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Long-time behavior for suspension bridge equations with time delay

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Abstract. In this paper, we consider suspension bridge equations with time delay of the form

$$u_{tt}(x,t) + \Delta^2 u(x,t) + ku^+(x,t) + a_0 u_t(x,t) + a_1 u_t(x,t-\tau) + f(u(x,t)) = g(x).$$

Many researchers have studied well-posedness, decay rates of energy, and existence of attractors for suspension bridge equations without delay effects. But, as far as we know, there is no work about suspension equations with time delay. In addition, there are not many studies on attractors for other delayed systems. Thus we first provide well-posedness for suspension equations with time delay. And then show the existence of global attractors and the finite dimensionality of the attractors by establishing energy functionals which are related to the norm of the phase space to our problem.

Mathematics Subject Classification. 35L70, 35B41.

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1. Introduction

We study the following suspension bride equations with time delay

$$u_{tt} + \Delta^2 u + ku^+ + a_0 u_t(x,t) + a_1 u_t(x,t-\tau) + f(u) = g \text{ in } \Omega \times \mathbb{R}^+, \tag{1.1}$$

$$u = \Delta u = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \tag{1.2}$$

$$u(0) = u_0, \;\; u_t(0) = u_1 \text{ on } \Omega, \tag{1.3}$$

$$u_t(x,t) = j_0(x,t) \quad \text{for } (x,t) \in \Omega \times (-\tau, 0), \tag{1.4}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, $k > 0$ is spring constant, $u^+ = \max\{u, 0\}$ is the positive part of u, the term $-ku^+$ models a restoring force due to the cables, which is different from zero only when they are being stretched. The constants a_0 and a_1 are real numbers, $\tau > 0$ is time delay, and f and g are forcing terms.

The suspension bridge equations were introduced by Lazer and McKenna [7] to describe the transverse deflection of the roadbed in the vertical plane and they were regarded as new problems in the field of nonlinear analysis. In the absence of delay, that is, when $a_1 = 0$ in (1.1), problem (1.1)–(1.3) was intensively studied about well-posedness, uniqueness of solutions, and attractors (see, e.g., [1,7,10,11,21] and references therein). An [1] obtained the existence and uniqueness of a weak solution for $k > -1$ and showed decay estimates of the solution, and Ma and Zhong [10] investigated the existence of global attractors in $H_0^2(\Omega) \times L^2(\Omega)$. Later, the authors of [21] improved the results of [10] by showing the existence of strong solutions and global attractors in $D(A) \times H_0^2(\Omega)$, making use of the norm-to-weak continuous semigroup scheme developed in [8]. Moreover, we refer [9,15,16] for works of suspension bridge equations. In this paper, we will study suspension bridge equations with time delay of the form (1.1)–(1.4).

Time delays arise in many applications depending not only on the present state but also on some past occurrences. The presence of delay may be a source of instability (see, e.g., [4,13]), and hence it affects the existence of attractors. Thus, partial differential equations with time delay effects have become an active area of research (see [12,14,18] and references therein). As regards wave equations with delay, Nicaise and Pignotti [13] investigated the wave equation with time delay

$$u_{tt}(x,t) - \Delta u(x,t) + a_0 u_t(x,t) + a_1 u_t(x,t-\tau) = 0. \tag{1.5}$$

They proved that the energy of the problem decays exponentially under the condition $0 < a_1 < a_0$. And then they extended the result to the time varying delay case in [14]. For the related works of equations with time delay, we also refer [6,17,20] and references therein.

Based on the above-mentioned research results, we investigate the existence of attractors and finite dimensionality of the attractors for problem (1.1)–(1.4). It is worth to mention that there are not much literature on attractors for delayed systems. Furthermore, as far as we are concerned, this is the first work in the literature that takes into account the global attractors for suspension equations with time delay. To obtain our desired results, we establish some functionals which are equivalent to the norm of the phase space to problem (1.1)–(1.4).

The outline of this paper is as follows. In Sect. 2, we give some notations and material needed for our work. In Sect. 3, we prove the existence of attractors for problem (1.1)–(1.4). Finally, in Sect. 4, we examine the finite dimensionality of the attractors.

2. Preliminaries

We first review some notations about function spaces and give hypothesis for problem (1.1)–(1.4). For a Banach space X, we denote the norm of X by $||\cdot||_X$. As usual, we denote the scalar product in $L^2(\Omega)$ by $(\cdot,\cdot)$ and $L^p(\Omega)$ norm by $||\cdot||_p$, respectively. For brevity, we denote $||\cdot||_2$ by $||\cdot||$.

Let λ be the best constant in the Poincaré-type inequality

$$\lambda ||u||^2 \le ||\Delta u||^2 \quad \text{for } u \in H^2(\Omega)\cap H_0^1(\Omega).$$

Let us introduce the phase space

$$\mathcal{H} = (H^2(\Omega)\cap H_0^1(\Omega)) \times L^2(\Omega) \times L^2(\Omega\times(0,1))$$

equipped with the norm

$$||(u,v,z)||_{\mathcal{H}}^2 = ||\Delta u||^2 + ||v||^2 + ||z||^2_{L^2(\Omega\times(0,1))}.$$

With regard to problem (1.1)–(1.4), we impose the following assumptions:

(H1) $f:\mathbb{R}\to\mathbb{R}$ satisfy

$$|f(u)-f(\tilde{u})| \le l(1+|u|^p+|\tilde{u}|^p)|u-\tilde{u}| \quad \text{for} \quad u,\tilde{u}\in\mathbb{R}, \tag{2.1}$$

$$-l_0 \le F(u) \le f(u)u \quad \text{for} \quad u\in\mathbb{R}, \tag{2.2}$$

here $l>0$, $l_0>0$, $F(u)=\int\limits_0^u f(s)\mathrm{d}s$, and $p>0$.

(H2) $g\in L^2(\Omega)$, $j_0\in L^2(\Omega\times(-\tau,0))$

(H3) The coefficients a_0 and a_1 satisfy

$$0<|a_1|<a_0.$$

As in [13], we introduce the function

$$z(x,\rho,t) = u_t(x,t-\rho\tau) \quad \text{for} \quad (x,\rho,t)\in\Omega\times(0,1)\times(0,\infty). \tag{2.3}$$

Then problem (1.1)–(1.4) is equivalent to

$$u_{tt} + \Delta^2 u + ku^+ + a_0u_t + a_1 z(x,1,t) + f(u) = g \text{ on } \Omega\times\mathbb{R}^+, \tag{2.4}$$

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \text{ for } (x,\rho,t)\in\Omega\times(0,1)\times(0,\infty), \tag{2.5}$$

$$u(x,t) = \Delta u(x,t) = 0 \quad \text{for } (x,t)\in\partial\Omega\times\mathbb{R}^+, \tag{2.6}$$

$$u(x,0)=u_0(x),\ \ u_t(x,0)=u_1(x) \text{ for } x\in\Omega, \tag{2.7}$$
$$z(x,\rho,0)=j_0(x,-\rho\tau):=z_0(x,\rho) \quad \text{for } (x,\rho)\in\Omega\times(0,1). \tag{2.8}$$

To obtain the global attractor of problem (2.4)–(2.8), we state the existence result:

Theorem 2.1. *Assume that (H1) and (H2) hold. Then we have:*

(i) *For every* $(u_0,u_1,z_0)\in\mathcal{H}$ *and* $T>0$, *there exists a weak solution* (u,u_t,z) *of problem* (2.4)–(2.8) *in the class*

$$u\in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)),\ \ u_t\in L^\infty(0,T;L^2(\Omega)),\ \ z\in L^\infty(0,T;L^2(\Omega\times(0,1)))$$

satisfying $(u,u_t,z)\in C([0,T];\mathcal{H})$.
Moreover, the solution is unique and depends continuously on the initial data $(u_0,u_1,z_0)\in\mathcal{H}$ *and* $g\in L^2(\Omega)$.

(ii) *Let* (u,u_t,z) *and* $(\tilde u,\tilde u_t,\tilde z)$ *be two weak solutions of problem* (2.4)–(2.8) *corresponding to initial data* (u_0,u_1,z_0) *and* $(\tilde u_0,\tilde u_1,\tilde z_0)$, *respectively. Then one gets*

$$||(u,u_t,z)-(\tilde u,\tilde u_t,\tilde z)||_{\mathcal{H}}\le e^{ct}||(u_0,u_1,z_0)-(\tilde u_0,\tilde u_1,\tilde z_0))||_{\mathcal{H}} \quad \textit{for some } c>0.$$

Proof. The proof can be established by combining arguments of [1,6,20,21]. □

The aim of this paper is to prove the existence of attractors for problem (1.1)–(1.4) and examine the finite dimensionality of the attractors. For this purpose, we present some basic concepts and abstract results on dynamical systems by following the Chueshov and Lasiecka's book [3] (or see, e.g., [2,5,19]).

Let $\mathcal{F}$ be a Banach space and B be a bounded subset of $\mathcal{F}$. We call a function $\phi(\cdot,\cdot)$ which defined on $\mathcal{F}\times\mathcal{F}$ is *a contractive function on* $B\times B$ if for any sequence $\{x_n\}_{n=1}^\infty\subset B$, there is a subsequence $\{x_{n_k}\}_{k=1}^\infty\subset\{x_n\}_{n=1}^\infty$ such that

$$\liminf_{k\to\infty}\liminf_{l\to\infty}\phi(x_{n_k},x_{n_l})=0.$$

Lemma 2.1. *Let* $\{S(t)\}_{t\ge0}$ *be a semigroup on a Banach space* $(\mathcal{F},||\cdot||_{\mathcal{F}})$ *and have a bounded absorbing set* B_0. *Assume that for any* $\epsilon>0$ *there exist* $T=T(B_0,\epsilon)$ *and a contractive function* $\phi_T(\cdot,\cdot)$ *on* $B_0\times B_0$ *such that*

$$||S(T)x-S(T)y||_{\mathcal{F}}\le\epsilon+\phi_T(x,y) \quad \textit{for all } x,y\in B_0,$$

where ϕ_T *depends on* T. *Then* $S(t)$ *is asymptotically smooth in* $\mathcal{F}$.

Lemma 2.2. *A dissipative dynamical system* $(S(t),\mathcal{F})$ *has a compact global attractor if and only if it is asymptotically smooth.*

Let X,Y,Z be three reflexive Banach spaces with X compactly embedded in Y, $\mathcal{F}=X\times Y\times Z$, and $(S(t),\mathcal{F})$ a dynamical system given by an evolution operator

$$S(t)x=(u(t),u_t(t),z(t)) \quad \text{for} \quad x=(u_0,u_1,z_0)\in\mathcal{F}, \tag{2.9}$$

where the functions u and z have regularity

$$u\in C(\mathbb{R}^+;X)\cap C(\mathbb{R}^+;Y),\ \ z\in C(\mathbb{R}^+;Z). \tag{2.10}$$

We call the dynamical system $(S(t),\mathcal{F})$ is *quasi-stable on* $B\subset\mathcal{F}$ if there exists a compact seminorm n_X on X and nonnegative scalar function $a(t)$ and $c(t)$, locally bounded in $[0,\infty)$, and $b(t)\in L^1(\mathbb{R}^+)$ with $\lim_{t\to\infty}b(t)=0$ such that

$$||S(t)x-S(t)y||^2_{\mathcal{F}}\le a(t)||x-y||^2_{\mathcal{F}} \tag{2.11}$$

and

$$||S(t)x-S(t)y||^2_{\mathcal{F}}\le b(t)||x-y||^2_{\mathcal{F}}+c(t)\sup_{0<s<t}[n_X(u(s)-\tilde u(s))]^2, \tag{2.12}$$

where $S(t)x=(u(t),u_t(t),z(t))$, $S(t)y=(\tilde u(t),\tilde u_t(t),\tilde z(t))$ and $x,y\in B$.

Lemma 2.3. *Let* $(S(t),\mathcal{F})$ *be given by* (2.9) *and satisfy* (2.10). *If* $(S(t),\mathcal{F})$ *has a compact global attractor* $\mathcal{A}$ *and is quasi-stable on* $\mathcal{A}$, *then the attractor* $\mathcal{A}$ *has finite fractional dimension.*

3. Existence of attractors

In this section, we prove the existence of global attractors for problem (1.1)–(1.4) adapting Lemma 2.2. To do this, we define a map $S(t) : \mathcal{H} \to \mathcal{H}$ by

$$S(t)(u_0, u_1, z_0) = (u(t), u_t(t), z(t)), \tag{3.1}$$

where $(u(t), u_t(t), z(t))$ is the unique weak solution of system (2.4)–(2.8) corresponding to initial data (u_0, u_1, z_0). Then, by Theorem 2.1, $\{S(t)\}_{t\geq 0}$ is a C^0-semigroup on $\mathcal{H}$.

Inspired by [14], let us define the energy of problem (2.4)–(2.8) by

$$E(t) = \frac{1}{2}||u_t||^2 + \frac{1}{2}||\Delta u||^2 + \frac{k}{2}||u^+||^2 + \int_\Omega F(u)\mathrm{d}x - (g, u) + \frac{\xi}{2}\int_{t-\tau}^{t} e^{\theta(s-t)}||u_t(s)||^2\mathrm{d}s, \tag{3.2}$$

where

$$|a_1| < \xi < 2a_0 - |a_1| \quad \text{and} \quad 0 < \theta < \frac{1}{\tau}\ln\frac{\xi}{|a_1|}. \tag{3.3}$$

Using integration by substitution $s = t - \rho\tau$, we find

$$\begin{aligned}\frac{\xi}{2}\int_{t-\tau}^{t} e^{\theta(s-t)}||u_t(s)||^2\mathrm{d}s &= -\frac{\xi\tau}{2}\int_1^0\int_\Omega e^{-\theta\rho\tau}u_t^2(x, t-\rho\tau)\mathrm{d}x\mathrm{d}\rho \\ &= \frac{\xi\tau}{2}\int_0^1\int_\Omega e^{-\theta\rho\tau}z^2(x,\rho,t)\mathrm{d}x\mathrm{d}\rho. \end{aligned}\tag{3.4}$$

From (2.2), we get

$$\int_\Omega F(u)\mathrm{d}x - (g, u) \geq -l_0|\Omega| - \frac{1}{4}||\Delta u||^2 - \frac{||g||^2}{\lambda}. \tag{3.5}$$

Applying these to (3.2), one sees that

$$\begin{aligned}E(t) &\geq \frac{1}{2}||u_t||^2 + \frac{1}{4}||\Delta u||^2 + \frac{k}{2}||u^+||^2 - l_0|\Omega| - \frac{||g||^2}{\lambda} + \frac{\xi\tau}{2}\int_0^1\int_\Omega e^{-\theta\tau\rho}z^2(x,\rho,t)\mathrm{d}x\mathrm{d}\rho \\ &\geq \frac{1}{2}||u_t||^2 + \frac{1}{4}||\Delta u||^2 + \frac{k}{2}||u^+||^2 - l_0|\Omega| - \frac{||g||^2}{\lambda} + \frac{\xi\tau e^{-\theta\tau}}{2}\int_0^1\int_\Omega z^2(x,\rho,t)\mathrm{d}x\mathrm{d}\rho \\ &\geq \frac{1}{c_0}(||u_t||^2 + ||\Delta u||^2 + \int_0^1\int_\Omega z^2(x,\rho,t)\mathrm{d}x\mathrm{d}\rho) - l_0|\Omega| - \frac{||g||^2}{\lambda}, \end{aligned}\tag{3.6}$$

where

$$\frac{1}{c_0} := \min\left\{\frac{1}{4}, \frac{\xi\tau e^{-\theta\tau}}{2}\right\}. \tag{3.7}$$

This yields that

$$||(u, u_t, z)||_{\mathcal{H}}^2 \leq c_0\Big(E(t) + l_0|\Omega| + \frac{||g||^2}{\lambda}\Big). \tag{3.8}$$

Lemma 3.1. *Assume that (H1), (H2), and (H3) hold. Then there exists positive constants* c_1 *and* c_2 *satisfying*

$$E'(t) \leq -c_1||u_t(t)||^2 - c_2||z(1,t)||^2 - \frac{\theta\xi}{2}\int_{t-\tau}^{t} e^{\theta(s-t)}||u_t(s)||^2 ds.$$

Proof. Multiplying u_t in (2.4), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{E(t) - \frac{\xi}{2}\int_{t-\tau}^{t} e^{\theta(s-t)}||u_t(s)||^2\mathrm{d}s\right\} = -a_0||u_t(t)||^2 - a_1(z(1,t), u_t(t)).$$

Thus, by direct calculation and Young's inequality, we get

$$\begin{aligned}
E'(t) = &-a_0||u_t(t)||^2 - a_1(z(1,t), u_t(t)) - \frac{\theta\xi}{2}\int_{t-\tau}^{t} e^{\theta(s-t)}||u_t(s)||^2\mathrm{d}s \\
&+ \frac{\xi}{2}||u_t(t)||^2 - \frac{\xi e^{-\theta\tau}}{2}||u_t(t-\tau)||^2 \\
\leq &-\Big(a_0 - \frac{|a_1|}{2} - \frac{\xi}{2}\Big)||u_t(t)||^2 - \Big(\frac{\xi e^{-\theta\tau}}{2} - \frac{|a_1|}{2}\Big)||z(1,t)||^2 \\
&- \frac{\theta\xi}{2}\int_{t-\tau}^{t} e^{\theta(s-t)}||u_t(s)||^2\mathrm{d}s.
\end{aligned} \tag{3.9}$$

From (3.3), the coefficients $a_0 - \frac{|a_1|}{2} - \frac{\xi}{2} := c_1$ and $\frac{\xi e^{-\theta\tau}}{2} - \frac{|a_1|}{2} := c_2$ are positive. This completes the proof. □

Next, let us define the perturbed functional by

$$L(t) = E(t) + \epsilon\Psi(t),$$

where $\Psi(t) = (u_t, u)$.

Lemma 3.2. *Let the conditions of Lemma 3.1 hold. Then, for* $\epsilon > 0$ *small enough there exist* $\alpha_i > 0$, $i = 1, 2$, *and* $c_3 > 0$ *such that*

$$\alpha_1 E(t) - c_3\Big(l_0|\Omega| + \frac{||g||^2}{\lambda}\Big) \leq L(t) \leq \alpha_2 E(t) + c_3\Big(l_0|\Omega| + \frac{||g||^2}{\lambda}\Big) \quad \textit{for} \quad t \geq 0. \tag{3.10}$$

Proof. Young's inequality and (3.8) give that

$$|\Psi(t)| \leq \frac{1}{2}||u_t||^2 + \frac{1}{2\lambda}||\Delta u||^2 \leq c_0 \max\left\{\frac{1}{2}, \frac{1}{2\lambda}\right\}\left(E(t) + l_0|\Omega| + \frac{||g||^2}{\lambda}\right).$$

Thus, we obtain

$$|L(t) - E(t)| \leq \epsilon c_0 \max\left\{\frac{1}{2}, \frac{1}{2\lambda}\right\}\Big(E(t) + l_0|\Omega| + \frac{||g||^2}{\lambda}\Big).$$

Choosing $\epsilon > 0$ small and letting $\alpha_1 = 1 - \epsilon c_0 \max\left\{\frac{1}{2}, \frac{1}{2\lambda}\right\}$, $\alpha_2 = 1 + \epsilon c_0 \max\{\frac{1}{2}, \frac{1}{2\lambda}\}$, $c_3 = \epsilon c_0 \max\{\frac{1}{2}, \frac{1}{2\lambda}\}$, we complete the proof. □

Lemma 3.3. *Let the conditions of Lemma 3.1 hold. Then, there exist positive constants* c_4 *and* c_5 *satisfying*

$$\Psi'(t) \leq c_4||u_t(t)||^2 - \frac{1}{2}||\Delta u(t)||^2 - k||u^+(t)||^2 + c_5||z(1,t)||^2 - \int_{\Omega} F(u)dx + (g, u(t)).$$

Proof. Using (2.4), we have

$$\Psi'(t) = ||u_t(t)||^2 - ||\Delta u(t)||^2 - k(u(t), u^+(t)) - a_0(u(t), u_t(t)) \\ - a_1(u(t), z(1,t)) - (f(u(t)), u(t)) + (g, u(t)). \tag{3.11}$$

Thanks to

$$\begin{aligned} &- k(u(t), u^+(t)) = -k||u^+(t)||^2, \\ &- a_0(u(t), u_t(t)) \le \frac{1}{4}||\Delta u(t)||^2 + \frac{a_0^2}{\lambda}||u_t(t)||^2, \\ &- a_1(u(t), z(1,t)) \le \frac{1}{4}||\Delta u(t)||^2 + \frac{a_1^2}{\lambda}||z(1,t)||^2, \\ &- (f(u(t)), u(t)) \le -\int_\Omega F(u)\mathrm{d}x, \end{aligned}$$

we have from (3.11) that

$$\Psi'(t) \le \Big(1 + \frac{a_0^2}{\lambda}\Big)||u_t(t)||^2 - \frac{1}{2}||\Delta u(t)||^2 - k||u^+(t)||^2 \\ + \frac{a_1^2}{\lambda}||z(1,t)||^2 - \int_\Omega F(u)\mathrm{d}x + (g, u(t)).$$

Putting $c_4 = 1 + \frac{a_0^2}{\lambda}$ and $c_5 = \frac{a_1^2}{\lambda}$, we complete the proof. □

Lemma 3.4. *Under the conditions of Lemma 3.1, the semigroup $\{S(t)\}_{t\ge 0}$ defined by (2.1) has a bounded absorbing set in $\mathcal{H}$.*

Proof. From Lemmas 3.1 and 3.3, we see that

$$\begin{aligned} L'(t) \le &- c_1||u_t(t)||^2 - c_2||z(1,t)||^2 - \frac{\theta\xi}{2}\int_{t-\tau}^{t} e^{\theta(s-t)}||u_t(s)||^2\mathrm{d}s \\ &+ \epsilon c_4||u_t(t)||^2 - \frac{\epsilon}{2}||\Delta u(t)||^2 - \epsilon k||u^+(t)||^2 \\ &+ \epsilon c_5||z(1,t)||^2 - \epsilon\int_\Omega F(u)\mathrm{d}x + \epsilon(g, u(t)) \\ = &- (c_1 - \epsilon c_4)||u_t(t)||^2 - \frac{\epsilon}{2}||\Delta u(t)||^2 - \epsilon k||u^+(t)||^2 - \epsilon\int_\Omega F(u)\mathrm{d}x + \epsilon(g, u(t)) \\ &- \frac{\theta\xi}{2}\int_{t-\tau}^{t} e^{\theta(s-t)}||u_t(s)||^2\mathrm{d}s - (c_2 - \epsilon c_5)||z(1,t)||^2. \end{aligned} \tag{3.12}$$

Choosing $\epsilon > 0$ small enough such that $c_1 - \epsilon c_4 > 0$, $c_2 - \epsilon c_5 > 0$, we deduce that

$$L'(t) \le -\alpha_3 E(t) \quad \text{for some } \alpha_3 > 0.$$

From this and (3.10), we have

$$L'(t) \le -\frac{\alpha_3}{\alpha_2}L(t) + \frac{\alpha_3 c_3}{\alpha_2}\Big(l_0|\Omega| + \frac{||g||^2}{\lambda}\Big),$$

and hence

$$L(t) \le \left\{L(0) - c_3\left(l_0|\Omega| + \frac{||g||^2}{\lambda}\right)\right\} e^{-\frac{\alpha_3 t}{\alpha_2}} + c_3\left(l_0|\Omega| + \frac{||g||^2}{\lambda}\right). \tag{3.13}$$

Applying (3.10) to (3.13), we deduce that

$$E(t) \leq \frac{\alpha_2}{\alpha_1} E(0) e^{-\frac{\alpha_3 t}{\alpha_2}} + 2c_3 \Big(l_0 |\Omega| + \frac{||g||^2}{\lambda} \Big). \tag{3.14}$$

Consequently, we conclude from (3.8) and (3.14) that

$$||(u(t), u_t(t), z)||^2_{\mathcal{H}} \leq \frac{c_0 \alpha_2}{\alpha_1} E(0) e^{-\frac{\alpha_3 t}{\alpha_2}} + c_0 (2c_3 + 1) \Big(l_0 |\Omega| + \frac{||g||^2}{\lambda} \Big).$$

This shows that any closed ball $B_0 = \overline{B}(0, R)$ with $R > \sqrt{c_0(2c_3+1)\Big(l_0|\Omega| + \frac{||g||^2}{\lambda}\Big)}$ is a bounded absorbing set of $(S(t), \mathcal{H})$. □

Lemma 3.5. *Assume the conditions (H1)–(H3) hold and $0 < k < \lambda$. Let B_0 be a bounded absorbing set obtained in Lemma 3.4; $S(t)y_0 = (u, u_t, z)$ and $S(t)\tilde{y_0} = (\tilde{u}, \tilde{u}_t, \tilde{z})$ be two weak solutions of problem (2.4)–(2.8) corresponding to initial data $y_0 = (u_0, u_1, z_0) \in B_0$ and $\tilde{y_0} = (\tilde{u}_0, \tilde{u}_1, \tilde{z}_0) \in B_0$, respectively. Then,*

$$||S(t)y_0 - S(t)\tilde{y_0}||^2_{\mathcal{H}} \leq c e^{-\omega t} ||y_0 - \tilde{y_0}||^2_{\mathcal{H}} + C(B_0) \int_0^t e^{-\omega(t-s)} ||u(s) - \tilde{u}(s)||^2_{2(p+1)} ds, \tag{3.15}$$

where $c > 0$, $\omega > 0$, and $C(B_0)$ is a constant depending on the size of B_0.

Proof.

Step 1. Let $w(t) = u(t) - \tilde{u}(t)$, $q(x, \rho, t) = z(x, \rho, t) - \tilde{z}(x, \rho, t)$. Then from (2.4)–(2.8), w and q satisfy

$$\begin{cases} w_{tt} + \Delta^2 w + k(u^+ - \tilde{u}^+) + a_0 w_t + a_1 q(x, 1, t) + f(u) - f(\tilde{u}) = 0 \text{ on } \Omega \times \mathbb{R}^+, \\ \tau q_t(x, \rho, t) + q(x, \rho, t) = 0 \text{ for } (x, \rho, t) \in \Omega \times (0,1) \times (0, \infty), \\ w = \Delta w = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ w(0) = u_0 - \tilde{u}_0, \ \ w_t(0) = u_1 - \tilde{u}_1 \text{ on } \Omega, \\ q(x, \rho, 0) = z_0(x, \rho) - \tilde{z}_0(x, \rho) := q_0 \quad \text{for } (x, \rho) \in \Omega \times (0,1). \end{cases} \tag{3.16}$$

Adapting the same arguments used to get (3.9), we can easily see that

$$\begin{aligned} E'_w(t) \leq & -\Big(a_0 - \frac{|a_1|}{2} - \frac{\xi}{2} \Big) ||w_t(t)||^2 - \Big(\frac{\xi e^{-\theta\tau}}{2} - \frac{|a_1|}{2} \Big) ||q(1, t)||^2 \\ & - \frac{\theta \xi}{2} \int_{t-\tau}^t e^{\theta(s-t)} ||w_t(s)||^2 \mathrm{d}s - k(u^+ - \tilde{u}^+, w_t) - (f(u) - f(\tilde{u}), w_t), \end{aligned} \tag{3.17}$$

where

$$E_w(t) = \frac{1}{2} ||w_t(t)||^2 + \frac{1}{2} ||\Delta w(t)||^2 + \frac{\xi}{2} \int_{t-\tau}^t e^{\theta(s-t)} ||w_t(s)||^2 \mathrm{d}s,$$

here θ and ξ are as given in (3.3).

Observing that $|u^+ - \tilde{u}^+| \leq |u - \tilde{u}|$ and $||w||^2 \leq \tilde{\lambda} ||w||^2_{2(p+1)}$, where $\tilde{\lambda}$ is the embedding constant, we have

$$-k(u^+ - \tilde{u}^+, w_t) \leq \frac{\tilde{\lambda} k^2}{\eta} ||w(t)||^2_{2(p+1)} + \frac{\eta}{4} ||w_t||^2.$$

Also, we have from (2.1) that

$$\begin{aligned} -(f(u) - f(\tilde{u}), w_t) & \leq l C(p) (|\Omega|^{\frac{p}{2(p+1)}} + ||u||^p_{2(p+1)} + ||\tilde{u}||^p_{2(p+1)}) ||w||_{2(p+1)} ||w_t|| \\ & \leq C(B_0) ||w||_{2(p+1)} ||w_t|| \end{aligned}$$

$$\leq \frac{C(B_0)}{\eta}||w||_{2(p+1)}^2 + \frac{\eta}{4}||w_t||^2,$$

here and after $C(\cdot)$ denotes a generic constant, which depends on the variable, different from line to line or even in the same line.

Substituting these into (3.17), we find that

$$E_w'(t) \leq -\Big(a_0 - \frac{|a_1|}{2} - \frac{\xi}{2} - \frac{\eta}{2}\Big)||w_t(t)||^2 - \Big(\frac{\xi e^{-\theta\tau}}{2} - \frac{|a_1|}{2}\Big)||q(1,t)||^2 - \frac{\theta\xi}{2}\int_{t-\tau}^{t} e^{\theta(s-t)}||w_t(s)||^2 \mathrm{d}s + \Big(\frac{\tilde{\lambda}k^2}{\eta} + \frac{C(B_0)}{\eta}\Big)||w||_{2(p+1)}^2. \tag{3.18}$$

On the other hand, it can be observed that E_w is equivalent to $||(w, w_t, q)||_{\mathcal{H}}^2$, that is, there exist positive constants α_4 and α_5 such that

$$\alpha_4 E_w(t) \leq ||(w, w_t, q)||_{\mathcal{H}}^2 \leq \alpha_5 E_w(t). \tag{3.19}$$

Indeed, integration by substitution $s = t - \rho\tau$ gives

$$\begin{aligned} E_w(t) &= \frac{1}{2}||w_t(t)||^2 + \frac{1}{2}||\Delta w(t)||^2 + \frac{\xi\tau}{2}\int_0^1 e^{-\theta\rho\tau}||w_t(t-\rho\tau)||^2 \mathrm{d}\rho \\ &\geq \frac{1}{2}||w_t(t)||^2 + \frac{1}{2}||\Delta w(t)||^2 + \frac{\xi\tau e^{-\theta\tau}}{2}\int_0^1\int_\Omega q^2(x,\rho,t)\mathrm{d}x\mathrm{d}\rho \\ &\geq \min\left\{\frac{1}{2}, \frac{\xi\tau e^{-\theta\tau}}{2}\right\}||(w, w_t, q)||_{\mathcal{H}}^2. \end{aligned} \tag{3.20}$$

Moreover it holds that

$$\begin{aligned} E_w(t) &= \frac{1}{2}||w_t(t)||^2 + \frac{1}{2}||\Delta w(t)||^2 + \frac{\xi\tau}{2}\int_0^1\int_\Omega e^{-\theta\rho\tau}q^2(x,\rho,t)\mathrm{d}x\mathrm{d}\rho \\ &\leq \frac{1}{2}||w_t(t)||^2 + \frac{1}{2}||\Delta w(t)||^2 + \frac{\xi\tau}{2}\int_0^1\int_\Omega q^2(x,\rho,t)\mathrm{d}x\mathrm{d}\rho \\ &= \max\left\{\frac{1}{2}, \frac{\xi\tau}{2}\right\}||(w, w_t, q)||_{\mathcal{H}}^2. \end{aligned} \tag{3.21}$$

Step 2. Now, let us define

$$L_w(t) = E_w(t) + \varepsilon\psi(t),$$

where $\psi(t) = (w_t(t), w(t))$. It can be easily shown that for appropriately small $\varepsilon > 0$ there exist positive constants α_6 and α_7 satisfying

$$\alpha_6 E_w(t) \leq L_w(t) \leq \alpha_7 E_w(t). \tag{3.22}$$

From (3.16), it follows

$$\begin{aligned} \psi'(t) = ||w_t||^2 - ||\Delta w||^2 - k(u^+ - \tilde{u}^+, w) - a_0(w_t, w) \\ - a_1(q(1,t), w) - (f(u) - f(\tilde{u}), w). \end{aligned} \tag{3.23}$$

Since

$$-k(u^+ - \tilde{u}^+, w) \leq k||w||^2 \leq \frac{k}{\lambda}||\Delta w||^2,$$

$$- a_0(w_t, w) \le \frac{\eta}{4}||\Delta w||^2 + \frac{a_0^2}{\eta\lambda}||w_t||^2,$$
$$- a_1(q(1,t), w) \le \frac{\eta}{4}||\Delta w||^2 + \frac{a_1^2}{\eta\lambda}||q(1,t)||^2,$$

and

$$-(f(u) - f(\tilde{u}), w) \le C(B_0)||w||_{2(p+1)}||w|| \le \frac{\eta}{4}||\Delta w||^2 + \frac{C(B_0)}{\eta\lambda}||w||^2_{2(p+1)},$$

we get from (3.23) that

$$\psi'(t) \le \Big(1 + \frac{a_0^2}{\eta\lambda}\Big)||w_t(t)||^2 - \Big(1 - \frac{k}{\lambda} - \frac{3\eta}{4}\Big)||\Delta w||^2 + \frac{a_1^2}{\eta\lambda}||q(1,t)||^2 + \frac{C(B_0)}{\eta\lambda}||w||^2_{2(p+1)}. \tag{3.24}$$

Combining (3.18) and (3.24), we have

$$\begin{aligned} L_w'(t) \le& -\Big\{a_0 - \frac{|a_1|}{2} - \frac{\xi}{2} - \frac{\eta}{2} - \varepsilon\Big(1 + \frac{a_0^2}{\eta\lambda}\Big)\Big\}||w_t(t)||^2 \\ &- \varepsilon\Big(1 - \frac{k}{\lambda} - \frac{3\eta}{4}\Big)||\Delta w||^2 - \frac{\theta\xi}{2}\int_{t-\tau}^{t} e^{\theta(s-t)}||w_t(s)||^2 \mathrm{d}s \\ &- \Big(\frac{\xi e^{-\theta\tau}}{2} - \frac{|a_1|}{2} - \varepsilon\frac{a_1^2}{\eta\lambda}\Big)||q(1,t)||^2 + \Big(\frac{\varepsilon C(B_0)}{\eta\lambda} + \frac{\tilde{\lambda}k^2}{\eta} + \frac{C(B_0)}{\eta}\Big)||w||^2_{2(p+1)}. \end{aligned} \tag{3.25}$$

Taking $\eta > 0$ and $\varepsilon > 0$ sufficiently small, we arrive at

$$L_w'(t) \le -c_6 E_w(t) + C(B_0)||w||^2_{2(p+1)} \quad \text{for some } c_6 > 0.$$

We have from this and (3.22) that

$$L_w'(t) \le -\omega L_w(t) + C(B_0)||w||^2_{2(p+1)} \quad \text{for some } \omega > 0.$$

This and (3.22) ensure that

$$E_w(t) \le c_7 e^{-\omega t}E_w(0) + C(B_0)\int_0^t e^{-\omega(t-s)}||w(s)||^2_{2(p+1)}\mathrm{d}s \quad \text{for some } c_7 > 0.$$

This and (3.19) complete the proof.

□

Lemma 3.6. *Assume (H1)–(H3) hold and* $0 < k < \lambda$. *Then, the semigroup* $\{S(t)\}_{t\ge 0}$ *defined by* (2.1) *is asymptotically smooth in* $\mathcal{H}$.

Proof. We apply Lemmas 3.5 and 2.2. Let B_0 be a bounded absorbing set obtained in Lemma 3.4, and $S(t)y_0 = (u, u_t, z)$ and $S(t)\tilde{y_0} = (\tilde{u}, \tilde{u}_t, \tilde{z})$ be two weak solutions of problem (2.4)–(2.8) corresponding to initial data $y_0 = (u_0, u_1, z_0) \in B_0$ and $\tilde{y_0} = (\tilde{u}_0, \tilde{u}_1, \tilde{z}_0) \in B_0$, respectively. Let $\varepsilon > 0$. Then, from (3.15), for every $\varepsilon > 0$ there exists $T = T(B_0, \varepsilon) > 0$ such that

$$||S(T)y_0 - S(T)\tilde{y_0}||_{\mathcal{H}} \le \varepsilon + C(B_0)\left(\int_0^T ||u(s) - \tilde{u}(s)||^2_{2(p+1)}\mathrm{d}s\right)^{\frac{1}{2}}. \tag{3.26}$$

Gagliardo–Nirenberg inequality implies that

$$||u(s) - \tilde{u}(s)||_{2(p+1)} \le c||\Delta u(s) - \Delta\tilde{u}(s)||^{\sigma}||u(s) - \tilde{u}(s)||^{1-\sigma} \le C(B_0)||u(s) - \tilde{u}(s)||^{1-\sigma},$$

where $c>0$ and $\sigma=\frac{p}{2(p+1)}$.
This and (3.26), we observe that

$$||S(T)y_0-S(T)\tilde{y_0}||_{\mathcal{H}}^2\le\varepsilon+\phi_T(y_0,\tilde{y_0}), \tag{3.27}$$

where

$$\phi_T(y_0,\tilde{y_0})=C(B_0)\left(\int_0^T||u(s)-\tilde{u}(s)||^{2(1-\sigma)}\mathrm{d}s\right)^{\frac{1}{2}}. \tag{3.28}$$

Now, it remains to show that ϕ_T defined in (3.28) is a contractive function on $B_0\times B_0$. Let (u^n,u_t^n,z^n) be the corresponding solution for the initial data $y_0^n=(u_0^n,u_1^n,z_0^n)\in B_0$, $n=1,2,\ldots$. Since B_0 is a bounded positively invariant set in $\mathcal{H}$ with respect to $S(t)$, without loss of generality, we assume that

$$u^n\to u \quad \text{weakly star in } L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \tag{3.29}$$

$$u_t^n\to u_t \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega)), \tag{3.30}$$

$$z^n\to\ z \text{ weakly star in } L^\infty(0,T;L^2(\Omega\times(0,1))). \tag{3.31}$$

These and Aubin–Lions lemma give that

$$u^n\to u \quad \text{stongly in } L^{2(1-\sigma)}(0,T;L^2(\Omega)). \tag{3.32}$$

This implies that

$$\lim_{n\to\infty}\lim_{m\to\infty}\phi_T(y_0^n,y_0^m)=C(B_0)\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T||u^n(s)-u^m(s)||^{2(1-\sigma)}\mathrm{d}s=0.$$

Hence, ϕ_T is a contractive function on $B_0\times B_0$. From Lemma 2.1, the proof is finished. □

Our main result of this section reads as:

Theorem 3.1. *Under the condition of Lemma 3.6, the semigroup $\{S(t)\}_{t\ge0}$ corresponding to problem (2.4)–(2.8) has a global attractor in $\mathcal{H}$.*

Proof. Lemmas 3.4, 3.6, and 2.2 ensure the existence of a global attractor. □

4. Finite-dimensional attractor

In this section, we prove the finite dimensionality of the attractors given in Theorem 3.1 making use of Lemma 2.3.

Lemma 4.1. *Let the conditions of Theorem 3.1 hold. Then, the dynamical system $(S(t),\mathcal{H})$ defined by (2.1) is quasi-stable on any bounded positively invariant set $B\subset\mathcal{H}$.*

Proof. Theorem 2.1 (i) ensures that the dynamical system $(S(t),\mathcal{H})$ satisfies (2.9) and (2.10) by considering $X=H^2(\Omega)\cap H_0^1(\Omega)$, $Y=L^2(\Omega)$, and $Z=L^2(\Omega\times(0,1))$. Furthermore, we observe from Theorem 2.1 (ii) that $(S(t),\mathcal{H})$ satisfies (2.11). Now, it remains to show that $(S(t),\mathcal{H})$ satisfies (2.12). Let $B_0\subset\mathcal{H}$ be a bounded set positively invariant with respect to $S(t)$. Let $S(t)y_0=(u,u_t,z)$ and $S(t)\tilde{y_0}=(\tilde{u},\tilde{u}_t,\tilde{z})$ for $y_0\in B_0$ and $\tilde{y_0}\in B_0$, respectively. Define the seminorm

$$n_X(u)=||u||_{2(p+1)},$$

then $n_X(\cdot)$ is a compact seminorm on $X=H^2(\Omega)\cap H_0^1(\Omega)$ because the embedding $H^2(\Omega)\cap H_0^1(\Omega)\hookrightarrow L^{2(p+1)}(\Omega)$ is compact. Hence, (3.15) can be rewritten as

$$||S(t)y_0-S(t)\tilde{y_0}||_{\mathcal{H}}^2\le b(t)||y_0-\tilde{y_0}||_{\mathcal{H}}^2+c(t)\sup_{0<s<t}(n_X(u(s)-\tilde{u}(s)))^2,$$

where $b(t) = ce^{-\omega t}$ and $c(t) = C(B_0) \int_0^t e^{-\omega(t-s)} ds$. Moreover we see that $b \in L^1(\mathbb{R}^+)$, $\lim_{t\to\infty} b(t) = 0$, and $c(t)$ is locally bounded on $[0, \infty)$ because B_0 is bounded. □

Our desired result of this section is following:

Theorem 4.1. *Let the conditions of Theorem 3.1 hold. Then, the global attractor $\mathcal{A}$ given in Theorem 3.1 has finite fractal dimension.*

Proof. Since the global attractor $\mathcal{A}$ given in Theorem 3.1 is a bounded positively invariant set of $\mathcal{H}$, Lemma 4.1 yields that the dynamical system $(S(t), \mathcal{H})$ defined (2.1) is quasi-stable on $\mathcal{A}$. Thus, Lemma 2.3 implies that $\mathcal{A}$ has finite fractal dimension. □

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