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Large deformations of 1D microstructured systems modeled as generalized Timoshenko beams

A. Battista, A. Della Corte, F. dell'Isola and P. Seppecher

Abstract. In the present paper we study a natural nonlinear generalization of Timoshenko beam model and show that it can describe the homogenized deformation energy of a 1D continuum with a simple microstructure. We prove the well posedness of the corresponding variational problem in the case of a generic end load, discuss some regularity issues and evaluate the critical load. Moreover, we generalize the model so as to include an additional rotational spring in the microstructure. Finally, some numerical simulations are presented and discussed.

Mathematics Subject Classification. 74B20, 49J45.

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1. Introduction

The main aim of this paper is to study the problem of large deformations of a microstructured beam, which can be described at the macroscopic level by means of a generalized Timoshenko model. We focus on the case of a beam clamped at one endpoint and submitted to a concentrated load at the other endpoint.

Timoshenko proposed the first beam model going beyond the classical Euler model. The latter was formulated around mid-18th century [1], and scientists of the caliber of Daniel and Jakob Bernoulli, as well as Lagrange, gave important contributions to its theory [2–4]. The model has been rigorously deduced from 3D elasticity [5,6] and it is widely applied in many problems of structural mechanics. (A useful reference book is [7].)

Let $\mathcal{E}$ be the affine euclidean plane and $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ an orthonormal basis. The deformation energy of an inextensible Euler beam can be written as:

$$\int_0^L \frac{k_b}{2}\boldsymbol{\chi}''(s)\cdot\boldsymbol{\chi}''(s)\mathrm{d}s = \int_0^L \frac{k_b}{2}\eta^2(s)\mathrm{d}s \tag{1}$$

under the inextensibility constraint

$$||\boldsymbol{\chi}'(s)|| = 1 \quad \forall s \in [0, L] \tag{2}$$

where L is the length of the beam (assumed straight and parallel to $\boldsymbol{e}_1$ in the reference configuration), $\boldsymbol{\chi}$ is the placement (with components χ_i), i.e., a bijective continuous function mapping material points of the beam (labeled with $s \in [0, L]$) into $\mathcal{E}$, and k_b is a material parameter accounting for the bending stiffness. The dot indicates the usual scalar product of $\mathbb{R}^2$, and the related norm is understood. In the present paper, $(.)'$ denotes differentiation with respect to s. In particular, $\eta(s) = ||\boldsymbol{\chi}''(s)||$ is the absolute value of the curvature of the current shape of the beam. Note that, although the deformation energy (1) is a quadratic form, the inextensibility constraint (2) is not convex.

Since in most of applications to structural mechanics deflections are small when compared to the length of the beams, it is very usual to consider the approximated linearized model, instead of the previous one. In this case, constraint (2) becomes $\chi_1 = s$ and the deformation energy reads:

$$\int_0^L \frac{k_b}{2}(\chi_2'')^2 \mathrm{d}s, \tag{3}$$

Static problems for the commonly employed linearized model lead to fourth-order linear ODEs, while the nonlinear model originally formulated by Euler leads to semilinear fourth-order ODEs.
The nonlinear Euler model can be reformulated in terms of the variable θ satisfying

$$\boldsymbol{\chi}' = \cos\theta \boldsymbol{e}_1 + \sin\theta \boldsymbol{e}_2 \tag{4}$$

The energy becomes:

$$\int_0^L \frac{k_b}{2}\theta'^2(s)\mathrm{d}s \tag{5}$$

The beam is said to be clamped at the left extremum if $\boldsymbol{\chi}(0) = \mathbf{0}$ and $\boldsymbol{\chi}'(0) = \mathbf{e}_1$. An external potential $\boldsymbol{b(s)}\cdot\boldsymbol{\chi(s)}$, where $\boldsymbol{b}(s)$ is a distributed load, can also be written in terms of θ by performing an integration by parts. The total energy of the system becomes:

$$E(\theta) = \int_0^L \left[\frac{k_b}{2}\theta'^2(s) - B_1\cos\theta(s) - B_2(s)\sin\theta(s)\right]\mathrm{d}s \tag{6}$$

where $\boldsymbol{B} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2$ denotes the primitive of $\boldsymbol{b}$ verifying $\boldsymbol{B}(L) = 0$. It is natural to search for minimizers of the energy (6) in the set of functions belonging to $H^1 := W^{1,2}$ verifying $\theta(0) = 0$ in the sense of traces. We remark that the angle $\theta(s)$ is uniquely determined by Eq. (4) if one takes into account the clamp condition $\theta(0) = 0$ and the fact that H^1 functions have continuous representatives. This reformulation automatically takes into account the inextensibility constraint and the non-convexity of the minimization problem appears clearly.
Timoshenko beam model was introduced to describe in a more precise way the shear deformation of the beam. (The problem is addressed in an original way in [8].) It was developed much later, in the early 1920s [9], and motivated by several applications. Specifically, for the static case, it was needed for describing beams that were not so slender, in which case shear deformation is no more negligible. This is also the case with sandwich composite beams ([10]). Timoshenko beam model is still an important tool for current research: the possibility of very precisely manufacturing the inner architecture of beams (e.g., by means of 3D printing) makes it now possible to produce slender objects which display a richer behavior than what can be captured by Euler beam model. (See, e.g., [11–13] for interesting examples, [14–16] for cases in which dynamical/instability problems are addressed and [17–20] for an approach using asymptotic justification; a review of complex structures employing fibers that can be modeled as generalized beams is [21].)
The original model from Timoshenko was established in a linear framework. The deformation energy of an inextensible linear Timoshenko beam reads:

$$\int_0^L \left[\frac{K_1}{2}(\phi'(s))^2 + \frac{K_2}{2}(\phi(s) - u_2'(s))^2\right]\mathrm{d}s \tag{7}$$

where ϕ is an independent kinematic descriptor and K_1 and K_2 are material parameters. In the original interpretation of the model, ϕ was thought to measure the angle between the cross section of the beam and a reference axis. Therefore the model can be seen as a particular case of Cosserat/micropolar continua. The interested reader can see [22] for a historically important reference; a general introduction on the

topic is given, e.g., in [23–25], while some interesting results using Γ-convergence arguments are provided in [26] (concerning 2D plate-like models) and [27] (concerning rod-like models).
Clearly there are many possible nonlinear generalizations of the previous linearized energy model. A natural generalization can be obtained by replacing, similarly to the Euler case, the term u_2' by the angle θ defined as above:

$$\int_0^L \left[\frac{K_1}{2}(\phi'(s))^2 + \frac{K_2}{2}(\phi(s)-\theta(s))^2\right] \mathrm{d}s \tag{8}$$

We will obtain this energy model with a formal homogenization starting from a microstructure consisting of articulated parallelograms and rotational springs.

The paper is organized as follows: in Sect. 2, various kinds of microstructures are considered and in particular a novel one is introduced which leads to the deformation energy (8); moreover, an additional rotational spring is also considered to introduce a regularizing term which further generalizes the model. In Sect. 3, the well posedness of the variational problem given by the equilibria of a clamped beam under end load is addressed, some regularity issues are discussed, and the critical load is evaluated. In Sect. 4, numerical simulations are presented and discussed. In Sect. 5, some conclusions are provided as well as possible directions for future researches.

2. Microscopic interpretation of a Timoshenko beam

2.1. The linearized case

A traditional way of introducing Timoshenko model in structural mechanics courses is to provide a microscopical interpretation of it by means of a discrete system of rotational and extensional springs. Nowadays, the possibility of accurately 3D printing the microstructures makes these discretizations more than academic examples, but rather possibilities to concretely implement mechanical systems with certain desired properties.
Let us consider the system illustrated in Fig. 1. The length of the deformed extensional springs connecting the points $\boldsymbol{t}_i^+$ and $\boldsymbol{t}_{i+1}^+$ can be computed as:

$$\|\boldsymbol{t}_{i+1}^+ - \boldsymbol{t}_i^+\|^2 = \varepsilon^2\left\{1 + 2d\left[\sin(\theta_i - \phi_{i+1}) + \sin(\phi_i - \theta_i)\right] + 2d^2\left[1 - \cos\left(\phi_{i+1} - \phi_i\right)\right]\right\} \tag{9}$$

Now if we assume that the quantities θ and ϕ, which measure the deformation of the system, remain small, using basic trigonometric identities and approximating at the first order the trigonometric functions and the square root, we can write the following approximate identity:

$$\|\boldsymbol{t}_{i+1}^+ - \boldsymbol{t}_i^+\| \approx \varepsilon\left[1 + d(\phi_i - \phi_{i-1})\right]$$

Let us assume that all the extensional springs are linear (with elastic constant $k_1/(2d^2)$). Performing a formal passage to the limit we have:

$$\frac{1}{2}\frac{k_1}{2d^2}\left(\|\boldsymbol{t}_{i+1}^+ - \boldsymbol{t}_i^+\| - \varepsilon\right)^2 \approx \frac{1}{2}\frac{k_1}{2}\phi'^2\varepsilon^4 \tag{10}$$

An analogous contribution in the energy will come from the spring connecting $\boldsymbol{t}_i^-$ and $\boldsymbol{t}_{i+1}^-$. The potential energy of the rotational spring connecting $\boldsymbol{t}_i^-\boldsymbol{t}_i^+$ and $\boldsymbol{p}_i\boldsymbol{p}_{i+1}$ is assumed to be:

$$\frac{1}{2}k_2(\phi_i - \theta_i)^2 \tag{11}$$

where k_2 is the elastic constant of the spring.

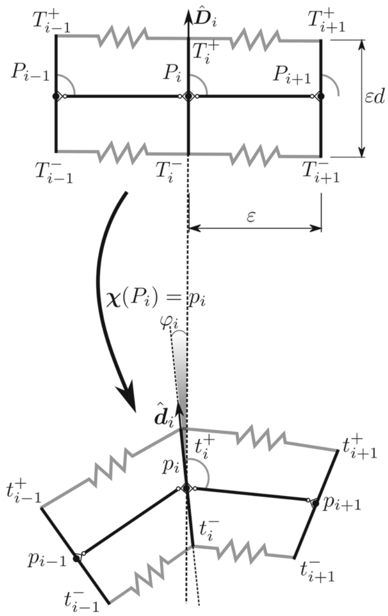

FIG. 1. The bar-spring microstructure usually introduced to get the linear Timoshenko model valid for small deformations (top: reference configuration; bottom: actual configuration). For finite deformations, a heuristic homogenization for this microstructure, leads to the energy (14)

The deformation energy of a microstructured system composed of N elementary cells like the two represented in Fig. 1, in the linearized case, can be therefore written as:

$$\sum_{i=1}^{i=N}\left[\frac{1}{2}k_1\phi'^2\varepsilon^4+\frac{1}{2}k_2(\phi_i-\theta_i)^2\right]. \tag{12}$$

In the limit for $\varepsilon \to 0$, the stiffnesses have to be suitably rescaled: $k_1 = K_1\varepsilon^{-4}$ and $k_2 = K_2$. Recalling that, in our hypotheses, θ is small, we can approximate it by means of u_2', obtaining for the homogenized energy formula (7).

2.2. The nonlinear case

If one does not want to assume that the angles ϕ and θ remain small, the energy of the previous discrete system takes a more complicated form.

Indeed, the continuous form of (9) is:

$$\frac{1}{2}\frac{k_1}{2d^2}\left(\|\boldsymbol{t}^+_{i+1}-\boldsymbol{t}^+_i\|-\varepsilon\right)^2 \approx \frac{1}{2}k_1\phi'^2\cos^2(\theta-\phi)\varepsilon^4 \tag{13}$$

and the homogenized energy, with the same scaling as before, becomes

$$\int_0^L\left[\frac{1}{2}K_1(\phi')^2\cos^2(\theta-\phi)+\frac{1}{2}K_2(\phi-\theta)^2\right] \tag{14}$$

As we can see, the last expression does not coincide with the form of Timoshenko deformation energy given in formula (8) because of the factor $\cos^2(\theta-\phi)$. While the microstructure relative to Fig.1 has an intrinsic physical interest, the homogenized energy (14) has a more complex structure than (8) and we

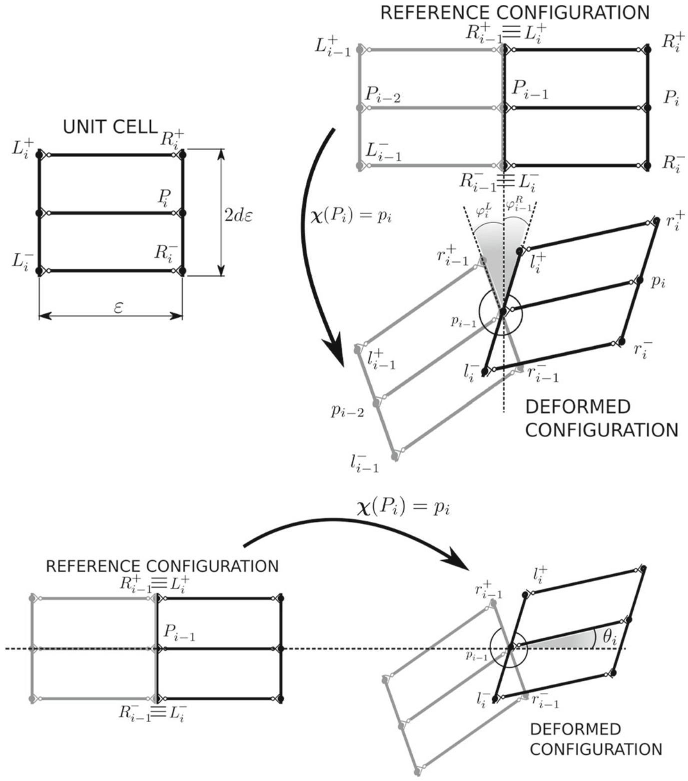

FIG. 2. Another possible microstructure for shear deformable beams producing (after linearization) the Timoshenko beam model. On top: the unit cell and a graphical representation of the variation of the cross-sectional angle ϕ. At the bottom: graphical representation of the variation of angle θ formed by the tangent to the deformed shape). A heuristic homogenization of this microstructure leads to the energy (8)

consider more suitable to attack the simplest problem first. Indeed, we can recover the energy model (8) by means of a different microstructure (see Fig. 2).

We start by considering an articulated parallelogram in which all the sides are rigid bars, and we add another rigid bar connecting the middle points of two sides. Then we organize them in series (see Fig. 2). The points R^+_{i-1} and L^+_i, R^-_{i-1} and L^-_i as well as P^r_i and P^ℓ_{i-1} coincide in the reference configuration, so that the two bars $\overrightarrow{R^+_{i-1}R^-_{i-1}}$ and $\overrightarrow{L^+_iL^-_i}$ are superposed. We describe the mechanical interaction between the bars by means of rotational springs. We introduce two rotational springs:

1. one between the directions of $\overrightarrow{R_iP_i}$ and $\overrightarrow{P_{i+1}P_i}$
2. one between the directions of $\overrightarrow{R_iP_i}$ and $\overrightarrow{L_{i+1}P_{i+1}}$

   Assuming that the rotational springs are linear in the angle, the deformation energy takes the form:

$$\sum_{i=1}^{N}\left[\frac{k_1}{2}(\phi_i-\phi_{i-1})^2+\frac{k_2}{2}(\phi_i-\theta_i)^2\right] \tag{15}$$

   and setting $K_1 := k_1\varepsilon^4$, $K_2 := k_2$, formula (8) is recovered with a formal homogenization procedure. In the following, we will call the energy model (8) *Nonlinear Timoshenko model*.

   Finally, we can introduce an additional rotational spring, i.e.:

3. between the directions of $\overrightarrow{P_iP_{i-1}}$ and $\overrightarrow{P_{i+1}P_i}$.

In this case (assuming that the new rotational spring has stiffness k_3 and setting $K_3 := k_3\varepsilon^4$), an analogous formal homogenization procedure leads to the following energy functional:

$$\int_0^L \left[\frac{K_1}{2}(\phi')^2 + \frac{K_2}{2}(\phi-\theta)^2 + \frac{K_3}{2}(\theta')^2\right] \mathrm{d}s \tag{16}$$

In the following, we will call the energy model (16) *Regularized Timoshenko model.*

The passage from the discrete system to its homogenized form can be made rigorous in various ways, the most natural being Γ-convergence arguments. This is the subject of a separated investigation, and here we just observe that in our context the passage to the Γ-limit is not trivial, due to the geometrically nonlinear character of the problem which in particular allows finite rotations, possibly around axes that lie in the plane. For instance, the possible overlapping (referring to Fig. 2) of the points R^+_{I-1} and L^-_I as well as R^-_{I-1} and L^+_I has to be considered. Similarly, one has to consider the possible overlapping of points indexed by i and points indexed by $i-2$.

In the following section, we investigate rigorously the proposed heuristic homogenized form to establish the well posedness of the variational problem and regularity properties of the solutions.

3. Properties of the minimization problems

3.1. Nonlinear Timoshenko model

Let us consider the deformation energy (8) in the model case in which a concentrated load $\boldsymbol{F} = F_1\boldsymbol{e}_1 + F_2\boldsymbol{e}_2$ is applied at the free endpoint. We are led to the following variational problem:

$$\begin{aligned}\min \quad &\Bigg\{E(\phi,\theta) := \int_0^L \left[\frac{K_1}{2}\phi'^2 + \frac{K_2}{2}(\phi-\theta)^2 + F_1\cos\theta + F_2\sin\theta\right]\mathrm{d}s\\ &\theta\in L^2[0,L],\quad \phi\in H^1[0,L],\quad \phi(0)=0\Bigg\}\end{aligned} \tag{17}$$

3.1.1. Well posedness. We prove in this section that problem (17) is well posed. Let us first rewrite the problem in a non-dimensional form. A change of length and energy units leads to

$$\begin{aligned}\min \quad &\Bigg\{\tilde{E}(\phi,\theta) := \int_0^1 \left[\frac{1}{2}\phi'^2 + \tilde{K}_2\left(\frac{1}{2}(\phi-\theta)^2 + \tilde{F}\cos(\theta-\gamma)\right)\right]\mathrm{d}s\\ &\theta\in L^2[0,1],\quad \phi\in H^1[0,1],\quad \phi(0)=0\Bigg\}\end{aligned} \tag{18}$$

where $\tilde{K}_2 := \frac{K_2L^2}{K_1}$ and $(\tilde{F},\gamma)\in[0,+\infty)\times[0,2\pi[$ are defined by:

$$\tilde{F}(\cos(\gamma),\sin(\gamma)) = K_2^{-1}(F_1,F_2).$$

Proposition 1. *Problem* (18) *admits a solution.*

Proof. Let $G(\theta) := \frac{\theta^2}{2} + \tilde{F}\cos(\theta-\gamma)$ and $G^*(z)$ its convex conjugate, defined by $G^*(z) := \max_\theta[z\theta - G(\theta)]$. Since G is continuous and coercive, there exists $\bar{\theta}$ solving the max problem and $\bar{\theta}(z)$ belongs to the subdifferential $\partial G^*(z)$. We note that $G^*(z)$ is not differentiable in correspondence of intervals in which G does not coincide with its lower convex envelop.

Hence:

$$\begin{aligned}\inf_{\theta,\phi} \tilde{E}(\theta,\phi) &= \inf_{\theta,\phi} \int_0^1 \left[\frac{1}{2}\phi'^2 + \tilde{K}_2\left(\frac{1}{2}\phi^2 + (-\phi\theta + G(\theta))\right)\right] \mathrm{d}s \\ &= \inf_{\phi} \int_0^1 \left[\frac{1}{2}\phi'^2 + \tilde{K}_2\left(\frac{1}{2}\phi^2 + \inf_{\theta}(-\phi\theta + G(\theta))\right)\right] \mathrm{d}s \\ &= \inf_{\phi} \int_0^1 \left[\frac{1}{2}\phi'^2 + \tilde{K}_2\left(\frac{1}{2}\phi^2 - \sup_{\theta}(\phi\theta - G(\theta))\right)\right] \mathrm{d}s = \inf_{\phi} \mathcal{A}(\phi)\end{aligned}$$

where

$$\mathcal{A}(\phi) := \int_0^1 \left(\frac{1}{2}\phi'^2 + \tilde{K}_2\, h(\phi)\right) \mathrm{d}s \tag{19}$$

with

$$h(\phi) := \frac{1}{2}\phi^2 - G^*(\phi).$$

Let (θ_n, ϕ_n) be a minimizing sequence. Then $\tilde{E}(\theta_n, \phi_n)$ is a bounded sequence and thus $\int_0^1 \frac{1}{2}(\phi_n')^2$ is also bounded and $\phi_n(0) = 0$. Thus ϕ_n is bounded in H^1 : there exists a subsequence (still denoted ϕ_n) weakly converging in H^1 to some function $\bar{\phi}$. Remind that in the one-dimensional case H^1-weak convergence imply uniform convergence and note also that G^* is convex, and therefore it is locally Lipschitz (see Theorem 3.7.3 in [28]). Thus $\int_0^1 [\phi_n^2/2 - G^*(\phi_n)]\mathrm{d}s$ converges to $\int_0^1 [\bar{\phi}^2/2 - G^*(\bar{\phi})]\mathrm{d}s$. On the other hand, as convexity and weak convergence implies $\liminf_n \int_0^1 \phi_n'^2 \mathrm{d}s \geq \int_0^1 \bar{\phi}'^2 \mathrm{d}s$, the functional $\mathcal{A}$ is H^1-lower semicontinuous, and $\bar{\phi}$ is a global minimizer for $\mathcal{A}$. Finally let us remark that the constraint $\phi_n(0) = 0$ passes to the uniform limit: $\bar{\phi}(0) = 0$.

The function $\bar{\theta}(s)$, which solves:

$$\max_{\theta} [\bar{\phi}(s)\theta - G(\theta)] \tag{20}$$

is uniquely defined everywhere on $[0, 1]$, by the equation

$$\frac{\mathrm{d}G^{**}(\theta)}{\mathrm{d}\theta} = \bar{\phi}(s) \tag{21}$$

or equivalently by

$$\theta = \frac{\mathrm{d}G^*(\bar{\phi}(s))}{\mathrm{d}z} \tag{22}$$

except at the points s such that $G^*(\bar{\phi}(s))$ is not differentiable.
Since $\bar{\phi} \in H^1(0, 1)$ is continuous in $[0, 1]$, it attains a maximum $\phi_{\max}$ and a minimum $\phi_{\min}$, and therefore we are interested in the differentiability of G^* only in the interval $[\phi_{\min}, \phi_{\max}]$.

It is easy to check that the convex conjugate of G is piecewise C^1 and that the set $\{\phi_1, \ldots, \phi_k\}$ of the points which belong to $[\phi_{\min}, \phi_{\max}]$ and where G^* is not differentiable is finite. Let us now prove by contradiction that $\bar{\phi}$ takes the values $\phi_1, \ldots, \phi_k$ only on a subset of $[0, 1]$ of measure zero. Otherwise there would exist an integer $i \in \{1, \ldots, k\}$ such that the measure of $\bar{\phi}^{-1}(\phi_i)$ is positive. Adapting known monotonicity results for autonomous variational problems (see, e.g., Theorem 3.1 in [30]), we know that $\bar{\phi}$ is monotonic[1] and thus that $\bar{\phi}^{-1}(\phi_i)$ is an interval. The following lemma states that for this reason G^* is differentiable at ϕ_i, which contradicts the definition of ϕ_i. In conclusion, $\bar{\theta}$ is uniquely determined

[1] The cited theorem is stated for a problem with values prescribed at both ends, but is true also in the present case. One just has to apply it to a problem having $\phi(1) = \bar{\phi}(1)$.

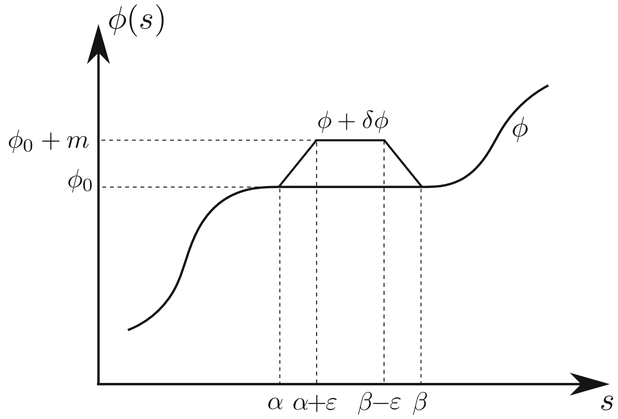

Fig. 3. Representation of ϕ and of the considered variation $\delta\phi$

almost everywhere, it is bounded because G^* is Lipschitz on $[\phi_{\min}, \phi_{\max}]$, and therefore, it belongs to $L^2[0,1]$. □

Lemma 1. *Let $\bar{\phi}$ be a minimizer of $\mathcal{A}$. Suppose that there exists an interval having positive measure $[\alpha, \beta] \subset [0,1]$ in which $\bar{\phi}(s) = \phi_0$ is constant. Then G^* is differentiable at ϕ_0, and $\frac{\partial G^*}{\partial \phi}(\phi_0) = \phi_0$.*

Proof. We first observe that since G^* is convex, it admits left and right derivatives and therefore for every ϕ there exist both the right and left derivatives $\frac{\partial h}{\partial \phi}^+(\phi)$ and $\frac{\partial h}{\partial \phi}^-(\phi)$.

Now let ε be such that $0 < \varepsilon < \frac{\beta - \alpha}{2}$ and let m and $c > 0$ be real numbers such that $\varepsilon = \sqrt{|m|/2c} < (\beta - \alpha)/2$.

Let us consider a variation $\delta\phi$ defined as (see Fig. 3):

$$\begin{cases} \delta\phi(s) := 0 \quad \text{for} \quad s \in [0, \alpha) \cup (\beta, 1] \\ \delta\phi(s) := \frac{m}{\varepsilon} s - \frac{m}{\varepsilon}\alpha \quad \text{for} \quad s \in [\alpha, \alpha + \varepsilon] \\ \delta\phi(s) := -\frac{m}{\varepsilon} s + \frac{m}{\varepsilon}\beta \quad \text{for} \quad s \in [\beta - \varepsilon, \beta] \\ \delta\phi(s) := m \quad \text{for} \quad s \in (\alpha + \varepsilon, \beta - \varepsilon) \end{cases}$$

Clearly $[\bar{\phi} + \delta\phi] \in H^1$ and $[\bar{\phi} + \delta\phi](0) = 0$. Let us set $\Delta h(\delta\phi) := h(\bar{\phi} + \delta\phi) - h(\bar{\phi})$. We have:

$$\begin{aligned} 0 \leq \Delta\mathcal{A} &:= \mathcal{A}(\bar{\phi} + \delta\phi) - \mathcal{A}(\bar{\phi}) \\ &= \int_\alpha^\beta \frac{1}{2}\delta\phi'^2 \mathrm{d}s + \tilde{K}_2 \left(\int_\alpha^{\alpha+\varepsilon} \Delta h(\delta\phi)\mathrm{d}s + \int_{\alpha+\varepsilon}^{\beta-\varepsilon} \Delta h(\delta\phi)\mathrm{d}s + \int_{\beta-\varepsilon}^{\beta} \Delta h(\delta\phi)\mathrm{d}s \right) \end{aligned}$$

Using that h is c-Lipschitz on the compact set $[\phi_{\min} - |m|, \phi_{\max} + |m|]$, one gets:

$$0 \leq \frac{m^2}{\varepsilon} + \tilde{K}_2 \left((\beta - \alpha - 2\varepsilon)(h(\phi_0 + m) - h(\phi_0)) + 2c\varepsilon|m| \right)$$

The last inequality implies:

$$h(\phi_0 + m) - h(\phi_0) \geq -\frac{2\sqrt{2c}}{\tilde{K}_2(\beta - \alpha - 2\varepsilon)}|m|^{3/2} \tag{23}$$

In terms of G^*, the inequality (23) can be written as:

$$G^*(\phi_0 + m) - G^*(\phi_0) \leq \frac{m^2}{2} + \phi_0 m + \frac{2\sqrt{2c}}{\tilde{K}_2(\beta - \alpha - 2\varepsilon)}|m|^{3/2} \tag{24}$$

The left hand side of (24) is a convex function of m, while the right hand side is a C^1 function of m. Both sides coincide when $m = 0$. It is easily seen that a convex function bounded from above by a C^1 function is differentiable at any coinciding point. Therefore G^* is differentiable at $\phi = \phi_0$.

Dividing by m both sides of (24) and letting $m \to 0^+$ or $m \to 0^-$, we obtain:

$$\left(\frac{\partial G^*}{\partial \phi}\right)^+(\phi_0) \leq \phi_0 \quad \text{and} \quad \left(\frac{\partial G^*}{\partial \phi}\right)^-(\phi_0) \geq \phi_0. \tag{25}$$

As a consequence

$$\frac{\partial G^*}{\partial \phi}(\phi_0) = \phi_0.$$

□

3.1.2. Regularity. We address now the properties of the function h. In order to establish its regularity properties, we first have to provide some preliminary results.

Let us first remark that h is a 2π-periodic function. Indeed, for any $z \in \mathbb{R}$, let us denote $[z]$ the integer part of $\frac{z-\gamma}{2\pi}$, and let us set $\tilde{z} := z - 2\pi[z]$. We can write

$$\begin{aligned}
G^*(z) &= \max_\theta \left[z\theta - \frac{\theta^2}{2} - F\cos(\theta - \gamma)\right] \\
&= \max_\theta \left[2\pi^2[z]^2 + 2\pi[z]\tilde{z} + \tilde{z}(\theta - 2\pi[z]) - \frac{1}{2}(\theta - 2\pi[z])^2 - F\cos(\theta - \gamma)\right] \\
&= \max_\alpha \left[2\pi^2[z]^2 + 2\pi[z]\tilde{z} + \tilde{z}\alpha - \frac{1}{2}\alpha^2 - F\cos(\alpha - \gamma)\right] \\
&= 2\pi^2[z]^2 + 2\pi[z]\tilde{z} + G^*(\tilde{z}).
\end{aligned}$$

where we introduced the variable $\alpha := \theta - 2\pi[z]$. Hence:

$$\begin{aligned}
h(z) &= \frac{z^2}{2} - G^*(z) \\
&= \frac{\tilde{z}^2}{2} + 2\pi^2[z]^2 + 2\pi[z]\tilde{z} - (2\pi^2[z]^2 + 2\pi[z]\tilde{z} + G^*(\tilde{z})) \\
&= \frac{\tilde{z}^2}{2} - G^*(\tilde{z}) = h(\tilde{z}).
\end{aligned}$$

We address now the dependence of h on γ. To this aim let us define

$$G_\gamma(\theta) := \frac{\theta^2}{2} + \tilde{F}\cos(\theta - \gamma) \tag{26}$$

and

$$h_\gamma(z) = \frac{z^2}{2} - G^*_\gamma(z). \tag{27}$$

We have

$$\begin{aligned} h_\gamma(z) &= \frac{z^2}{2} - \max_\theta \left[\theta z - \frac{\theta^2}{2} - \tilde{F}\cos(\theta-\gamma)\right] \\ &= \frac{z^2}{2} - \max_\alpha \left[(\alpha+\gamma)z - \frac{1}{2}(\alpha+\gamma)^2 - \tilde{F}\cos\alpha\right] \\ &= \frac{z^2}{2} - \gamma z + \frac{\gamma^2}{2} - \max_\alpha \left[(z-\gamma)\alpha - \frac{\alpha^2}{2} - \tilde{F}\cos\alpha\right] \\ &= \frac{(z-\gamma)^2}{2} - G_0^*(z-\gamma) = h_0(z-\gamma). \end{aligned}$$

Therefore the regularity properties of h_γ result from those of h_0. Remarking that h_0 like G_0^* is clearly even, we are reduced to study h_0 on the interval $[0,\pi]$. We can now address the regularity problem:

Lemma 2. *(i) For $\tilde{F} < 1$, h_0 is a C^∞ function,*

(ii) For $\tilde{F} \geq 1$, h_0 is a C^∞ function everywhere but at 0 where the jump of derivative is $-2\,a(\tilde{F})$ with $a(\tilde{F})$ the first positive solution of the equation

$$a(\tilde{F}) = \tilde{F}\sin(a(\tilde{F})) \tag{28}$$

(Note that a is an increasing function from $[1,+\infty)$ onto $[0,\pi]$ and $a(\pi/2)=\pi/2$),

(iii) In any case h_0 is decreasing on the interval $[0,\pi]$. Setting $V(\tilde{F}) := h(0) - h(\pi)$, we have:

$$V(\tilde{F}) = \begin{cases} 2\tilde{F} & \text{if } \tilde{F} < 1 \\ a(\tilde{F})^2/2 + \tilde{F} + \sqrt{\tilde{F}^2 - (a(\tilde{F}))^2} & \text{if } 1 \leq \tilde{F} \leq \pi/2 \\ a(\tilde{F})^2/2 + \tilde{F} - \sqrt{\tilde{F}^2 - (a(\tilde{F}))^2} & \text{if } \pi/2 < \tilde{F} \end{cases}$$

and

$$h_0'(z) = -\sqrt{2}\sqrt{\tilde{F} - h_0(z)}\,\frac{\sqrt{\tilde{F} + 1 - \sqrt{1 + \tilde{F}^2 - 2\,h_0(z)}}}{\sqrt{\tilde{1} - F + \sqrt{1 + \tilde{F}^2 - 2\,h_0(z)})}}. \tag{29}$$

This lemma is illustrated in Fig. 4.

Proof. Let us start by preliminary remarks. The problem

$$\max_\theta \Big(z\,\theta - G_0(\theta)\Big) \tag{30}$$

admits at least one solution $\theta(z)$ which has to satisfy $z = G_0'(\theta(z))$ that is

$$z = \theta(z) - \tilde{F}\sin(\theta(z)) \tag{31}$$

When this solution is unique and when $G_0''(\theta(z)) > 0$, local inversion theorem states that $\theta(z)$ is an increasing C^∞ function. Therefore $G_0^*(z) = z\theta(z) - G_0(\theta(z))$ and consequently h_0 are of class C^∞. In that case, we have

$$\begin{aligned} (G_0^*)'(z) &= \theta(z) + (z - G_0'(\theta(z)))\theta'(z) \\ &= \theta(z) + (z - \theta(z) + \tilde{F}\sin(\theta(z)))\theta'(z) = \theta(z) \end{aligned}$$

and so

$$h_0'(z) = z - (G_0^*)'(z) = z - \theta(z) = -F\sin(\theta(z)) \leq 0. \tag{32}$$

We also have

$$h_0(z) = \frac{z^2}{2} - (G_0^*)'(z) = \frac{(z-\theta(z))^2}{2} - \tilde{F}\cos(\theta(z)) \tag{33}$$

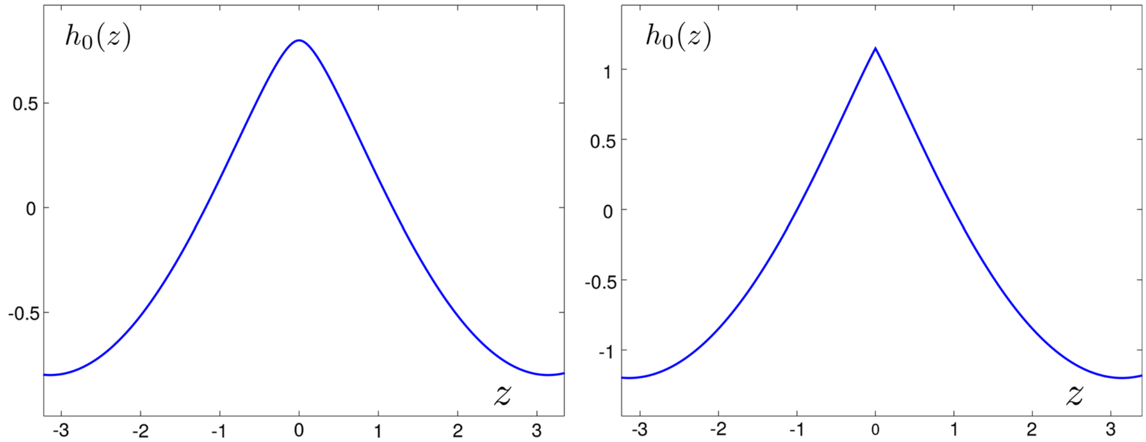

Fig. 4. Plot of the function h_0 in case of $\tilde{F} = 0.8$ (left) and $\tilde{F} = 1.2$ (right). It can be seen that the derivative jumps at $s = 0$ in the second case

and, using (31),

$$h_0(z) = \frac{\tilde{F}^2\,(1-\cos^2(\theta(z)))}{2} + \tilde{F}\,\cos(\theta(z)) \tag{34}$$

Therefore[2]

$$\tilde{F}\,\cos(\theta(z)) = 1 - \sqrt{1+\tilde{F}^2 - 2\,h_0(z)} \tag{35}$$

and so[3]

$$\begin{aligned} h_0'(z) &= -\sqrt{\tilde{F}^2 - (1-\sqrt{1+\tilde{F}^2-2\,h_0(z)})^2} \\ &= -\sqrt{2}\sqrt{\tilde{F}-h_0(z)}\,\frac{\sqrt{\tilde{F}+1-\sqrt{1+\tilde{F}^2-2\,h_0(z)}}}{\sqrt{\tilde{1}-F+\sqrt{1+\tilde{F}^2-2\,h_0(z)})}}. \end{aligned} \tag{36}$$

When $\tilde{F} < 1$, previous remarks apply because the function G_0 is a strictly convex C^∞ function. Point (i) and point (iii) for the case $\tilde{F} < 1$ come directly: we only have to focus on the case $\tilde{F} \geq 1$.

Let us now prove that the restriction of G on the interval $(a(\tilde{F}), \pi)$ is strictly convex. As $G''(\theta) := 1-\tilde{F}\cos(\theta)$ is strictly increasing, it is enough to check that $G''(a(\tilde{F})) \geq 0$. This is trivial if $\tilde{F} \geq \pi/2$ because

[2] The justification of the minus sign before the square root involves some cumbersome computations.

[3]

$$h_0'(z) = -\sqrt{\tilde{F}-1+\sqrt{1+\tilde{F}^2-2\,h_0(z))}}\sqrt{\tilde{F}+1-\sqrt{1+\tilde{F}^2-2\,h_0(z)}}.$$

$$h_0'(z) = -\frac{\sqrt{(\tilde{F}-1)^2-(1+\tilde{F}^2-2\,h_0(z)))}}{\sqrt{\tilde{F}-1-\sqrt{1+\tilde{F}^2-2\,h_0(z))}}}\sqrt{\tilde{F}+1-\sqrt{1+\tilde{F}^2-2\,h_0(z)}}.$$

$$h_0'(z) = -\sqrt{2}\sqrt{-h_0(z)+\tilde{F}}\,\frac{\sqrt{\tilde{F}+1-\sqrt{1+\tilde{F}^2-2\,h_0(z)}}}{\sqrt{\tilde{-}F+1+\sqrt{1+\tilde{F}^2-2\,h_0(z))}}}.$$

$a(\tilde{F}) \in (\pi/2, \pi)$ and so $\cos(a(\tilde{F})) < 0$. This is a bit more complicated if $1 \leq \tilde{F} \leq \pi/2$: $a(\tilde{F}) \in (\pi/2, \pi)$ and we can write

$$G''(a(\tilde{F})) = 1 - \tilde{F}\cos(a(\tilde{F})) = 1 - \sqrt{\tilde{F}^2 - (a(\tilde{F}))^2}$$

The results follow[4] from the properties of Eq. (28).

We notice now that, by definition of $a(\tilde{F})$,

$$G(\theta) \geq 0 \times \theta + G(a(\tilde{F})),$$

for any θ. On the other hand, $T(\theta) := \pi(\theta - \pi) + G(\pi)$ is tangent to G at $\theta = \pi$ and satisfies, for any $\theta \neq \pi$,

$$G(\theta) - T(\theta) = \frac{1}{2}(\theta - \pi)^2 + \tilde{F}\,(\cos(\theta) + 1) > 0.$$

As a consequence, owing to strict convexity, the problem (30) admits, for any $z \in (0, \pi]$, a unique solution $\theta(z)$. This solution belongs to $(a(\tilde{F}), \pi]$ and the preliminary remarks apply. The function h_0 is of class C^∞ on $(0, \pi]$ and, as $(G^*)'(0^+) = a(\tilde{F})$, we get the right derivative $h_0'(0^+) = -a(\tilde{F})$. By symmetry, h_0 is also of class C^∞ on the interval $[-\pi, 0)$ and therefore the function h_0 admits a unique singularity at $z = 0$ on the periodicity interval $[-\pi, \pi]$ with a jump of derivative $-2\,a(\tilde{F})$ at $z = 0$. [5] Point (ii) is proven.

It remains to compute the variation of h_0 in the case $\tilde{F} \geq 1$. We have $\theta(0) = a(\tilde{F})$ and $\theta(\pi) = \pi$. From (34), we directly deduce

$$V(\tilde{F}) := h(0) - h(\pi) = a(\tilde{F})^2/2 + \tilde{F}\,(\cos(a(\tilde{F})) + 1)$$

and we get the desired result by taking into account the sign of $\cos(a(\tilde{F}))$. □

[4] In order to prove that $G''(a(\tilde{F})) > 0$, we have to prove that $a(\tilde{F}) > \sqrt{\tilde{F}^2 - 1}$. Owing to the properties of Eq. (28), it is enough to check that $\sqrt{\tilde{F}^2 - 1} < \tilde{F}\sin(\sqrt{\tilde{F}^2 - 1})$ or equivalently that for any $\alpha \in (0, \sqrt{\pi^2/4 - 1})$,

$$\sin(\alpha) - \frac{\alpha}{\sqrt{1 + \alpha^2}} > 0$$

We first notice that, on the interval $(0, \pi)$ owing to Taylor–Lagrange expansion, $\sin(\alpha) \geq \alpha - \frac{\alpha^3}{6}$ and then we are reduced to study the inequality

$$\alpha - \frac{\alpha^3}{6} > \frac{\alpha}{\sqrt{1 + \alpha^2}}$$

since $\alpha < \sqrt{6}$ on the interval $(0, \sqrt{\pi/4 - 1})$, the previous relation is equivalent to:

$$1 + \frac{\alpha^4}{36} - \frac{\alpha^2}{3} > \frac{1}{1 + \alpha^2}$$

or

$$0 < 24 - 11\alpha^2 + \alpha^4.$$

The result is ensured by the fact that this inequality is true for any $\alpha^2 \in (0, 3)$.

[5] It may also be useful to remark that, as

$$\theta'(z) = 1/G''(\theta(z)) = 1/(1 - \tilde{F}\cos(\theta(z))) \geq 0,$$

we have

$$h''(z) = 1 - 1/(1 - \tilde{F}\cos(\theta(z))) = -\tilde{F}\cos(\theta(z))/(1 - \tilde{F}\cos(\theta(z))).$$

Hence $h''(z)$ has the opposite sign to $\cos(\theta(z))$. It remains positive on the interval $(a(\tilde{F}), \pi)$ if and only if $a(\tilde{F}) \geq \pi/2$, in otherwords, if $\tilde{F} \geq \pi/2$. In that case h is convex between two successive singularities. If $1 < \tilde{F} < \pi/2$, there exists an inflexion point, and h is concave in a vicinity of the singularities. (We remark that it not very easy to appreciate the inflection point in Fig. 4, right, because the graph of h_0 is there very similar to a straight line.)

3.1.3. Buckling load. When $\gamma = 0$, the constant function $\phi = 0$ is a critical point of Problem (18). When F is small enough, this trivial solution is the minimizer of the energy but when F increases, other critical solutions may exist. This phenomenon is known as the buckling of the beam under compressive load. We determine in this section the value of the load above which several critical solutions exist.

Lemma 3. *The number of critical solutions of the problem*

$$\min_{\phi} \left\{ \int_0^1 \left(\frac{1}{2}(\phi'(s))^2 + \tilde{K}_2 h_0(\phi(s)) \right) \mathrm{d}s;\ \phi(0) = 0 \right\} \tag{37}$$

with

$$h_0(z) = \frac{(z - \theta(z))^2}{2} + \tilde{F}\,\cos(\theta(z)) \tag{38}$$

and

$$z = \theta(z) - \tilde{F}\sin(\theta(z)) \tag{39}$$

is larger than one if and only if $\tilde{F} > \frac{1}{1+\frac{4\tilde{K}_2}{\pi^2}}$.

Proof. It is easy to check that critical solutions of (37) are the solutions, for some constant C, of the differential equation

$$\frac{1}{2}(\phi'(s))^2 - \tilde{K}_2 h_0(\phi(z)) = C \tag{40}$$

with the boundary conditions $\phi(0) = 0$ and $\phi'(1) = 0$.

Let us set $C = \tilde{K}_2\left(\frac{k^2}{2} - h_0(0)\right)$ or equivalently $|\phi'(0)| = k\sqrt{\tilde{K}_2}$ and let us study for which values of k the solution of (40) with initial conditions $\phi(0) = 0$, $\phi'(0) = k\sqrt{\tilde{K}_2}$ satisfies $\phi'(1) = 0$. If $k^2/2 > V(F) = h_0(0) - h_0(\pi)$, it is clearly impossible.

We let apart the trivial solution $k = 0$, $\phi = 0$, and by symmetry, we focus on the case $\phi'(0) = k\sqrt{\tilde{K}_2} > 0$. We call ϕ_k the solution of (40) and we denote $L(k)$ the smallest positive value of s such that $\phi_k'(s) = 0$. We also denote $z_k = \phi_k(L(k))$ which implies

$$h_0(z_k) = h_0(0) - \frac{k^2}{2}. \tag{41}$$

As ϕ_k is monotonic on $[0, L(k)]$, we can write

$$L(k) = \int_0^{z_k} ((\phi_k)^{(-1)})'(z)\,\mathrm{d}z = \frac{1}{\sqrt{\tilde{K}_2}} \int_0^{z_k} \frac{1}{\sqrt{k^2 + 2\,(h_0(z) - h_0(0))}}\,\mathrm{d}z$$

Let us first remark that L is a continuous (not necessarily monotonic) function of k which tends to infinity when k tends to $\sqrt{2(h_0(0) - h_0(\pi))}$. Indeed the integral becomes singular when k tends to π. Things are less clear when k tends to zero. Using the change of variable

$$v(z) := \arcsin\left(\frac{\sqrt{2\,(h_0(0) - h_0(z))}}{k}\right),$$

we get

$$\sqrt{\tilde{K}_2}\,L(k) = \int_0^{\frac{\pi}{2}} j(k\sin(v))\,\mathrm{d}v \tag{42}$$

with

$$j(u) := -\frac{u}{h_0'(h_0^{(-1)}(h_0(0) - \frac{u^2}{2}))} = -\frac{u}{h_0'(z_u)}$$

which becomes, using the expressions of h_0' given by Lemma 2 and Eq. (41),

$$j(u) = \frac{u}{\sqrt{2\tilde{F} - 2h_0(z_u)}} \sqrt{\frac{1 - \tilde{F} + \sqrt{1 + \tilde{F}^2 - 2\,h_0(z_u)}}{\tilde{F} + 1 - \sqrt{1 + \tilde{F}^2 - 2\,h_0(z_u)}}}$$

$$= \frac{u}{\sqrt{2(\tilde{F} - h_0(0)) + u^2}} \sqrt{\frac{1 - \tilde{F} + \sqrt{1 + \tilde{F}^2 - 2\,h_0(0) + u^2}}{\tilde{F} + 1 - \sqrt{1 + \tilde{F}^2 - 2\,h_0(0) + u^2}}}$$

When $\tilde{F} > 1$, one can easily check that $h_0(0) < \tilde{F}$. The function j has no singularities, $j(0) = 0$ and we can pass to the limit in (42). We obtain $L(0^+) = 0$. The theorem of intermediate values ensures the existence of a k in $(0, +\infty)$ such that $L(k) = 1$;

When $F \leq 1$, we have $h_0(0) = \tilde{F}$ and the expression of j can be simplified

$$j(u) = \sqrt{\frac{1 - \tilde{F} + \sqrt{(1 - \tilde{F})^2 + u^2}}{\tilde{F} + 1 - \sqrt{(1 - \tilde{F})^2 + u^2}}}$$

We see now that j is strictly increasing : L takes values in $(L(0), +\infty)$ and a solution such that $L(k) = 1$ exists if and only if

$$1 > \frac{1}{\sqrt{\tilde{K}_2}} L(0) = \frac{1}{\sqrt{\tilde{K}_2}} \int_0^{\frac{\pi}{2}} j(0)\,\mathrm{d}v = \frac{1}{\sqrt{\tilde{K}_2}} \frac{\pi}{2} \sqrt{\frac{1 - \tilde{F}}{\tilde{F}}}$$

or equivalently

$$\tilde{F} > \tilde{F}_c := \frac{1}{1 + \frac{4\tilde{K}_2}{\pi^2}}.$$

□

It is interesting to write this critical load in terms of the original parameters. It becomes

$$\boxed{F \geq F_c \frac{1}{1 + \frac{F_c}{K_2}} \quad \text{with} \quad F_c := \frac{\pi^2 K_1}{4L^2}}$$

The quantity F_c is known to be the critical load for a clamped-free Euler beam. (The historical reference is [1]; for interesting recent developments see also [36–40].) The previous formula provides the correction to be applied for a Timoshenko beam. As expected the critical value tends to F_c when the Timoshenko coupling parameter K_2 is very large.

Note also that the number of critical solutions increases with $\tilde{F}$. It passes from 1 to 3 when $\tilde{F}$ reaches the critical value $\tilde{F}_c$. It can be shown that more generally, it passes from $2n-1$ to $2n+1$ when $\tilde{F}$ reaches the value

$$\frac{1}{1 + \frac{4(2n-1)^2 \tilde{K}_2}{\pi^2}}.$$

To our knowledge, this is the first estimate for the critical load of a beam with energy model (8).

3.1.4. Euler–Lagrange equations. A formal computation starting from (18) leads to the following Euler–Lagrange equations for an inextensible Timoshenko beam under the end load $\boldsymbol{F} = F_1\boldsymbol{e}_1 + F_2\boldsymbol{e}_2$:

$$\begin{cases} \phi''(s) = \dfrac{K_2}{K_1}(\phi(s) - \theta(s)) \\ \phi(s) = \theta(s) - \dfrac{F_1}{K_2}\sin(\theta(s)) + \dfrac{F_2}{K_2}\cos(\theta(s)) \\ \phi(0) = \phi'(1) = 0 \end{cases} \tag{43}$$

Substituting ϕ in the first equation, we obtain the boundary value problem for $\theta(s)$:

$$\begin{cases} \theta''(s) = \dfrac{(F_2\cos\theta - F_1\sin\theta)\left(K_2 + K_1\theta'^2\right)}{K_1(K_2 - F_1\cos\theta - F_2\sin\theta)} \\ \theta(0) = -\dfrac{F_1}{K_2}\sin(\theta(0)) + \dfrac{F_2}{K_2}\cos(\theta(0)) \\ \theta'(1) = 0 \end{cases} \tag{44}$$

Of course a minimizer of the problem (18) will not have, in general, enough regularity to solve (43) in a strong sense. A formal computation starting from the functional $\mathcal{A}(\phi)$ defined in (19) leads to the differential equation:

$$\phi'' = \phi - \tilde{K}_2\frac{dG^*}{d\phi} \tag{45}$$

with $\phi(0) = \phi'(1) = 0$. Clearly, ϕ is at least C^2 on the intervals where G^* is differentiable. (The regularity of G^* is addressed in Lemma 2.) On the other hand, a stationary pair (ϕ, θ) solves the integral equation:

$$\phi(s) = \int_0^s \mathrm{d}t \int_t^1 \frac{K_2}{K_1}(\phi - \theta)\mathrm{d}x \tag{46}$$

In the intervals on which ϕ is C^2, the previous integral representation implies C^0 regularity for θ. Hence, the second of Eq. (43) implies $\theta \in C^2$, that together with (46) implies C^∞ regularity for both ϕ and θ almost everywhere, except at the points where G^* is not differentiable. In conclusion, θ solves Eq. (44) in a strong sense on all the intervals in which G^* is differentiable, and we already showed that in case of $\tilde{F} < 1$, this condition holds in all $\mathbb{R}$.

3.2. Regularized Timoshenko model

Since the integrand in (16) is convex and coercive with respect to the highest order derivatives, by a well-known result of calculus of variations (see [31]), the problem:

$$\begin{cases} \min \quad E(\phi,\theta) := \int_0^1 \left[\frac{K_1}{2}\phi'^2 + \frac{K_2}{2}(\phi-\theta)^2 + \frac{K_3}{2}\theta'^2 + F_1\cos\theta + F_2\sin\theta\right]\mathrm{d}s \\ \\ \theta \in H^1[0,1], \quad \phi \in H^1[0,1], \quad \theta(0) = \phi(0) = 0 \end{cases} \tag{47}$$

admits at least one solution, say $(\bar{\phi}, \bar{\theta})$. Moreover, the minimizer $(\bar{\phi}, \bar{\theta})$ verifies in a weak form the relative Euler–Lagrange equations:

$$\begin{cases} \phi'' = \frac{K_2}{K_1}(\phi - \theta) \\ \theta'' = \frac{K_2}{K_3}(\theta - \phi) - \frac{F_1}{K_3}\sin\theta + \frac{F_2}{K_3}\cos\theta \end{cases} \tag{48}$$

which means that a minimizing pair $(\bar{\phi}, \bar{\theta})$ is a fixed point for the following nonlinear integral operator:

$$(\phi, \theta) \longmapsto \mathbb{T}(\phi, \theta) := \Big(\mathbb{T}_1(\phi, \theta), \mathbb{T}_2(\phi, \theta)\Big)$$

$$\mathbb{T}_1(\phi, \theta) := \frac{K_2}{K_1} \int_0^s \mathrm{d}x \int_1^x \mathrm{d}t[\phi(t) - \theta(t)]$$

$$\mathbb{T}_2(\phi, \theta) := \frac{1}{K_3} \int_0^s \mathrm{d}x \int_1^x \mathrm{d}t[K_2(\theta(t) - \phi(t)) + F_1 \sin\theta - F_2 \cos\theta]$$

Clearly $\mathbb{T}(\phi, \theta)$ is in $C^2 \times C^2$ if $(\phi, \theta) \in H^1 \times H^1$. Therefore, $(\bar{\phi}, \bar{\theta})$ verify (48) in a strong (pointwise) sense. Finally, since $(\bar{\phi}, \bar{\theta})$ is a fixed point for $\mathbb{T}$, the previous argument can be applied recursively, which implies C^∞ regularity for the minimizer.

4. Numerical simulations

4.1. Results on the absolute minimizer

In this section, we want to show some numerical results on the absolute minimizers of energy models (18) and (47). The main aim of these numerical investigations is to show the effect of the introduction of the regularizing term in θ'^2 in the deformation energy, and indeed we will "perturb" the problem (18) by means of a term of the type $\varepsilon\theta'^2$ with $\varepsilon \ll 1$.

The numerical procedure to find the minimizer is different in the two cases of models (18) and (47). In the first case, we used a direct numerical minimization (based on finite elements) to find a solution of the problem P (18). We searched for $\tilde{\theta}$ and $\tilde{\phi}$, respectively, in the set of piecewise constant and piecewise linear functions (which are dense, respectively, in L^2 and H^1). As an internal consistency test, we also checked that the solution $\tilde{\theta}$ coincides with the one obtained from $\tilde{\phi}$ by means of (20).

On the other hand, in the case of the problem (47), the regularity proven in Sect. 3.2 allowed us to use the Euler–Lagrange equations. The corresponding boundary value problem was solved by means of a two-dimensional shooting technique. Specifically, an explicit Euler method (with an integration step $\eta = 10^{-3}$) was implemented to solve a set of initial value problems parameterized by means of two parameters:

$$\begin{cases} \phi'' = \frac{K_2}{K_1}(\phi - \theta), \qquad \theta'' = \frac{K_2}{K_3}(\theta - \phi) - \frac{F_1}{K_3}\sin\theta + \frac{F_2}{K_3}\cos\theta, \\ \phi(0) = 0, \qquad \theta(0) = 0, \\ \phi'(0) = A_1, \qquad \theta'(0) = A_2 \end{cases} \tag{49}$$

We studied the behavior of the vector of boundary values $\boldsymbol{p} := (\phi'(1), \theta'(1))$ as a function of A_1 and A_2, and employed a direct minimizer to find the points in the space of parameters where $||\boldsymbol{p}||$ is minimal. More precisely, we introduced a square grid (with step length $\ell \approx 10^{-3}$) on the plane (A_1, A_2) and searched for the cells of the grid in which the local minima of $||\boldsymbol{p}||$ are situated.

In the following graphs, we show some relevant results obtained with these two procedures. The parameters employed in the various cases are indicated in the captions (Figs. 5, 6, 7 and 8).

4.2. Results on other stationary points

In this section, we want to show some curled equilibria of the nonlinear Timoshenko beam model (8) and of the regularized model (16), which resemble the ones presented for an inextensible Euler beam (in case

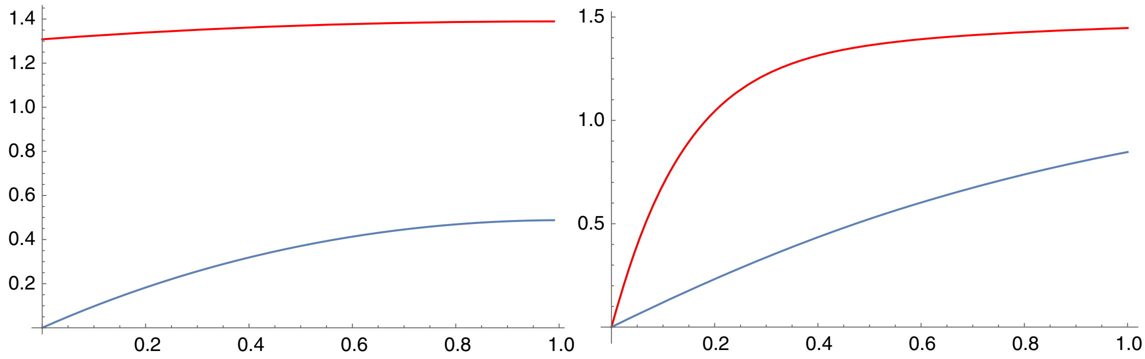

FIG. 5. The minimizing pair $(\tilde{\phi}, \tilde{\theta})$ in case of transverse end load for a nonlinear Timoshenko energy model (left) and regularized model obtained adding to the previous a term in θ'^2 with a small coefficient (right). The parameters are $F_1 = 0$, $F_2 = 10$, $K_1/2 = 1, K_2/2 = 1$ (left) and the same with $K_3/2 = 0.1$ (right). Notice that on the left we have $\tilde{\theta}(0) \neq 0$

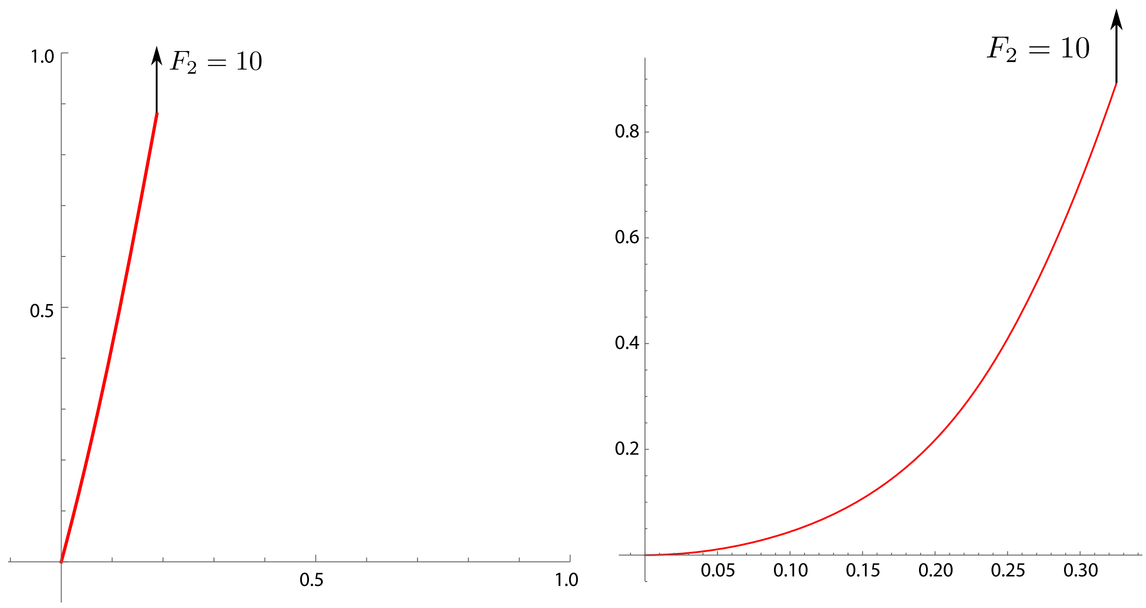

FIG. 6. The deformed shape of the beam corresponding to the case of transverse end load for a nonlinear Timoshenko energy model (left) and regularized model obtained adding to the previous a term in θ'^2 with a small coefficient (right). The parameters are the same as in the previous figure

of distributed load) in [32]. In doing so, we assume that stationary points different from the absolute minimizer solve Euler–Lagrange Eq. (43).

Solving the boundary value problem by means of a shooting technique (similarly to what explained in the previous section), we found the curled solutions shown in Fig. 9. It is interesting to notice that, passing from the simple Euler model to nonlinear Timoshenko and regularized Timoshenko, the property of having such a kind of curled solutions is still kept. The complete study of local minima of the nonlinear Timoshenko beam model is not trivial and would probably require ideas from the theory of phase transitions (see [43,44]).

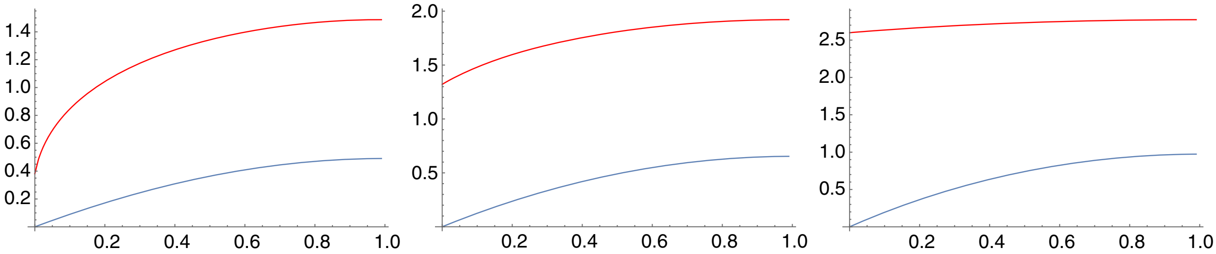

FIG. 7. The minimizing pair $(\tilde{\phi}, \tilde{\theta})$ corresponding to three different compressive end loads for a nonlinear Timoshenko energy model. The parameters are $F_2 = 0$, $K_1/2 = 1, K_2/2 = 1$ in the three plots, while $F_1 = 2$ (left), $F_1 = 2.7$ (center) and $F_1 = 10$ (right). Notice that the absolute value of $\tilde{\theta}(0)$ increases with F_1

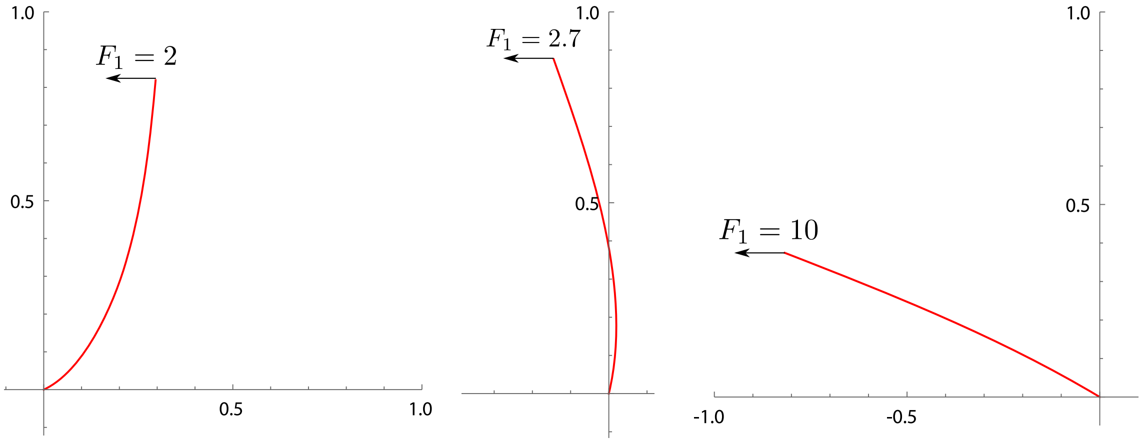

FIG. 8. The deformed shape of the beam corresponding to three different compressive end loads for a nonlinear Timoshenko energy model. The parameters are $F_2 = 0$, $K_1/2 = 1, K_2/2 = 1$ in the three plots, while $F_1 = 2$ (left), $F_1 = 2.7$ (center) and $F_1 = 10$ (right)

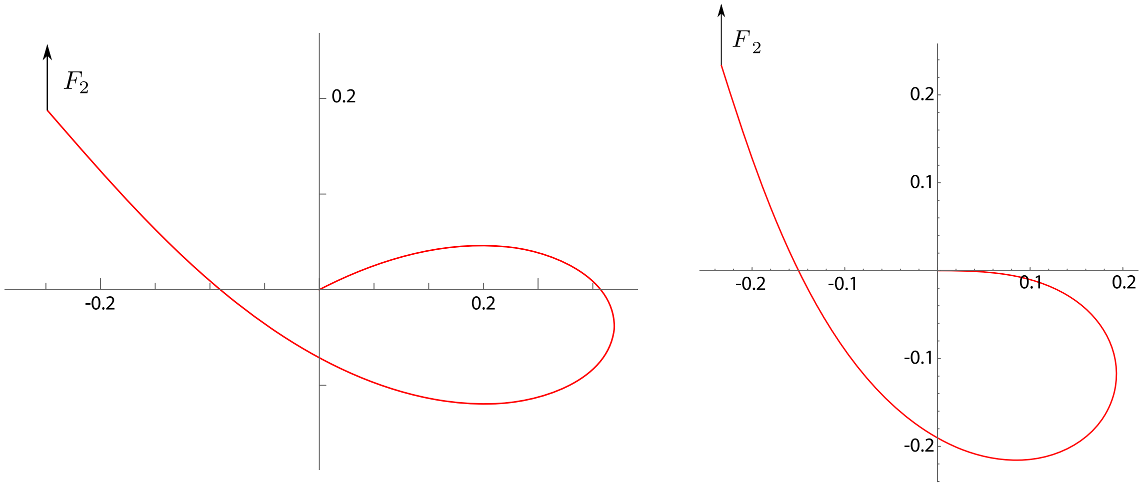

FIG. 9. A curled equilibrium shape for nonlinear Timoshenko model (left); the parameters are: $K_1/2 = 1$, $K_2/2 = 20$, $F_2 = 25$. The same kind of equilibrium shape for a regularized Timoshenko model (right); the parameters are: $K_1/2 = 1$, $K_2/2 = 1$, $K_3/2 = 0.1$, $F_2 = 10$

5. Conclusions

The importance of geometrical nonlinearities is increasingly important in modern structural mechanics (see, e.g., [42]) and in general in elasticity theory (see [45–50]). On the other hand, beam theory, and especially generalized beam theory, is particularly interesting nowadays in view of applications to lattice systems [33,51,52], and in this context, pantographic structures are naturally leading to the problem of large deformations of the fibers (see for instance [53–59]).

This paper dealt with the problem of geometrically nonlinear deformations of generalized Timoshenko beam models, obtained by means of a formal homogenization starting from a microstructured 1D system. We considered both a straightforward generalization of the customary linearized model (7) and the model obtained introducing in the microstructure an additional rotational spring entailing a term in θ'^2 in the deformation energy density. We proved well posedness of the variational problem concerning a clamped beam with generic end load, as well as some properties of the minimizers, and presented and discussed some numerical simulations.

The main open problem connected with the content of the work is the generalization of the results to the case of a distributed load, which would allow the use of the result in the modeling of lattice fibrous systems with beams connected each other with finely spaced pivots ([34,35,53]). However, this generalization is not trivial as it leads to a non-autonomous variational problem. We remark indeed that the given demonstrations do not generalize trivially to the case of a distributed load, as the monotonicity properties of the minimizer discussed in [30] are not available in general.

Finally, we want to mention that the nonlinear differential equations describing the nonlinear model introduced here are suitable to account for many different phenomena. For instance, they can describe the motion of a pendulum consisting of a weight linked by means of a Hooke's spring to a charge of negligible mass oscillating in an electromagnetic field; in case of distributed load, the electromagnetic field would be time-dependent (see [60]). Also, aspects of the theory of interface instabilities can be interpreted as shear elastic problems and can lead to variational problems presenting similarities with the ones discussed herein (see [61]). Therefore, the interest of studying the nonlinear energy model also in case of distributed load goes probably well beyond 1D elasticity.

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A. Battista
Uiversité de La Rochelle
La Rochelle
France
e-mail: antoniobattista1986@gmail.com

A. Battista, A. Della Corte, F. dell'Isola and P. Seppecher
M&MoCS, Research Center
University of L'Aquila
L'Aquila
Italy
e-mail: alessandro.dellacorte.memocs@gmail.com

F. dell'Isola
e-mail: francesco.dellisola@uniroma1.it

P. Seppecher
e-mail: seppecher@imath.fr

A. Della Corte and F. dell'Isola
University La Sapienza
Rome
Italy

P. Seppecher
IMATH-Université de Toulon
Toulon
France

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