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Boundedness and global stability of the two-predator and one-prey models with nonlinear prey-taxis

Jianping Wang and Mingxin Wang

Abstract. This paper concerns the reaction–diffusion systems modeling the population dynamics of two predators and one prey with nonlinear prey-taxis. We first investigate the global existence and boundedness of the unique classical solution for the general model. Then, we study the global stabilities of nonnegative spatially homogeneous equilibria for an explicit system with type I functional responses and density-dependent death rates for the predators and logistic growth for the prey. Moreover, the convergence rates are also established.

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1. Introduction

The dynamical relationship between the predator and prey has been investigated widely in recent years due to its universal existence and importance in mathematical biology and ecology. In the spatial predator–prey interaction, in addition to the random diffusion of predator and prey, the predator has the tendency to move toward the area with higher density of prey population. Kareiva and Odell [11] first derived a prey-taxis model to describe the predator aggregation in high prey density areas. Since then, various reaction–diffusion models have been proposed to interpret the prey-taxis phenomenon [1,7,19]. The general prey-taxis model with one predator and one prey reads as follows

$$\begin{cases} u_t - \Delta u + \nabla\cdot(\phi(u,w)\nabla w) = bug(w) - uh(u), \\ w_t - d\Delta w = f(w) - ug(w), \end{cases} \tag{1.1}$$

where $u(t,x)$ denotes the predator density at position x and time $t > 0$ and $w = w(x,t)$ the prey population density; the term $\nabla\cdot(\phi(u,w)\nabla w)$ describes the tendency of the predator moving toward the increasing prey gradient direction and d is the prey diffusion rate. The function $ug(w)$ represents the inter-specific interaction, functions $uh(u)$ and $f(w)$ account for the intra-specific interactions. The parameter $b > 0$ denotes the intrinsic predation rate. This system has been studied by many authors. Lee et al. [13] studied the pattern formation of (1.1) in a bounded interval with zero Neumann boundary conditions. When $\phi(u,w) = \chi q(u)$ with positive constant χ, Wu et al. [33] investigated the global existence and boundedness of solutions, and the existence of a global attractor and the uniform persistence of (1.1) in a bounded domain with zero Neumann boundary conditions under a smallness assumption on χ. When $\phi(u,w) = \chi u$ for positive constant χ, Jin and Wang [10] proved the global boundedness of solution and stabilities of nonnegative spatially homogeneous equilibria of (1.1) in a two-dimensional bounded domain with zero Neumann boundary conditions. For more related works, we refer the readers to [8,17,22,30].

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Before stating our model, we would like to recall some works on two-predator and one-prey model without prey-taxis. In [18], the emergence of stationary patterns for a strongly coupled system of partial differential equations which models the dynamics of a two-predator-one-prey ecosystem is demonstrated. For the other related works, please refer to [15,16,26]. In this paper, we consider the following two-predator and one-prey model with nonlinear prey-taxis:

$$\begin{cases} u_t - \Delta u + \nabla\cdot(u\phi_1(w)\nabla w) = b_1 u g_1(w) - u h_1(u), & x\in\Omega,\ t>0,\\ v_t - \Delta v + \nabla\cdot(v\phi_2(w)\nabla w) = b_2 v g_2(w) - v h_2(v), & x\in\Omega,\ t>0,\\ w_t - d\Delta w = f(w) - u g_1(w) - v g_2(w), & x\in\Omega,\ t>0,\\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x),\ w(x,0)=w_0(x), & x\in\Omega. \end{cases} \tag{1.2}$$

In this model, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\partial_\nu = \frac{\partial}{\partial\nu}$ and ν is the unit outward normal vector of $\partial\Omega$. Functions u and v are population densities of two predators, and w is the population density of the prey. Here d, b_1, b_2 are positive constants. It is assumed that the prey-tactic coefficients $\phi_1(w)$ and $\phi_2(w)$ depend only upon w. Functions $uh_1(u)$ and $vh_2(v)$ describe the population kinetics of u and v, respectively. Functions $g_1(w)$ and $g_2(w)$ are the functional responses accounting for the intake rates of predators u and v as functions of prey density. The function $f(w)$ is the growth function of the prey. Problem (1.2) had been studied by Wang et al. ([28]) for the following special case:

$$\phi_1(s) = \chi\phi_2(s),\ g_i(s) = \beta_i s,\ h_i(s) = -\alpha_i(1-s),\ i=1,2,\quad f(s) = \alpha_3 s(1-s)$$

with positive constants $\chi, \beta_1, \beta_2, \alpha_1, \alpha_2, \alpha_3$. They discussed the global existence of solutions when $n=2$, and the stationary and time-periodic patterns.

In the present paper, we suppose that ϕ_i, h_i, g_i and f satisfy the following hypotheses:

(A1) Functions $\phi_i \in C^2([0,\infty))$, $\phi_i \ge 0$, $i=1,2$. The well-known examples are

$$\text{(i) } \phi_i(s) = \chi_i,\quad \text{(ii) } \phi_i(s) = \frac{\chi_i}{s+\varepsilon_i},\quad \text{(iii) } \phi_i(s) = \frac{\chi_i}{(s+\varepsilon_i)^2},$$

with positive constants χ_i, ε_i, $i=1,2$.

(A2) Functions $g_i \in C^2([0,\infty))$, $g_i(0)=0$, $g_i(s)>0$ in $(0,\infty)$, $i=1,2$. The typical examples are

$$\text{(type I) } g_i(s) = \gamma_i s,\quad \text{(type II) } g_i(s) = \frac{\gamma_i s}{l_i + s},$$

$$\text{(type III) } g_i(s) = \frac{\gamma_i s^{\kappa_i}}{l_i^{\kappa_i} + s^{\kappa_i}},\quad \text{(Ivlev type) } g_i(s) = \gamma_i(1-e^{-l_i s}),$$

where γ_i, l_i, κ_i are positive constants, $i=1,2$.

(A3) Functions $h_i \in C^2([0,\infty))$ and there exist two constants a_i and $\mu_i > 0$ such that $h_i(s) \ge a_i$ and $h_i'(s) \ge \mu_i$ in $[0,\infty)$, $i=1,2$. It then follows that $-h_i(s) \le |a_i| - \mu_i s$ for $s\in[0,\infty)$, $i=1,2$. The typical example is $h_i(s) = a_i + \mu_i s$, in which a_i may be positive or negative.

(A4) The function $f \in C^2([0,\infty))$ satisfying $f(0)=0$, and there exist two positive constants q, K such that $f(s) \le qs$ for $s\ge 0$, $f(K)=0$ and $f(s)<0$ for $s>K$. Some examples are

$$\text{(logistic) } f(s) = qs\left(1-\frac{s}{K}\right),\quad \text{(Allee effect) } f(s) = q's\left(1-\frac{s}{K}\right)\left(\frac{s}{N}-1\right)$$

with $0<N<K$ and $q' = \frac{4KN}{(K-N)^2}q$.

The initial data u_0, v_0, w_0 are supposed to satisfy

$$u_0,\ v_0,\ w_0 \ge, \not\equiv 0 \quad \text{and} \quad u_0,\ v_0,\ w_0 \in W^{1,\theta}(\Omega) \ \text{ for some } \theta > \max\{n,2\}.$$

Remark 1.1. In the assumption $(A3)$, our assumption $\mu_i > 0$ is a technique requirement (see Lemma 2.2). Compared with the condition (H1) in [10], we remove the monotonicity assumptions on g_i.

Remark 1.2. It is easy to see that, in contrast to the model proposed in [28], system (1.2) is a more general one.

Throughout this paper, we denote $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$ for the simplicity. And denote the general constants by c_i.

The first result of this paper shows that the solution of the prey-taxis system (1.2) in two dimensions exists globally. In [28], the boundedness of solutions is obtained by introducing several entropy-like inequalities (see also [10]). However, we apply a different method ([3]) to prove our existence result in two dimensions. It seems worth pointing out that, in contrast to the existence results in [28], we relax the regularity assumptions on the initial data.

Theorem 1.1. *Let $n \le 2$. Then, (1.2) has a unique nonnegative and bounded global solution (u, v, w), and*

$$u, v, w \in C(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)). \tag{1.3}$$

Remark 1.3. The methods in the proof of Theorem 1.1 (and the following Theorem 1.2) can be applied to the model discussed by [28].

The following theorem asserts that, in higher dimension ($n \ge 3$), the solution of (1.2) exists globally when the prey-tactic coefficients $\phi_1(w)$ and $\phi_2(w)$ are small.

Theorem 1.2. *Let $n \ge 3$. Define $m = \max\{\|w_0\|_\infty, K\}$. If*

$$\|\phi_i\|_{L^\infty(0,m)} \le \frac{2d}{nm\left(1 + 2\sqrt{(d+1)^2 - d(n-2)/n}\right)}, \quad i = 1, 2, \tag{1.4}$$

then (1.2) has a unique nonnegative and bounded global solution (u, v, w), and (1.3) holds.

Remark 1.4. Inspired by the works in [24,32], we expect that if μ_1, μ_2 are sufficiently large, then system (1.2) has a unique global-in-time classical solution that is bounded in $\Omega \times (0, \infty)$. We are currently working on such issue in a separate paper.

The second goal of this paper is to understand the role of prey-taxis in the global stabilities of nonnegative spatially homogeneous equilibria to a three-species predator–prey model. Since it is hopeless to construct Lyapunov functionals for the general system, we would like to consider an explicit model. We consider type I function responses and density-dependent death rates for the predators, and logistic growth for the prey, reading as

$$h_i(s) = a_i + \mu_i s, \;\; g_i(s) = s, \;\; i = 1, 2, \quad f(s) = qs(1 - s/K),$$

where a_1, a_2, μ_1, μ_2, q, K are positive constants (the case of $a_1, a_2 < 0$ had been studied in [28]). For simplicity, we suppose that $\mu_1 = \mu_2 = K = 1$. Then, problem (1.2) becomes

$$\begin{cases} u_t - \Delta u + \nabla \cdot (u\phi_1(w)\nabla w) = b_1 uw - u(a_1 + u), & x \in \Omega, \; t > 0, \\ v_t - \Delta v + \nabla \cdot (v\phi_2(w)\nabla w) = b_2 vw - v(a_2 + v), & x \in \Omega, \; t > 0, \\ w_t - d\Delta w = qw(1 - w) - (u + v)w, & x \in \Omega, \; t > 0, \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & x \in \partial\Omega, \; t > 0, \\ u(x, 0) = u_0(x), \; v(x, 0) = v_0(x), \; w(x, 0) = w_0(x), & x \in \Omega. \end{cases} \tag{1.5}$$

In this model, the coefficients a_1 and b_1 represent the death rate and predation rate of the predator u, respectively (a_2 and b_2 represent the death rate and predation rate of the predator v, respectively), and q is the intrinsic birth rate of the prey w. We set from now on

$$A = (b_1 - a_1)q + a_2 b_1 - a_1 b_2, \quad B = (b_2 - a_2)q + a_1 b_2 - a_2 b_1. \tag{1.6}$$

When $A, B > 0$, one can find a positive solution (u_*, v_*, w_*) of the linear algebraic system

$$\begin{cases} b_1 w_* - a_1 - u_* = 0, \\ b_2 w_* - a_2 - v_* = 0, \\ q(1 - w_*) - u_* - v_* = 0. \end{cases}$$

The direct computation gives

$$u_* = \frac{A}{q+b_1+b_2}, \quad v_* = \frac{B}{q+b_1+b_2}, \quad w_* = \frac{q+a_1+a_2}{q+b_1+b_2}. \tag{1.7}$$

In this case, we shall show that if the prey-tactic coefficients $\phi_1(w)$ and $\phi_2(w)$ are small or the diffusion coefficient d of the prey is large, then the positive spatially homogeneous equilibrium (u_*, v_*, w_*) is global stable.

Theorem 1.3. *Assume $A, B > 0$ and set*

$$m = \max\{\|w_0\|_{L^\infty(\Omega)}, 1\}, \quad L_i = \|\phi_i\|_{L^\infty(0,m)}, \quad i = 1, 2.$$

Let (u, v, w) be a bounded global classical solution of (1.5). If

$$\frac{dw_*}{m^2} - \frac{L_1^2 u_*}{4b_1} - \frac{L_2^2 v_*}{4b_2} > 0, \tag{1.8}$$

then there exist constants C_1, $\lambda_1 > 0$ such that

$$\|u - u_*\|_\infty + \|v - v_*\|_\infty + \|w - w_*\|_\infty \le C_1 e^{-\lambda_1 t}, \quad \forall\, t > 0. \tag{1.9}$$

Remark 1.5. It is easy to see that $A, B > 0$ $(u_*, v_* > 0)$ implies $w_* < 1$, which yields $a_1 + a_2 < b_1 + b_2$. Therefore, $b_i \le a_i, i = 1, 2$ implies that $A \le 0$ or $B \le 0$.

In the case of $A \le 0$, $B > 0$ and $a_2 < b_2$. Let

$$\tilde{v} = \frac{(b_2 - a_2)q}{q + b_2}, \quad \tilde{w} = \frac{q + a_2}{q + b_2}. \tag{1.10}$$

Then $(0, \tilde{v}, \tilde{w})$ is a semi-trivial spatially homogeneous equilibrium. The following theorem shows that if the prey-tactic coefficient $\phi_2(w)$ is small or the diffusion coefficient d of the prey is large, then the semi-trivial spatially homogeneous equilibrium $(0, \tilde{v}, \tilde{w})$ is global stable.

Theorem 1.4. *Let m, L_2 be as in Theorem 1.3 and $A \le 0$, $B > 0$, $a_2 < b_2$. Let (u, v, w) be a bounded global classical solution of (1.5). Under the condition*

$$\frac{d\tilde{w}}{m^2} - \frac{L_2^2 \tilde{v}}{4b_2} > 0, \tag{1.11}$$

we have

(i) *If $a_1 > b_1\tilde{w}$, then there exist constants C_2, $\lambda_2 > 0$ such that*

$$\|u\|_\infty + \|v - \tilde{v}\|_\infty + \|w - \tilde{w}\|_\infty \le C_2 e^{-\lambda_2 t}, \quad \forall\, t > 0. \tag{1.12}$$

(ii) *If $a_1 = b_1\tilde{w}$, then there exist constants C_3, $\lambda_3 > 0$ such that*

$$\|u\|_\infty + \|v - \tilde{v}\|_\infty + \|w - \tilde{w}\|_\infty \le C_3 (t+1)^{-\lambda_3}, \quad \forall\, t > 0. \tag{1.13}$$

Similarly, when $A > 0$, $B \le 0$ and $a_1 < b_1$, let

$$\hat{u} = \frac{(b_1 - a_1)q}{q + b_1}, \quad \hat{w} = \frac{q + a_1}{q + b_1}.$$

Then $(\hat{u}, 0, \hat{w})$ is a semi-trivial spatially homogeneous equilibrium. The following theorem shows that if the prey-tactic coefficient $\phi_1(w)$ is small or the diffusion coefficient d of the prey is large then the semi-trivial spatially homogeneous equilibrium $(\hat{u}, 0, \hat{w})$ is global stable.

Theorem 1.5. *Let m, L_1 be as in Theorem 1.3 and $A > 0$, $B \le 0$, $a_1 < b_1$. Let (u, v, w) be a bounded global classical solution of (1.5). Under the condition*

$$\frac{d\hat{w}}{m^2} - \frac{L_1^2 \hat{u}}{4b_1} > 0,$$

we have

(i) *When* $a_2 > b_2\hat{w}$, *we can find constants* C_4, $\lambda_4 > 0$ *such that*

$$\|u-\hat{u}\|_\infty + \|v\|_\infty + \|w-\hat{w}\|_\infty \le C_4 e^{-\lambda_4 t}, \quad \forall\, t > 0.$$

(ii) *When* $a_2 = b_2\hat{w}$, *we can find constants* C_5, $\lambda_5 > 0$ *such that*

$$\|u-\hat{u}\|_\infty + \|v\|_\infty + \|w-\hat{w}\|_\infty \le C_5 (t+1)^{-\lambda_5}, \quad \forall\, t > 0.$$

In the case of $b_i \le a_i, i = 1, 2$, we have the following theorem.

Theorem 1.6. *Let* $b_i \le a_i$, $i = 1, 2$, *and* (u, v, w) *be a bounded global classical solution of* (1.5).

(i) *When* $b_i < a_i$, $i = 1, 2$, *we can find constants* C_6, $\lambda_6 > 0$ *such that*

$$\|u\|_\infty + \|v\|_\infty + \|w-1\|_\infty \le C_6 e^{-\lambda_6 t}, \quad \forall\, t > 0. \tag{1.14}$$

(ii) *When* $b_i = a_i$, $i = 1, 2$, *we can find constants* C_7, $\lambda_7 > 0$ *such that*

$$\|u\|_\infty + \|v\|_\infty + \|w-1\|_\infty \le C_7 (t+1)^{-\lambda_7}, \quad \forall\, t > 0. \tag{1.15}$$

From Theorem 1.3, we see that if the predation of both the two predators is strong, in the sense of $A, B > 0$, the coexistence steady state (u_*, v_*, w_*) can be reached if ϕ_1, ϕ_2 is small. Theorems 1.4 and 1.5 show that the prey and the predator with strong predation and weak prey-taxis will reach to positive constant steady state, while the predator with weak predation will go extinct (with any prey-taxis). Theorem 1.6 tells us that if the predation of both the predators are weak, in the sense of $b_i \le a_i$ $(i = 1, 2)$, the prey-only steady state $(0, 0, 1)$ will be attained and both the predators will go extinct.

In order to better understand the model (1.5), it seems worthwhile to mention the following two-species chemotaxis model

$$\begin{cases} u_t = d_1\Delta u - \chi_1 \nabla\cdot(u\nabla w) + \theta_1 u(1-u-k_1 v), & x\in\Omega,\ t>0,\\ v_t = d_2\Delta v - \chi_2 \nabla\cdot(v\nabla w) + \theta_2 v(1-k_2 u - v), & x\in\Omega,\ t>0,\\ \tau w_t = d_3\Delta w + \xi u + \rho v - \gamma v, & x\in\Omega,\ t>0,\\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x),\ w(x,0)=\tau w_0(x), & x\in\Omega, \end{cases} \tag{1.16}$$

where $\tau = 0, 1$. The coefficients $d_1, d_2, d_3, \chi_1, \chi_2, \theta_1, \theta_2, \xi, \rho, \gamma$ are positive constants and $k_1, k_2 \ge 0$. Here $u = u(x,t)$ and $v = v(x,t)$ denote the population densities of two species and $w = w(x,t)$ represents the concentration of the chemoattractant. When $\tau = 0$ and $k_1, k_2 \in [0, 1)$, Tello and Winkler [25] proved that (1.16) possesses a uniquely determined spatially homogeneous positive equilibrium

$$(u^*, v^*, w^*) = \left(\frac{1-k_1}{1-k_1k_2}, \frac{1-k_2}{1-k_1k_2}, \frac{\xi(1-k_1)+\rho(1-k_2)}{\gamma(1-k_1k_2)}\right), \tag{1.17}$$

provided

$$2(\chi_1+\chi_2) + k_1\theta_2 < \theta_1 \text{ and } 2(\chi_1+\chi_2) + k_2\theta_1 < \theta_2. \tag{1.18}$$

Black et al. [6] replaced the condition (1.18) by a smallness condition on χ_i/θ_i $(i = 1, 2)$. For $\tau = 0$ and $k_1 > 1 > k_2 \ge 0$, Stinner et al. [20] proved that if both χ_1/θ_1 and χ_2/θ_2 are sufficiently small then competitive exclusion occurs for any solution (u, v, w) with $v \not\equiv 0$. For $\tau = 1$, in the 2-dimensional case Bai and Winkler [3] obtained global existence of solution to system (1.3). Moreover, for $k_1, k_2 \in (0, 1)$, they proved that

$$\|u(\cdot,t)-u^*\|_{L^\infty(\Omega)} + \|v(\cdot,t)-v^*\|_{L^\infty(\Omega)} + \|w(\cdot,t)-w^*\|_{L^\infty(\Omega)} \le Ce^{-\lambda t}, \quad \forall\, t>0,$$

for some positive constants C, λ, where u^*, v^*, w^* is given by (1.17),under the conditions

$$\mu_1 > \frac{d_2\chi_1^2 u^*}{\frac{4k_1\gamma(1-k_1k_2)d_1d_2d_3}{(k_1\xi^2+k_2\rho^2-2k_1k_2\xi\rho)} - \frac{d_1k_1\chi_2^2 v^*}{4\theta_2 k_2}}, \quad \mu_2 > \frac{\chi_2^2 v^*(k_1\xi^2+k_2\rho^2-2k_1k_2\xi\rho)}{16d_2d_3k_2\gamma(1-k_1k_2)}.$$

When $k_1 \geq 1 > k_2 > 0$, they show that

$$\lim_{t\to\infty}\left(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)-1\|_{L^\infty(\Omega)} + \|w(\cdot,t)-\rho/\gamma\|_{L^\infty(\Omega)}\right) = 0,$$

under the conditions that there is $k_1' \in [1, k_1]$ such that $k_1' k_2 < 1$ and

$$\theta_2 > \frac{\chi_2^2(k_1'\xi^2 + k_2\rho^2 - 2k_1'k_2\xi\rho)}{16 d_2 d_3 k_2 \gamma (1 - k_1' k_2)}.$$

Recently, this result is improved by Mizukami [17]. In the 3-dimensional case, Lin and Mu [14] obtained similar results if θ_1 and θ_2 are sufficiently large.

The article is organized as follows. Section 2 provides the uniqueness, global existence and boundedness of the classical solution of (1.2). Section 3 is devoted to prove the global stability results in Theorems 1.3–1.6. The last section is a brief discussion.

Before ending this section, we should mention that if the forms of the two functions g_1 and g_2 in system (1.2) are very different from each other, then describing stability properties of spatially homogeneous equilibria for the general system (1.2) would be a challenging open topic.

2. Existence, uniqueness and boundedness of global solutions

2.1. Existence and uniqueness of local solutions, some preliminaries

We first give a claim concerning the local-in-time existence of the classical solutions to (1.2).

Lemma 2.1. *There exists a $\hat{T} \in (0, \infty]$ and a unique nonnegative solution (u, v, w) of (1.2) defined in $[0, \hat{T})$ and satisfies*

$$u, v, w \in C(\bar{\Omega} \times [0, \hat{T})) \cap C^{2,1}(\bar{\Omega} \times (0, \hat{T})),$$

and

$$u,\ v > 0, \quad 0 < w \leq m := \max\{\|w_0\|_\infty,\ K\} \quad \text{in} \quad \Omega \times (0, \hat{T}). \tag{2.1}$$

Moreover, the "existence time $\hat{T}$" can be chosen maximal: either $\hat{T} = \infty$, or $\hat{T} < \infty$ and

$$\limsup_{t\to\hat{T}} (\|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty) = \infty.$$

Proof. The local-in-time existence and uniqueness of the classical solutions to problem (1.2) follows from Amann's theorem [2, Theorem 7.3 and Corollary 9.3]. The estimates (2.1) can be derived by the maximum principle. □

Without loss of generality, we may suppose that $\hat{T} > 1$ from now on.

Lemma 2.2. *The unique solution (u, v, w) of (1.2) satisfies*

$$\int_\Omega u\mathrm{d}x \leq m_1 := \max\left\{\int_\Omega u_0\mathrm{d}x,\ \frac{(b_1k_1 + |a_1|)|\Omega|}{\mu_1}\right\}, \quad t \in (0, \hat{T}), \tag{2.2}$$

$$\int_\Omega v\mathrm{d}x \leq m_2 := \max\left\{\int_\Omega v_0\mathrm{d}x,\ \frac{(b_2k_2 + |a_2|)|\Omega|}{\mu_2}\right\}, \quad t \in (0, \hat{T}) \tag{2.3}$$

and

$$\int_t^{t+1}\int_\Omega (u^2 + v^2)\mathrm{d}x\mathrm{d}s \leq M, \quad t \in (0, \hat{T} - 1), \tag{2.4}$$

where $k_i = \|g_i\|_{C([0,m])}$, $i = 1, 2$, *and* $M = \frac{m_1}{\mu_1}(b_1 k_1 + |a_1| + 1) + \frac{m_2}{\mu_2}(b_2 k_2 + |a_2| + 1)$.

Proof. From (A2) and Lemma 2.1, we have $0 \le g_1(w) \le k_1$. By the condition (A3), $\mu_1 > 0$. Integrating the first equation in (1.2) over Ω and using the Cauchy–Schwarz inequality we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u &= b_1\int_\Omega u g_1(w)\mathrm{d}x - \int_\Omega u h_1(u)\mathrm{d}x\\ &\le (b_1k_1 + |a_1|)\int_\Omega u\mathrm{d}x - \mu_1\int_\Omega u^2\mathrm{d}x\\ &\le (b_1k_1 + |a_1|)\int_\Omega u\mathrm{d}x - \frac{\mu_1}{|\Omega|}\left(\int_\Omega u\mathrm{d}x\right)^2, \quad t\in(0,\hat{T}),\end{aligned} \tag{2.5}$$

where we have used (A3) to derive $-h_1(u) \le |a_1| - \mu_1 u$. By an ODE comparison principle, we derive (2.2). Inequality (2.3) can be derived similarly.

Integrating inequality (2.5) over $(t, t+1)$ and using (2.2) yields that

$$\mu_1\int_t^{t+1}\int_\Omega u^2\mathrm{d}x\mathrm{d}s \le \int_\Omega u\mathrm{d}x + (b_1k_1 + |a_1|)\int_t^{t+1}\int_\Omega u\mathrm{d}x\mathrm{d}s \le m_1(b_1k_1 + |a_1| + 1), \quad t\in(0,\hat{T}-1).$$

This combined with a similar argument for v yields (2.4). □

Next we provide a lemma containing a general statement on extensibility and regularity of solutions known to be bounded in $L^\infty((0,\hat{T}); L^p(\Omega))$ for some $p > n/2$.

Lemma 2.3. *Let* $n \ge 1$ *and* (u, v, w) *be the unique solution of* (1.2) *in* $\Omega \times (0, \hat{T})$. *Suppose that there exists a number* $p \ge 1$ *and* $p > n/2$ *for which*

$$\sup_{t\in(0,\hat{T})}(\|u(\cdot,t)\|_p + \|v(\cdot,t)\|_p) < \infty. \tag{2.6}$$

Then $\hat{T} = \infty$ *and*

$$\sup_{t>0}(\|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty + \|w(\cdot,t)\|_\infty) < \infty. \tag{2.7}$$

Furthermore, there exists $\alpha \in (0, 1)$ *and* $C > 0$ *such that*

$$\|u,\ v,\ w\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}\times[1,\infty))} \le C. \tag{2.8}$$

Proof. Let $\Phi(u, v, w) = f(w) - ug_1(w) - vg_2(w)$. Making use of (2.1) we have

$$|\phi_i(w)| \le \|\phi_i\|_{L^\infty(0,m)}, \quad |\Phi(u,v,w)| \le K(u + v + 1)$$

for all $t \in (0, \hat{T})$, where the positive constant K is independent of t. Thanks to [5, Lemma 3.2], it is easy to deduce that $\hat{T} = \infty$ and (2.7) holds. For the details, please refer to [10, Lemma 3.1].

Take advantage of (2.7), inequality (2.8) can be deduced by the standard parabolic regularity theory ([12]). In details, by the same arguments as those in [29, Theorem 2.1] and [27, Theorems 2.1 and 2.2] we can show the regularity $u, v, w \in C(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))$, similar to the discussions in [27, Theorem 3.1] and [29, Theorem 2.1] we can get the estimate (2.8). □

2.2. Proof of Theorem 1.1

In order to prove Theorem 1.1, we first give a lemma.

Lemma 2.4. *Let $n = 2$, (u, v, w) be the unique solution of (1.2). Then there exists $C > 0$ so that*

$$\|u(\cdot,t)\|_2 + \|v(\cdot,t)\|_2 \leq C, \quad t \in (0, \hat{T}).$$

Proof. This proof is inspired by [3, Lemma 2.5]. For convenience, let us denote $k_0 = \|f(w) + w\|_{L^\infty(0,m)}$. Without loss of generality we suppose $\hat{T} > 1$. We first show that

$$\int_t^{t+1}\int_\Omega |\Delta w(x,s)|^2 \mathrm{d}x\mathrm{d}s$$

is bounded with respect to $t \in (0, \hat{T}-1)$. Multiplying the third equation in (1.2) by $-\Delta w$, and integrating the results over Ω and using Young's inequality, we derive that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla w|^2 \mathrm{d}x + d\int_\Omega |\Delta w|^2 \mathrm{d}x + \int_\Omega |\nabla w|^2 \mathrm{d}x \\
&= -\int_\Omega (f(w) + w)\Delta w \mathrm{d}x + \int_\Omega u g_1(w)\Delta w \mathrm{d}x + \int_\Omega v g_2(w)\Delta w \mathrm{d}x \\
&\leq \frac{d}{2}\int_\Omega |\Delta w|^2 \mathrm{d}x + \frac{3}{2d}\int_\Omega \left[(f(w)+w)^2 + u^2 g_1^2(w) + v^2 g_2^2(w)\right] \mathrm{d}x \\
&\leq \frac{d}{2}\int_\Omega |\Delta w|^2 \mathrm{d}x + c_1\left(1 + \int_\Omega (u^2+v^2)\mathrm{d}x\right), \quad t \in (0, \hat{T}),
\end{aligned}$$

where $c_1 = \frac{3}{2d}\max\left\{k_0^2|\Omega|,\, k_1^2,\, k_2^2\right\}$, $k_i = \|g_i\|_{C([0,m])}$, $i = 1, 2$. Let

$$y(t) = \int_\Omega |\nabla w|^2 \mathrm{d}x, \quad g(t) = 2c_1\left(1 + \int_\Omega (u^2+v^2)\mathrm{d}x\right), \quad t \in (0, \hat{T}).$$

Then $y(t)$ satisfies

$$y'(t) + 2y(t) + d\int_\Omega |\Delta w|^2 \mathrm{d}x \leq g(t), \quad t \in (0, \hat{T}). \tag{2.9}$$

Taking advantage of (2.4) we have

$$\int_t^{t+1} g(s)\mathrm{d}s \leq c_2 = 2c_1(1+M), \quad t \in (0, \hat{T}-1).$$

In view of [3, Lemma 2.3] it can be shown that

$$y(t) = \int_\Omega |\nabla w|^2 \mathrm{d}x \leq c_3 = \max\left\{\int_\Omega |\nabla w_0|^2 \mathrm{d}x + c_2,\, \frac{5}{2}c_2\right\}, \quad t \in (0, \hat{T}).$$

Notice $y(t) \geq 0$. Integrating (2.9) over $(t, t+1)$ we have

$$d\int_t^{t+1}\int_\Omega |\Delta w|^2 \mathrm{d}x\mathrm{d}s \leq y(t) + \int_t^{t+1} g(s)\mathrm{d}s \leq c_3 + c_2 := c_4, \quad t \in (0, \hat{T}-1). \tag{2.10}$$

Now we study the L^2 estimates for u and v. By the first equation in (1.2) and (A3) we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2\mathrm{d}x+\int_\Omega|\nabla u|^2\mathrm{d}x=\int_\Omega u\phi_1(w)\nabla u\cdot\nabla w\mathrm{d}x+\int_\Omega u^2(b_1g_1(w)-h_1(u))\mathrm{d}x$$
$$\le\int_\Omega u\phi_1(w)\nabla u\cdot\nabla w\mathrm{d}x+(b_1k_1+|a_1|)\int_\Omega u^2\mathrm{d}x \tag{2.11}$$

for all $t\in(0,\hat{T})$. It follows by the Hölder's inequality that, for some $c_5>0$,

$$\begin{aligned}\int_\Omega u\phi_1(w)\nabla u\cdot\nabla w\mathrm{d}x&=\frac{1}{2}\int_\Omega\phi_1(w)\nabla(u^2)\cdot\nabla w\mathrm{d}x\\&=-\frac{1}{2}\int_\Omega u^2\phi_1'(w)|\nabla w|^2\mathrm{d}x-\frac{1}{2}\int_\Omega u^2\phi_1(w)\Delta w\mathrm{d}x\\&\le\frac{1}{2}\|\phi_1'\|_{L^\infty(0,m)}\|u\|_4^2\||\nabla w|^2\|_2+\frac{k_1}{2}\|u\|_4^2\,\|\Delta w\|_2\\&\le c_5\|u\|_4^2\big(\|\nabla w\|_4^2+\|\Delta w\|_2\big),\quad t\in(0,\hat{T}).\end{aligned}\tag{2.12}$$

Thanks to the G–N (Gagliardo–Nirenberg) inequality ([23]) and (2.2) we can find $c_6>0$ such that

$$\|u\|_4^2\le c_6\|\nabla u\|_2\|u\|_2+c_6\|u\|_1^2\le c_6\|\nabla u\|_2\|u\|_2+c_6m_1^2,\quad t\in(0,\hat{T}).\tag{2.13}$$

Again, using the G–N inequality and (2.1) we have

$$\|\nabla w\|_4^2\le c_7\|\Delta w\|_2\|w\|_\infty+c_7\|w\|_\infty^2\le c_7(m\|\Delta w\|_2+m^2),\quad t\in(0,\hat{T}),\tag{2.14}$$

for some $c_7>0$.

Substituting (2.13) and (2.14) into (2.12) and applying the Young inequality, we get

$$\begin{aligned}\int_\Omega u\phi_1(w)\nabla u\cdot\nabla w\mathrm{d}x&\le c_8(\|\nabla u\|_2\|u\|_2+1)(\|\Delta w\|_2+1)\\&\le\|\nabla u\|_2^2+c_9\big(\|u\|_2^2\|\Delta w\|_2^2+\|u\|_2^2+\|\Delta w\|_2^2+1\big),\end{aligned}$$

where $c_8,c_9>0$. Inserting this into (2.11) yields that there exists $c_{10}>0$ fulfilling

$$\begin{aligned}z'(t)=\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2\mathrm{d}x\ &\le\ c_{10}\left(\int_\Omega u^2\mathrm{d}x+1\right)\left(\int_\Omega|\Delta w|^2\mathrm{d}x+1\right)\\&:=c_{10}z(t)h(t),\quad t\in(0,\hat{T}),\end{aligned}\tag{2.15}$$

where

$$z(t)=\int_\Omega u^2\mathrm{d}x+1,\quad h(t)=\int_\Omega|\Delta w|^2\mathrm{d}x+1.$$

Clearly, $z(t)$ is bounded in $[0,1]$. Fix $t\in(1,\hat{T})$. By (2.4), there exists $t_0\in[t-1,t]$ such that

$$z(t_0)=\int_\Omega u^2(x,t_0)\mathrm{d}x+1\le M+1.$$

In view of (2.10) we have

$$\int_{t_0}^t h(s)\mathrm{d}s=\int_{t_0}^t\left(\int_\Omega|\Delta w|^2\mathrm{d}x+1\right)\mathrm{d}s\le\int_{t-1}^t\left(\int_\Omega|\Delta w|^2\mathrm{d}x+1\right)\mathrm{d}s\le 1+c_4/d.$$

Now an integration of (2.15) over (t_0, t) shows that

$$z(t) \leq z(t_0) e^{c_{10} \int_{t_0}^{t} h(s)\mathrm{d}s} \leq (M+1) e^{c_{10}(1+c_4/d)}, \quad 1 < t < \hat{T}.$$

Thus $\|u(\cdot,t)\|_2 \leq C$ in $(0,\hat{T})$. Similarly, $\|v(\cdot,t)\|_2 \leq C$ in $(0,\hat{T})$. □

Proof of Theorem 1.1. Using (2.2), (2.3) and Lemma 2.3 with $p = 1$ when $n = 1$; using Lemma 2.3 with $p = 2$ and Lemma 2.4 when $n = 2$, the conclusion of Theorem 1.1 is obtained immediately. The proof is complete. □

2.3. Proof of Theorem 1.2

Theorem 1.2 can be derived by Lemma 2.3 and the following lemma.

Lemma 2.5. *Under the conditions of Theorem 1.2, there exist $k > n/2$, $C > 0$ so that*

$$\|u(\cdot,t)\|_k + \|v(\cdot,t)\|_k \leq C, \quad t \in (0,\hat{T}).$$

Proof. This proof is inspired by [21,31,33]. From (1.4), there exists $k > n/2$ such that

$$\|\phi_i\|_{L^\infty(0,m)} \leq \frac{d}{km\left(1 + 2\sqrt{(d+1)^2 - \frac{d(k-1)}{k}}\right)}, \quad i = 1, 2, \tag{2.16}$$

Let us define $L_1 = \|\phi_1\|_{C([0,m])}$, $k_1 = \|g_1\|_{C([0,m])}$ and a weight function

$$\rho(w) = e^{\beta w^2}, \quad 0 \leq w \leq m,$$

where $\beta > 0$ will be chosen later, $m = \max\{\|w_0\|_\infty, K\}$ and K is given by the condition (A4). Then $1 \leq \rho(w) \leq e^{\beta m^2} := r$ for $0 \leq w \leq m$. The direct calculation yields

$$\begin{aligned}
\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^k \rho(w)\mathrm{d}x &= \int_\Omega u^{k-1}\rho(w)u_t\mathrm{d}x + \frac{1}{k}\int_\Omega u^k \rho'(w) w_t \mathrm{d}x \\
&= \int_\Omega u^{k-1}\rho(w)[\Delta u - \nabla\cdot(u\phi_1(w)\nabla w) + b_1 u g_1(w) - u h_1(u)]\mathrm{d}x \\
&\quad + \frac{1}{k}\int_\Omega u^k \rho'(w)[d\Delta w + f(w) - u g_1(w) - v g_2(w)]\mathrm{d}x \\
&\leq -(k-1)\int_\Omega u^{k-2}\rho(w)|\nabla u|^2\mathrm{d}x - \int_\Omega u^{k-1}\rho'(w)\nabla u\cdot\nabla w\mathrm{d}x \\
&\quad + (k-1)\int_\Omega u^{k-1}\phi_1(w)\rho(w)\nabla u\cdot\nabla w\mathrm{d}x + \int_\Omega u^k \phi_1(w)\rho'(w)|\nabla w|^2\mathrm{d}x \\
&\quad + (b_1 k_1 + |a_1|)\int_\Omega u^k \rho(w)\mathrm{d}x - \frac{d}{k}\int_\Omega u^k \rho''(w)|\nabla w|^2\mathrm{d}x \\
&\quad - d\int_\Omega u^{k-1}\rho'(w)\nabla u\cdot\nabla w\mathrm{d}x + \frac{2q\beta}{k}\int_\Omega u^k \rho(w) w^2\mathrm{d}x,
\end{aligned}$$

which implies that

$$\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega} u^k\rho(w)\mathrm{d}x+(k-1)\int\limits_{\Omega} u^{k-2}\rho(w)|\nabla u|^2\mathrm{d}x+\frac{d}{k}\int\limits_{\Omega} u^k\rho''(w)|\nabla w|^2\mathrm{d}x$$
$$\le -(d+1)\int\limits_{\Omega} u^{k-1}\rho'(w)\nabla u\cdot\nabla w\mathrm{d}x+L_1(k-1)\int\limits_{\Omega} u^{k-1}\rho(w)|\nabla u\cdot\nabla w|\mathrm{d}x$$
$$+L_1\int\limits_{\Omega} u^k\rho'(w)|\nabla w|^2\mathrm{d}x+c_1\int\limits_{\Omega} u^k\rho(w)\mathrm{d}x, \tag{2.17}$$

where $c_1=b_1k_1+|a_1|+2q\beta m^2/k$. It is easy to check that

$$\frac{d}{k}\int\limits_{\Omega} u^k\rho''(w)|\nabla w|^2\mathrm{d}x=\frac{2d\beta}{k}\int\limits_{\Omega}\left(1+2\beta w^2\right)\rho(w)u^k|\nabla w|^2\mathrm{d}x. \tag{2.18}$$

Take advantage of Young's inequality, we have

$$\left|\int\limits_{\Omega} u^{k-1}\rho'(w)\nabla u\cdot\nabla w\mathrm{d}x\right|\le\frac{k-1}{4(d+1)}\int\limits_{\Omega} u^{k-2}\rho(w)|\nabla u|^2\mathrm{d}x+\frac{d+1}{k-1}\int\limits_{\Omega} u^k\frac{(\rho'(w))^2}{\rho(w)}|\nabla w|^2\mathrm{d}x, \tag{2.19}$$

$$L_1\int\limits_{\Omega} u^{k-1}\rho(w)|\nabla u\cdot\nabla w|\mathrm{d}x\le\frac{1}{4}\int\limits_{\Omega} u^{k-2}\rho(w)|\nabla u|^2\mathrm{d}x+L_1^2\int\limits_{\Omega} u^k\rho(w)|\nabla w|^2\mathrm{d}x. \tag{2.20}$$

Plugging (2.18)–(2.20) into (2.17), we get

$$\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega} u^k\rho(w)\mathrm{d}x+\frac{k-1}{2}\int\limits_{\Omega} u^{k-2}\rho(w)|\nabla u|^2\mathrm{d}x+\frac{2d\beta}{k}\int\limits_{\Omega}\left(1+2\beta w^2\right)\rho(w)u^k|\nabla w|^2\mathrm{d}x$$
$$\le\frac{(d+1)^2}{k-1}\int\limits_{\Omega}\frac{(\rho'(w))^2}{\rho(w)}u^k|\nabla w|^2\mathrm{d}x+L_1^2(k-1)\int\limits_{\Omega}\rho(w)u^k|\nabla w|^2\mathrm{d}x$$
$$+L_1\int\limits_{\Omega}\rho'(w)u^k|\nabla w|^2\mathrm{d}x+c_1\int\limits_{\Omega} u^k\rho(w)\mathrm{d}x$$
$$=\int\limits_{\Omega}\left(\frac{(d+1)^2}{k-1}\frac{(\rho'(w))^2}{\rho(w)}+L_1^2(k-1)\rho(w)+L_1\rho'(w)\right)u^k|\nabla w|^2\mathrm{d}x+c_1\int\limits_{\Omega} u^k\rho(w)\mathrm{d}x$$
$$=\int\limits_{\Omega}\left(\frac{4(d+1)^2\beta^2}{k-1}w^2+L_1^2(k-1)+2L_1\beta w\right)\rho(w)u^k|\nabla w|^2\mathrm{d}x+c_1\int\limits_{\Omega} u^k\rho(w)\mathrm{d}x. \tag{2.21}$$

Next we shall show that there exists $\beta>0$ so as to

$$\frac{4(d+1)^2\beta^2}{k-1}w^2+L_1^2(k-1)+2L_1\beta w\le\frac{2d\beta}{k}\left(1+2\beta w^2\right),$$

i.e.,

$$4w^2\left(\frac{(d+1)^2}{k-1}-\frac{d}{k}\right)\beta^2+2\left(L_1w-\frac{d}{k}\right)\beta+L_1^2(k-1)\le 0. \tag{2.22}$$

It is sufficient to prove that there exists $\beta>0$ such that

$$J(\beta):=4m^2\left(\frac{(d+1)^2}{k-1}-\frac{d}{k}\right)\beta^2+2\left(L_1m-\frac{d}{k}\right)\beta+L_1^2(k-1)\le 0.$$

To this purpose, let us investigate the coefficients of $J(\beta)$. Set

$$j_1 = 4m^2\left(\frac{(d+1)^2}{k-1} - \frac{d}{k}\right), \quad j_2 = 2\left(L_1 m - \frac{d}{k}\right), \quad j_3 = (k-1)L_1^2.$$

Thanks to (2.16), we have $j_2 < 0$. Besides,

$$\begin{aligned} j_2^2 - 4j_1j_3 &= 4\left(L_1 m - d/k\right)^2 - 16m^2L_1^2\left[(d+1)^2 - d(k-1)/k\right] \\ &= 4m^2\left[1 - 4(d+1)^2 + 4d(k-1)/k\right]L_1^2 - (8dm/k)L_1 + 4d^2/k^2 \\ &= 4m^2(1-\gamma^2)L_1^2 - (8dm/k)L_1 + 4d^2/k^2 \\ &:= aL_1^2 + bL_1 + c, \end{aligned}$$

where $\gamma = 2\sqrt{(d+1)^2 - d(k-1)/k}$. It is easy to see that $1-\gamma^2 < 0$, i.e., $a < 0$, and

$$b^2 - 4ac = 64d^2m^2/k^2 - 64d^2m^2(1-\gamma^2)/k^2 = (8dm\gamma/k)^2 > 0.$$

Therefore, the quadratic polynomial $aL_1^2 + bL_1 + c$ has two roots $-\frac{d}{mk(\gamma-1)}$ and $\frac{d}{mk(\gamma+1)}$. Hence, $j_2^2 - 4j_1j_3 \geq 0$ if and only if

$$-\frac{d}{mk(\gamma-1)} \leq L_1 \leq \frac{d}{mk(\gamma+1)}.$$

Due to (2.16), $j_2^2 - 4j_1j_3 \geq 0$ holds. Recalling $j_2 < 0$, we can find a constant $\beta > 0$ so that $J(\beta) \leq 0$. Thereby, inequality (2.22) holds.

It follows from (2.21) and (2.22) that

$$\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^k\rho(w)\mathrm{d}x + \frac{k-1}{2}\int_\Omega u^{k-2}\rho(w)|\nabla u|^2\mathrm{d}x \leq c_1\int_\Omega u^k\rho(w)\mathrm{d}x. \tag{2.23}$$

Taking advantage of the G–N inequality and Poincaré inequality ([9]) firstly, and using (2.2) secondly, we have

$$\begin{aligned} \int_\Omega u^k\rho(w)\mathrm{d}x &\leq r\int_\Omega u^k = r\|u^{k/2}\|_2^2 \\ &\leq rc_2\|u^{k/2}\|_{W^{1,2}(\Omega)}^{2s}\|u^{k/2}\|_{2/k}^{2(1-s)} \\ &\leq rc_3\left(\|\nabla u^{k/2}\|_2 + \|u^{k/2}\|_{2/k}\right)^{2s}\|u^{k/2}\|_{2/k}^{2(1-s)} \\ &= rc_3\left(\|\nabla u^{k/2}\|_2 + \|u\|_1^{k/2}\right)^{2s}\|u\|_1^{k(1-s)} \\ &\leq rc_3\left(\|\nabla u^{k/2}\|_2 + m_1^{k/2}\right)^{2s}m_1^{k(1-s)} \\ &\leq c_4\left(\|\nabla u^{k/2}\|_2^2 + 1\right)^s \end{aligned} \tag{2.24}$$

with some constants $c_2, c_3, c_4 > 0$ and

$$s = \frac{k/2 - 1/2}{k/2 + 1/n - 1/2} \in (0,1).$$

As $\rho(w) \geq 1$, by use of (2.24) we have

$$\int_\Omega u^{k-2}\rho(w)|\nabla u|^2\mathrm{d}x \geq \int_\Omega u^{k-2}|\nabla u|^2\mathrm{d}x = \frac{4}{k^2}\int_\Omega |\nabla u^{k/2}|^2\mathrm{d}x$$
$$\geq \frac{4}{k^2 c_4^{1/s}}\left(\int_\Omega u^k\rho(w)\mathrm{d}x\right)^{1/s} - \frac{4}{k^2}. \tag{2.25}$$

Inserting (2.25) into (2.23) we have

$$\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^k\rho(w)\mathrm{d}x \leq -\frac{2(k-1)}{k^2 c_4^{1/s}}\left(\int_\Omega u^k\rho(w)\mathrm{d}x\right)^{1/s} + c_1\int_\Omega u^k\rho(w)\mathrm{d}x + \frac{2(k-1)}{k^2}$$

for all $t \in (0, \hat{T})$. Recall that $s \in (0,1)$ and $\rho(w) \geq 1$. By an ODE comparison principle, there exists $c_5 > 0$ such that

$$\|u(\cdot,t)\|_k \leq \left(\int_\Omega u^k\rho(w)\mathrm{d}x\right)^{1/k} \leq c_5, \quad t \in (0, \hat{T}).$$

This completes the proof. □

3. Global stability

We shall use the ideas in [3] to prove Theorems 1.3–1.6. Let us first recall a basic result.

Lemma 3.1. (Barbălat's Lemma *[4]*) *Suppose that* $h : [1,\infty) \to \mathbb{R}$ *is uniformly continuous and that* $\lim_{t\to\infty}\int_1^t h(s)\mathrm{d}s$ *exists. Then* $\lim_{t\to\infty} h(t) = 0$ *holds.*

Next we give a lemma which will play the important roles in our later discussions.

Lemma 3.2. *Let* $(\underline{u}, \underline{v}, \underline{w})$ *be one solution of*

$$\begin{cases} (b_1\underline{w} - a_1 - \underline{u})\underline{u} = 0, \\ (b_2\underline{w} - a_2 - \underline{v})\underline{v} = 0, \\ [q(1-\underline{w}) - \underline{u} - \underline{v}]\underline{w} = 0, \end{cases} \tag{3.1}$$

and (u,v,w) *be a bounded global classical solution of* (1.5). *Suppose that there exist two decreasing functions* $G_1(t)$ *and* $G_2(t)$ *defined in* $[t_0,\infty)$ *for some* $t_0 > 0$ *such that*

$$\|u-\underline{u}\|_2^2 + \|v-\underline{v}\|_2^2 + \|w-\underline{w}\|_2^2 \leq G_1(t), \quad t > t_0, \tag{3.2}$$
$$\left(\int_{t-1}^t\int_\Omega |\nabla w|^2\mathrm{d}x\mathrm{d}s\right)^{\frac{1}{2n+2}} \leq G_2(t), \quad t > t_0+1.$$

Then, there exists a constant $C > 0$ *so as to*

$$\|u-\underline{u}\|_\infty + \|v-\underline{v}\|_\infty + \|w-\underline{w}\|_\infty \leq C[G_1^{\frac{1}{2n}}(t-1) + G_2(t)], \quad t > t_0+2. \tag{3.3}$$

Proof. Note that (u, v, w) exists globally and is bounded. It follows from (3.2) and the Hölder inequality: $\|\varphi\|_{2n} \le \|\varphi\|_{\infty}^{\frac{n-1}{n}} \|\varphi\|_{2}^{\frac{1}{n}}$ that there exists a positive constant $\hat{C}$ such that

$$\|u - \underline{u}\|_{2n} + \|v - \underline{v}\|_{2n} + \|w - \underline{w}\|_{2n} \le \hat{C} G_1^{\frac{1}{2n}}(t), \quad t > t_0.$$

Evidently, $\hat{G}_1(t) := \hat{C} G_1^{\frac{1}{2n}}(t)$ is a decreasing function in $[t_0, \infty)$ for $t_0 > 0$. By use of [3, Lemma 3.6] we can get the estimate (3.3). □

3.1. Global stability of (u_*, v_*, w_*): proof of Theorem 1.3

In this subsection, we always assume that A, $B > 0$, (u, v, w) is a bounded global solution of (1.5) and (1.8) holds. The constants A, B are given by (1.6), and (u_*, v_*, w_*) is given by (1.7).

Lemma 3.3. *There is $\varepsilon > 0$ such that functions $E_1(t)$, $F_1(t)$ defined by*

$$E_1(t) = \int_\Omega \left[\frac{1}{b_1} \left(u - u_* - u_* \ln \frac{u}{u_*} \right) + \frac{1}{b_2} \left(v - v_* - v_* \ln \frac{v}{v_*} \right) + w - w_* - w_* \ln \frac{w}{w_*} \right] dx,$$

$$F_1(t) = \int_\Omega \left[(u - u_*)^2 + (v - v_*)^2 + (w - w_*)^2 + |\nabla w|^2 \right] \mathrm{d}x$$

satisfies

$$E_1'(t) \le -\varepsilon F_1(t), \quad t > 0. \tag{3.4}$$

Proof. For the convenience, we set

$$A_1(t) = \frac{1}{b_1} \int_\Omega \left(u - u_* - u_* \ln \frac{u}{u_*} \right) \mathrm{d}x,$$

$$B_1(t) = \frac{1}{b_2} \int_\Omega \left(v - v_* - v_* \ln \frac{v}{v_*} \right) \mathrm{d}x,$$

$$D_1(t) = \int_\Omega \left(w - w_* - w_* \ln \frac{w}{w_*} \right) \mathrm{d}x.$$

Evidently, $A_1(t), B_1(t), D_1(t) \ge 0$. The straightforward calculation gives

$$A_1'(t) = \int_\Omega \left(-\frac{u_*}{b_1} \frac{|\nabla u|^2}{u^2} + \frac{u_*}{b_1} \phi_1(w) \frac{\nabla u}{u} \cdot \nabla w - \frac{1}{b_1}(u - u_*)^2 + (u - u_*)(w - w_*) \right) \mathrm{d}x,$$

$$B_1'(t) = \int_\Omega \left(-\frac{v_*}{b_2} \frac{|\nabla v|^2}{v^2} + \frac{v_*}{b_2} \phi_2(w) \frac{\nabla v}{v} \cdot \nabla w - \frac{1}{b_2}(v - v_*)^2 + (v - v_*)(w - w_*) \right) \mathrm{d}x,$$

$$D_1'(t) = \int_\Omega \left(-d w_* \frac{|\nabla w|^2}{w^2} - q(w - w_*)^2 - (u - u_*)(w - w_*) - (v - v_*)(w - w_*) \right) \mathrm{d}x.$$

Thus, we have

$$E_1'(t) = I_1(t) + I_2(t),$$

where

$$I_1(t) = \int_\Omega \left[\frac{u_*}{b_1}\left(\phi_1(w)\frac{\nabla u}{u}\cdot\nabla w - \frac{|\nabla u|^2}{u^2}\right) + \frac{v_*}{b_2}\left(\phi_2(w)\frac{\nabla v}{v}\cdot\nabla w - \frac{|\nabla v|^2}{v^2}\right) - dw_*\frac{|\nabla w|^2}{w^2}\right]\mathrm{d}x,$$

$$I_2(t) = -\int_\Omega \left(\frac{1}{b_1}(u-u_*)^2 + \frac{1}{b_2}(v-v_*)^2 + q(w-w_*)^2\right)\mathrm{d}x.$$

Set $\varepsilon_1 = \min\{1/b_1,\, 1/b_2,\, q\}$. Then, it is clear that

$$I_2(t) \le -\varepsilon_1 \int_\Omega \left[(u-u_*)^2 + (v-v_*)^2 + (w-w_*)^2\right]\mathrm{d}x.$$

We claim that there exists $\varepsilon_2 > 0$ such that

$$I_1(t) \le -\varepsilon_2 \int_\Omega |\nabla w|^2 \mathrm{d}x. \tag{3.5}$$

Once this is done, by choosing $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ we then get (3.4).

Now we prove (3.5). Firstly, as $w \le m$ (cf. (2.1)), we have

$$I_1(t) \le \int_\Omega \left[\frac{u_*}{b_1}\left(\phi_1(w)\frac{\nabla u}{u}\cdot\nabla w - \frac{|\nabla u|^2}{u^2}\right) + \frac{v_*}{b_2}\left(\phi_2(w)\frac{\nabla v}{v}\cdot\nabla w - \frac{|\nabla v|^2}{v^2}\right) - \frac{dw_*}{m^2}|\nabla w|^2\right]\mathrm{d}x.$$

As above, we let $L_i = \|\phi_i\|_{L^\infty(0,m)}$, $i = 1, 2$. An application of the Young inequality yields

$$\int_\Omega \phi_1(w)\frac{\nabla u}{u}\cdot\nabla w \mathrm{d}x \le L_1 \int_\Omega \frac{|\nabla u\cdot\nabla w|}{u}\mathrm{d}x \le \int_\Omega \frac{|\nabla u|^2}{u^2}\mathrm{d}x + \frac{L_1^2}{4}\int_\Omega |\nabla w|^2 \mathrm{d}x,$$

$$\int_\Omega \phi_2(w)\frac{\nabla v}{v}\cdot\nabla w \mathrm{d}x \le L_2 \int_\Omega \frac{|\nabla v\cdot\nabla w|}{v}\mathrm{d}x \le \int_\Omega \frac{|\nabla v|^2}{v^2}\mathrm{d}x + \frac{L_2^2}{4}\int_\Omega |\nabla w|^2 \mathrm{d}x.$$

Consequently,

$$I_1(t) \le -\left(\frac{dw_*}{m^2} - \frac{L_1^2 u_*}{4b_1} - \frac{L_2^2 v_*}{4b_2}\right)\int_\Omega |\nabla w|^2\mathrm{d}x := -\varepsilon_2 \int_\Omega |\nabla w|^2 \mathrm{d}x.$$

The assumption (1.8) implies $\varepsilon_2 > 0$, and so (3.5) holds. $\square$

Lemma 3.4. *The following asymptotic behavior holds:*

$$\|u-u_*\|_\infty + \|v-v_*\|_\infty + \|w-w_*\|_\infty \to 0 \quad \text{as } t\to\infty. \tag{3.6}$$

Proof. Let

$$f_1(t) = \int_\Omega \left[(u-u_*)^2 + (v-v_*)^2 + (w-w_*)^2\right]\mathrm{d}x.$$

Clearly, $0 \le f_1(t) \le F_1(t)$. Hence, by (3.4), $E_1'(t) \le -\varepsilon F_1(t) \le -\varepsilon f_1(t)$. Since $E_1(t) \ge 0$, we have $\int_1^\infty f_1(t)dt \le \frac{1}{\varepsilon}E_1(1) < \infty$. It follows from the regularity of u, v, w that $f_1(t)$ is uniformly continuous in $[1,\infty)$. An application of Lemma 3.1 yields

$$\int_\Omega \left[(u-u_*)^2 + (v-v_*)^2 + (w-w_*)^2\right]\mathrm{d}x = f_1(t) \to 0 \quad \text{as } t\to\infty. \tag{3.7}$$

Note that (u, v, w) is a bounded global solution of (1.5). By the standard parabolic regularity theory, we can get the estimate (2.8), which implies that, in the space $W^{1,\infty}(\Omega)$, $u(\cdot, t)$, $v(\cdot, t)$ and $w(\cdot, t)$ are bounded for $t > 1$. Apply the G–N inequality

$$\|\psi\|_\infty \le c\|\psi\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}}\|\psi\|_2^{\frac{2}{n+2}}, \quad \forall\ \psi \in W^{1,\infty}(\Omega)$$

to $u - u_*$, $v - v_*$ and $w - w_*$, respectively, the limit (3.6) is deduced by (3.7). □

Proof of Theorem 1.3. For the given positive constant y_*, we define $h(y) = y - y_* \ln y$ for $y > 0$. By L'Hôpital's rule, one can easily check that

$$\lim_{y\to y_*} \frac{h(y) - h(y_*)}{(y - y_*)^2} = \lim_{y\to y_*} \frac{h'(y)}{2(y - y_*)} = \frac{1}{2y_*}.$$

Remember the limit (3.6), it follows that there is $t_0 > 0$ such that

$$\frac{1}{4u_*}\int_\Omega (u - u_*)^2 \mathrm{d}x \le \int_\Omega \left(u - u_* - u_* \ln \frac{u}{u_*}\right) \mathrm{d}x \le \frac{1}{u_*}\int_\Omega (u - u_*)^2 \mathrm{d}x, \tag{3.8}$$

$$\frac{1}{4u_*}\int_\Omega (v - v_*)^2 \mathrm{d}x \le \int_\Omega \left(v - v_* - v_* \ln \frac{v}{v_*}\right) \mathrm{d}x \le \frac{1}{v_*}\int_\Omega (v - v_*)^2 \mathrm{d}x, \tag{3.9}$$

$$\frac{1}{4w_*}\int_\Omega (w - w_*)^2 \mathrm{d}x \le \int_\Omega \left(w - w_* - w_* \ln \frac{w}{w_*}\right) \mathrm{d}x \le \frac{1}{w_*}\int_\Omega (w - w_*)^2 \mathrm{d}x \tag{3.10}$$

for all $t > t_0$. Recall the definitions of $E_1(t)$ and $F_1(t)$, it follows from the right inequalities in (3.8)–(3.10) that $E_1(t) \le c_1 F_1(t)$ for all $t > t_0$ and some $c_1 > 0$. Inserting this into (3.4) we get $E_1'(t) \le -\varepsilon F_1(t) \le -\frac{\varepsilon}{c_1} E_1(t)$ for $t > t_0$. Thus, $E_1(t) \le c_2 e^{-\sigma t}$ for $t > t_0$ and some c_2, $\sigma > 0$. In view of the left inequalities in (3.8)–(3.10), there exist c_3, $c_4 > 0$ such that

$$\int_\Omega (u - u_*)^2 + \int_\Omega (v - v_*)^2 + \int_\Omega (w - w_*)^2 \le c_3 E_1(t) \le c_4 e^{-\sigma t}, \quad t > t_0.$$

Besides, there is $c_5 > 0$ such that

$$\begin{aligned}\int_{t-1}^{t}\int_\Omega |\nabla w|^2 &\le \int_{t-1}^{t} F_1(s)\mathrm{d}s \le -\frac{1}{\varepsilon}\int_{t-1}^{t} E_1'(s)\mathrm{d}s \\ &\le \frac{1}{\varepsilon} E_1(t-1) \le c_5 e^{-\sigma t}, \quad t > t_0 + 1.\end{aligned}$$

By Lemma 3.2, we can find C_1', $\lambda_1 > 0$ such that

$$\|u - u_*\|_\infty + \|v - v_*\|_\infty + \|w - w_*\|_\infty \le C_1' e^{-\lambda_1 t}, \quad t > t_0 + 2.$$

Thus (1.9) holds, and the proof is complete. □

3.2. Global stability of $(0, \tilde{v}, \tilde{w})$: Proof of Theorem 1.4

Throughout this subsection, we always assume that $A \le 0$, $B > 0$, $a_2 < b_2$, and (u, v, w) is a bounded global classical solution of (1.5) and (1.11) holds. Constants $\tilde{v}$, $\tilde{w}$ are given by (1.10).

Lemma 3.5. *Assume $a_1 \geq b_1\tilde{w}$. Then there is $\varepsilon > 0$ such that functions $E_2(t)$, $F_2(t)$ defined by*

$$E_2(t) = \int_\Omega \left[\frac{1}{b_1}u + \frac{1}{b_2}\left(v - \tilde{v} - \tilde{v}\ln\frac{v}{\tilde{v}}\right) + w - \tilde{w} - \tilde{w}\ln\frac{w}{\tilde{w}}\right] \mathrm{d}x,$$

$$F_2(t) = \int_\Omega \left[u^2 + (v-\tilde{v})^2 + (w-\tilde{w})^2 + |\nabla w|^2\right] \mathrm{d}x$$

satisfies

$$E_2'(t) \leq -\varepsilon F_2(t) - \left(\frac{a_1}{b_1} - \tilde{w}\right)\int_\Omega u dx, \quad t > 0. \tag{3.11}$$

Proof. Similar to the proof of Lemma 3.3, by a series of calculations we can get

$$E_2'(t) \leq I_3(t) + I_4(t) - \left(\frac{a_1}{b_1} - \tilde{w}\right)\int_\Omega u \mathrm{d}x,$$

where

$$I_3(t) = \int_\Omega \left(\frac{\tilde{v}}{b_2}\phi_2(w)\frac{\nabla v}{v}\cdot\nabla w - \frac{\tilde{v}}{b_2}\frac{|\nabla v|^2}{v^2} - d\tilde{w}\frac{|\nabla w|^2}{w^2}\right)\mathrm{d}x,$$

$$I_4(t) = -\int_\Omega \left(\frac{1}{b_1}u^2 + \frac{1}{b_2}(v-\tilde{v})^2 + q(w-\tilde{w})^2\right)\mathrm{d}x.$$

Let $\varepsilon_1 = \min\{1/b_1, 1/b_2, q\}$. Then

$$I_4(t) \leq -\varepsilon_1 \int_\Omega \left[u^2 + (v-\tilde{v})^2 + (w-\tilde{w})^2\right]\mathrm{d}x. \tag{3.12}$$

In treating $I_3(t)$, we apply Young's inequality and (2.1) to derive that

$$\begin{aligned} I_3(t) &\leq \int_\Omega \left(-\frac{\tilde{v}}{b_2}\frac{|\nabla v|^2}{v^2} + \frac{L_2\tilde{v}}{b_2}\frac{|\nabla v\cdot\nabla w|}{v} - \frac{d\tilde{w}}{m^2}|\nabla w|^2\right)\mathrm{d}x \\ &\leq -\left(\frac{d\tilde{w}}{m^2} - \frac{L_2^2\tilde{v}}{4b_2}\right)\int_\Omega |\nabla w|^2\mathrm{d}x \\ &:= -\varepsilon_2\int_\Omega |\nabla w|^2\mathrm{d}x, \end{aligned}$$

where $m = \max\{\|w_0\|_\infty, 1\}$ and $L_2 = \|\phi_2\|_{L^\infty(0,m)}$. By (1.11), $\varepsilon_2 > 0$. This combines with (3.12) gives (3.11). □

Proof of Theorem 1.4 (i). Suppose that $a_1 > b_1\tilde{w}$. Remember that (u, v, w) is a global bounded solution of (1.5). By the same argument as in Lemma 3.4 we can get

$$\|u\|_\infty + \|v-\tilde{v}\|_\infty + \|w-\tilde{w}\|_\infty \to 0 \ \text{ as } t \to 0. \tag{3.13}$$

Using the fact $\lim\limits_{y\to 0}\frac{y}{y^2+y}=1$, we assert that there exists $t_0>0$ such that

$$\frac{1}{2}\int\limits_\Omega u^2\mathrm{d}x+\frac{1}{2}\int\limits_\Omega u\mathrm{d}x\le\int\limits_\Omega u\mathrm{d}x\le 2\int\limits_\Omega u^2\mathrm{d}x+2\int\limits_\Omega u\mathrm{d}x,\quad t>t_1, \tag{3.14}$$

$$\frac{1}{4\tilde v}\int\limits_\Omega (v-\tilde v)^2\mathrm{d}x\le\int\limits_\Omega\left(v-\tilde v-\tilde v\ln\frac{v}{\tilde v}\right)\mathrm{d}x\le\frac{1}{\tilde v}\int\limits_\Omega (v-\tilde v)^2\mathrm{d}x,\quad t>t_0, \tag{3.15}$$

$$\frac{1}{4\tilde w}\int\limits_\Omega (w-\tilde w)^2\mathrm{d}x\le\int\limits_\Omega\left(w-\tilde w-\tilde w\ln\frac{w}{\tilde w}\right)\mathrm{d}x\le\frac{1}{\tilde w}\int\limits_\Omega (w-\tilde w)^2\mathrm{d}x,\quad t>t_0. \tag{3.16}$$

In view of the definitions of $E_2(t), F_2(t)$ and the right inequalities in (3.14)–(3.16), we get

$$E_2(t)\le c_1\left(F_2(t)+\int\limits_\Omega u\mathrm{d}x\right),\quad t>t_0,$$

where $c_1=\max\left\{2/b_1,\,1/(b_2\tilde v),\,1/\tilde w\right\}$. It follows that

$$-F_2(t)\le-\frac{E_2(t)}{c_1}+\int\limits_\Omega u\mathrm{d}x,\quad t>t_0. \tag{3.17}$$

Note that $a_1>b_1\tilde w$, without loss of generality we can choose $\varepsilon<a_1/b_1-\tilde w$ in inequality (3.11). Plugging (3.17) into (3.11) we get

$$\begin{aligned}E_2'(t)&\le-\varepsilon F_2(t)-\left(\frac{a_1}{b_1}-\tilde w\right)\int\limits_\Omega u\mathrm{d}x\\&\le-\frac{\varepsilon}{c_1}E_2(t)-\left(\frac{a_1}{b_1}-\tilde w-\varepsilon\right)\int\limits_\Omega u\mathrm{d}x\\&\le-\frac{\varepsilon}{c_1}E_2(t),\quad t>t_0.\end{aligned}$$

This implies that there exist $c_2,\,\sigma>0$ such that $E_2(t)\le c_2e^{-\sigma t}$ for $t>t_0$. Hence, by the left inequalities in (3.14)–(3.16) we have

$$\int\limits_\Omega\left[u^2+(v-\tilde v)^2+(w-\tilde w)^2\right]\mathrm{d}x\le c_3E_2(t)\le c_4e^{-\sigma t},\quad t>t_0$$

with some $c_3,c_4>0$. Moreover,

$$\begin{aligned}\int\limits_{t-1}^{t}\int\limits_\Omega|\nabla w|^2\mathrm{d}x\mathrm{d}s&\le\int\limits_{t-1}^{t}F_2(s)\mathrm{d}s\le-\frac{1}{\varepsilon}\int\limits_{t-1}^{t}\frac{d}{\mathrm{d}s}E_2(s)\mathrm{d}s\\&\le\frac{1}{\varepsilon}E_2(t-1)\le c_5e^{-\sigma t},\quad t>t_0+1\end{aligned}$$

with some $c_5>0$. In light of Lemma 3.2, there exists $C_2',\,\lambda_2>0$ such that

$$\|u\|_\infty+\|v-\tilde v\|_\infty+\|w-\tilde w\|_\infty\le C_2'e^{-\lambda_2 t},\quad t>t_0+2.$$

This implies (1.12). Theorem 1.4 (i) is proved. □

Proof of Theorem 1.4 (ii). Suppose $a_1 = b_1\tilde{w}$. In this case, we have $(a_1 - b_1\tilde{w})\int\limits_\Omega u\mathrm{d}x = 0$. And so, (3.13)–(3.16) hold. Let $E_2(t)$, $F_2(t)$ be given by Lemma 3.5, and $t_0 > 0$ be as in the proof of Theorem 1.4 (i). Using (3.15) and (3.16) firstly, and the boundedness of (u, v, w) secondly, we can find $c_6 > 0$ such that

$$\begin{aligned}
E_2(t) &\le \frac{1}{b_1}\int\limits_\Omega u\mathrm{d}x + \frac{1}{b_2\tilde{v}}\int\limits_\Omega (v-\tilde{v})^2\mathrm{d}x + \frac{1}{\tilde{w}}\int\limits_\Omega (w-\tilde{w})^2\mathrm{d}x \\
&\le c_6\left(\int\limits_\Omega u^2\mathrm{d}x\right)^{1/2} + c_6\left(\int\limits_\Omega (v-\tilde{v})^2\mathrm{d}x\right)^{1/2} + c_6\left(\int\limits_\Omega (w-\tilde{w})^2\mathrm{d}x\right)^{1/2} \\
&\le \sqrt{3}c_6\left(\int\limits_\Omega \left[u^2 + (v-\tilde{v})^2 + (w-\tilde{w})^2\right]\mathrm{d}x\right)^{1/2} \\
&= c_6\sqrt{3F_2(t)}, \quad t > t_0.
\end{aligned}$$

This combined with (3.11) enable us to find $c_7 > 0$ fulfilling $E_2'(t) \le -c_7E_2(t)^2$ for $t > t_0$. Therefore, $E_2(t) \le \frac{c_8}{t+1}$ for $t > t_0$ and some $c_8 > 0$. Hence, we obtain by the left inequalities in (3.14)–(3.16) that there exists $c_9 > 0$ such that

$$\int\limits_\Omega \left[u^2 + (v-\tilde{v})^2 + (w-\tilde{w})^2\right]\mathrm{d}x \le c_9E_2(t) \le \frac{c_8c_9}{t+1}, \quad t > t_0.$$

On the other hand, it follows from (3.11) that

$$\begin{aligned}
\int\limits_{t-1}^{t}\int\limits_\Omega |\nabla w|^2 \le \int\limits_{t-1}^{t} F_2(t) &\le -\frac{1}{\varepsilon}\int\limits_{t-1}^{t}\frac{\mathrm{d}}{\mathrm{d}s}E_2(s) \\
&\le \frac{1}{\varepsilon}E_2(t-1) \le \frac{c_{10}}{\varepsilon(t+1)} \quad t > t_0 + 1
\end{aligned}$$

with some $c_{10} > 0$. Recall that (u, v, w) is a global bounded solution of (1.5). In view of Lemma 3.2, there exists C_3', $\lambda_3 > 0$ such that

$$\|u\|_\infty + \|v-\tilde{v}\|_\infty + \|w-\tilde{w}\|_\infty \le C_3'(t+1)^{-\lambda_3}, \quad t > t_0 + 2.$$

This implies (1.13) and the proof of Theorem 1.4 (ii) is complete. □

3.3. Global stability of the prey-only steady state: proof of Theorem 1.6

Lemma 3.6. *Let $b_i \le a_i$, $i = 1, 2$, and (u, v, w) be a bounded global classical solution of (1.5). Then there is $0 < \varepsilon < \min\left\{(a_1 - b_1)/b_1,\ (a_2 - b_2)/b_2\right\}$ such that the nonnegative functions $E_3(t)$ and $F_3(t)$ defined by*

$$\begin{aligned}
E_3(t) &= \int\limits_\Omega \left(\frac{1}{b_1}u + \frac{1}{b_2}v + w - 1 - \ln w\right)\mathrm{d}x, \\
F_3(t) &= \int\limits_\Omega \left(u^2 + v^2 + (w-1)^2 + |\nabla w|^2\right)\mathrm{d}x
\end{aligned}$$

satisfies

$$E_3'(t) \leq -\varepsilon F_3(t) - \int\limits_\Omega \left(\frac{a_1 - b_1}{b_1} u + \frac{a_2 - b_2}{b_2} v \right) \mathrm{d}x, \quad t > 0. \tag{3.18}$$

Proof. Similar to the proof of Lemma 3.3, by a series of carefully calculations we have

$$E_3'(t) = -\int\limits_\Omega \left(d\frac{|\nabla w|^2}{w^2} + \frac{u^2}{b_1} + \frac{v^2}{b_2} + q(w-1)^2 \right) \mathrm{d}x - \int\limits_\Omega \left(\frac{a_1 - b_1}{b_1} u + \frac{a_2 - b_2}{b_2} v \right) \mathrm{d}x.$$

Remember $0 < w \leq m$. Take $0 < \varepsilon < \min\limits_{i=1,2}\{(a_i - b_i)/b_i,\, 1/b_i,\, q,\, d/m^2\}$, then (3.18) is followed. □

Proof of Theorem 1.6 (i). Assume that $b_i < a_i$, $i = 1, 2$, and (u, v, w) is a bounded global classical solution of (1.5). Similar to the proof of Lemma 3.4, we have $\|u\|_\infty + \|v\|_\infty + \|w - 1\|_\infty \to 0$ as $t \to 0$. The same as the proof of Theorem 1.4 (i), there exists $t_0 > 0$ such that

$$\frac{1}{2}\int\limits_\Omega u^2 \mathrm{d}x + \frac{1}{2}\int\limits_\Omega u \mathrm{d}x \leq \int\limits_\Omega u \mathrm{d}x \leq 2\int\limits_\Omega u^2 \mathrm{d}x + 2\int\limits_\Omega u \mathrm{d}x, \quad t > t_0, \tag{3.19}$$

$$\frac{1}{2}\int\limits_\Omega v^2 \mathrm{d}x + \frac{1}{2}\int\limits_\Omega v \mathrm{d}x \leq \int\limits_\Omega v \mathrm{d}x \leq 2\int\limits_\Omega v^2 \mathrm{d}x + 2\int\limits_\Omega v \mathrm{d}x, \quad t > t_0, \tag{3.20}$$

$$\frac{1}{4}\int\limits_\Omega (w-1)^2 \mathrm{d}x \leq \int\limits_\Omega (w - 1 - \ln w)\, \mathrm{d}x \leq \int\limits_\Omega (w-1)^2 \mathrm{d}x, \quad t > t_0. \tag{3.21}$$

In view of the definitions of $E_3(t), F_3(t)$ and the right inequalities in (3.19)–(3.21), we get

$$E_3(t) \leq c_1 \left(F_3(t) + \int\limits_\Omega (u+v) \mathrm{d}x \right), \quad t > t_0,$$

where $c_1 = 2\max\left\{1/b_1,\, 1/b_2,\, 1\right\}$. It follows that

$$-F_3(t) \leq -\frac{E_3(t)}{c_1} + \int\limits_\Omega (u+v) \mathrm{d}x, \quad t > t_0. \tag{3.22}$$

Note that $0 < \varepsilon < \min\left\{(a_1 - b_1)/b_1,\, (a_2 - b_2)/b_2\right\}$. Plugging (3.22) into (3.18) we get

$$\begin{aligned} E_3'(t) &\leq -\varepsilon F_3(t) - \frac{a_1 - b_1}{b_1}\int\limits_\Omega u \mathrm{d}x - \frac{a_2 - b_2}{b_2}\int\limits_\Omega v \mathrm{d}x \\ &\leq -\frac{\varepsilon}{c_1} E_3(t) - \left(\frac{a_1 - b_1}{b_1} - \varepsilon \right) \int\limits_\Omega u \mathrm{d}x - \left(\frac{a_2 - b_2}{b_2} - \varepsilon \right) \int\limits_\Omega v \mathrm{d}x \\ &\leq -\frac{\varepsilon}{c_1} E_3(t), \quad t > t_0. \end{aligned}$$

This implies that there exist $c_2, \sigma > 0$ such that $E_3(t) \leq c_2 e^{-\sigma t}$ for $t > t_0$. By the left inequalities in (3.19)–(3.21) we have

$$\int\limits_\Omega u^2 \mathrm{d}x + \int\limits_\Omega v^2 \mathrm{d}x + \int\limits_\Omega (w-1)^2 \mathrm{d}x \leq c_3 E_3(t) \leq c_4 e^{-\sigma t}, \quad t > t_0$$

for some $c_3, c_4 > 0$. Moreover,

$$\int_{t-1}^{t}\int_{\Omega}|\nabla w|^2\mathrm{d}x\mathrm{d}s \le \int_{t-1}^{t} F_3(s)\mathrm{d}s \le -\frac{1}{\varepsilon}\int_{t-1}^{t}\frac{d}{\mathrm{d}s}E_3(s)\mathrm{d}s$$
$$\le \frac{1}{\varepsilon}E_3(t-1) \le c_5 e^{-\sigma t}, \quad t > t_0+1$$

for some $c_5 > 0$. Note that (u, v, w) is a global bounded solution of (1.5). In light of Lemma 3.2, there exist C_6', $\lambda_6 > 0$ such that

$$\|u\|_\infty + \|v\|_\infty + \|w-1\|_\infty \le C_6' e^{-\lambda_6 t}, \quad t > t_0+2,$$

which implies (1.14). The proof of Theorem 1.6 (i) is complete. □

Proof of Theorem 1.6 (ii). Assume that $b_i = a_i$, $i = 1, 2$, and (u, v, w) is a bounded global classical solution of (1.5). Let $t_0 > 0$ be as in the proof of Theorem 1.6 (i). Using (3.21), the Cauchy–Schwarz inequality and boundedness of (u, v, w) we can find $c_6 > 0$ such that

$$\begin{aligned}
E_3(t) &\le \frac{1}{b_1}\int_{\Omega} u\mathrm{d}x + \frac{1}{b_2}\int_{\Omega} v\mathrm{d}x + \int_{\Omega}(w-1)^2\mathrm{d}x \\
&\le c_6\left(\int_{\Omega} u^2\mathrm{d}x\right)^{1/2} + c_6\left(\int_{\Omega} v^2\mathrm{d}x\right)^{1/2} + c_6\left(\int_{\Omega}(w-1)^2\mathrm{d}x\right)^{1/2} \\
&\le \sqrt{3}c_6\left(\int_{\Omega}\left[u^2+v^2+(w-1)^2\right]\mathrm{d}x\right)^{1/2} \\
&= c_6\sqrt{3F_3(t)}, \quad t > t_0.
\end{aligned}$$

This combined with (3.18) enable us to find $c_7 > 0$ such that $E_3'(t) \le -c_7 E_3^2(t)$ for $t > t_0$. Thus, $E_3(t) \le \frac{c_8}{t+1}$ for $t > t_0$ and some $c_8 > 0$. In view of the left inequalities in (3.19)–(3.21), we can find $c_9 > 0$ such that

$$\int_{\Omega}\left[u^2+v^2+(w-1)^2\right]\mathrm{d}x \le c_9 E_3(t) \le \frac{c_8 c_9}{t+1}, \quad t > t_0.$$

On the other hand, it follows from (3.18) that, for some $c_{10} > 0$,

$$\int_{t-1}^{t}\int_{\Omega}|\nabla w|^2\mathrm{d}x\mathrm{d}s \le \int_{t-1}^{t} F_3(s)\mathrm{d}s \le -\frac{1}{\varepsilon}\int_{t-1}^{t}\frac{d}{\mathrm{d}s}E_3(s)\mathrm{d}s$$
$$\le \frac{1}{\varepsilon}E_3(t-1) \le \frac{c_{10}}{\varepsilon(t+1)}, \quad t > t_0+1.$$

Recall that (u, v, w) is a global bounded solution of (1.5). In view of Lemma 3.2, there exists $C_7' > 0$ and $\lambda_7 > 0$ such that

$$\|u\|_\infty + \|v\|_\infty + \|w-1\|_\infty \le C_7'(t+1)^{-\lambda_7}, \quad t > t_0+2.$$

This implies (1.15) and the proof is finished. □

4. Discussion

In this paper, we investigated the reaction–diffusion systems modeling the population dynamics of two predators and one prey with nonlinear prey-taxis. The following realistic conclusions are obtained.

(i) When the dimension $n \leq 2$ or the prey-tactic coefficients $\phi_1(w)$ and $\phi_2(w)$ are small, problem (1.2) has a unique nonnegative and bounded global classical solution.

(ii) The global stabilities of the positive and semi-trivial spatially homogeneous equilibria and the convergence rates are established.

The coefficients a_1 and b_1 represent the death rate and predation rate of the predator u, respectively (a_2 and b_2 represent the death rate and predation rate of the predator v, respectively), and q is the intrinsic birth rate of the prey w. The quantity $a_1b_2 - a_2b_1$ can be regarded as the difference of roles (death rates and predation rates) of the two predators u and v on the systems.

The quantities

$$(b_1 - a_1)q + a_2b_1 - a_1b_2 > 0, \quad (b_2 - a_2)q + a_1b_2 - a_2b_1 > 0$$

mean that the predator's predation rates should be larger than their death rates, and the intrinsic birth rate of the prey should be large enough. In this case, system (1.5) has a unique positive spatially homogeneous equilibrium (u_*, v_*, w_*) and it is globally stable provided that the prey-tactic coefficients $\phi_1(w)$ and $\phi_2(w)$ are small or the diffusion coefficient d of the prey is large.

The conditions

$$(b_1 - a_1)q + a_2b_1 - a_1b_2 \leq 0, \quad (b_2 - a_2)q + a_1b_2 - a_2b_1 > 0, \quad a_2 < b_2$$

show that the role of the predator u is weaker than that of the predator v. In such a situation, the predator u will eventually disappear, system (1.5) has a semi-trivial spatially homogeneous equilibrium $(0, \tilde{v}, \tilde{w})$ and it is globally stable if the prey-tactic coefficient $\phi_2(w)$ is small or the diffusion coefficient d of the prey is large.

Symmetrically, the conditions

$$(b_1 - a_1)q + a_2b_1 - a_1b_2 > 0, \quad (b_2 - a_2)q + a_1b_2 - a_2b_1 \leq 0, \quad a_1 < b_1$$

show that the role of the predator v is weaker than that of the predator u. In such a situation, the predator v will eventually disappear, system (1.5) has a semi-trivial spatially homogeneous equilibrium $(\hat{u}, 0, \hat{w})$ and it is globally stable if the prey-tactic coefficient $\phi_1(w)$ is small or the diffusion coefficient d of the prey is large.

The conditions $b_i \leq a_i$, $i = 1, 2$ mean that the predator's predation rates are less than their death rates, and so the two predators will eventually disappear and the prey will stabilize at its unique positive equilibrium state.

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Jianping Wang and Mingxin Wang
Department of Mathematics
Harbin Institute of Technology
Harbin 150001
People's Republic of China
e-mail: mxwang@hit.edu.cn

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