}) \tag{4.31}$$

and

$$\limsup_{|\beta|\to\infty} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{X}} < \infty \tag{4.32}$$

Our main result reads as follows:

Theorem 4.2. *The semigroup of system* (3.8) *decays exponentially as*

$$\|e^{t\mathcal{A}}V_0\|_{\mathcal{H}} \leq Ce^{-\gamma t}\|V_0\|_{D(\mathcal{A})}, \quad \forall\, V_0 \in D(\mathcal{A}),\ t > 0 \tag{4.33}$$

Proof. The proof is splinted into two parts: the first part consists to prove (4.31) which is equivalent to prove the following two assertions

1. If β is a real number, then $(i\beta I - \mathcal{A})$ is injective and
2. If β is a real number, then $(i\beta I - \mathcal{A})$ is surjective.

It is the objective of the two following lemmas.

Lemma 4.3. *If* β *is a real number, then* $i\beta$ *is not an eigenvalue of* $\mathcal{A}$*.*

Proof. We will show that the equation

$$\mathcal{A}Z = i\beta Z \tag{4.34}$$

with $Z = (u, \upsilon, \omega, \eta)^{\mathsf{T}} \in D(\mathcal{A})$ and $\beta \in \mathbb{R}$ has only the trivial solution. Equation (4.34) can be written as

$$i\beta u - \upsilon = 0 \tag{4.35}$$

$$i\beta\upsilon - (1 - g_0)\Delta u - \int_0^\infty g(s)\Delta\eta(s)\mathrm{d}s = 0 \tag{4.36}$$

$$i\beta\omega + \frac{\partial u}{\partial \nu} + \int_0^\infty g(s)\frac{\partial \omega(s)}{\partial \nu}\mathrm{d}s = 0 \tag{4.37}$$

$$i\beta\eta + \frac{\partial \eta}{\partial s} - \upsilon = 0 \tag{4.38}$$

By taking the inner product of (4.34) with $Z \in D(\mathcal{A})$ and using (3.27), we get:

$$\begin{aligned}\Re(\langle \mathcal{A}Z, Z\rangle_{\mathcal{H}}) &\leq \int_0^\infty g'(s)\|\nabla\eta(s)\|^2\mathrm{d}s \\ &\leq -\int_0^\infty g(s)\|\nabla\eta(s)\|^2\mathrm{d}s \\ &= -\|\eta\|^2_{\mathcal{M}} \\ &\leq 0\end{aligned} \tag{4.39}$$

Thus, we obtain that $\eta = 0$; moreover, as η satisfies (4.38), by integration, we obtain

$$\eta(s) = \left(\int_0^s \mathrm{e}^{i\beta y}\upsilon(y))\mathrm{d}y\right)\mathrm{e}^{-i\beta s}.$$

Since $\eta = 0$, we deduce that $\upsilon = 0$, and from (4.35), we have $u = 0$. Moreover, as $\omega = \gamma_1(u) = u_0(.,0)$, we obtain also $\omega = 0$. Thus, the only solution of (4.34) is the trivial one. Hence, the proof is completed. □

Next, we show that $\mathcal{A}$ has no continuous spectrum on the imaginary axis.

Lemma 4.4. *If β is a real number, then $i\beta$ lies in the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$.*

Proof. In view of Lemma 4.3, it is enough to show that $i\beta I - \mathcal{A}$ is surjective.

In fact, for $F = (f_1, f_2, f_3, f_4)^\mathsf{T} \in \mathcal{H}$, let $V = (u, \upsilon, \omega, \eta)^\mathsf{T} \in D(\mathcal{A})$ solution of

$$(i\beta I - \mathcal{A})V = F \tag{4.40}$$

which is

$$\begin{cases} i\beta u - \upsilon = f_1 \\ -(1-g_0)\Delta u + i\beta\upsilon - \int\limits_0^\infty g(s)\Delta\eta(s)\mathrm{d}s = f_2 \\ i\beta\omega + \dfrac{\partial u}{\partial\nu} + \int\limits_0^\infty g(s)\dfrac{\partial\omega(s)}{\partial\nu}\mathrm{d}s = f_3 \\ -\upsilon + i\beta\eta + \dfrac{\partial\eta}{\partial s} = f_4. \end{cases} \tag{4.41}$$

The first equation in (4.41) gives

$$\upsilon = i\beta\omega_1 - f_1. \tag{4.42}$$

The last equation in (4.41) with $\eta(0) = 0$ has unique solution

$$\omega_4(s) = \left(\int\limits_0^s \mathrm{e}^{i\beta y}(f_4(y) + \omega_2(y))\mathrm{d}y\right)\mathrm{e}^{-i\beta s} \tag{4.43}$$

Another time, from the first and the second equation in (4.41), we can deduce the following

$$(i\beta)^2\omega_1 - (1-g_0)\Delta\omega_1 = (f_2 + i\beta f_1) + \int\limits_0^\infty g(s)\omega_4(s)\mathrm{d}s \tag{4.44}$$

If we take $\omega_1 + \int\limits_0^\infty g(s)\omega_3(s)\mathrm{d}s = \bar{u}$, then from Eq. (4.44) $\bar{u}$ must satisfy

$$\begin{aligned} (i\beta)^2\bar{u} - (1-g_0)\Delta\bar{u} = (i\beta)^2\int\limits_0^\infty g(s)\omega_3(s)\mathrm{d}s - (1-g_0)\int\limits_0^\infty g(s)\Delta\omega_3(s)\mathrm{d}s \\ + (f_2 + i\beta f_1) + \int\limits_0^\infty g(s)\omega_4(s)\mathrm{d}s \end{aligned} \tag{4.45}$$

with the boundary conditions

$$\bar{u} = 0 \quad \text{on } \Gamma_0 \tag{4.46}$$

$$\frac{\partial\bar{u}}{\partial\nu} = f_3 - i\beta\bar{u} + i\beta u_0(x)(1-\ell) \quad \text{on } \Gamma_1. \tag{4.47}$$

It is sufficient to prove that (4.45) has a solution $\bar{u}$ in $H^2 \cap H^1_{\Gamma_0}(\Omega)$, and then, we replace in (4.42) and (4.43) to conclude that (4.40) has a solution $V \in D(\mathcal{A})$. Then, we multiply (4.45) by a test function $\varphi \in H^1_{\Gamma_0}(\Omega)$ and we integrate by parts, obtaining the following variational formulation of (4.45):

$$b(\bar{u}, \varphi) = l(\varphi) \quad \forall\, \varphi \in H^1_{\Gamma_0}(\Omega) \tag{4.48}$$

where

$$b(\bar{u},\varphi)=\int_{\Omega}\left[(i\beta)^2\bar{u}.\varphi+(1-g_0)\,\nabla\bar{u}.\nabla\varphi\right]\mathrm{d}x+i\beta\int_{\Gamma_1}\bar{u}(\sigma)\varphi(\sigma)\mathrm{d}\sigma \tag{4.49}$$

and

$$\begin{aligned} l(\varphi)=&\int_{\Omega}\left[(i\beta)^2\int_0^\infty g(s)\omega_3(s)\mathrm{d}s\varphi\mathrm{d}x+(1-g_0)\int_0^\infty g(s)\nabla\omega_3(s)\mathrm{d}s\nabla\varphi\mathrm{d}x+(f_2+i\beta f_1)\varphi\mathrm{d}x\right]\\ &+\int_{\Omega}\int_0^\infty g(s)\omega_4(s)\mathrm{d}s\varphi\mathrm{d}x+i\beta\int_{\Gamma_1}u_0(\sigma)\varphi(\sigma)\mathrm{d}\sigma \end{aligned} \tag{4.50}$$

It is clear that b is a bilinear and continuous form on $H^1_{\Gamma_0}(\Omega)$ and l is linear and continuous form on $H^1_{\Gamma_0}(\Omega)$. On the other hand, (4.49) implies that there exists a positive constant C_0 such that

$$\begin{aligned} b(\bar{u},\bar{u})=&\int_{\Omega}(i\beta)^2|\bar{u}|^2\mathrm{d}x+(1-g_0)\int_{\Omega}|\nabla\bar{u}|^2\mathrm{d}x+i\beta\int_{\Gamma_1}|\bar{u}(\sigma)|^2\mathrm{d}\sigma\\ &\geq C_0\|\bar{u}\|_2^2\quad\forall\bar{u}\in H^1_{\Gamma_0}(\Omega), \end{aligned} \tag{4.51}$$

which implies that b is coercive. Therefore, using the Lax–Milgram theorem, we conclude that (4.45) has a unique solution $\bar{u}$ in $H^1_{\Gamma_0}(\Omega)$. By classical regularity arguments, we conclude that the solution $\bar{u}$ of (4.42) belongs into $H^2(\Omega)\cap H^1_{\Gamma_0}(\Omega)$. Consequently, using (4.42) and (4.43), we deduce that (4.34) has a unique solution $V\in D(\mathcal{A})$. This proves that $(i\beta-\mathcal{A})$ is surjective. □

Lemma 4.5. *The resolvent operator of $\mathcal{A}$ satisfies* (4.32).

Proof. Suppose that condition (4.32) is false. By Banach–Steinhaus theorem [7, Theorem A.3.19], there exists a sequence of real numbers $\beta_n\to+\infty$ and a sequence of vectors

$$Z_n=(u_n,v_n,\omega_n,\eta_n)^{\mathsf{T}}\in D(\mathcal{A})\quad\text{with }\|Z_n\|_{\mathcal{H}}=1 \tag{4.52}$$

such that

$$\|(i\beta_nI-\mathcal{A})Z_n\|_{\mathcal{H}}\to 0\text{ as }n\to\infty. \tag{4.53}$$

That's

$$(i\beta_nu_n-v_n)\equiv f_n\to 0,\ \text{ in }H^1_{\Gamma_0}(\Omega) \tag{4.54}$$

$$\left(i\beta_nv_n-(1-g_0)\Delta u_n-\int_0^\infty g(s)\Delta\eta_n(s)\mathrm{d}s\right)\equiv g_n\to 0,\ \text{ in }L^2(\Omega) \tag{4.55}$$

$$\left(i\beta_n\omega_n+\frac{\partial u_n}{\partial\nu}+\int_0^\infty g(s)\frac{\partial\omega_n(s)}{\partial\nu}\mathrm{d}s\right)\equiv h_n\to 0,\ \text{ in }L^2(\Gamma_1) \tag{4.56}$$

$$\left(i\beta_n\eta_n+\frac{\partial\eta_n}{\partial s}-v_n\right)\equiv k_n\to 0,\ \text{ in }\mathcal{M}. \tag{4.57}$$

Our aim is to derive from (4.53) that $\|Z_n\|_{\mathcal{H}}$ converges to zero; thus, there is a contradiction.

$$|\Re\left\langle(i\beta_nI-\mathcal{A})Z_n,Z_n\right\rangle_{\mathcal{H}}|\leq\|(i\beta_nI-\mathcal{A})Z_n\|_{\mathcal{H}}. \tag{4.58}$$

Using the hypotheses on g , we find that

$$\eta_n\to 0\quad\text{in}\quad L^2_g(\mathbb{R}_+;H^1_{\Gamma_0}(\Omega)) \tag{4.59}$$

and

$$\eta_n(s) = \left(\int_0^s e^{i\beta y} k_n(y)\right) e^{-i\beta s} + \left(\int_0^s e^{i\beta y} \upsilon_n(y) dy\right) e^{-i\beta s}. \tag{4.60}$$

By exploiting the convergence (4.59) and (4.60), we can deduce from (4.54) that

$$\upsilon_n \to 0 \quad \text{in} \quad L^2(\Omega) \text{ and } u_n \to 0 \quad \text{in} \quad L^2(\Omega). \tag{4.61}$$

Now, multiplying equation (4.54) by υ_n and (4.55) by u_n, adding them and taking the real parts , we obtain

$$-\|\upsilon_n\|_2^2 + (1-g_0)\|\nabla u_n\|_2^2 + \int_0^\infty g(s)\nabla\eta_n(s)\nabla u_n(t) ds \to 0 \text{ in } L^2(\Omega). \tag{4.62}$$

According to Young's inequality, we have for any $\delta > 0$

$$\begin{aligned} \int_0^\infty g(s) \int_\Omega \nabla\eta_n(s)\nabla u_n(t) dx ds &\leq \int_0^\infty g(s)\left(\frac{1}{4\delta}\|\nabla\eta_n(s)\|_2^2 + \delta\|\nabla u_n\|_2^2\right) ds \\ &\leq \delta\left(\int_0^\infty g(s) ds\right)\|\nabla u_n\|_2^2 + \frac{1}{4\delta}\int_0^\infty g(s)\|\nabla\eta_n(s)\|_2^2 ds \\ &= \delta g_0\|\nabla u_n\|_2^2 + \frac{1}{4\delta}\|\eta\|_{L_g^2}^2. \end{aligned} \tag{4.63}$$

Replacing the last inequality in (4.62), for δ sufficiently small, we get

$$\nabla u_n \to 0 \quad \text{in} \quad L^2(\Omega). \tag{4.64}$$

Consequently, we have

$$u_n \to 0 \quad \text{in} \quad H^1_{\Gamma_0}(\Omega). \tag{4.65}$$

By using (4.56) and trace theorem, we get

$$\omega_n \to 0 \quad \text{in} \quad L^2(\Gamma_1) \tag{4.66}$$

which contradicts (4.52). Thus, (4.32) is proved. □

□

References

[1] Ammari, K., Gerbi, S.: Interior feedback stabilisation of wave equations with dynamic boundary delay. Z. Anal. Anwend. **36**(03), 297–327 (2017)

[2] Andrews, Kevin T.: Kuttler, K.L., Shillor, M.: Second order evolution equations with dynamic boundary conditions. J. Math. Anal. Appl. **197**(3), 781–795 (1996)

[3] Budak, B.M., Samarskii, A.A., Tikhonov, A.N.: A Collection of Problems on Mathematical Physics (Translated by A.R.M. Robson). The Macmillan Co., New York (1964)

[4] Caroll, R.W., Showalter, R.E.: Singular and Degenerate Cauchy Problems. Academic Press, New York (1976)

[5] Cavalcanti, M.M., Khemmoudj, A., Medjden, M.: Uniform stabilisation of the damped-Ventcel problem with variable coefficients and dynamic boundary conditions. J. Math. Anal. Appl. **328**(1), 900–930 (2007)

[6] Conrad, F., Morgul, O.: stabilization of a flexible beam with a tip mass. SIAM J. Control Optim. **36**(6), 1962–1986 (1998)

[7] Curtain, R.F., Zwart, H.J.: An Introduction to Infinite Dimensional Linear Systems Theory. Springer, New York (1995)

[8] Dafermos, C.M.: Asymptotic stability in viscoelasticity. Arch. Ration. Mech. Anal. **37**, 297–308 (1970)

[9] Gerbi, S., Said-Houari, B.: Asymptotic stability and blow up for a semilinear damped wave equation with dynamic boundary conditions. Nonlinear Anal. **74**(18), 7137–7150 (2011)

[10] Gerbi, S., Said-Houari, B.: Local existence and exponential growth for a semilinear damped wave equation with dynamic boundary conditions. Adv. Differ. Equ. **13**, 1051–1074 (2008)
[11] Gerbi, S., Said-Houari, B.: Global existence and exponential growth for visceolastic wave equation with dynamic boundary conditions. Adv. Nonlinear Anal. **2**, 163–193 (2013)
[12] Gearhart, L.: Spectral theory for contraction semigroups on Hilbert spaces. Trans. Am. Math. Soc. **236**, 385–394 (1978)
[13] Grobbelaar-van Dalsen, M.: On fractional powers of a closed pair of operators and a damped wave equation with dynamic boundary conditions. Appl. Anal. **53**, 41–54 (1994)
[14] Grobbelaar-van Dalsen, M.: On the initial-boundary-value problem for the extensible beam with attached load. Math. Methods Appl. Sci. **19**(12), 943–957 (1996)
[15] Grobbelaar-van Dalsen, M.: On the solvability of the boundary-value problem for the elastic beam with attached load. Math. Models Methods Appl. Sci. **4**(1), 89–105 (1994)
[16] Huang, F.L.: Characteristic condition for exponential stability of linear dynamical systems in Hilbert spaces. Ann. Differ. Equ. **1**, 43–56 (1985)
[17] Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential. Springer, New York (1983)
[18] Prüss, J.: On the spectrum of C_0-semigroups. Trans. Am. Math. Soc. **284**(2), 847–857 (1984)

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