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Strong solutions for an incompressible Navier–Stokes/Allen–Cahn system with different densities

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Abstract. In this paper, we investigate a coupled Navier–Stokes/Allen–Cahn system describing a diffuse interface model for two-phase flow of viscous incompressible fluids with different densities in a bounded domain $\Omega \subset \mathbb{R}^N (N = 2, 3)$. We prove the existence and uniqueness of local strong solutions to the initial boundary value problem when the initial density function ρ_0 has a positive lower bound.

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1. Introduction

In this paper, we are interested in a diffusive interface model, which describes the motion of a mixture of two viscous incompressible fluids with different densities. This thermodynamically and mechanically consistent model has many interesting features, thus representing an important development in fluid mechanics. In fact, this model describes two-phase mixture of fluids undergoing phase transitions, where sharp interfaces are replaced by narrow transition layers. The latter feature has the advantage to deal with interfaces that merge, reconnect, and hit conditions. This is in contrast to sharp interface models which usually fail in these situations. A phase field variable χ is introduced and a mixing energy is defined in terms of χ and its spatial gradient. The model consists of Navier–Stokes equations governing the fluid velocity coupled with a convective Allen–Cahn equation for the change of the concentration caused by diffusion. It is evident that, the change of the concentration is effected by the velocity of the fluids, and the velocity of the fluids is also related with the concentration because of the surface tension. Actually, the phase field variable χ defined by concentration difference can also be assumed to satisfy different variants of Cahn–Hilliard or other types of dynamics, see [9,17,33].

In this paper, we investigate the following coupled Navier–Stokes/Allen–Cahn system for viscous incompressible fluids with different densities

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla p = \mathrm{div}(2\eta(\chi) Du) - \delta \mathrm{div}(\nabla \chi \otimes \nabla \chi), \\ \mathrm{div} u = 0, \\ (\rho \chi)_t + \mathrm{div}(\rho u \chi) = -\mu, \\ \rho \mu = -\delta \Delta \chi + \rho \dfrac{\partial f}{\partial \chi} \end{cases} \tag{1.1}$$

for $(x, t) \in \Omega \times (0, +\infty)$, where Ω is a bounded domain in $\mathbb{R}^N (N = 2, 3)$ with smooth boundary $\partial\Omega$, $\rho \geq 0$ is the total density, u denotes the mean velocity of the fluid mixture, $Du = \frac{1}{2}(\nabla u + \nabla u^T)$, p is the pressure, χ represents the concentration difference of the two fluids, μ is the chemical potential, $\eta(\chi) > 0$

is the viscosity of the mixture, the free energy density satisfies double-well structure $f(\chi) = \frac{1}{\delta}(\frac{\chi^4}{4} - \frac{\chi^2}{2})$, positive constant δ denotes the width of the interface. The usual Kronecker product is denoted by $\otimes$, i.e., $(a \otimes b)_{ij} = a_i b_j$ for $a, b \in \mathbb{R}^N$. The equations $(1.1)_{1-3}$ are nonhomogeneous incompressible Navier–Stokes equations, which have an extra term $\nabla\chi \otimes \nabla\chi$ describing capillary effect related to the free energy

$$F(\rho, \chi) = \int_\Omega \left(\rho f(\chi) + \frac{\delta}{2}|\nabla\chi|^2 \right) \mathrm{d}x.$$

The system (1.1) is a highly nonlinear system coupling hyperbolic equations with parabolic equations. Here, we point out some special cases of this coupled system:

(i) When the densities of the two fluids are the same or at least very close ("matched densities"), the total density ρ is assumed to be constant, then (1.1) reduces to an incompressible Navier–Stokes/Allen–Cahn system. From another point of view, it is also closely related to liquid crystal model, Magnetohydrodynamics (MHD) equations, and viscoelastic system with infinite Weissenberg number, see [40].

(ii) When χ is a constant, the system (1.1) becomes a nonhomogeneous incompressible Navier–Stokes equations. It has been paid many attentions, see Antontsev and Kazhikov [8], Kazhikov [25], Simon [35], Lions [28], Choe and Kim [13], and the references therein.

(iii) When ρ and χ are constants, the system (1.1) reduces to classical incompressible Navier–Stokes equation, which is the fundamental equation to describe Newtonian fluids. It has attracted great interests, see Lions [28] and Feireisl [37] for survey of important developments.

(iv) When ρ is a constant and $u = 0$, the system (1.1) turns out to be the Allen–Cahn equation, which was originally introduced by Allen and Cahn [4] to describe the motion of antiphase boundaries in crystalline solids. This type of equation has been extensively studied, see [14,22,36] for example.

The diffuse interface models for two-phase flow of incompressible viscous fluids with "matched densities" have been extensively studied. For incompressible Navier–Stokes/Allen–Cahn system, Xu et al. [40] discussed the axisymmetric solutions in 3D. They prove the global regularity of the constructed solutions in both large viscosity and small initial data cases. Zhao et al. [42] investigated the vanishing viscosity limit. They proved that the solutions of the Navier–Stokes/Allen–Cahn system converge to that of the Euler/Allen–Cahn system in a proper small time interval. Gal and Grasselli [20] showed the existence of the trajectory attractor. For another type of diffuse interface model—Navier–Stokes/Cahn–Hilliard system, Boyer [10] studied the existence of global weak and strong solutions in 2D, the existence of unique strong solution in 3D and the stability of the stationary solutions. For the studies on well-posedness, asymptotic behavior, attractor, etc, see [1,20,21] and the references cited therein. Moreover, for numerical simulations, such as jet pinching-off and drop formation, we refer the readers to [11,29,38,41].

It is evident that, the densities in two fluids are often quite different. Therefore, the investigations on the phase field models for two-phase flow with non-matched densities are significant. To our knowledge, there are only a few theoretical results available to compressible models. For compressible Navier–Stokes/Allen–Cahn system, Feireisl et al. [18] proved the existence of weak solutions in 3D. Kotschote [26] got the existence of unique local strong solutions. In [16], we obtained the global well-posedness in 1D with constant mobility. We prove the existence of the initial boundary value problem in various regularity classes, as well as uniqueness for strong solutions. For compressible Navier–Stokes/Cahn–Hilliard system, Abels and Feireisl [6] derived the existence of weak solutions. Kotschote and Zacher [27] established the local existence of unique strong solutions. There are also some investigations on incompressible fluids with different densities under the assumption of $\rho = \rho(\chi)$. Helmut Abels has done a series of researches on it. In [5], he studied the existence of weak solutions of a modified Navier–Stokes/Cahn–Hilliard system which had been obtained by himself in [7]. A coupled system of a nonhomogeneous generalized Navier–Stokes system and a Cahn–Hilliard equation was considered in [3], and he proved the existence of weak solutions by using L^∞-truncation method and Galerkin approximation. For related studies of Helmut Abels, one can find in [2,3,5,7] and the references therein. In 2015, Chun Liu [32] deduced another type of Navier–Stokes/Cahn–Hilliard equations by the energetic variational approaches, and carried out numerical experiments to validate the model and the schemes for problems with large density and viscosity ratios. A similar Navier–Stokes/Allen–Cahn system has been derived by Jie Jiang et al. [23], they got the existence of weak solutions in 3D as well as the well-posedness of strong solutions in 2D, and then investigated the longtime behavior of the 2D strong solutions.

In this paper, we investigate the Navier–Stokes/Allen–Cahn system for two fluids with non-matched densities, $\rho = \rho(x, t)$ is an unknown function, but the velocity u satisfies the divergence-free condition $\mathrm{div} u = 0$, i.e., the fluids are incompressible and with different densities. As far as we know, there are not any results about this type of model until now. This paper concerned with the existence and uniqueness of local strong solutions. The main mechanism for possible breakdown of such a local strong solution has been investigated in our another work [30].

We supplement the system (1.1) with the following initial conditions

$$(\rho, u, \chi)\Big|_{t=0} = (\rho_0, u_0, \chi_0), \quad x \in \Omega, \tag{1.2}$$

the usual no-slip boundary condition on the velocity and Neumann boundary condition on the phase field variable

$$\Big(u, \frac{\partial \chi}{\partial \mathbf{n}}\Big)\Big|_{\partial\Omega} = (0, 0), \quad t \geq 0, \tag{1.3}$$

where $\mathbf{n}$ is the unit outward normal vector of $\partial\Omega$.

Notations For $p \geq 1$, denote $L^p = L^p(\Omega)$ as the L^p space with the norm $\|\cdot\|_{L^p}$. For $k \geq 1$ and $p \geq 1$, denote $W^{k,p} = W^{k,p}(\Omega)$ for a Sobolev space, whose norm is denoted by $\|\cdot\|_{W^{k,p}}$, and specially $H^k = W^{k,2}(\Omega)$.

Definition 1.1. For $T > 0$, $(\rho,\, u,\, p,\, \chi,\, \mu)$ is called a strong solution of the coupled Navier–Stokes/Allen–Cahn system (1.1) in $\Omega \times (0, T]$, if

$$\begin{aligned}
&\rho \in L^\infty(0, T; W^{2,6}), \quad \rho_t \in L^\infty(0, T; W^{1,6}), \quad 0 < c^{-1} \leq \rho \leq c,\\
&u \in L^\infty\big(0, T; H^2 \cap H_0^1\big) \cap L^2(0, T; W^{2,6}), \quad u_t \in L^\infty(0, T; L^2) \cap L^2(0, T; H_0^1),\\
&p \in L^\infty(0, T; H^1) \cap L^2(0, T; W^{1,6}),\\
&\chi \in L^\infty(0, T; H^3) \cap L^2(0, T; H^4),\ \chi_t \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2),\ \chi_{tt} \in L^2(0, T; L^2),\\
&\mu \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2), \quad \mu_t \in L^2(0, T; L^2),
\end{aligned}$$

and $(\rho,\, u,\, p,\, \chi,\, \mu)$ satisfies (1.1) a.e. in $\Omega \times (0, T]$.

Our main result is the existence and uniqueness of local strong solutions.

Theorem 1.1. *Assume that $\rho_0 \in W^{2,6}(\Omega)$ satisfies $0 < c_0^{-1} \leq \rho_0 \leq c_0$ for some constant c_0, $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $\chi_0 \in H^3(\Omega)$ and $u_0|_{\partial\Omega} = 0$, $\mathrm{div} u_0 = 0$ for $x \in \Omega$. Then there exist a time $T_* > 0$, a constant $c = c(c_0, T_*)$ and a unique strong solution (ρ, u, p, χ, μ) of the problem (1.1)–(1.3) in $\Omega \times (0, T_*]$.*

Remark. (i) To our knowledge, there are only a few theoretical investigations on Navier–Stokes/Allen–Cahn system for two-phase flow with different densities. For compressible fluids, the first result addressing solvability is due to Able and Feireisl [6], in which the authors proved the existence of global weak solutions in 3D, but not uniqueness. In another paper [16], we proved global well-posedness in 1D with constant viscosity coefficients. This paper is concerned with incompressible fluids with different densities, and the viscosity coefficient depends on phase variable χ, which is of interest from the physical point of view.
(ii) Noticing that if the density ρ do not appear in Allen–Cahn equation $(1.1)_{4,5}$ and the viscous coefficient is a constant, the system (1.1) reduces to the Ginzburg–Landau approximation model of nonhomogeneous incompressible nematic liquid crystals. Global existence of weak solutions to this type of model has been proved in [24,34]. Wen and Ding [39] establish the global existence and uniqueness of solution with small initial data to original nonhomogeneous incompressible nematic liquid crystals in 2D. The system

(1.1) becomes nontrivial because of the appearance of the density ρ. Within our knowledge, there are no results for this type of model even for local existence or global solutions with small initial data in 2D. The existence of global solutions is closely related to the estimate for $\|\nabla\rho\|_{L^\infty(Q_T)}$, which is the main difficulty we can not handle. But for the results about blow-up criterion, we refer the readers to our another work [30].

Since the constant δ plays no role in the analysis, we assume henceforth that $\delta = 1$. Throughout this paper, we assume that $\eta(s) \in C^1(\mathbb{R})$ and there exist positive constants $\underline{\eta}$, $\overline{\eta}$ and $\tilde{\eta}$, such that

$$0 < \underline{\eta} \le \eta(s) \le \overline{\eta}, \quad |\eta^{'}(s)| \le \tilde{\eta}. \tag{1.4}$$

In the next section, we prove the local existence of unique strong solution by using the technique of iteration. Firstly, we introduce an auxiliary problem for nonhomogeneous incompressible Navier–Stokes equations. The proof of the existence and uniqueness of solution to this auxiliary problem is similar to that in [39]. If the initial density is more regular, the density is also regular too, see Lemma 2.2. This is an important character of the density. Based on these results and classical theory, we construct the approximate solutions and begin to do iterate. At last, in terms of the estimates for the approximate solutions, we derive the desired local strong solution by taking limits.

2. Proof of our main result

Let T be a fixed time with $0 < T < 1$. Denote

$$V_{T,K_1} = \left\{ v \,\middle|\, v(x,0) = u_0(x),\ v\big|_{\partial\Omega} = 0,\ \|v\|_V \le K_1 \right\},$$

$$\Phi_{T,K_2} = \left\{ \varphi \,\middle|\, \varphi(x,0) = \chi_0(x),\ \frac{\partial\varphi}{\partial\mathbf{n}}\Big|_{\partial\Omega} = 0,\ \|\varphi\|_\Phi \le K_2 \right\},$$

where

$$\|v\|_V = \|v_t\|_{L^\infty(0,T;L^2)} + \|v_t\|_{L^2(0,T;H^1)} + \|v\|_{L^\infty(0,T;H^2)} + \|v\|_{L^2(0,T;W^{2,6})},$$

$$\|\varphi\|_\Phi = \|\varphi_t\|_{L^\infty(0,T;H^1)} + \|\varphi_t\|_{L^2(0,T;H^2)} + \|\varphi\|_{L^\infty(0,T;H^3)} + \|\varphi\|_{L^2(0,T;H^4)},$$

and the constants $K_1, K_2 > 1$ will be determined later.

Firstly, before proving local existence of strong solutions, we consider the following auxiliary problem with $(v,\varphi) \in V_{T,K_1} \times \Phi_{T,K_2}$

$$\begin{cases} \rho_t + (v\cdot\nabla)\rho = 0, \\ \rho u_t + \nabla p = \operatorname{div}(2\eta(\varphi)Du) + \rho f_1 + f_2, \\ \operatorname{div} u = 0 \end{cases} \tag{2.1}$$

subject to the initial and boundary conditions

$$(\rho,u)\big|_{t=0} = (\rho_0,u_0), \quad x\in\Omega, \quad u\big|_{\partial\Omega} = 0, \quad t\ge 0. \tag{2.2}$$

There exists a unique strong solution to the problem (2.1)–(2.2).

Lemma 2.1. *Assume that $\rho_0 \in H^1(\Omega)\cap L^\infty(\Omega)$, $\rho_0 \ge 0$, $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$, $f_i \in L^2(0,T;H^1(\Omega))$, $f_{it} \in L^2(Q_T)$ $(i = 1,2)$, and the following compatible conditions are valid*

$$\operatorname{div}(2\eta(\chi_0)Du_0) - \nabla p_0(x) + f_2(x,0) = \rho_0^{1/2} g(x) \quad \text{and} \quad \operatorname{div} u_0(x) = 0, \quad \text{in } \Omega,$$

for $(p_0,g) \in H^1(\Omega)\times L^2(\Omega)$. Then for any $T > 0$, the problem (2.1)–(2.2) admits a unique solution (ρ,u,p) such that

$$\rho \in L^\infty(0,T;H^1)\cap L^\infty(Q_T), \quad \rho_t \in L^\infty(0,T;L^2),$$

$$u \in L^\infty(0,T;H^2\cap H_0^1)\cap L^2(0,T;W^{2,6}),$$

$$u_t \in L^2(0,T;H_0^1), \quad \sqrt{\rho}u_t \in L^\infty(0,T;L^2),$$
$$p \in L^\infty(0,T;H^1) \cap L^2(0,T;W^{1,6}).$$

When $\mathrm{div}(2\eta(\varphi)Du)$ is substituted by Δu in (2.1), this result has been obtained by Wen and Ding [39]. The treatment of the coefficient $\eta(\varphi)$ is similar to the proof of step 3 in this section, so we omit the proof of this lemma here.

Lemma 2.2. *In addition to the conditions in Lemma 2.1, if $\rho_0 \in W^{2,6}(\Omega)$, $0 < c_0^{-1} \le \rho \le c_0$ for some constant c_0, then we also have*

$$\rho \in L^\infty(0,T;W^{2,6}), \quad \rho_t \in L^\infty(0,T;W^{1,6}), \quad 0 < c^{-1} \le \rho \le c.$$

Proof. The existence and uniqueness of strong solutions to the hyperbolic equation $(2.1)_1$ is well known. Moreover, from [13], the solution satisfies the following estimates

$$0 < c^{-1} \le \rho \le c, \quad \text{in } Q_T, \tag{2.3}$$
$$\sup_{0\le t\le T}\left(\|\rho\|_{H^1} + K_1^{-1}\|\rho_t\|_{L^2}\right) \le c\exp\{cK_1T^{1/2}\}. \tag{2.4}$$

Differentiating $(2.1)_1$ with respect to x, multiplying by $r|\nabla\rho|^{r-2}\nabla\rho$ $(1 \le r < +\infty)$ and integrating the result with respect to x over Ω, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\rho|^r\mathrm{d}x &= -\int_\Omega (v\cdot\nabla)(|\nabla\rho|^r)\mathrm{d}x - r\int_\Omega |\nabla\rho|^{r-2}\nabla\rho\cdot\nabla(v\cdot\nabla)\rho\,\mathrm{d}x \\ &= \int_\Omega \mathrm{div}v|\nabla\rho|^r\mathrm{d}x - r\int_\Omega |\nabla\rho|^{r-2}\nabla\rho\cdot\nabla(v\cdot\nabla)\rho\,\mathrm{d}x \\ &\le (1+r)\|\nabla v\|_{L^\infty}\int_\Omega |\nabla\rho|^r\mathrm{d}x.\end{aligned}$$

From which we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_{L^r} \le \frac{r+1}{r}\|\nabla v\|_{L^\infty}\|\nabla\rho\|_{L^r}.$$

Then Gronwall's inequality implies

$$\|\nabla\rho\|_{L^r} \le \|\nabla\rho_0\|_{L^r}\exp\left\{\frac{r+1}{r}\int_0^T \|\nabla v\|_{W^{1,6}}\mathrm{d}t\right\}.$$

Sending $r \to +\infty$, recalling $\rho_0 \in W^{2,6}(\Omega)$ and using Hölder's inequality yield

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^\infty} \le c\exp\{cK_1T^{1/2}\}. \tag{2.5}$$

Differentiating $(2.1)_1$ with respect to x twice and multiplying the above equation by $l|\nabla^2\rho|^{l-2}\nabla^2\rho$ $(2 \le l \le 6)$, and integrating the result over Ω, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla^2\rho|^l\mathrm{d}x &= -l\int_\Omega |\nabla^2\rho|^{l-2}\nabla^2\rho : \nabla^2(v\cdot\nabla)\rho\mathrm{d}x + \int_\Omega \mathrm{div}v|\nabla^2\rho|^l\mathrm{d}x \\ &\quad - 2l\int_\Omega |\nabla^2\rho|^{l-2}\nabla^2\rho : \nabla(v\cdot\nabla)\nabla\rho\mathrm{d}x \\ &\le c\|\nabla^2 v\|_{L^6}\|\nabla\rho\|_{L^\infty}\left(\int_\Omega |\nabla^2\rho|^{\frac{6}{5}(l-1)}\mathrm{d}x\right)^{5/6} + c\|\nabla v\|_{L^\infty}\int_\Omega |\nabla^2\rho|^l\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\leq c\|\nabla^2 v\|_{L^6}\|\nabla\rho\|_{L^\infty}\left(\int_\Omega |\nabla^2\rho|^l \mathrm{d}x\right)^{(l-1)/l} + c\|\nabla v\|_{L^\infty}\int_\Omega |\nabla^2\rho|^l \mathrm{d}x \\
&\leq c\|v\|_{W^{2,6}}(\|\nabla\rho\|_{L^\infty}+1)\int_\Omega |\nabla^2\rho|^l \mathrm{d}x + c\|v\|_{W^{2,6}}\|\nabla\rho\|_{L^\infty},
\end{aligned}
$$

where we have used Hölder's inequality for $2 \leq l \leq 6$ in the third step. Applying Gronwall's inequality, we obtain

$$
\int_\Omega |\nabla^2\rho|^l \mathrm{d}x \leq c\left(1+K_1T^{1/2}\exp\{cK_1T^{1/2}\}\right)\exp\left\{cK_1T^{1/2}(\exp\{cK_1T^{1/2}\}+1)\right\}. \tag{2.6}
$$

Furthermore, differentiating $(2.1)_1$ with respect to x, we derive that

$$
\begin{aligned}
\|\nabla\rho_t\|_{L^l} &\leq c\|\nabla v\|_{L^l}\|\nabla\rho\|_{L^\infty} + c\|v\|_{L^\infty}\|\nabla^2\rho\|_{L^l} \leq c\|v\|_{H^2}(\|\nabla\rho\|_{L^\infty}+\|\nabla^2\rho\|_{L^l}) \\
&\leq cK_1\left(1+K_1T^{1/2}\exp\{cK_1T^{1/2}\}\right)\exp\left\{cK_1T^{1/2}(\exp\{cK_1T^{1/2}\}+1)\right\}.
\end{aligned} \tag{2.7}
$$

Then Lemma 2.2 follows from (2.3), (2.4), (2.6) and (2.7). ☐

Next, we consider the following linearized problem

$$
\begin{cases}
\rho_t + (v\cdot\nabla)\rho = 0, \\
\rho u_t + \nabla p = \mathrm{div}(2\eta(\varphi)Du) - \rho(v\cdot\nabla)v - \mathrm{div}(\nabla\chi\otimes\nabla\chi), \\
\mathrm{div} u = 0, \\
\chi_t = \dfrac{1}{\rho^2}\Delta\chi - (v\cdot\nabla)\varphi - \dfrac{1}{\rho}(\varphi^3-\varphi)
\end{cases} \tag{2.8}
$$

with the initial boundary conditions (1.2) and (1.3), where $(v,\varphi)\in V_{T,K_1}\times\Phi_{T,K_2}$. Recalling Lemma 2.2 and the definition of V_{T,K_1}, Φ_{T,K_2}, we have $-(v\cdot\nabla)\varphi - \frac{1}{\rho}(\varphi^3-\varphi)\in W_2^{2,1}(Q_T)$. It follows from classical arguments [31] that Eq. $(2.8)_4$ subject to the corresponding initial boundary value conditions admits a unique solution such that

$$
\chi\in W_2^{4,2}(Q_T)\cap L^\infty(0,T;H^3), \quad \chi_t\in L^\infty(0,T;H^1).
$$

Moreover, by Lemma 2.1, the problem $(2.8)_{1-3}$ with the corresponding initial boundary value conditions has a unique solution (ρ,u,p) and the regularities like that in Lemma 2.1–2.2.

Therefore, we have a solution (ρ^1,u^1,p^1,χ^1) of the problem (2.8) with (v,φ) replaced by (u^0,χ^0), where $(u^0,\chi^0)\in V_{T,K_1}\times\Phi_{T,K_2}$. Suppose $(u^{k-1},\chi^{k-1})\in V_{T,K_1}\times\Phi_{T,K_2}$ for $k\geq 1$, then we can construct an approximate solution (ρ^k,u^k,p^k,χ^k) satisfying the following problem

$$
\begin{cases}
\rho_t^k + (u^{k-1}\cdot\nabla)\rho^k = 0, \\
\rho^k u_t^k + \nabla p^k = \mathrm{div}(2\eta(\chi^{k-1})Du^k) - \rho^k(u^{k-1}\cdot\nabla)u^{k-1} - \mathrm{div}(\nabla\chi^k\otimes\nabla\chi^k), \\
\mathrm{div} u^k = 0, \\
\chi_t^k = \dfrac{1}{(\rho^k)^2}\triangle\chi^k - (u^{k-1}\cdot\nabla)\chi^{k-1} - \dfrac{1}{\rho^k}(\chi^{k-1})^3 + \dfrac{1}{\rho^k}\chi^{k-1}
\end{cases} \tag{2.9}
$$

supplemented with initial and boundary conditions

$$
\begin{aligned}
&(\rho^k,u^k,\chi^k)\big|_{t=0} = (\rho_0,u_0,\chi_0), \quad x\in\Omega, \\
&\left(u^k,\frac{\partial\chi^k}{\partial\mathbf{n}}\right)\Big|_{\partial\Omega} = (0,0), \quad t\geq 0.
\end{aligned}
$$

In what follows, we prove Theorem 1.1 by iteration. Throughout this paper, we denote by $A \lesssim B$ if there exists a constant C such that $A \le CB$. Moreover, in step 1-3, we denote by C a constant whose value may be different from line to line but independent of K_1 and K_2.

Step 1 It holds that

$$0 < C^{-1} \le \rho^k \le C, \tag{2.10}$$

$$\|\rho^k\|_{L^\infty(0,T;W^{2,6})} \le C, \tag{2.11}$$

$$\|\rho_t^k\|_{L^\infty(0,T;W^{1,6})} \le CK_1. \tag{2.12}$$

Taking $0 < T < T_1 := \dfrac{1}{K_1^2}$, then the estimates (2.10)–(2.12) are the direct deductions of (2.3), (2.4), (2.6) and (2.7).

Step 2 We will prove

$$\|\chi^k\|_\Phi = \|\chi_t^k\|_{L^\infty(0,T;H^1)} + \|\chi_t^k\|_{L^2(0,T;H^2)} + \|\chi^k\|_{L^\infty(0,T;H^3)} + \|\chi^k\|_{L^2(0,T;H^4)} \le K_2. \tag{2.13}$$

We rewrite Eq. $(2.9)_4$ as follows:

$$(\rho^k)^2\chi_t^k - \Delta\chi^k = -(\rho^k)^2(u^{k-1}\cdot\nabla)\chi^{k-1} - \rho^k(\chi^{k-1})^3 + \rho^k\chi^{k-1}. \tag{2.14}$$

Differentiating (2.14) with respect to t, multiplying the result by χ_t^k, then integrating over Ω yield

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\rho^k\chi_t^k|^2\mathrm{d}x + \int_\Omega |\nabla\chi_t^k|^2\mathrm{d}x \\
&= -\int_\Omega \rho^k\rho_t^k|\chi_t^k|^2\mathrm{d}x - \int_\Omega 2\rho^k\rho_t^k(u^{k-1}\cdot\nabla)\chi^{k-1}\chi_t^k\mathrm{d}x - \int_\Omega (\rho^k)^2(u_t^{k-1}\cdot\nabla)\chi^{k-1}\chi_t^k\mathrm{d}x \\
&\quad - \int_\Omega (\rho^k)^2(u^{k-1}\cdot\nabla)\chi_t^{k-1}\chi_t^k\mathrm{d}x - \int_\Omega \rho_t^k(\chi^{k-1})^3\chi_t^k\mathrm{d}x - \int_\Omega 3\rho^k(\chi^{k-1})^2\chi_t^{k-1}\chi_t^k\mathrm{d}x \\
&\quad + \int_\Omega \rho_t^k\chi^{k-1}\chi_t^k\mathrm{d}x + \int_\Omega \rho^k\chi_t^{k-1}\chi_t^k\mathrm{d}x \\
&\lesssim \|\rho_t^k\|_{L^\infty}\|\rho^k\chi_t^k\|_{L^2}^2 + \|\rho_t^k\|_{L^\infty}\|u^{k-1}\|_{L^6}\|\nabla\chi^{k-1}\|_{L^3}\|\rho^k\chi_t^k\|_{L^2} + \|u_t^{k-1}\|_{L^2}\|\nabla\chi^{k-1}\|_{L^\infty}\|\rho^k\chi_t^k\|_{L^2} \\
&\quad + \|u^{k-1}\|_{L^\infty}\|\nabla\chi_t^{k-1}\|_{L^2}\|\rho^k\chi_t^k\|_{L^2} + \|\rho_t^k\|_{L^\infty}\|\chi^{k-1}\|_{L^6}^3\|\rho^k\chi_t^k\|_{L^2} \\
&\quad + \|\chi^{k-1}\|_{L^\infty}^2\|\chi_t^{k-1}\|_{L^2}\|\rho^k\chi_t^k\|_{L^2} + \|\rho_t^k\|_{L^\infty}\|\chi_t^{k-1}\|_{L^2}\|\rho^k\chi_t^k\|_{L^2} + \|\chi_t^{k-1}\|_{L^2}\|\rho^k\chi_t^k\|_{L^2} \\
&\lesssim (\|\rho_t^k\|_{W^{1,6}} + 1)\int_\Omega |\rho^k\chi_t^k|^2\mathrm{d}x + \|\rho_t^k\|_{W^{1,6}}^2\|u^{k-1}\|_{H^1}^2\|\nabla\chi^{k-1}\|_{H^1}^2 + \|u_t^{k-1}\|_{L^2}^2\|\nabla\chi^{k-1}\|_{H^2}^2 \\
&\quad + \|u^{k-1}\|_{H^2}^2\|\nabla\chi_t^{k-1}\|_{L^2}^2 + \|\rho_t^k\|_{W^{1,6}}^2\|\chi^{k-1}\|_{H^1}^6 + \|\chi^{k-1}\|_{H^2}^4\|\chi_t^{k-1}\|_{L^2}^2 \\
&\quad + \|\rho_t^k\|_{W^{1,6}}^2\|\chi_t^{k-1}\|_{L^2}^2 + \|\chi_t^{k-1}\|_{L^2}^2 \\
&\lesssim (K_1+1)\int_\Omega |\rho^k\chi_t^k|^2\mathrm{d}x + K_1^4K_2^2 + K_1^2K_2^2 + K_1^2K_2^6 + K_2^6 + K_2^2.
\end{aligned}$$

Taking $T_2 := \min\left\{T_1, \dfrac{1}{K_1^4K_2^2 + K_1^2K_2^6}\right\}$ and using Gronwall's inequality, for any $0 < T < T_2$, we have

$$\sup_{0\le t\le T}\int_\Omega |\rho^k\chi_t^k|^2\mathrm{d}x + \int_0^T\int_\Omega |\nabla\chi_t^k|^2\mathrm{d}x\mathrm{d}t \le C. \tag{2.15}$$

Differentiating (2.9)$_4$ with respect to t, multiplying by $\Delta\chi_t^k$ and differentiating the result over Ω, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega}|\nabla\chi_t^k|^2\mathrm{d}x+\int\limits_{\Omega}\frac{1}{(\rho^k)^2}|\Delta\chi_t^k|^2\mathrm{d}x\\
&\quad=\int\limits_{\Omega}\frac{2\rho_t^k}{(\rho^k)^3}\triangle\chi^k\Delta\chi_t^k\mathrm{d}x+\int\limits_{\Omega}(u_t^{k-1}\cdot\nabla)\chi^{k-1}\Delta\chi_t^k\mathrm{d}x+\int\limits_{\Omega}(u^{k-1}\cdot\nabla)\chi_t^{k-1}\Delta\chi_t^k\mathrm{d}x\\
&\qquad-\int\limits_{\Omega}\frac{\rho_t^k}{(\rho^k)^2}(\chi^{k-1})^3\Delta\chi_t^k\mathrm{d}x+\int\limits_{\Omega}\frac{3}{\rho^k}(\chi^{k-1})^2\chi_t^{k-1}\Delta\chi_t^k\mathrm{d}x\\
&\qquad+\int\limits_{\Omega}\frac{\rho_t^k}{(\rho^k)^2}\chi^{k-1}\Delta\chi_t^k\mathrm{d}x-\int\limits_{\Omega}\frac{1}{\rho^k}\chi_t^{k-1}\Delta\chi_t^k\mathrm{d}x\\
&\quad\lesssim\|\rho_t^k\|_{L^\infty}\|\Delta\chi^k\|_{L^2}\|\Delta\chi_t^k\|_{L^2}+\|u_t^{k-1}\|_{L^2}\|\nabla\chi^{k-1}\|_{L^\infty}\|\Delta\chi_t^k\|_{L^2}\\
&\qquad+\|u^{k-1}\|_{L^\infty}\|\nabla\chi_t^{k-1}\|_{L^2}\|\Delta\chi_t^k\|_{L^2}+\|\rho_t^k\|_{L^\infty}\|\chi^{k-1}\|_{L^6}^3\|\Delta\chi_t^k\|_{L^2}\\
&\qquad+\|\chi^{k-1}\|_{L^\infty}^2\|\chi_t^{k-1}\|_{L^2}\|\Delta\chi_t^k\|_{L^2}+\|\rho_t^k\|_{L^\infty}\|\chi^{k-1}\|_{L^2}\|\Delta\chi_t^k\|_{L^2}+\|\chi_t^{k-1}\|_{L^2}\|\Delta\chi_t^k\|_{L^2}\\
&\quad\lesssim\frac{1}{2}\int\limits_{\Omega}\frac{1}{(\rho^k)^2}|\Delta\chi_t^k|^2\mathrm{d}x+\|\rho_t^k\|_{W^{1,6}}^2\|\Delta\chi^k\|_{L^2}^2+\|u_t^{k-1}\|_{L^2}^2\|\nabla\chi^{k-1}\|_{H^2}^2+\|u^{k-1}\|_{H^2}^2\|\nabla\chi_t^{k-1}\|_{L^2}^2\\
&\qquad+\|\rho_t^k\|_{W^{1,6}}^2\|\chi^{k-1}\|_{H^1}^6+\|\chi^{k-1}\|_{H^2}^4\|\chi_t^{k-1}\|_{L^2}^2+\|\rho_t^k\|_{W^{1,6}}^2\|\chi^{k-1}\|_{L^2}^2+\|\chi_t^{k-1}\|_{L^2}^2.
\end{aligned}$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega}|\nabla\chi_t^k|^2\mathrm{d}x+\int\limits_{\Omega}\frac{1}{(\rho^k)^2}|\Delta\chi_t^k|^2\mathrm{d}x\lesssim K_1^2\|\Delta\chi^k\|_{L^2}^2+K_1^2K_2^6+K_2^6+K_1^2K_2^2+K_2^2. \tag{2.16}$$

From Eq. (2.14) and the estimate (2.15), we see that

$$\begin{aligned}
\|\Delta\chi^k\|_{L^2}^2&\lesssim\|\chi_t^k\|_{L^2}^2+\|u^{k-1}\|_{L^6}^2\|\nabla\chi^{k-1}\|_{L^3}^2+\|\chi^{k-1}\|_{L^6}^6+\|\chi^{k-1}\|_{L^2}^2\\
&\lesssim\|\chi_t^k\|_{L^2}^2+\|u^{k-1}\|_{H^1}^2\|\nabla\chi^{k-1}\|_{H^1}^2+\|\chi^{k-1}\|_{H^1}^6+\|\chi^{k-1}\|_{L^2}^2\\
&\lesssim 1+K_1^2K_2^2+K_2^6+K_2^2.
\end{aligned}\tag{2.17}$$

Substituting (2.17) into (2.16) and integrating the result over $(0,T)$, for any $0<T<T_2$, we obtain

$$\|\chi_t^k\|_{L^\infty(0,T;H^1)}\le C. \tag{2.18}$$

By (2.15)–(2.18) and the elliptic estimates for Neumann boundary value problem

$$\|\nabla^2\chi_t^k\|_{L^2}\lesssim\|\Delta\chi_t^k\|_{L^2}+\|\nabla\chi_t^k\|_{L^2},$$

for any $0<T<T_2$, we get

$$\|\chi_t^k\|_{L^2(0,T;H^2)}\le C. \tag{2.19}$$

On the other hand, for any $0<T<T_2$, using the elliptic estimates for Eq. (2.14) gives

$$\begin{aligned}
\|\chi^k\|_{H^3}^2&\lesssim\|(\rho^k)^2\chi_t^k\|_{H^1}^2+\|(\rho^k)^2(u^{k-1}\cdot\nabla)\chi^{k-1}\|_{H^1}^2+\|\rho^k(\chi^{k-1})^3\|_{H^1}^2+\|\rho^k\chi^{k-1}\|_{H^1}^2+\|\chi_0\|_{H^3}^2\\
&:=\sum_{i=1}^4 I_i+\|\chi_0\|_{H^3}^2.
\end{aligned}$$

In what follows, we deal with the terms on the right hand side one by one. By (2.18), we get

$$I_1\lesssim\|(\rho^k)^2\chi_t^k\|_{L^2}^2+\|\nabla\left((\rho^k)^2\chi_t^k\right)\|_{L^2}^2\lesssim\|\chi_t^k\|_{L^2}^2+\|\nabla\chi_t^k\|_{L^2}^2+\|\nabla\rho^k\|_{L^\infty}^2\|\chi_t^k\|_{L^2}^2\le C.$$

For I_2, we have

$$I_2 \lesssim \int\limits_\Omega |u^{k-1}|^2|\nabla\chi^{k-1}|^2\mathrm{d}x + \int\limits_\Omega |\nabla u^{k-1}|^2|\nabla\chi^{k-1}|^2\mathrm{d}x + \int\limits_\Omega |u^{k-1}|^2|\nabla^2\chi^{k-1}|^2\mathrm{d}x \\ + \|\nabla\rho^k\|_{L^\infty}^2 \int\limits_\Omega |u^{k-1}|^2|\nabla\chi^{k-1}|^2\mathrm{d}x := \sum_{i=1}^4 J_i,$$

where

$$\begin{aligned} J_1 + J_4 &\lesssim \int\limits_\Omega |u^{k-1}|^2|\nabla\chi^{k-1}|^2\mathrm{d}x \lesssim \int\limits_\Omega |u^{k-1}-u_0|^2|\nabla\chi^{k-1}|^2\mathrm{d}x + \int\limits_\Omega |\nabla\chi^{k-1}|^2\mathrm{d}x \\ &\lesssim \|\nabla\chi^{k-1}\|_{L^\infty}^2 \int\limits_\Omega \left|\int\limits_0^t u_t^{k-1}(x,s)\mathrm{d}s\right|^2 \mathrm{d}x + \int\limits_\Omega |\nabla(\chi^{k-1}-\chi_0)|^2\mathrm{d}x + 1 \\ &\lesssim K_2^2 T \int\limits_0^T\int\limits_\Omega |u_t^{k-1}|^2\mathrm{d}x\mathrm{d}t + K_2^{-2}\int\limits_\Omega |\nabla^2(\chi^{k-1}-\chi_0)|^2\mathrm{d}x \\ &\quad + K_2^2\int\limits_\Omega |\chi^{k-1}-\chi_0|^2\mathrm{d}x + 1 \\ &\lesssim K_1^2K_2^2T^2 + K_2^2\int\limits_\Omega \left|\int\limits_0^t \chi_t^{k-1}(x,s)\mathrm{d}s\right|^2 \mathrm{d}x + 1 \\ &\lesssim K_1^2K_2^2T^2 + K_2^2 T\int\limits_0^T\int\limits_\Omega |\chi_t^{k-1}|^2\mathrm{d}x\mathrm{d}t + 1 \lesssim K_1^2K_2^2T^2 + K_2^2T^2 + 1, \end{aligned}$$

$$\begin{aligned} J_2 &= \int\limits_\Omega |\nabla u^{k-1}|^2|\nabla\chi^{k-1}|^2\mathrm{d}x \\ &\lesssim K_2^2\int\limits_\Omega |\nabla(u^{k-1}-u_0)|^2\mathrm{d}x + \int\limits_\Omega |\nabla(\chi^{k-1}-\chi_0)|^2\mathrm{d}x + 1 \\ &\lesssim K_1^{-2}\int\limits_\Omega |\nabla^2(u^{k-1}-u_0)|^2\mathrm{d}x + K_1^2K_2^2\int\limits_\Omega |u^{k-1}-u_0|^2\mathrm{d}x + K_2^4T^2 + 1 \\ &\lesssim K_1^2K_2^2T\int\limits_0^T\int\limits_\Omega |u_t^{k-1}|^2\mathrm{d}x\mathrm{d}t + K_2^4T^2 + 1 \lesssim K_1^4K_2^2T^2 + K_2^4T^2 + 1, \end{aligned}$$

and

$$\begin{aligned} J_3 &= \int\limits_\Omega |u^{k-1}|^2|\nabla^2\chi^{k-1}|^2\mathrm{d}x \\ &\lesssim \int\limits_\Omega |u^{k-1}-u_0|^2|\nabla^2\chi^{k-1}|^2\mathrm{d}x + \int\limits_\Omega |\nabla^2(\chi^{k-1}-\chi_0)|^2\mathrm{d}x + 1 \end{aligned}$$

$$\lesssim \|u^{k-1}-u_0\|_{L^6}^2\|\nabla^2\chi^{k-1}\|_{L^3}^2 + K_2^{-2}\int_\Omega |\nabla^3(\chi^{k-1}-\chi_0)|^2\mathrm{d}x$$
$$+ K_2^4\int_\Omega |\chi^{k-1}-\chi_0|^2\mathrm{d}x + 1$$
$$\lesssim K_2^2\int_\Omega |\nabla(u^{k-1}-u_0)|^2\mathrm{d}x + K_2^6T^2 + 1 \lesssim K_1^4K_2^2T^2 + K_2^6T^2 + 1.$$

Similarly, we deduce that

$$\begin{aligned} I_3 &= \|\rho^k(\chi^{k-1})^3\|_{H^1}^2 \lesssim \|\rho^k(\chi^{k-1})^3\|_{L^2}^2 + \|3\rho^k(\chi^{k-1})^2\nabla\chi^{k-1}\|_{L^2}^2 + \|\nabla\rho^k(\chi^{k-1})^3\|_{L^2}^2 \\ &\lesssim \|\chi^{k-1}\|_{L^6}^6 + \|\chi^{k-1}\|_{L^6}^4\|\nabla\chi^{k-1}\|_{L^6}^2 + \|\nabla\rho^k\|_{L^\infty}^2\|\chi^{k-1}\|_{L^6}^6 \\ &\lesssim \|\chi^{k-1}\|_{H^1}^6 + \|\chi^{k-1}\|_{H^1}^4\|\nabla^2\chi^{k-1}\|_{L^2}^2 \\ &\lesssim (K_2^2T^2+K_2^4T^2+1)^3 + K_2^4T^2 + K_2^6T^2 + K_2^8T^2 + K_2^{12}T^2 + 1, \end{aligned}$$

where we have used

$$\begin{aligned} &\|\nabla\chi^{k-1}\|_{L^2}^4\|\nabla^2\chi^{k-1}\|_{L^2}^2 \\ &\quad\lesssim \|\nabla(\chi^{k-1}-\chi_0)\|_{L^2}^2\|\nabla\chi^{k-1}\|_{L^2}^2\|\nabla^2\chi^{k-1}\|_{L^2}^2 \\ &\qquad + \|\nabla(\chi^{k-1}-\chi_0)\|_{L^2}^2\|\nabla^2\chi^{k-1}\|_{L^2}^2 + \|\nabla^2\chi^{k-1}\|_{L^2}^2 \\ &\quad\lesssim K_2^4\|\nabla(\chi^{k-1}-\chi_0)\|_{L^2}^2 + K_2^2\|\nabla(\chi^{k-1}-\chi_0)\|_{L^2}^2 + \|\nabla^2(\chi^{k-1}-\chi_0)\|_{L^2}^2 + 1 \\ &\quad\lesssim K_2^{-2}\|\nabla^2(\chi^{k-1}-\chi_0)\|_{L^2}^2 + K_2^{-2}\|\nabla^3(\chi^{k-1}-\chi_0)\|_{L^2}^2 \\ &\qquad + (K_2^{10}+K_2^6+K_2^4)\|\chi^{k-1}-\chi_0\|_{L^2}^2 + 1 \\ &\quad\lesssim (K_2^{10}+K_2^6+K_2^4)T\int_0^T\int_\Omega |\chi_t^{k-1}|^2\mathrm{d}x\mathrm{d}t + 1 \\ &\quad\lesssim (K_2^{10}+K_2^6+K_2^4)K_2^2T^2 + 1. \end{aligned}$$

At last,

$$\begin{aligned} I_4 &= \|\rho^k\chi^{k-1}\|_{H^1}^2 \lesssim \|\chi^{k-1}\|_{L^2}^2 + \|\nabla\chi^{k-1}\|_{L^2}^2 + \|\nabla\rho^k\|_{L^\infty}^2\|\chi^{k-1}\|_{L^2}^2 \\ &\lesssim \|\chi^{k-1}\|_{H^1}^2 \lesssim K_2^2T^2 + K_2^4T^2 + 1. \end{aligned}$$

Putting all the above estimates together, for any $0<T<T_2$, we obtain

$$\|\chi^k\|_{L^\infty(0,T;H^3)} \le C. \tag{2.20}$$

From Eq. (2.14), we can also derive that

$$\begin{aligned} \|\nabla^4\chi\|_{L^2}^2 &\lesssim \|\nabla^2\left((\rho^k)^2\chi_t^k\right)\|_{L^2}^2 + \|\nabla^2\left((\rho^k)^2(u^{k-1}\cdot\nabla)\chi^{k-1}\right)\|_{L^2}^2 \\ &\quad + \|\nabla^2\left(\rho^k(\chi^{k-1})^3\right)\|_{L^2}^2 + \|\nabla^2\left(\rho^k\chi^{k-1}\right)\|_{L^2}^2 + 1 \\ &\lesssim \|\chi_t^k\|_{H^2}^2 + K_1^2K_2^2 + K_2^6 + K_2^2 + 1. \end{aligned}$$

From which and (2.19), for any $0<T<T_2$, we get

$$\|\chi^k\|_{L^2(0,T;H^4)} \le C. \tag{2.21}$$

Then (2.13) follows from (2.18), (2.19), (2.20) and (2.21) by choosing $K_2 \ge C$.
Step 3 We prove that

$$\|u^k\|_V = \|u_t^k\|_{L^\infty(0,T;L^2)} + \|u_t^k\|_{L^2(0,T;H^1)} + \|u^k\|_{L^\infty(0,T;H^2)} + \|u^k\|_{L^2(0,T;W^{2,6})} \le K_1, \tag{2.22}$$

$$\|p^k\|_{L^\infty(0,T;H^1)} + \|p^k\|_{L^2(0,T;W^{1,6})} \le C. \tag{2.23}$$

Differentiating $(2.9)_2$ with respect to t, multiplying the result by u_t^k, and integrating over Ω, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega \rho^k|u_t^k|^2\mathrm{d}x + \int\limits_\Omega 2\eta(\chi^{k-1})|Du_t^k|^2\mathrm{d}x \\
&\quad= -\frac{1}{2}\int\limits_\Omega \rho_t^k|u_t^k|^2\mathrm{d}x - \int\limits_\Omega 2\eta^{'}(\chi^{k-1})\chi_t^{k-1}Du^k : Du_t^k\mathrm{d}x - \int\limits_\Omega \rho_t^k(u^{k-1}\cdot\nabla)u^{k-1}\cdot u_t^k\mathrm{d}x \\
&\qquad - \int\limits_\Omega \rho^k(u_t^{k-1}\cdot\nabla)u^{k-1}\cdot u_t^k\mathrm{d}x - \int\limits_\Omega \rho^k(u^{k-1}\cdot\nabla)u_t^{k-1}\cdot u_t^k\mathrm{d}x \\
&\qquad - \int\limits_\Omega \mathrm{div}(\nabla\chi_t^k\otimes\nabla\chi^k)\cdot u_t^k\mathrm{d}x - \int\limits_\Omega \mathrm{div}(\nabla\chi^k\otimes\nabla\chi_t^k)\cdot u_t^k\mathrm{d}x \\
&\quad\lesssim \|\rho_t^k\|_{L^\infty}\|u_t^k\|_{L^2}^2 + \|\chi_t^{k-1}\|_{L^6}\|Du^k\|_{L^3}\|Du_t^k\|_{L^2} + \|\rho_t^k\|_{L^\infty}\|u^{k-1}\|_{L^\infty}\|\nabla u^{k-1}\|_{L^2}\|u_t^k\|_{L^2} \\
&\qquad + \|u_t^{k-1}\|_{L^6}\|\nabla u^{k-1}\|_{L^3}\|u_t^k\|_{L^2} + \|u^{k-1}\|_{L^\infty}\|\nabla u_t^{k-1}\|_{L^2}\|u_t^k\|_{L^2} \\
&\qquad + \|\nabla^2\chi_t^k\|_{L^2}\|\nabla\chi^k\|_{L^\infty}\|u_t^k\|_{L^2} + \|\nabla\chi_t^k\|_{L^6}\|\nabla^2\chi^k\|_{L^3}\|u_t^k\|_{L^2} \\
&\quad\lesssim (\|\rho_t^k\|_{W^{1,6}} + K_1^4 + K_2^4)\int\limits_\Omega \rho^k|u_t^k|^2\mathrm{d}x + \underline{\eta}\int\limits_\Omega |Du_t^k|^2\mathrm{d}x + \|\chi_t^{k-1}\|_{H^1}^2\|u^k\|_{H^2}^2 \\
&\qquad + \|\rho_t^k\|_{W^{1,6}}^2\|u^{k-1}\|_{H^2}^2\|\nabla u^{k-1}\|_{L^2}^2 + K_1^{-4}\|u_t^{k-1}\|_{H^1}^2\|u^{k-1}\|_{H^2}^2 + K_2^{-4}\|\nabla^2\chi_t^k\|_{L^2}^2\|\nabla\chi^k\|_{H^2}^2.
\end{aligned}$$

By using (1.4), it follows that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega \rho^k|u_t^k|^2\mathrm{d}x + \int\limits_\Omega |Du_t^k|^2\mathrm{d}x \\
&\quad\lesssim (K_1^4 + K_2^4 + 1)\int\limits_\Omega \rho^k|u_t^k|^2\mathrm{d}x + K_2^2\|u^k\|_{H^2}^2 + K_1^6 + K_1^{-2}\|u_t^{k-1}\|_{H^1}^2 + K_2^{-2}\|\chi_t^k\|_{H^2}^2.
\end{aligned} \tag{2.24}$$

It remains for us to deal with the term $\|u^k\|_{H^2}^2$. We rewrite Eq. $(2.9)_2$ as

$$-\mathrm{div}(2\eta(\chi^{k-1})Du^k) + \nabla p = -\rho^k u_t^k - \rho^k(u^{k-1}\cdot\nabla)u^{k-1} - \mathrm{div}(\nabla\chi^k\otimes\nabla\chi^k).$$

It follows from (1.4) and the estimates for the stationary Stokes equation [19] that

$$\|u^k\|_{H^2}^2 + \|p^k\|_{H^1}^2 \lesssim \|\rho^k u_t^k\|_{L^2}^2 + \|\rho^k(u^{k-1}\cdot\nabla)u^{k-1}\|_{L^2}^2 + \|\mathrm{div}(\nabla\chi^k\otimes\nabla\chi^k)\|_{L^2}^2,$$

where

$$\begin{aligned}
\|\rho^k(u^{k-1}\cdot\nabla)u^{k-1}\|_{L^2}^2 &\lesssim \int\limits_\Omega |u^{k-1}-u_0|^2|\nabla u^{k-1}|^2\mathrm{d}x + \int\limits_\Omega |\nabla(u^{k-1}-u_0)|^2\mathrm{d}x + 1 \\
&\lesssim \int\limits_\Omega \left|\int\limits_0^t u_t^{k-1}(x,s)\mathrm{d}s\right|^2 |\nabla u^{k-1}|^2\mathrm{d}x + K_1^{-2}\int\limits_\Omega |\nabla^2(u^{k-1}-u_0)|^2\mathrm{d}x \\
&\quad + K_1^2\int\limits_\Omega |u^{k-1}-u_0|^2\mathrm{d}x + 1 \\
&\lesssim T\int\limits_0^t\int\limits_\Omega |u_t^{k-1}(x,s)|^2|\nabla u^{k-1}(x,t)|^2\mathrm{d}x\mathrm{d}s + K_1^2 T\int\limits_0^T\int\limits_\Omega |u_t^{k-1}|^2\mathrm{d}x\mathrm{d}t + 1
\end{aligned}$$

$$\lesssim T\int_0^T \|u_t^{k-1}(\cdot,s)\|_{L^6}^2\|\nabla u^{k-1}\|_{L^3}^2\mathrm{d}s + K_1^4T^2+1$$
$$\lesssim T\|\nabla u^{k-1}\|_{H^1}^2\int_0^T \|u_t^{k-1}(\cdot,s)\|_{H^1}^2\mathrm{d}s + K_1^4T^2+1 \lesssim K_1^4T + K_1^4T^2+1.$$

From (2.13), we can deduce that

$$\|\mathrm{div}(\nabla\chi^k\otimes\nabla\chi^k)\|_{L^2}^2 \lesssim \int_\Omega |\nabla(\chi^k-\chi_0)|^2|\nabla^2\chi^k|^2\mathrm{d}x + \int_\Omega |\nabla^2(\chi^k-\chi_0)|^2\mathrm{d}x + 1$$
$$\lesssim K^{12}T^2 + K_2^6T^2+1.$$

Therefore, for any $0<T<T_2$, we obtain

$$\|u^k\|_{L^\infty(0,T;H^2)}^2 + \|p^k\|_{L^\infty(0,T;H^1)}^2 \lesssim \|\sqrt{\rho^k}u_t^k\|_{L^2}^2+1. \tag{2.25}$$

Substituting (2.25) into (2.24) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^k|u_t^k|^2\mathrm{d}x + \int_\Omega |Du_t^k|^2\mathrm{d}x$$
$$\lesssim (K_1^4+K_2^4+1)\int_\Omega \rho^k|u_t^k|^2\mathrm{d}x + K_2^2 + K_1^6 + K_1^{-2}\|u_t^{k-1}\|_{H^1}^2 + K_2^{-2}\|\chi_t^k\|_{H^2}^2.$$

Taking $T_3 := \min\left\{T_2, \dfrac{1}{K_1^6}\right\}$, applying Gronwall's inequality and recalling Eq. $(2.9)_2$, for any $0<T<T_3$, we get

$$\sup_{0\le t\le T}\int_\Omega \rho^k|u_t^k|^2\mathrm{d}x + \int_0^T\int_\Omega |Du_t^k|^2\mathrm{d}x\mathrm{d}t \le C.$$

The well-known Korn's inequality [12,15] implies that, for bounded connected open domain $\Omega\subset\mathbb{R}^d$ $(N=2,3)$, there exists a (generic) positive constant C_Ω such that

$$\|\nabla v\|_{L^2(\Omega)} \le C_\Omega(\|Dv\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}), \quad \forall\, v\in \left(H^1(\Omega)\right)^N. \tag{2.26}$$

Hence, for any $0<T<T_3$, we have

$$\|u_t^k\|_{L^\infty(0,T;L^2)} + \|u_t^k\|_{L^2(0,T;H^1)} \le C. \tag{2.27}$$

Moreover, the estimates for the stationary Stokes equation [19] also implies that

$$\|u^k\|_{W^{2,6}} + \|p^k\|_{W^{1,6}} \lesssim \|\rho^k u_t^k\|_{L^6} + \|\rho^k(u^{k-1}\cdot\nabla)u^{k-1}\|_{L^6} + \|\mathrm{div}(\nabla\chi^k\otimes\nabla\chi^k)\|_{L^6}$$
$$\lesssim \|u_t^k\|_{L^6} + \|u^{k-1}\|_{L^\infty}\|\nabla u^{k-1}\|_{L^6} + \|\nabla\chi^k\|_{L^\infty}\|\nabla^2\chi^k\|_{L^6}$$
$$\lesssim \|u_t^k\|_{H^1} + \|u^{k-1}\|_{H^2}^2 + \|\chi^k\|_{H^3}^2 \lesssim \|u_t^k\|_{H^1} + K_1^2+1.$$

Then for any $0<T<T_3$, it holds that

$$\|u^k\|_{L^2(0,T;W^{2,6})} + \|p^k\|_{L^2(0,T;W^{1,6})} \le C. \tag{2.28}$$

Here, we have normalized p as $\int_\Omega p(x,t)\mathrm{d}x = 0$. Choosing $K_1\ge C$, then (2.22) and (2.23) follows from (2.25), (2.27) and (2.28).

Step 4 Taking limits.

Here, we denote by $\tilde{C}$ a constant whose value may be different from line to line depending on K_1, K_2 and other known constants. Denote $\overline{\rho}^{k+1} = \rho^{k+1} - \rho^k$, $\overline{u}^{k+1} = u^{k+1} - u^k$, $\overline{p}^{k+1} = p^{k+1} - p^k$, $\overline{\chi}^{k+1} = \chi^{k+1} - \chi^k$. Then from (2.9), we have the following system

$$\begin{cases} \overline{\rho}_t^{k+1} + (u^k \cdot \nabla)\overline{\rho}^{k+1} = -(\overline{u}^k \cdot \nabla)\rho^k, \\ \rho^{k+1}\overline{u}_t^{k+1} + \nabla \overline{p}^{k+1} = \operatorname{div}(2\eta(\chi^k)D\overline{u}^{k+1}) + \operatorname{div}(2\eta'(\theta\overline{\chi}^k)\overline{\chi}^k Du^k) - \overline{\rho}^{k+1}u_t^k - \overline{\rho}^{k+1}(u^k\cdot\nabla)u^k \\ \ -\rho^k(\overline{u}^k\cdot\nabla)u^k - \rho^k(u^{k-1}\cdot\nabla)\overline{u}^k - \operatorname{div}(\nabla\overline{\chi}^{k+1}\otimes\nabla\chi^{k+1}) - \operatorname{div}(\nabla\chi^k\otimes\nabla\overline{\chi}^{k+1}), \\ \operatorname{div}\overline{u}^{k+1} = 0, \\ \overline{\chi}_t^{k+1} = \dfrac{1}{(\rho^{k+1})^2}\Delta\overline{\chi}^{k+1} - \overline{\rho}^{k+1}\dfrac{\rho^{k+1}+\rho^k}{(\rho^{k+1}\rho^k)^2}\Delta\chi^k - (\overline{u}^k\cdot\nabla)\chi^k - (u^{k-1}\cdot\nabla)\overline{\chi}^k \\ \ +\overline{\rho}^{k+1}\dfrac{(\chi^{k-1})^3 - \chi^{k-1}}{\rho^{k+1}\rho^k} - \dfrac{(\chi^k)^2 + \chi^k\chi^{k-1} + (\chi^{k-1})^2 - 1}{\rho^{k+1}}\overline{\chi}^k, \end{cases} \tag{2.29}$$

where $0 < \theta < 1$ is a constant. The above system is supplemented with the initial boundary conditions

$$\left(\overline{\rho}^{k+1}, \overline{u}^{k+1}, \overline{\chi}^{k+1}\right)\Big|_{t=0} = (0,0,0), \quad x \in \Omega,$$
$$\left(\overline{u}^{k+1}, \frac{\partial \overline{\chi}^{k+1}}{\partial \mathbf{n}}\right)\Big|_{\partial\Omega} = (0,0), \quad t \geq 0.$$

Multiplying $(2.29)_1$ by $\overline{\rho}^{k+1}$ and integrating the result over Ω yield

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\overline{\rho}^{k+1}|^2\mathrm{d}x &= -2\int_\Omega (\overline{u}^k\cdot\nabla)\rho^k\overline{\rho}^{k+1}\mathrm{d}x \lesssim \|\overline{u}^k\|_{L^2}\|\nabla\rho^k\|_{L^\infty}\|\overline{\rho}^{k+1}\|_{L^2} \\ &\lesssim \|\overline{u}^k\|_{L^2}\|\nabla\rho^k\|_{W^{1,6}}\|\overline{\rho}^{k+1}\|_{L^2} \lesssim \|\sqrt{\rho^k}\,\overline{u}^k\|_{L^2}^2 + \|\overline{\rho}^{k+1}\|_{L^2}^2. \end{aligned} \tag{2.30}$$

Multiplying $(2.29)_2$ by $\overline{u}^{k+1}$ and integrating over Ω, we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega \rho^{k+1}|\overline{u}^{k+1}|^2\mathrm{d}x + \int_\Omega 2\eta(\chi^k)|D\overline{u}^{k+1}|^2\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega \rho_t^{k+1}|\overline{u}^{k+1}|^2\mathrm{d}x - 2\int_\Omega \eta'(\theta\overline{\chi}^k)\overline{\chi}^k Du^k : D\overline{u}^{k+1}\mathrm{d}x - \int_\Omega \overline{\rho}^{k+1}u_t^k\cdot\overline{u}^{k+1}\mathrm{d}x \\ &\quad - \int_\Omega \overline{\rho}^{k+1}(u^k\cdot\nabla)u^k\cdot\overline{u}^{k+1}\mathrm{d}x - \int_\Omega \rho^k(\overline{u}^k\cdot\nabla)u^k\cdot\overline{u}^{k+1}\mathrm{d}x - \int_\Omega \rho^k(u^{k-1}\cdot\nabla)\overline{u}^k\cdot\overline{u}^{k+1}\mathrm{d}x \\ &\quad + \int_\Omega (\nabla\overline{\chi}^{k+1}\otimes\nabla\chi^{k+1}) : \nabla\overline{u}^{k+1}\mathrm{d}x + \int_\Omega (\nabla\chi^k\otimes\nabla\overline{\chi}^{k+1}) : \nabla\overline{u}^{k+1}\mathrm{d}x \\ &\lesssim \|\rho_t^{k+1}\|_{L^\infty}\|\overline{u}^{k+1}\|_{L^2}^2 + \|\overline{\chi}^k\|_{L^3}\|Du^k\|_{L^6}\|D\overline{u}^{k+1}\|_{L^2} + \|\overline{\rho}^{k+1}\|_{L^2}\|u_t^k\|_{L^3}\|\overline{u}^{k+1}\|_{L^6} \\ &\quad + \|\overline{\rho}^{k+1}\|_{L^2}\|u^k\|_{L^\infty}\|\nabla u^k\|_{L^3}\|\overline{u}^{k+1}\|_{L^6} + \|\overline{u}^k\|_{L^6}\|\nabla u^k\|_{L^3}\|\overline{u}^{k+1}\|_{L^2} \\ &\quad + \|u^{k-1}\|_{L^\infty}\|\nabla\overline{u}^k\|_{L^2}\|\overline{u}^{k+1}\|_{L^2} + \|\nabla\overline{\chi}^{k+1}\|_{L^2}\|\nabla\chi^{k+1}\|_{L^\infty}\|\nabla\overline{u}^{k+1}\|_{L^2} \\ &\quad + \|\nabla\chi^k\|_{L^\infty}\|\nabla\overline{\chi}^{k+1}\|_{L^2}\|\nabla\overline{u}^{k+1}\|_{L^2}. \end{aligned}$$

Together with Sobolev embedding theorem and Korn's inequality (2.26), we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega \rho^{k+1}|\overline{u}^{k+1}|^2\mathrm{d}x + \int_\Omega 2\eta(\chi^k)|D\overline{u}^{k+1}|^2\mathrm{d}x \\ &\quad\lesssim \|\rho_t^{k+1}\|_{W^{1,6}}\|\overline{u}^{k+1}\|_{L^2}^2 + \left(\|\overline{\chi}^k\|_{L^2} + \|\nabla\overline{\chi}^k\|_{L^2}\right)\|u^k\|_{H^2}\|D\overline{u}^{k+1}\|_{L^2} \\ &\qquad + \left(\|\overline{\rho}^{k+1}\|_{L^2}\|u_t^k\|_{H^1} + \|\overline{\rho}^{k+1}\|_{L^2}\|u^k\|_{H^2}\|\nabla u^k\|_{H^1}\right)\left(\|D\overline{u}^{k+1}\|_{L^2} + \|\overline{u}^{k+1}\|_{L^2}\right) \end{aligned}$$

$$
\begin{aligned}
&\quad + \left(\|\nabla u^k\|_{H^1}\|\overline{u}^{k+1}\|_{L^2} + \|u^{k-1}\|_{H^2}\|\overline{u}^{k+1}\|_{L^2}\right)\left(\|D\overline{u}^k\|_{L^2} + \|\overline{u}^k\|_{L^2}\right) \\
&\quad + \left(\|\nabla\overline{\chi}^{k+1}\|_{L^2}\|\nabla\chi^{k+1}\|_{H^2} + \|\nabla\chi^k\|_{H^2}\|\nabla\overline{\chi}^{k+1}\|_{L^2}\right)\left(\|D\overline{u}^{k+1}\|_{L^2} + \|\overline{u}^{k+1}\|_{L^2}\right) \\
&\lesssim \|\overline{u}^{k+1}\|_{L^2}^2 + \left(\|\overline{\chi}^k\|_{L^2} + \|\nabla\overline{\chi}^k\|_{L^2}\right)\|D\overline{u}^{k+1}\|_{L^2} + \left(\|D\overline{u}^k\|_{L^2} + \|\overline{u}^k\|_{L^2}\right)\|\overline{u}^{k+1}\|_{L^2} \\
&\quad + \left(\|\overline{\rho}^{k+1}\|_{L^2}\left(\|u_t^k\|_{H^1} + 1\right) + \|\nabla\overline{\chi}^{k+1}\|_{L^2}\right)\left(\|D\overline{u}^{k+1}\|_{L^2} + \|\overline{u}^{k+1}\|_{L^2}\right).
\end{aligned}
$$

By using Cauchy inequality and (1.4), it follows that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^{k+1}|\overline{u}^{k+1}|^2\mathrm{d}x + \int_\Omega |D\overline{u}^{k+1}|^2\mathrm{d}x \\
&\lesssim \|\sqrt{\rho^{k+1}}\overline{u}^{k+1}\|_{L^2}^2 + (\|u_t^k\|_{H^1}^2 + 1)\|\overline{\rho}^{k+1}\|_{L^2}^2 + \|\nabla\overline{\chi}^{k+1}\|_{L^2}^2 \\
&\quad + \|\rho^k\overline{\chi}^k\|_{L^2}^2 + \|\nabla\overline{\chi}^k\|_{L^2}^2 + \|\sqrt{\rho^k}\overline{u}^k\|_{L^2}^2 + \varepsilon\|D\overline{u}^k\|_{L^2}^2.
\end{aligned}
\tag{2.31}
$$

Multiplying $(2.29)_4$ by $\Delta\overline{\chi}^{k+1}$ and integrating over Ω, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\overline{\chi}^{k+1}|^2\mathrm{d}x + \int_\Omega \frac{1}{(\rho^{k+1})^2}|\Delta\overline{\chi}^{k+1}|^2\mathrm{d}x \\
&\lesssim \|\overline{\rho}^{k+1}\|_{L^2}\|\Delta\chi^k\|_{L^\infty}\|\Delta\overline{\chi}^{k+1}\|_{L^2} + \|\overline{\rho}^{k+1}\|_{L^2}\left(\|\chi^{k-1}\|_{L^\infty}^3 + \|\chi^{k-1}\|_{L^\infty}\right)\|\Delta\overline{\chi}^{k+1}\|_{L^2} \\
&\quad + \|\overline{u}^k\|_{L^2}\|\nabla\chi^k\|_{L^\infty}\|\Delta\overline{\chi}^{k+1}\|_{L^2} + \|u^{k-1}\|_{L^\infty}\|\nabla\overline{\chi}^k\|_{L^2}\|\Delta\overline{\chi}^{k+1}\|_{L^2} \\
&\quad + \left(\|\chi^k\|_{L^\infty}^2 + \|\chi^k\|_{L^\infty}\|\chi^{k-1}\|_{L^\infty} + \|\chi^{k-1}\|_{L^\infty}^2 + 1\right)\|\overline{\chi}^k\|_{L^2}\|\Delta\overline{\chi}^{k+1}\|_{L^2} \\
&\lesssim \frac{1}{2}\int_\Omega \frac{1}{(\rho^{k+1})^2}|\Delta\overline{\chi}^{k+1}|^2\mathrm{d}x + \|\overline{\rho}^{k+1}\|_{L^2}^2\|\Delta\chi^k\|_{H^2}^2 + \|\overline{\rho}^{k+1}\|_{L^2}^2\left(\|\chi^{k-1}\|_{H^2}^6 + \|\chi^{k-1}\|_{H^2}^2\right) \\
&\quad + \|\overline{u}^k\|_{L^2}^2\|\nabla\chi^k\|_{H^2}^2 + \|u^{k-1}\|_{H^2}^2\|\nabla\overline{\chi}^k\|_{L^2}^2 \\
&\quad + \left(\|\chi^k\|_{H^2}^4 + \|\chi^k\|_{H^2}^2\|\chi^{k-1}\|_{H^2}^2 + \|\chi^{k-1}\|_{H^2}^4 + 1\right)\|\overline{\chi}^k\|_{L^2}^2.
\end{aligned}
$$

From which we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\overline{\chi}^{k+1}|^2\mathrm{d}x + \int_\Omega \frac{1}{(\rho^{k+1})^2}|\Delta\overline{\chi}^{k+1}|^2\mathrm{d}x \\
&\lesssim \left(\|\chi^k\|_{H^4}^2 + 1\right)\|\overline{\rho}^{k+1}\|_{L^2}^2 + \|\sqrt{\rho^k}\,\overline{u}^k\|_{L^2}^2 + \|\nabla\overline{\chi}^k\|_{L^2}^2 + \|\rho^k\overline{\chi}^k\|_{L^2}^2.
\end{aligned}
\tag{2.32}
$$

Multiplying $(2.29)_4$ by $(\rho^{k+1})^2\overline{\chi}^{k+1}$ and integrating over Ω, we can deduce that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\rho^{k+1}\overline{\chi}^{k+1}|^2\mathrm{d}x + \int_\Omega |\nabla\overline{\chi}^{k+1}|^2\mathrm{d}x \\
&\lesssim \|\rho^{k+1}\overline{\chi}^{k+1}\|_{L^2}^2 + \left(\|\chi^k\|_{H^4}^2 + 1\right)\|\overline{\rho}^{k+1}\|_{L^2}^2 + \|\sqrt{\rho^k}\,\overline{u}^k\|_{L^2}^2 + \|\nabla\overline{\chi}^k\|_{L^2}^2 + \|\rho^k\overline{\chi}^k\|_{L^2}^2.
\end{aligned}
\tag{2.33}
$$

Putting (2.30)–(2.33) together gives

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\overline{\rho}^{k+1}\|_{L^2}^2 + \|\sqrt{\rho^{k+1}}\overline{u}^{k+1}\|_{L^2}^2 + \|\nabla\overline{\chi}^{k+1}\|_{L^2}^2 + \|\rho^{k+1}\overline{\chi}^{k+1}\|_{L^2}^2\right) \\
&\quad + \|D\overline{u}^{k+1}\|_{L^2}^2 + \left\|(\rho^{k+1})^{-1}\Delta\overline{\chi}^{k+1}\right\|_{L^2}^2 \\
&\lesssim \left(\|u_t^k\|_{H^1}^2 + \|\chi^k\|_{H^4}^2 + 1\right)\left(\|\overline{\rho}^{k+1}\|_{L^2}^2 + \|\sqrt{\rho^{k+1}}\overline{u}^{k+1}\|_{L^2}^2 + \|\nabla\overline{\chi}^{k+1}\|_{L^2}^2 + \|\rho^{k+1}\overline{\chi}^{k+1}\|_{L^2}^2\right) \\
&\quad + \left(\|\sqrt{\rho^k}\overline{u}^k\|_{L^2}^2 + \|\nabla\overline{\chi}^k\|_{L^2}^2 + \|\rho^k\overline{\chi}^k\|_{L^2}^2\right) + \varepsilon\|D\overline{u}^k\|_{L^2}^2.
\end{aligned}
$$

Set

$$\begin{aligned}
A^k(t) &= \|\overline{\rho}^k(\cdot,t)\|_{L^2}^2 + \|\sqrt{\rho^k}\,\overline{u}^k(\cdot,t)\|_{L^2}^2 + \|\nabla\overline{\chi}^k(\cdot,t)\|_{L^2}^2 + \|\rho^k\overline{\chi}^k(\cdot,t)\|_{L^2}^2,\\
B^k(t) &= \|D\overline{u}^k(\cdot,t)\|_{L^2}^2 + \left\|(\rho^k)^{-1}\Delta\overline{\chi}^k(\cdot,t)\right\|_{L^2}^2\\
c^k(t) &= \|u_t^k(\cdot,t)\|_{H^1}^2 + \|\chi^k(\cdot,t)\|_{H^4}^2 + \|\chi_t^k(\cdot,t)\|_{H^2}^2.
\end{aligned}$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}A^{k+1}(t) + B^{k+1}(t) \le \tilde{C}\big(c^k(t)+1\big)A^{k+1}(t) + \tilde{C}A^k(t) + \varepsilon B^k(t),$$

where $\int_0^T c^k(t)\mathrm{d}t \le \tilde{C}$. By using Gronwall's inequality, we get

$$\sup_{0\le t\le T} A^{k+1}(t) \le \left(\tilde{C}\int_0^T A^k(t)\mathrm{d}t + \varepsilon\int_0^T B^k(t)\mathrm{d}t\right)\exp\left\{\tilde{C}(1+T)\right\}.$$

Hence

$$\begin{aligned}
&\sup_{0\le t\le T} A^{k+1}(t) + \int_0^T B^{k+1}(t)\mathrm{d}t\\
&\le \left(\tilde{C}\,T\sup_{0\le t\le T} A^k(t) + \varepsilon\int_0^T B^k(t)\mathrm{d}t\right)\left(\tilde{C}(1+T)\exp\{\tilde{C}(1+T)\}+1\right).
\end{aligned}$$

Choosing $T_4 := \left\{T_3,\ \dfrac{1}{4\tilde{C}(2\tilde{C}\exp\{2\tilde{C}\}+1)}\right\}$, $\varepsilon = \dfrac{1}{4(2\tilde{C}\exp\{2\tilde{C}\}+1)}$ and recalling $0<T<1$, then for any $0<T<T_4$ and $k\ge 1$, we have

$$\sup_{0\le t\le T} A^{k+1}(t) + \int_0^T B^{k+1}(t)\mathrm{d}t \le \frac{1}{4}\left(\sup_{0\le t\le T} A^k(t) + \int_0^T B^k(t)\mathrm{d}t\right).$$

By iteration, we derive that

$$\sup_{0\le t\le T} A^{k+1}(t) + \int_0^T B^{k+1}(t)\mathrm{d}t \le \frac{1}{4^{k-1}}\left(\sup_{0\le t\le T} A^2(t) + \int_0^T B^2(t)\mathrm{d}t\right).$$

Together with Korn's inequality, we have

$$\begin{aligned}
&\left\|\overline{\rho}^{k+1}\right\|_{L^\infty(0,T;L^2)} + \left\|\overline{u}^{k+1}\right\|_{L^\infty(0,T;L^2)} + \left\|\overline{\chi}^{k+1}\right\|_{L^\infty(0,T;H^1)}\\
&\quad + \left\|\overline{u}^{k+1}\right\|_{L^2(0,T;H^1)} + \left\|\overline{\chi}^{k+1}\right\|_{L^2(0,T;H^2)} \le \frac{1}{2^{k-1}}\tilde{C}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
&\sum_{k=2}^{\infty}\left\|\overline{\rho}^k\right\|_{L^\infty(0,T;L^2)} < \infty,\\
&\sum_{k=2}^{\infty}\left(\left\|\overline{u}^{k+1}\right\|_{L^\infty(0,T;L^2)} + \left\|\overline{u}^{k+1}\right\|_{L^2(0,T;H^1)}\right) < \infty,\\
&\sum_{k=2}^{\infty}\left(\left\|\overline{\chi}^{k+1}\right\|_{L^\infty(0,T;H^1)} + \left\|\overline{\chi}^{k+1}\right\|_{L^2(0,T;H^2)}\right) < \infty.
\end{aligned}$$

Therefore, as $k \to \infty$, we have

$$\rho^k \to \rho^1 + \sum_{k=2}^{\infty} \overline{\rho}^k, \quad \text{in } L^\infty(0,T;L^2),$$

$$u^k \to u^1 + \sum_{k=2}^{\infty} \overline{u}^k, \quad \text{in } L^\infty(0,T;L^2) \cap L^2(0,T;H^1), \tag{2.34}$$

$$\chi^k \to \chi^1 + \sum_{k=2}^{\infty} \overline{\chi}^k, \quad \text{in } L^\infty(0,T;H^1) \cap L^2(0,T;H^2). \tag{2.35}$$

By (2.10)–(2.13), (2.22), (2.23), after taking possible subsequences (denoted by itself for convenience), sending $k \to \infty$, we have

$$\begin{aligned}
&\rho^k \to \rho, \quad \text{strongly in } C(0,T;H^1),\\
&(\nabla\rho^k,\, \rho^k_t) \rightharpoonup (\nabla\rho,\, \rho_t), \quad \text{weak}^* \text{ in } L^\infty(0,T;L^2),\\
&(\nabla^2\rho^k,\, \nabla\rho^k_t) \rightharpoonup (\nabla^2\rho,\, \nabla\rho_t), \quad \text{weak}^* \text{ in } L^\infty(0,T;L^6),\\
&u^k \to u, \quad \text{strongly in } C(0,T;H^1),\\
&(\nabla u^k,\, \nabla^2 u^k,\, u^k_t) \rightharpoonup (\nabla u,\, \nabla^2 u,\, u_t), \quad \text{weak}^* \text{ in } L^\infty(0,T;L^2),\\
&\nabla^2 u^k \rightharpoonup \nabla^2 u, \quad \text{weakly in } L^2(0,T;L^6),\\
&\nabla u^k_t \rightharpoonup \nabla u_t, \quad \text{weakly in } L^2(0,T;L^2),\\
&(p^k,\, \nabla p^k) \rightharpoonup (p,\, \nabla p), \quad \text{weak}^* \text{ in } L^\infty(0,T;L^2),\\
&\nabla p^k \rightharpoonup \nabla p, \quad \text{weakly in } L^2(0,T;L^6),\\
&\chi^k \to \chi, \quad \text{strongly in } C(0,T;H^2),\\
&(\nabla\chi^k,\, \nabla^2\chi^k,\, \nabla^3\chi^k,\, \chi^k_t,\, \nabla\chi^k_t) \rightharpoonup (\nabla\chi,\, \nabla^2\chi,\, \nabla^3\chi,\, \chi_t,\, \nabla\chi_t), \quad \text{weak}^* \text{ in } L^\infty(0,T;L^2),\\
&(\nabla^4\chi^k,\, \nabla^2\chi_t) \rightharpoonup (\nabla^4\chi,\, \nabla^2\chi_t), \quad \text{weakly in } L^2(0,T;L^2).
\end{aligned}$$

By lower semi-continuity, we derive

$$\|\rho\|_{L^\infty(0,T;W^{2,6})} + \|\rho_t\|_{L^\infty(0,T;W^{1,6})} + \|u\|_V + \|p\|_{L^\infty(0,T;H^1)} + \|p\|_{L^2(0,T;W^{1,6})} + \|\chi\|_\Phi \le \tilde{C}.$$

By the uniqueness of the limits, we get $\rho = \rho^1 + \sum_{k=2}^\infty \overline{\rho}^k$, $u = u^1 + \sum_{k=2}^\infty \overline{u}^k$, $\chi = \chi^1 + \sum_{k=2}^\infty \overline{\chi}^k$. On the other hand, (2.34) and (2.35) also imply

$$\begin{aligned}
&u^{k-1} \to u, \quad \text{in } L^\infty(0,T;L^2) \cap L^2(0,T;H^1),\\
&\chi^{k-1} \to \chi, \quad \text{in } L^\infty(0,T;H^1) \cap L^2(0,T;H^2),
\end{aligned}$$

as $k \to \infty$.

Taking limits in (2.9), we see that (ρ, u, p, χ) is accurately a solution of the problem (1.1) with the regularities like in Theorem 1.1. The uniqueness of the solution can be obtained by the standard energy method similar to step 4. Therefore, the proof of Theorem 1.1 is complete. □

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