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An investigation on a two-dimensional problem of Mode-I crack in a thermoelastic medium

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Abstract. In this work, we consider a two-dimensional dynamical problem of an infinite space with finite linear Mode-I crack and employ a recently proposed heat conduction model: an exact heat conduction with a single delay term. The thermoelastic medium is taken to be homogeneous and isotropic. However, the boundary of the crack is subjected to a prescribed temperature and stress distributions. The Fourier and Laplace transform techniques are used to solve the problem. Mathematical modeling of the present problem reduces the solution of the problem into the solution of a system of four dual integral equations. The solution of these equations is equivalent to the solution of the Fredholm's integral equation of the first kind which has been solved by using the regularization method. Inverse Laplace transform is carried out by using the Bellman method, and we obtain the numerical solution for all the physical field variables in the physical domain. Results are shown graphically, and we highlight the effects of the presence of crack in the behavior of thermoelastic interactions inside the medium in the present context, and its results are compared with the results of the thermoelasticity of type-III.

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1. Introduction

The classical coupled dynamical theory of thermoelasticity was developed by Biot [1] by assuming that the elastic changes have an effect on the temperature and vice versa. However, this theory was based on Fourier law of heat conduction. Therefore, when this theory was combined with the law of conservation of energy, a parabolic-type heat conduction equation was obtained and hence predicted an infinite speed of thermal signal which contradicted the physical fact. In order to resolve this so-called paradox in the Biot's theory [1], the topic generalized theory of thermoelasticity has come into existence and attracted the several researchers during last few decades. The generalized theories were specially formulated to account the finite speed of propagation of thermal signals which was termed as second sound effect, and in this respect, we would like to mention here the one of the earliest development of the second sound theory for thermoelasticity by Fox [2] in which he applied the principles of modern continuum thermodynamics. Further, the two well-established and well-studied generalized thermoelasticity theories were also developed by Lord and Shulman [3] and Green and Lindsay [4]. Two thermal relaxation time parameters were introduced in the theory developed by Green and Lindsay [4], and one relaxation time was considered in the theory proposed by Lord and Shulman [3]. Later on, Chandrashekhariah [5,6] and Hetnarski and Ignaczak [7] also reported some review articles. The book by Ignaczak and Ostoja-Starzewaski [8] also addressed a detailed study of the generalized thermoelasticity theory.

Green and Naghdi [9–11] proposed the new thermoelasticity theory which was considered to be an alternative formulation of heat propagation during the period 1992–1995, and they included the thermal pulse transmission in this theory in very consistently manner. This theory includes three models, and these are subsequently known as thermoelastic models of type GN-I, GN-II, and GN-III. In this theory, the temperature gradient and thermal displacement gradient are assumed as the constitutive variables, and the main concept of its formulation is that an entropy balance law is used in the place of usual entropy production inequality (see Chandrasekharaiah [6]). Out of these three models, first two models are the special cases of the GN-III model, whereas GN-I model is closely related to the theory of classical thermoelasticity theory, and therefore, it suffers from the paradox of infinite heat propagation speed. In GN-II model, there is no dissipation of thermal energy which is caused by no change in internal energy. Therefore, it implies that the internal rate of production of entropy is almost zero which is not obtained in the general case, i.e., GN-III model. The proposed heat conduction law under GN-III theory can be written as given below:

$$\vec{q}(x,\,t) = -\left[\kappa\vec{\nabla}\theta(x,\,t) + \kappa^*\vec{\nabla}\upsilon(x,\,t)\right],$$

where $\vec{q}$ is the heat flux vector. υ is the thermal displacement defined by $\dot{\upsilon} = \theta$, where θ being the temperature and $\kappa > 0$, $\kappa^* > 0$ are the material parameters which are known as the thermal conductivity and conductivity rate, respectively.

Tzou [12] developed the two phase-lag model and the microstructural effects in heat transport process. We must mention a detailed analysis of this model was reported by Quintanilla [13]. Later on, Hetnarski and Ignaczak [7] had also studied the various models in a survey article by mentioning the theoretical significance of these models, and the domain of influence theorem was also explained by them for the Lord–Shulman and Green–Naghdi theories with an initial boundary value problems.

Chandrashekhariah [6] proposed a review article in which he has extended the dual phase-lag heat conduction model proposed by Tzou [12] to a hyperbolic thermoelastic dual phase-lag model using its Taylor series expansion. Subsequently, Roychoudhuri [14] introduced an alternative extension of GN-III model in the frame of dual phase-lag model which is called as three phase-lag model in which three different phase-lag parameters are introduced in the constitutive variables of the equation of heat conduction under GN-III model which can be written in following form:

$$\vec{q}(x,\,t+\tau_q) = -\left[\kappa\vec{\nabla}\theta(x,\,t+\tau_\theta) + \kappa^*\vec{\nabla}\upsilon(x,\,t+\tau_\upsilon)\right],$$

where τ_q, τ_θ and τ_υ are the material parameters which are known as time relaxation parameters.

Further, Quintanilla and Racke [15] discussed the stability of three phase-lag heat conduction equation. Dreher et al. [16] gave an analysis on dual phase-lag and three phase-lag heat conduction theories in which they have found the ill-posed behavior of the problem in Hadamard sense. In order to find the acceptable results, Quintanilla [17] developed the theory to reformulate the three phase-lag heat conduction model and proposed an alternative heat conduction model with single delay term. Again, Quintanilla and Leseduarte [18] re-examined the model proposed by Quintanilla [17], and they discussed the stability and spatial behavior of the solutions under this model. They assumed $\tau_\upsilon > \tau_q = \tau_\theta$ and $\tau_0 = \tau_\nu - \tau_q$, and the constitutive law of heat conduction equation has been written as

$$\vec{q}(x,\,t) = -\left[\kappa\vec{\nabla}\theta(x,\,t) + \kappa^*\vec{\nabla}\upsilon(x,\,t+\tau_0)\right].$$

By using the above constitutive equation, they discussed the spatial behavior of the solutions for this theory. Further, Quintanilla and Leseduarte [18] studied the Taylor series approximation until order l of the above equation in forward sense which is given in the following form:

$$\vec{q}(x,\,t) = -\left[\kappa\vec{\nabla}T(x,\,t) + \kappa^*\left\{\vec{\nabla}\upsilon(x,\,t) + \tau_0\vec{\nabla}\dot{\upsilon}(x,\,t) + \cdots + \frac{\tau_0^l}{l(l-1)\ldots 1}\vec{\nabla}\upsilon^{(l)}(x,\,t)\right\}\right].$$

The energy equation which imply heat conduction equation is obtained in the form:

$$c_v\dot{\theta}(x,\,t) = -\left[\kappa\nabla^2\theta(x,\,t) + \kappa^*\left\{\nabla^2\upsilon(x,\,t) + \tau_0\nabla^2\theta(x,\,t) + \cdots + \frac{\tau_0^l}{l(l-1)\ldots 1}\nabla^2\theta^{(l-1)}(x,\,t)\right\}\right],$$

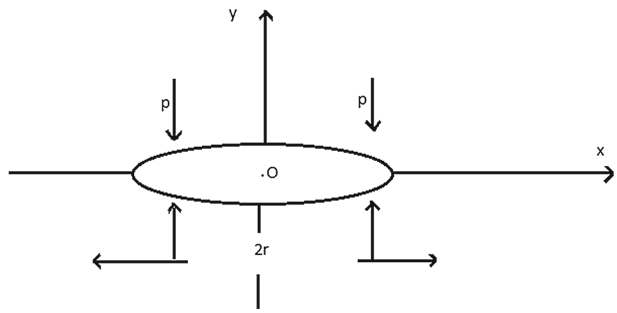

FIG. 1. Displacement of Mode-I crack

where c_v is the specific heat at constant strain and ∇^2 is the Laplacian operator.

Further, Quintanilla [17] has also shown that the solution of the above heat conduction equation is always stable whenever $l \leq 3$. In the case $l = 0$, the above equation can be written as

$$c_v \dot{T}(x,\, t) = -\left[\kappa \nabla^2 \theta(x,\, t) + \kappa^* \nabla^2 \upsilon(x,\, t)\right],$$

which is the heat conduction equation under GN-III model, and when we take $l = 2$, we obtain the following equation of heat conduction which we say new model-I (i.e., Quintanilla-I model):

$$c_v \dot{\theta}(x,\, t) = -\left[\kappa \nabla^2 \theta(x,\, t) + \kappa^* \left\{1 + \tau_0 \frac{\partial}{\partial t} + \frac{\tau_0^2}{2} \frac{\partial^2}{\partial t^2}\right\} \nabla^2 \upsilon(x,\, t)\right].$$

Now if we do not consider the term containing τ_0^2 due to very small value of τ_0, we obtain the following equation which we say new model-II (i.e., Quintanilla-II model):

$$c_v \dot{\theta}(x,\, t) = -\left[\kappa \nabla^2 \theta(x,\, t) + \kappa^* \left\{1 + \tau_0 \frac{\partial}{\partial t}\right\} \nabla^2 \upsilon(x,\, t)\right].$$

Recently, many researchers have seriously studied the cracks and failures in solid as it has the wide applications in the industry, particularly in the fabrication of electronic components, geophysics, earthquake engineering, etc. They have considered the two-dimensional Griffith crack problem represented by a line segment. In reality, it is a long flat ribbon-shaped cavity in a solid which stressed in such a way that stress patterns remain unaltered while passing in a direction parallel to the direction of the crack. It is noted that Griffith [19] has firstly studied the theory of the cracks in two-dimensional thermoelastic medium. The fracture mode of any material shows the separation geometrically. There are three basic problems of crack corresponding to three different modes, i.e., Mode-I, Mode-II, and Mode-III in terms of displacement in two-dimensional problems which is very useful to study. A Griffith crack having the length $2r$ in a solid medium in the case of Mode-I is shown in Fig 1, under the action of the tension which is in the direction perpendicular to the line of the crack. Mode-I shows a symmetric opening in the relative displacements of the medium being normal to the fracture surface which is given in Irwin's study [20]. One thing is also noted that the crack growth usually occurs in Mode-I or close to it.

Mallik and Kanoria [21] discussed an understanding of thermally induced stresses in solids which is necessary for a detailed study of the manufacturing stages. We would also like to mention here that the thermal stresses have a important role in the building structural elements. The flow-induced thermal stresses in the infinite isotropic solids have been studied by Florence and Goodier [22]. The crack problems in thermoelastic media are also discussed by Sih [23], Kassir and Bergman [24], Prasad and Aliabadi [25], Raveendra and Banerjee [26], Elfalaky and Abdel-Halim [27], Hosseini-Teherani and Eslami [28], Chaudhuri and Ray [29]. Sherief and El-Maghraby [30], Mallik and Kanoria [21], Abdel-Halim and Elfalaky [31] have also discussed the dynamical problems for an internal penny-shaped crack in an infinite thermoelastic solid. Recently, Sherief and El-Maghraby [32] and Prasad and Mukhopadhyay [33] have solved the mode-I crack problem of an infinite thermoelastic medium in the context of Lord–Shulman's theory [3] and Green–Naghdi's theory [34], respectively. However, to the best of authors' knowledge, the mode-I crack problem under impact loading in the thermoelasticity theory given by Quntanilla [17] has not yet been studied by any researcher.

In the present work, we have solved a two-dimensional dynamical problem in an isotropic homogeneous elastic medium having Mode-I crack. We have discussed the thermoelastic behavior inside the medium in the neighborhood of the crack in which we have used the thermoelasticity theory given by Quintanilla [17], namely Quintanilla-I and Quintanilla-II models and compared all its results with the results of type-III thermoelasticity theory of Green and Naghdi which is already discussed by Prasad and Mukhopadhyay [33]. We have formulated the problem in such way that all the three models (new model-I, II and GN-III model) can be written in a unified way, from which we can obtain every particular model. Two-dimensional equations of motion are given and constitutive relations are also given to describe the components of the stresses. Laplace and exponential Fourier transforms are used to solve the problem and we obtain the solution in the transformed domain. In Sect. 4, the prescribed boundary temperature and stress distributions are used to find four dual integral equations which are further reduced into two dual integral equations. These dual integral equations are solved using regularization method which is explained in "Appendix A." The method given by Bellman et al. [35] is used to invert the Laplace transform numerically which is also described in "Appendix B." Therefore, we obtain all the physical fields in the physical domain and all the numerical values of field variables are represented graphically. In Sect. 6, we discuss and compare all the findings regarding the behavior of physical fields near the crack region.

2. Formulation of the problem

Two-dimensional dynamical problem is considered in an infinite medium $-\infty < x < \infty$, $-\infty < y < \infty$ which has a Mode-I (opening-mode) crack defined by $|x| \leq r,\ y = 0$. The crack surface is subjected to the known temperature and normal stress distributions. We consider the basic governing equations of coupled thermoelasticity as follows:

Equations of motion:

$$(\lambda+\mu)\frac{\partial \Delta}{\partial x} + \mu\nabla^2 u - \beta\frac{\partial \theta}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2} \tag{1}$$

$$(\lambda+\mu)\frac{\partial \Delta}{\partial y} + \mu\nabla^2 v - \beta\frac{\partial \theta}{\partial y} = \rho\frac{\partial^2 v}{\partial t^2}. \tag{2}$$

The **heat conduction equation** under theory of thermoelasticity of type-III due to Green and Naghdi [34] is given by

$$\left(K^* + K\frac{\partial}{\partial t}\right)\nabla^2\theta = \frac{\partial^2}{\partial t^2}(\rho c_v\theta + \beta\theta_0\Delta). \tag{3}$$

We consider the **heat conduction equation** by Quintanilla [17] as

$$\left[K^*\left(1 + \tau_0\frac{\partial}{\partial t} + \frac{\tau_0^2}{2}\frac{\partial^2}{\partial t^2}\right) + K\frac{\partial}{\partial t}\right]\nabla^2\theta = \frac{\partial^2}{\partial t^2}(\rho c_v\theta + \beta\theta_0\Delta). \tag{4}$$

In the above equation, if we neglect the effect of higher-order terms containing τ_0, then we get

$$\left[K^*\left(1 + \tau_0\frac{\partial}{\partial t}\right) + K\frac{\partial}{\partial t}\right]\nabla^2\theta = \frac{\partial^2}{\partial t^2}(\rho c_v\theta + \beta\theta_0\Delta). \tag{5}$$

The **stress–strain–temperature** relations for the present case are given by

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} + \lambda\Delta - \beta(\theta - \theta_0) \tag{6}$$

$$\tau_{yy} = 2\mu \frac{\partial v}{\partial y} + \lambda \Delta - \beta(\theta - \theta_0) \tag{7}$$

$$\tau_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right). \tag{8}$$

Now, we aim to study the present problem by considering it as a problem of thermoelasticity in the contexts of three different forms of heat conduction equations as given by Eqs. (3–5). Hence, we combine them in the following manner:

$$\left[K^* \left(1 + \tau_0 \frac{\partial}{\partial t} + \tau_1 \frac{\partial^2}{\partial t^2} \right) + K \frac{\partial}{\partial t} \right] \nabla^2 \theta = \frac{\partial^2}{\partial t^2} (\rho c_v \theta + \beta \theta_0 \Delta). \tag{9}$$

From Eq. (9), we can get the different heat conduction equations in the following manner:

1. **Quintanilla model (new model-I):** $\tau_1 = \frac{\tau_0^2}{2}$, $\tau_0 \neq 0$
2. **Quintanilla model (new model-II):** $\tau_1 = 0$, $\tau_0 \neq 0$
3. **GN-III model:** $\tau_1 = 0$, $\tau_0 = 0$

In above Eqs. (1–9), u and v are the displacement components along the x and y directions, respectively, and t is the time. θ is the absolute temperature, θ_0 is the reference temperature, ρ is the density of the material, K is the thermal conductivity and K^* is the rate of the thermal conductivity. τ_{ij} are the stress components, c_v is the specific heat at constant strain or volume, λ and μ are the Lame's elastic constants however β is the material constants given by $\beta = (3\lambda + 2\mu)\alpha_t$, where, α_t is the coefficient of linear of thermal expansion. ∇^2 is Laplacian operator and Δ is the dilatation given by

$$\Delta = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \tag{10}$$

Now for simplicity, we use the following non-dimensional variables:

$x' = c\eta x$, $y' = c\eta y$,$r' = c\eta r$, $u' = c\eta u$, $v' = c\eta v$, $t' = c^2\eta t$, $\tau'_{ij} = \frac{\tau_{ij}}{\mu}$, $T = \frac{\theta - \theta_0}{\theta_0}$, $\tau'_1 = c^4\eta^2\tau_1$, $\tau'_0 = c^2\eta\tau_0$

where, $\eta = \frac{\rho c_v}{K}$, and $c = \sqrt{\frac{\lambda+2\mu}{\rho}}$, where c is the speed of the propagation of longitudinal elastic waves.

Now with the help of the above non-dimensional quantities, Eqs. (1-2), (6), and (7–9) are reduced in the following non-dimensional forms:

$$(m^2 - 1)\frac{\partial \Delta}{\partial x} + \nabla^2 u - a_2 \frac{\partial T}{\partial x} = m^2 \frac{\partial^2 u}{\partial t^2} \tag{11}$$

$$(m^2 - 1)\frac{\partial \Delta}{\partial y} + \nabla^2 v - a_2 \frac{\partial T}{\partial y} = m^2 \frac{\partial^2 v}{\partial t^2} \tag{12}$$

$$\left[a_1 \left(1 + \tau \frac{\partial}{\partial t} + \tau_1 \frac{\partial^2}{\partial t^2} \right) + \frac{\partial}{\partial t} \right] \nabla^2 T = \frac{\partial^2}{\partial t^2} (T + a_3 e) \tag{13}$$

$$\tau_{xx} = 2\frac{\partial u}{\partial x} + (m^2 - 2)\Delta - a_2 T \tag{14}$$

$$\tau_{yy} = 2\frac{\partial v}{\partial y} + (m^2 - 2)\Delta - a_2 T \tag{15}$$

$$\tau_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \tag{16}$$

where $a_1 = \frac{K*}{Kc^2\eta}$, $a_2 = \frac{\eta T_0}{\mu}$, $a_3 = \frac{\beta}{K\eta}$, $m^2 = \frac{\lambda+2\mu}{\mu}$.

Now removing u and v from Eqs. (11–12) using Eq. (10), we obtain

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2} \right) \Delta = a_4 \nabla^2 T \tag{17}$$

where, $a_4 = \frac{a_2}{m^2}$

3. Solution in the Laplace and Fourier transform domain

The Laplace transform of a function $g(x,\, y,\, t)$ can be defined as

$$\bar{g}(x,\, y,\, s) = L[g(x,\, y,\, t)] = \int_0^\infty e^{-st} g(x,\, y,\, t)\, \mathrm{d}t \quad s > 0,$$

where s is the Laplace transform parameter.

After taking the Laplace transform to both the sides of Eqs. (10–13) and (17), we obtain the following

$$\bar{\Delta} = \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \tag{18}$$

$$(1 - m^2)\frac{\partial \bar{\Delta}}{\partial x} + a_2 \frac{\partial \bar{T}}{\partial x} = (\nabla^2 - m^2 s^2)\bar{u} \tag{19}$$

$$(1 - m^2)\frac{\partial \bar{\Delta}}{\partial y} + a_2 \frac{\partial \bar{T}}{\partial y} = (\nabla^2 - m^2 s^2)\bar{v} \tag{20}$$

$$\left[\left\{a_1\left(1 + \tau_0\, s + \tau_1 s^2\right) + s\right\} \nabla^2 - s^2\right] \bar{T} = s^2 a_3 \bar{\Delta} \tag{21}$$

$$\left(\nabla^2 - s^2\right) \bar{\Delta} = a_4 \nabla^2 \bar{T} \tag{22}$$

Now eliminating $\bar{\Delta}$ from Eqs. (21–22), we obtain the partial differential equation which is satisfied by $\bar{T}$ given as

$$\left(\nabla^2 - k_1^2\right)\left(\nabla^2 - k_2^2\right) \bar{T} = 0, \tag{23}$$

where k_1^2 and k_2^2 are the roots of the following characteristic equation

$$k^4 - \frac{s^2\left\{1 + a_1\left(1 + \tau_0 s + \tau_1 s^2\right) + s + \varepsilon\right\}}{\left\{a_1\left(1 + \tau_0 s + \tau_1 s^2\right) + s\right\}} k^2 + \frac{s^4}{\left\{a_1\left(1 + \tau_0 s + \tau_1 s^2\right) + s\right\}} = 0, \tag{24}$$

where $\varepsilon = a_3 a_4$.

Now, we can obtain $\bar{T}$, the solution of Eq. (23), in the following form:

$$\bar{T} = \bar{T}_1 + \bar{T}_2,$$

where $\bar{T}_i$ is the solution of the equation given as

$$\left(\nabla^2 - k_i^2\right) \bar{T}_i = 0, \quad i = 1,\, 2. \tag{25}$$

The exponential Fourier transform of a function $\bar{g}(x,\, y,\, s)$ can be defined as

$$\bar{g}^*(q,\, y,\, s) = F\left[\bar{g}\left(x,\, y,\, s\right)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{g}(x,\, y,\, s)\, e^{-iqx}\, \mathrm{d}x,$$

where q is the Fourier transform parameters.

The inverse Fourier transform can be defined as

$$\bar{g}(x,\, y,\, s) = F^{-1}\left[\bar{g}^*(q,\, y,\, s)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{g}^*(q,\, y,\, s)\, e^{iqx}\, \mathrm{d}q.$$

Now, we apply the exponential Fourier transform to both sides of Eq. (25) which can be written as

$$\left(\frac{\partial^2}{\partial y^2} - k_i^2 + q^2\right) \bar{T}_i^* = 0, \quad i = 1,\, 2. \tag{26}$$

The solution of Eq. (26) is bounded at infinity which can be found in the following form:

$$\bar{T}_i^* = B_i(q,\, s)(k_i^2 - s^2)e^{-q_i|y|} \quad i = 1,\, 2,$$

where $q_i = \sqrt{q^2 + m_i^2}$ and $B_i(q,\, s)$ is the parameter which depends upon q and s only for $i = 1,\, 2$.

Due to symmetry of the problem, we take the case $y > 0$ only. Then the above equation can be written as

$$\bar{T}_i^* = B_i(q,\, s)(k_i^2 - s^2)e^{-q_i y}, \quad i = 1,\, 2. \tag{27}$$

In a similar manner, now eliminating $\bar{T}$ from Eqs. (21) and (22), we obtain $\bar{\Delta}^* = \bar{\Delta}_1^* + \bar{\Delta}_2^*$, where, $\bar{\triangle}_i^*,\ i = 1,\, 2$ can be written as

$$\bar{\Delta}_i^* = B_i'(q,\, s)(k_i^2 - s^2)e^{-q_i y}, \quad i = 1,\, 2, \tag{28}$$

where $B_i'(q,\, s),\ i = 1,\, 2$ are also which depend only on q and s.

Now, substituting Eqs. (27) and (28) into Eq.(22), we obtain the equation which relates the parameters $B_i(q,\, s)$ and $B_i'(q,\, s)$ for $i = 1,\, 2$ in the following form:

$$B_i'(q,\, s) = a_4 k_i^2 B_i(q,\, s), \quad i = 1,\, 2 \tag{29}$$

Therefore, using Eq.(29) and (28), we find

$$\bar{\Delta}_i^* = c a_4 k B_i(q,\, s)\,(k_i^2 - s^2)e^{-q_i y}, \quad i = 1,\, 2 \tag{30}$$

Now, we take the exponential Fourier transform of Eqs. (19) and (20), we get

$$\left(\frac{\partial^2}{\partial y^2} - q^2 - m^2 s^2\right)\bar{u}^* = \left(1 - m^2\right) i\, q\, \bar{\Delta}^* + i\, q\, a_2\, \bar{T}^* \tag{31}$$

$$\left(\frac{\partial^2}{\partial y^2} - q^2 - m^2 s^2\right)\bar{v}^* = \left(1 - m^2\right) \frac{\partial}{\partial y}\bar{\Delta}^* + a_2 \frac{\partial}{\partial y}\bar{T}^* \tag{32}$$

Using Eqs. (27) and (30), Eqs. (31–32) are rewritten as

$$\left(\frac{\partial^2}{\partial y^2} - q^2 - m^2 s^2\right)\bar{u}^* = i\, q\, a_4 \sum_{i=1}^{2}\left(k_i^2 - m^2 s^2\right) B_i(q,\, s)\, e^{-q_i s} \tag{33}$$

$$\left(\frac{\partial^2}{\partial y^2} - q^2 - m^2 s^2\right)\bar{v}^* = -a_4 \sum_{i=1}^{2}\left(k_i^2 - m^2 s^2\right) B_i(q,\, s)\, q_i\, e^{-q_i s} \tag{34}$$

The solution $\bar{u}^*$ of Eq. (33) can be written as

$$\bar{u}^* = i\, q\, a_4 \left(\sum_{i=1}^{2} B_i(q,\, s)e^{-q_i y} + H_1 e^{-\delta y}\right). \tag{35}$$

where $\delta = \sqrt{q^2 + m^2 s^2}$ and $H_1 = H_1(q,\, s)$ is a parameter which also depends on q and s only.

The exponential Fourier transform of Eq. (18) with respect to x can be written as

$$\frac{\partial \bar{v}^*}{\partial y} = \bar{\Delta}^* - i\, q\, \bar{u}^*. \tag{36}$$

Now, with the help of Eqs. (30) and (35) along with the integration with respect to y, Eq. (36) is rewritten as given below

$$\bar{v}^{*} = -a_4 \left(\sum_{i=1}^{2} B_i(q,\, s)\, q_i e^{-q_i s} + \frac{q^2 H_1(q,\, s)}{\delta} e^{-\delta y} \right). \tag{37}$$

Now, taking the Laplace and exponential Fourier transforms to the both sides of Eqs. (14–16) and using the results of Eqs. (27), (30), (35), (37), we can write the components of the stress tensor in the Laplace and Fourier transform domain in the following form:

$$\bar{\tau}^{*}_{xx} = a_4 \left[B_1 \left(m^2 s^2 - 2\, q_1^2 \right) e^{-q_1 y} + B_2 \left(m^2 s^2 - 2 q_2^2 \right) e^{-q_2 y} - 2 H_1 q^2 e^{-\delta y} \right] \tag{38}$$

$$\bar{\tau}^{*}_{yy} = a_4 \left[\left(m^2 s^2 + 2\, q^2 \right) \left(B_1 e^{-q_1 y} + B_2 e^{-q_2 y} \right) + 2 H_1 q^2 e^{-\delta y} \right] \tag{39}$$

$$\bar{\tau}^{*}_{xy} = -i a_4 q \left[2 \left(B_1 q_1 e^{-q_1 y} + B_2 q_2 e^{-q_2 y} \right) + \frac{q^2 + \delta^2}{\delta} H_1 e^{-\delta y} \right] \tag{40}$$

Now, taking the inverse Fourier transform of Eqs. (27), (30), (35), and (38–40), we find the solution in the Laplace transform domain as given below

$$\bar{T} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[a_2 (k_1^2 - s^2) e^{-q_1 y} + G_2 (k_2^2 - s^2) e^{-q_2 y} \right] e^{iqx}\, \mathrm{d}q \tag{41}$$

$$\bar{\Delta} = \frac{a_4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[B_1 k_1^2 e^{-q_1 y} + B_2 k_2^2 e^{-q_2 y} \right] e^{iqx}\, \mathrm{d}q \tag{42}$$

$$\bar{u} = \frac{i a_4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[B_1 e^{-q_1 y} + B_2 e^{-q_2 y} + H_1 e^{-\delta y} \right] q\, e^{iqx}\, \mathrm{d}q \tag{43}$$

$$\bar{v} = \frac{-a_4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[B_1 q_1 e^{-q_1 y} + B_2 q_2 e^{-q_2 y} + \frac{H_1 q^2}{\delta} e^{-\delta y} \right] e^{iqx}\, \mathrm{d}q \tag{44}$$

$$\bar{\tau}_{xx} = \frac{a_4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[B_1 \left(m^2 s^2 - 2\, q_1^2 \right) e^{-q_1 y} + B_2 \left(m^2 s^2 - 2 q_2^2 \right) e^{-q_2 y} - 2 H_1 q^2 e^{-\delta y} \right] e^{iqx}\, \mathrm{d}q \tag{45}$$

$$\bar{\tau}_{yy} = \frac{a_4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\left(m^2 s^2 + 2\, q^2 \right) \left(B_1 e^{-q_1 y} + B_2 e^{-q_2 y} \right) + 2 H_1 q^2 e^{-\delta y} \right] e^{iqx}\, \mathrm{d}q \tag{46}$$

$$\bar{\tau}_{xy} = \frac{-i a_4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[2 \left(B_1 q_1 e^{-q_1 y} + B_2 q_2 e^{-q_2 y} \right) + \frac{q^2 + \delta^2}{\delta} H_1 e^{-\delta y} \right] q e^{iqx}\, \mathrm{d}q \tag{47}$$

4. Boundary conditions and dual integral equations formulation

We assume the following boundary conditions at $y = 0$

$$\frac{\partial T}{\partial y} = 0, \quad |x| > r \tag{48}$$

$$v = 0, \quad |x| > r \tag{49}$$

$$T = H(t), \quad |x| < r \tag{50}$$

$$\tau_{yy} = -H(t), \quad |x| < r \tag{51}$$

$$\tau_{xy} = 0. \quad -\infty < x < \infty \tag{52}$$

where $H(.)$ is the Heaviside unit step function.

Now, using the boundary conditions given by Eqs. (48) and (50), Eq. (41) is rewritten as

$$\int_{-\infty}^{\infty} \left[B_1(k_1^2 - s^2) + B_2(k_2^2 - s^2)\right] e^{iqx}\, \mathrm{d}q = \frac{\sqrt{2\pi}}{s}, \quad |x| < r \tag{53}$$

$$\int_{-\infty}^{\infty} \left[B_1 q_1(k_1^2 - s^2) + B_2 q_2(k_2^2 - s^2)\right] e^{iqx}\, \mathrm{d}q = 0, \quad |x| > r \tag{54}$$

and using the boundary conditions given by Eqs. (49), (51) and (52), Eqs. (44), (46), and (47) are written as

$$\int_{-\infty}^{\infty} \left[B_1 q_1 + B_2 q_2 + \frac{H_1 q^2}{\delta}\right] e^{iqx}\, \mathrm{d}q = 0, \quad |x| > r \tag{55}$$

$$\int_{-\infty}^{\infty} \left[\left(m^2 s^2 + 2\,q^2\right)(B_1 + B_2) + 2H_1 q^2\right] e^{iqx}\, \mathrm{d}q = -\frac{\sqrt{2\pi}}{s\,a_4}, \quad |x| < r \tag{56}$$

$$\int_{-\infty}^{\infty} \left[2\,(B_1 q_1 + B_2 q_2) + \frac{q^2 + \delta^2}{\delta} H_1\right] q e^{iqx}\, \mathrm{d}q = 0, \quad -\infty < x < \infty \tag{57}$$

From Eq. (57), we have

$$H_1 = -\frac{2\delta\,(B_1 q_1 + B_2 q_2)}{q^2 + \delta^2} \tag{58}$$

Using Eq. (58) and the symmetry of the problem to consider x only in the intervals $[0,\ r]$ and $[r,\ \infty]$ (see Ref. [32]), Eqs. (53–56) are rewritten as

$$\sum_{i=1}^{2}(k_i^2 - s^2)\int_0^{\infty} B_i\,\cos(qx)\,\mathrm{d}q = \sqrt{\frac{\pi}{2}}\frac{1}{s}, \quad 0 < x < r \tag{59}$$

$$\sum_{i=1}^{2}(k_i^2 - s^2)\int_0^{\infty} B_i\,q\,\cos(qx)\,\mathrm{d}q = 0, \quad x > r \tag{60}$$

$$\sum_{i=1}^{2}\int_0^{\infty} \frac{B_i q_i}{m^2 s^2 + 2q^2}\cos(qx)\,\mathrm{d}q = 0, \quad x > r \tag{61}$$

$$\sum_{i=1}^{2}\int_0^{\infty} B_i \left[\frac{\left(m^2 s^2 + 2q^2\right)^2 - 4q^2 q_i \delta}{m^2 s^2 + 2q^2}\right]\cos(qx)\,\mathrm{d}q = -\sqrt{\frac{\pi}{2}}\frac{1}{s\,a_4}, \quad 0 < x < r \tag{62}$$

Equations (59–62) form a set of four dual integral equations. From these equations, we can obtain the unknown parameters B_1 and B_2, and in order to solve these dual integral equations, we follow the

Sherief and El-Maghraby [32], from which we assume the following:

$$B_i(q,\, s) = \int_0^r h_i(v,\, s)\, J_0(q\, v)\, \mathrm{d}v, \tag{63}$$

where, h_i, $i = 1,\, 2$ is the function of v and s only and $J_0(.)$ is the Bessel function of the first kind of order zero.

Now substituting the value of B_i from Eq. (63) into the Eq. (59) and after changing the order of integration, we obtain the following

$$\sum_{i=1}^{2} (k_i^2 - s^2) \int_0^a h_i(v,\, s)\, \mathrm{d}v \int_0^\infty \cos(qx)\, J_0(qv)\, \mathrm{d}q = \sqrt{\frac{\pi}{2}}\frac{1}{s}, \quad 0 < x < r. \tag{64}$$

The integral relation of Bessel function from Refs. [36,37] is given below

$$\int_0^\infty \cos(qx)\, J_0(qv)\, \mathrm{d}q = \begin{cases} \frac{1}{\sqrt{v^2 - x^2}}, & x < v \\ 0, & x > v \end{cases} \tag{65}$$

Using above substitution, Eq. (64) is written as given below

$$\sum_{i=1}^{2} (k_i^2 - s^2) \int_x^\infty \frac{h_i(u,\, s)}{\sqrt{u^2 - s^2}}\, \mathrm{d}u = \sqrt{\frac{\pi}{2}}\frac{1}{s}, \quad 0 < x < r$$

Multiply the above equation with $\frac{x}{\sqrt{x^2 - v^2}}$ and integrate with respect to x from v to r. After changing the order of integration and differentiating the resultant equation, we have the following

$$(k_1^2 - s^2) h_1(v,\, s) + (k_2^2 - s^2) h_2(v,\, s) = -\frac{N(v)}{s}, \; 0 < x < r, \tag{66}$$

where

$$N(v) = -\sqrt{\frac{2}{\pi}} \frac{v}{\sqrt{r^2 - v^2}} \tag{67}$$

Now, multiplying both sides of Eq. (66) by $J_0(qv)$ and integrating with respect to v from 0 to r, we obtain

$$B_2 = -\frac{1}{k_2^2 - s^2} \left[\frac{J(q)}{s} + \left(k_1^2 - s^2\right) B_1 \right], \qquad 0 < x < r \tag{68}$$

where

$$J(q) = \int_0^r N(v)\, J_0(qv)\, \mathrm{d}v \tag{69}$$

For obtaining the similar relation to Eq. (68) between B_1 and B_2 for the case $x > r$, we take the following

$$B_i(q,\, s) = \frac{1}{q_i} \int_r^\infty h_i(v,\, s)\, J_0(qv)\, \mathrm{d}v, \quad x > r, \;\; 1 = 1,\, 2 \tag{70}$$

Using the relation (65) into Eq. (60) and after changing the order of integration, we obtain the following

$$\sum_{i=1}^{2}(k_i^2 - s^2)\int_x^\infty \frac{h_i(u,\, s)}{\sqrt{u^2 - x^2}}\,\mathrm{d}u = 0, \quad x > r$$

Now, multiply both sides of the above relation by $\frac{x}{\sqrt{x^2 - v^2}}$ and integrate with respect to x from v to ∞. After changing the order of integration with the help of relation (70), we have the following

$$B_2 = -\frac{\left(k_1^2 - s^2\right) q_1}{\left(k_2^2 - s^2\right) q_2} B_1, \quad x > r \tag{71}$$

Now, substituting from Eq. (68) into Eq. (62), we have the following

$$\int_0^\infty \frac{B_1 q_1 L_1(q,\, s)}{m^2 s^2 + 2q^2}\cos(qx)\,\mathrm{d}q = \bar{L}_2(x,\, s), \quad x < r \tag{72}$$

where

$$L_1(q,\, s) = -\frac{\left(k_2^2 - k_1^2\right)\left(m^2 s^2 + 2q^2\right)^2 - 4q^2\delta\left[q_1\left(k_2^2 - s^2\right) - q_2\left(k_1^2 - s^2\right)\right]}{q_1}$$

$$\bar{L}_2(x,\, s) = -\sqrt{\frac{2}{\pi}}\frac{\left(k_2^2 - s^2\right)}{s\, a_4} + \frac{1}{s}\int_0^\infty J(q)\left[\frac{\left(m^2 s^2 + 2q^2\right)^2 - 4q^2\delta\, q_2}{\left(m^2 s^2 + 2q^2\right)}\right]\cos(qx)\,\mathrm{d}q, \quad x < r \tag{73}$$

Now substituting from Eq. (71) into Eq. (61), we obtain

$$\int_0^\infty \frac{B_1 q_1 cos(qx)}{\left(m^2 s^2 + 2q^2\right)}\mathrm{d}q = 0, \quad x > r \tag{74}$$

In this way, the original four dual integral Eqs. (59–62) having the parameters B_1 and B_2 are now changed into two dual integral Eqs. (72) and (74) in the single parameter B_1 only.

5. Solution of the dual integral equations

For solving the above two integral Eqs. (72) and (74), we assume the following substitution [32]:

$$B_1(q,\, s) = \frac{\left(m^2 s^2 + 2q^2\right)}{q_1}\phi(q,\, s). \tag{75}$$

Therefore, Eqs. (72) and (74) are reduced into the following forms:

$$\int_0^\infty L_1(q,\, s)\,\phi(q,\, s)\,\cos(qx)\,\mathrm{d}q = \bar{L}_2(x,\, s), \quad 0 < x < r \tag{76}$$

$$\int_0^\infty \phi(q,\, s)\,\cos(qx)\,\mathrm{d}q = 0, \quad x > r \tag{77}$$

In order to define for all the values of x, we are now extending the definition of the integral which is in Eq. (77) in the following manner:

$$\int_0^{\infty} \phi(q,\, s)\, \cos(qx)\, \mathrm{d}q = \begin{cases} \sqrt{2\pi}\frac{\mathrm{d}}{\mathrm{d}x}\left[x\int_x^r \frac{\varphi(z,\, s)\, \mathrm{d}z}{\sqrt{z^2-x^2}}\right], & 0<x<r \\ 0, & x>r \end{cases} \tag{78}$$

where $\varphi(z,\, s)$ is a function which has to be determined.

We see that the left side of Eq. (78) is just the Fourier cosine transform of $\phi(q,\, s)$. Therefore, by using the inverse Fourier cosine transform [32,38,39], we have the following

$$\phi(q,\, p) = \int_0^r \frac{\mathrm{d}}{\mathrm{d}x}\left[x\int_x^r \frac{\varphi(z,\, s)\, \mathrm{d}z}{\sqrt{z^2-x^2}}\right]\cos(qx)\, \mathrm{d}x \tag{79}$$

Now using integration by parts and followed by the changing the order of integration to solve the above equation, we have

$$\varphi(q,\, p) = q\int_0^r \phi(z,\, s)\, \mathrm{d}z \int_0^z \frac{x\, \sin(qx)\, \mathrm{d}x}{\sqrt{z^2-x^2}} \tag{80}$$

By using the formula from [36,37], we have

$$\int_0^z \frac{x\, \sin(qx)\, \mathrm{d}x}{\sqrt{z^2-x^2}} = \frac{\pi}{2} z J_1(qz)$$

Therefore, Eq. (80) is rewritten in the following form:

$$\phi(q,\, s) = \frac{\pi}{2} q \int_0^r z\, \varphi(z,\, s) J_1(qz)\mathrm{d}z, \tag{81}$$

Now putting the value of $\phi(q,\, s)$ from Eq. (81) into Eq. (76), we have the following

$$\int_0^r \bar{L}_1(z,\, x,\, s)\, \varphi(z,\, s)\, \mathrm{d}z = \bar{L}_2(x,\, s), \quad 0<x<r \tag{82}$$

where

$$\bar{L}_1(z,\, x,\, s) = \frac{\pi z}{2}\int_0^{\infty} q\, L_1(q,\, s)\, J_1(qz)\, \cos(qx)\, \mathrm{d}q$$

We see that Eq. (82) is the Fredholm's integral equation of the first kind in the unknown parameter function $\varphi(z,\, s)$ which can be obtained by solving numerically and then $\phi(q,\, s)$ can be obtained from Eq. (81). Therefore, using the value of $\phi(q,\, s)$ into Eq. (75), we can get the value of B_1. In this way, the expression for B_2 can be obtained using the value of B_1 for the case $x<r$ and $x>r$ with the help of Eqs. (68) and (71), respectively.

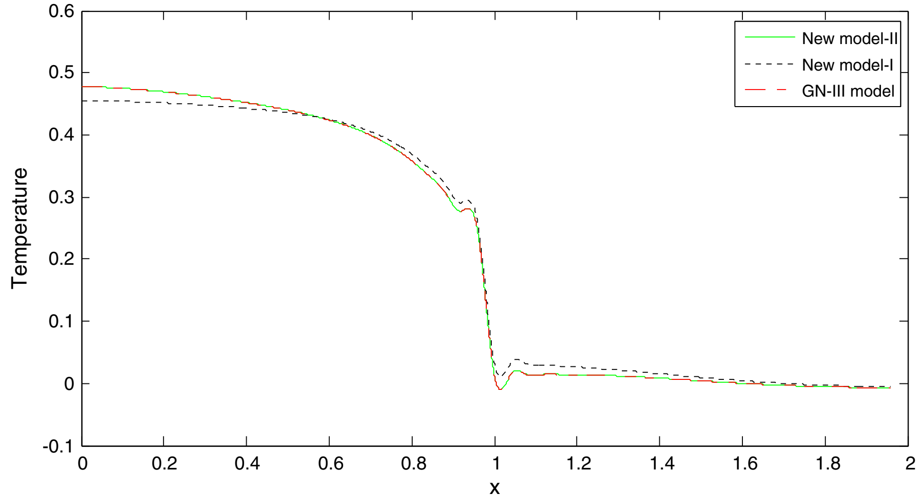

Fig. 2. Temperature distributions at the vertical distance 0.2

6. Numerical results and discussion

In order to obtain the final solution of the present problem in space-time domain, we proceed as follows. The method (see Delves and Mohammed [40]) described in "Appendix A" is used to solve the dual integral equations. However, the inversion of Laplace transform is carried out using the Bellman et al. [35] which is described in "Appendix B". In order to see the behavior of all the physical fields near the crack region, we have considered the copper material having the Mode-I crack with unit length. The material constants are taken as follows (see Sherief and El-Maghraby [32]):

$m = 2$, $\alpha_t = 1.78(10^{-5})\,\mathrm{K}^{-1}$, $c = 4.158(10^3)\,\mathrm{ms}^{-1}$, $a_4 = 0.01$, $\rho = 8954\,\mathrm{kgm}^{-3}$, $\eta = 8886\,\mathrm{sm}^{-2}$, $r = 1\,\mathrm{m}$, $c_e = 383.1\ \mathrm{JKkg}^{-1}$, $\lambda = 7.76(10^{10})\,\mathrm{Nm}^{-2}$, $\mu = 3.86(10^{10})\,\mathrm{Nm}^{-2}$, $T_0 = 293\,\mathrm{K}$, $a_2 = 0.042$, $\tau_0 = 0.02$, $\tau_1 = \frac{\tau_0^2}{2}$.

We carry out programming by using the software Mathematica-7 to find out the non-dimensional numerical values of all the different physical fields like temperature, vertical and horizontal stresses, vertical and horizontal displacements for different values of vertical distance y. We make an attempt to compare the predictions by all three models, namely new model-I, new model-II and GN-III model, and the graphical representation of our results is carried out for each physical field with respect to the horizontal distance, x. Due to symmetricity of the problem, we show the results for half length ($x \geq 0$) only. We specially observe the behavior of the physical fields in the vicinity of crack. Each physical field under all models is plotted for different values of y at non-dimensional time 1.2. Figures 2, 4, 6, 8, and 10 show the nature of the different physical fields at the non-dimensional vertical distance 0.2, whereas Figs. 3, 5, 7, 9, and 11 are showing the nature of different physical fields under all three models for non-dimensional vertical distance 0.3.

The temperature distribution is shown in Figs. 2 and 3 for non-dimensional vertical distances, 0.2 and 0.3, respectively. From Figs. 2 and 3, we note that the maximum value of temperature distribution under all three models is occurred at the beginning of the crack for both the vertical distances 0.2 and 0.3, and it decreases very slowly up to the middle of the crack region. Thereafter the decreasing rate increases. Further, at the end of the crack edge, it suddenly decreases which becomes zero after some distance. We further observe that the value of the temperature for lower vertical distance 0.2 at the beginning of the crack is more as compared to the values for the non-dimensional distance 0.3 under all three models.

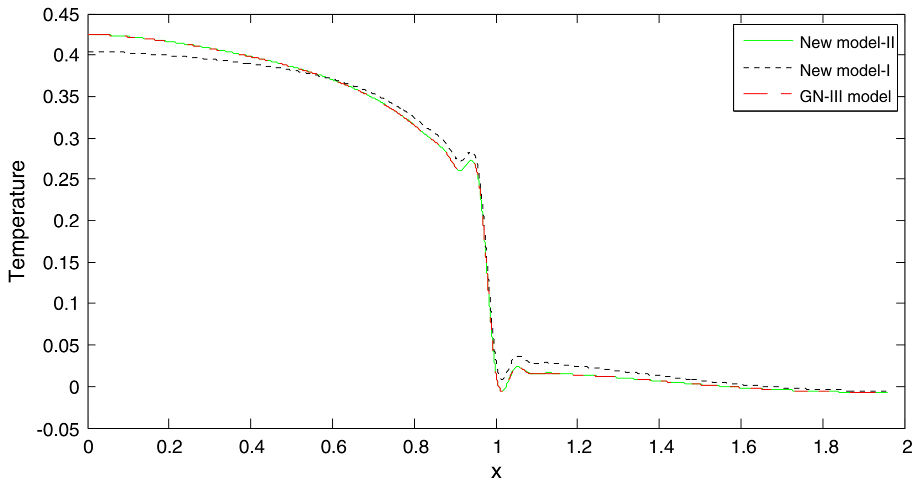

FIG. 3. Temperature distributions at the vertical distance 0.3

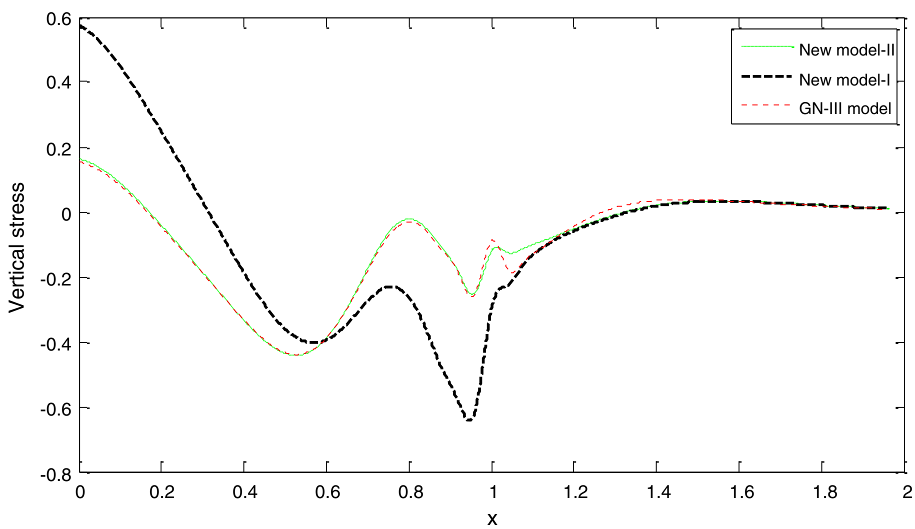

FIG. 4. Vertical stress distributions at the vertical distance 0.2

However, the values under GN-III model and new model-II are almost the same for both the vertical distances 0.2 and 0.3 which is significantly different from the values under new model-I.

The vertical stress distribution is shown in Figs. 4 and 5 for different non-dimensional vertical distances. We find that at the start of the crack, the values under all three models are maximum which are started to decrease up to middle of the crack. Then two local minima and one local maximum are occurred before the end of the crack edge. From there, it increases which becomes zero after some distance. We also find that the values under all three models for the non-dimensional vertical distance 0.2 is greater than the values for non-dimensional vertical distance 0.3. However, the values under GN-III model and new model-II are almost the same which are significantly different from the values obtained under the new model-I. Furthermore, we note that the value under new model-I is larger up to the middle of the crack

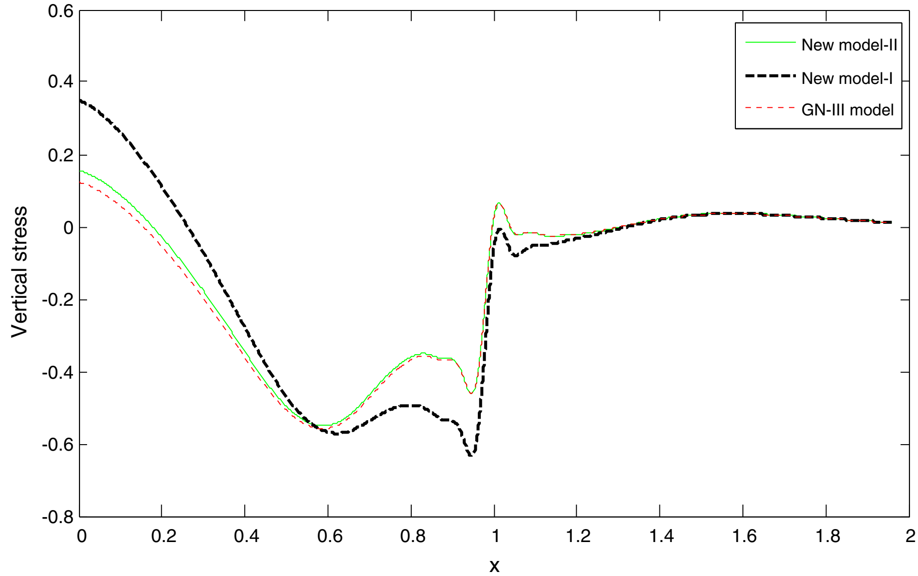

Fig. 5. Vertical stress distributions at the vertical distance 0.3

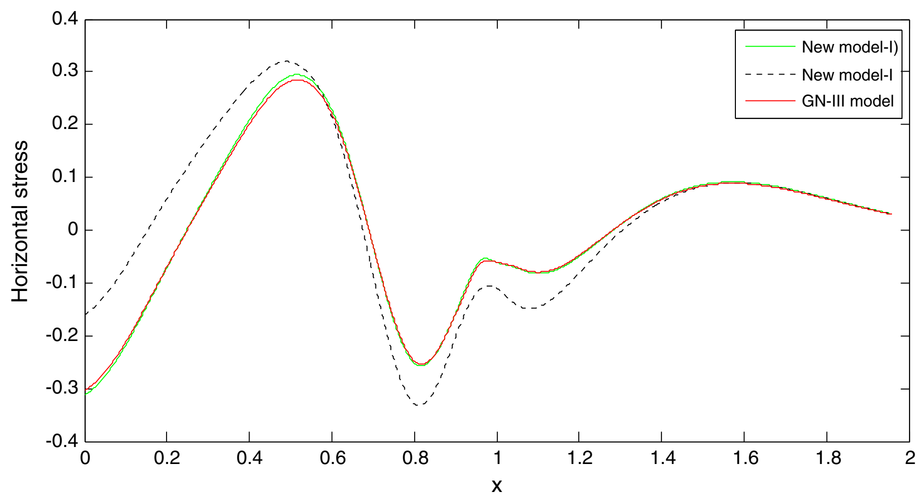

Fig. 6. Horizontal stress distributions at the vertical distance 0.2

edge as compared to the values under other two models for both the vertical distances 0.2 and 0.3. The nature of the vertical stress is oscillatory in nature near the end of the crack and it is more pronounced for lower vertical distance and for the GN-III model and new model-II.

Figures 6 and 7 show the behavior of the horizontal stress for the non-dimensional vertical distances 0.2 and 0.3, respectively. It is indicated that the value under the new model-I is significantly different from the values under new model-II and GN-III model at the beginning of the crack edge. However, under all models the horizontal stress increases when we move toward the end of the middle edge, and this field yields a maximum value at a point near the middle of the crack edge which is the same under all three models. Further, the values decrease up to the end of the crack edge, and from there, it again starts

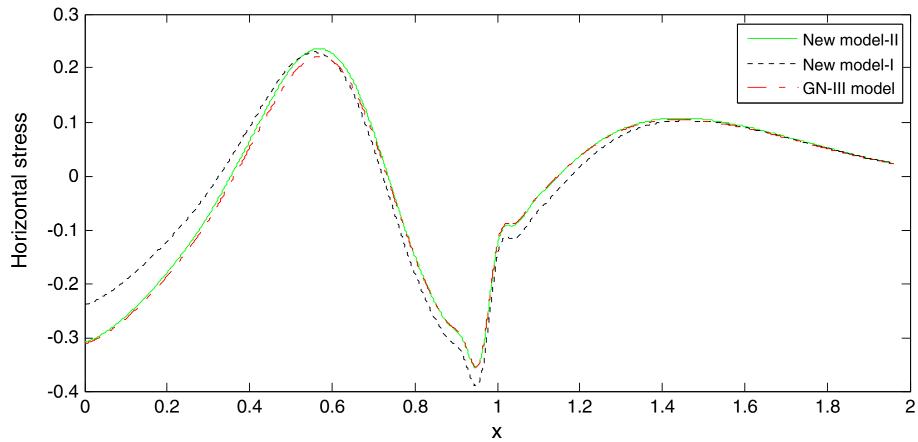

FIG. 7. Horizontal stress distributions at the vertical distance 0.3

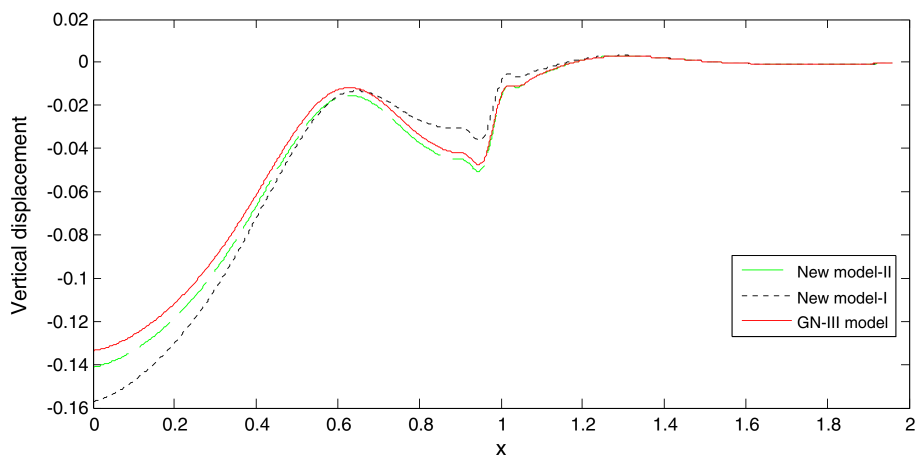

FIG. 8. Vertical displacement distributions at the vertical distance 0.2

to increase which finally become zero after some distance. Therefore, one local maximum and one local minimum are occurred for this field inside the crack edge. Furthermore, it is noted that near the middle of the crack edge, the value under the new model-I is significantly different with the values occurred under the GN-III model and new model-II. We further observe that the values under all three models are more for the vertical distance 0.2 as compared to the values found for the vertical distance 0.3. This implies that the horizontal stress decreases with the increase in vertical distance.

The nature of the vertical displacement distribution near the crack edge for the vertical distance 0.2 and 0.3 can be seen from the Figs. 8 and 9, respectively. We observe that the values are significantly different under all three models near the crack edge for both the vertical distances which is maintained up to the end of the crack edge. Further, we have also seen that one local maximum is occurred near the middle of the crack edge and showing a decreasing trend thereafter up to the end of crack edge. However, it again starts increasing after the end of the crack edge which finally vanishes after some distance. The values for the vertical distance 0.3 are smaller as compared to the values for the vertical distance 0.2

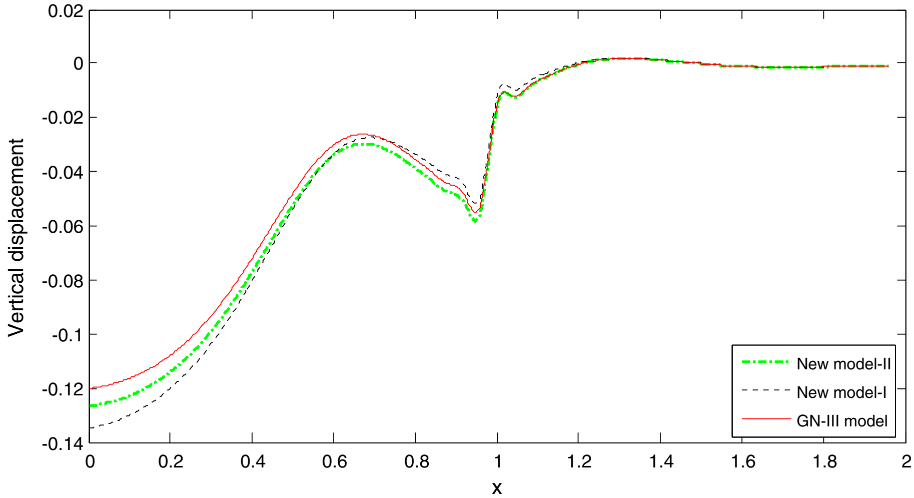

Fig. 9. Vertical displacement distributions at the vertical distance 0.3

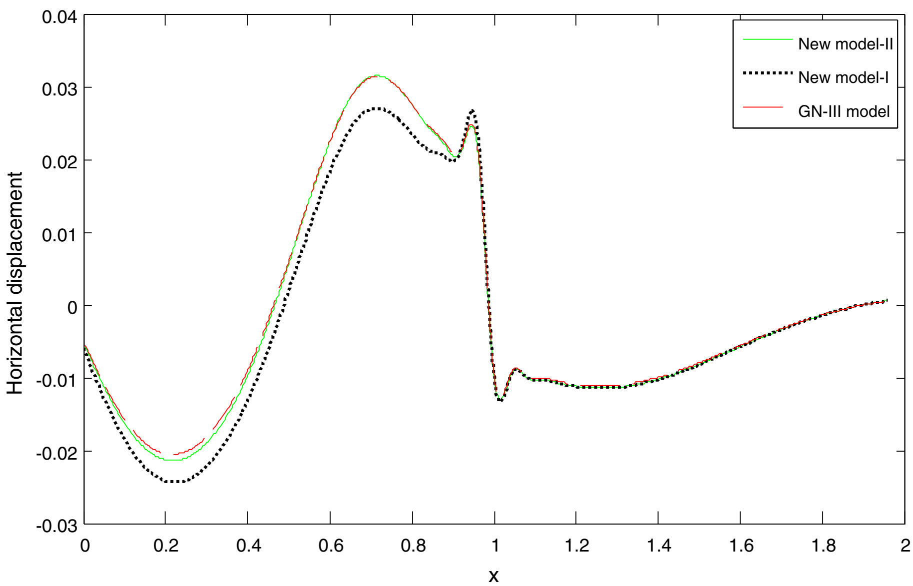

Fig. 10. Horizontal displacement distributions at the vertical distance 0.2

under all three models. The behavior of this physical field is same under all three models: new model-I, new model-II, and GN-III model in the case of both the vertical distances 0.2 and 0.3.

The horizontal displacement distribution is shown in Figs. 10 and 11. We see that the values are same at the beginning of the crack edge under all three models and decrease with vertical distance. There is a significant difference up to the middle of the crack edge for both the vertical distances. The value up to end of the crack edge under the new model-I shows a prominent difference with the values predicted by other two models. After the end of the crack, the horizontal displacement suddenly decreases to a local minimum value and increases thereafter. Finally, it becomes zero after some distance. It is also seen

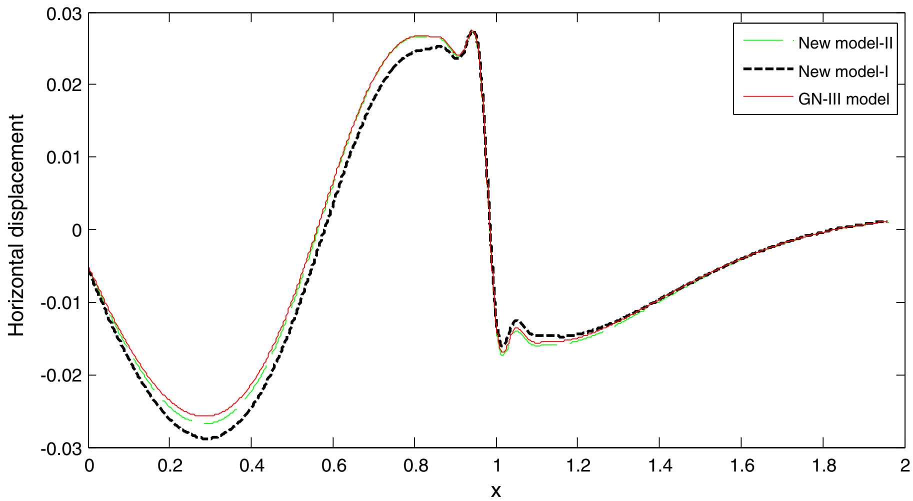

Fig. 11. Horizontal displacement distributions at the vertical distance 0.3

that within the crack edge, two local minima and one local maximum are occurred for both the vertical distances. It is also observed that the values of horizontal displacement for the vertical distance 0.2 are larger than the values obtained for the vertical distance 0.3 under all three models.

Therefore, it is clear from above discussion that all the physical fields under all three models: new model-I, new model-II, and GN-III model vanish after some distance from the end of the crack edge. There is a prominent difference in the predictions of different models for each field, and it is more prominent in the crack region. The vertical distance also plays a role in the behavior of each physical field. The value of each physical field decreases with the increase of the vertical distance in the crack region under each thermoelasticity theory.

7. Conclusion

In this paper, we have investigated a dynamical problem of an infinite two-dimensional elastic medium with a crack of Mode-I type in the contexts of thermoelasticity theories, namely, Quintanilla's theory [17] and Green–Naghdi theory [34]. The temperature and impact loading are considered at the boundary of the crack inside the medium. Laplace and exponential Fourier transform techniques are used to solve the problem. We obtain four dual integral equations which are further reduced into two dual integral equations. These dual integral equations are solved by using regularization method and a numerical method is used to invert the Laplace transform numerically to obtain the final solution of the problem. In order to compare the results under different models, we carry out computational work for finding the numerical values of all the physical field variables for different vertical distances. We observe the behavior of all the physical fields in the vicinity of the crack and concluded that under all models, each physical field shows the same nature throughout the domain which vanishes after some distance from the end edge of the crack. However, the value of each physical field decreases with the increase in the vertical distance from the end of the crack region under each thermoelasticity theory. The results under different models differ significantly, although the new model-II and GN-III model predict more similar results as compared to new model-I. This implies that there is a significant effect of single delay time parameter for the present crack problem.

Appendix A

The numerical method which is used to solve the Fredholm's integral equation of the first kind [40]

In this appendix, we are going to describe the method which is used to solve the Fredholm's integral equation of first kind. We assume an integral equation in the following form (see Delves and Mohammed [40], Sherief and El-Maghraby [32]):

$$\int_{r_1}^{r_2} \bar{L}_1(z,\, x,\, s)\, \varphi(z,\, s) \mathrm{d}z = \bar{L}_2(x,\, s) \tag{A.1}$$

We can write the above equation briefly in the following form:

$$\bar{L}_1 \varphi = \bar{L}_2 \tag{A.2}$$

Here, $\bar{L}_2(x,\, s)$ will have some finite accuracy k. Hence, we try to obtain $||\bar{L}_1\varphi - \bar{L}_2|| \leq k$. From all functions φ which satisfy this relation, we use only the smoothest function in the sense that for some linear operator L, $||L\varphi||$ has the minimum value which produce the constrained minimization problem as given below:

$$minimize_{\varphi} ||L\varphi|| \tag{A.3}$$

subject to $||\bar{L}_1\varphi - \bar{L}_2|| \leq k$.

The above minimizing problem can be solved in any given norm which is comparatively difficult to do so (see Delves and Mohammed [40]). Therefore, Eqs. (A.1)–(A.2) cannot be solved analytically in the easy way, but can be easily solved numerically in (see Delves and Mohammed [40], Sherief and El-Maghraby [32]).

We know that the value of objective function $||L\varphi||$ decreases with the increase in the value of k which means, as the constraint weaken. Therefore at the minimum of (A.1)–(A.2), the constraint will be binding that is $||\bar{L}_1\varphi - \bar{L}_2|| = k$. Now if we solve the unconstrained problem for any fixed α as given below

$$minimize_{\varphi} ||\bar{L}_1\varphi - \bar{L}_2||^2 + \alpha\, ||L\varphi||^2 \tag{A.4}$$

Now we will have to obtain the some minimum value β of $||\bar{L}_1\varphi - \bar{L}_2||$. If we take $\alpha \to 0,\ \beta \to 0$ provided that the solution of (A.1)–(A.2) exists, and for some value of α, $\beta = k$. The solution of the problem (A.4) is identical to the solution of the original problem (A.1)–(A.2) (see Delves and Mohammed [40]). We know that (A.4) is an unconstrained problem, it is easier to solve than (A.1)–(A.2). This problem is referred as the regularization problem and the method which is based on a numerical solution of (A.4) is known as the regularization method (see Delves and Mohammed [40]).

The most common possibility for the operator L can be taken as $L = 1;\ \frac{d}{dx};\ \frac{d^2}{dx^2}$. $L = 1$ and the natural norm L^2 have been taken for the computational work.

Now Eq. (A.4) is written as

$$minimize_{\varphi}\ Z(\varphi) = < \bar{L}_1\varphi - \bar{L}_2,\, \bar{L}_1\varphi - \bar{L}_2 > + \alpha < \varphi,\, \varphi > \tag{A.5}$$

where $< . >$ is the scalar product in L^2 norm.

After simplifying the above Eq. (A.5), we obtain the following

$$Z(\varphi) = < \varphi, \left\{ \bar{L}_1^{+}\, \bar{L}_1 + \alpha I \right\} \varphi > - < \varphi,\, \bar{L}_1^{+}\, \bar{L}_2 > - < \bar{L}_1^{+}\, \bar{L}_2,\, \varphi > + < \bar{L}_2,\, \bar{L}_2 >$$

where $\bar{L}_1^{+}$ is the Hermitian conjugate of $\bar{L}_1$.

Now we know that the following condition will be satisfied for the minimum value of Z at the point φ for any function g

$$\frac{\partial Z(\varphi + kg)}{\partial k}\,|_{k=0} = 0$$

using the above equation, we find that φ is a minimum point if the following is satisfied

$$\left\{\bar{L}_1^{+}\bar{L}_1 + \alpha I\right\}\varphi = \bar{L}_1^{+}\bar{L}_2 \tag{A.6}$$

Now simplify the above equation, we find

$$\int_{r_1}^{r_2} \hat{\bar{L}}_1(z,\, x,\, s)\,\varphi(z,\, s)\mathrm{d}z + \alpha\varphi(x,\, s) = \hat{\bar{L}}_2(x,\, s) \tag{A.7}$$

where

$$\hat{\bar{L}}_1(z,\, x,\, s) = \int_{r_1}^{r_2} \bar{L}_1^{*}(z,\, r,\, s)\,\bar{L}_1(z,\, r,\, s)\,\mathrm{d}r$$

$$\hat{\bar{L}}_2(x,\, s) = \int_{r_1}^{r_2} \bar{L}_1^{*}(z,\, x,\, s)\,\bar{L}_2(z,\, s)\,\mathrm{d}z$$

where the asterisk notation represents the complex conjugation and Eq. (A.7) is a Fredholm's integral equation of the second kind with iterated kernels and the parameter α can be chosen to be equal to 10^{-5} (see Sherief and El-Maghraby [32]).

We are now going to introduce the method which is used to solve the Fredholm's integral equation of the second kind. For this, we take an integral equation given below (see Delves and Mohammed [40], Sherief and El-Maghraby [32]):

$$w(u,\, s) + \int_{r_1}^{r_2} \bar{L}_1(u,\, v,\, s)\,w(v,\, s)\,\mathrm{d}v = \bar{L}_2(u,\, s) \tag{A.8}$$

The above equation can be approximated in the following form:

$$w(u,\, s) + \sum_{i=0}^{n} d_i\bar{L}_1(u,\, v_i,\, s)\,w(v_i,\, s) \approx \bar{L}_2(u,\, s) \tag{A.9}$$

where the points $v_i\,(i = 0,\, 1,\, 2,\, 3, \ldots, n)$ are the equally spaced points in the interval $[r_1,\, r_2]$ and d_i's are the corresponding weights. We know that two sides of the any equation must be equal at all the points of the domain. Therefore, from Eq. (A.9), we have the system of $n + 1$ linear equations for $j = 0,\, 1,\, 2,\, 3,\, \ldots,\, n$ which can be written as

$$w(v_j,\, s) + \sum_{i=0}^{n} d_i\bar{L}(v_j,\, v_i,\, s)\,w(v_i,\, s) \approx \bar{L}_2(v_j,\, s) \tag{A.10}$$

The above equation is having the $n+1$ unknowns, $w(v_0,\, s), \ldots, w(v_n,\, s)$, that denote the approximate values of the unknown function $w(v,\, s)$ at the $n + 1$ chosen points. We now introduce the following notations for simplification:

$$w_j = w(v_j,\, s),\;\; \bar{L_{2j}} = \bar{L}_1(v_j,\, s),\;\; \bar{L}_{1ji} = \bar{L}(v_j,\, v_i,\, s); \quad j,\, i = 0,\, 1,\, 2, \ldots, n$$

Therefore, Eq. (A.1) is written in the following form:

$$w_j + \sum_{i=0}^{n} d_i \bar{L}_{1ji}\, w_i \approx \bar{L}_{2j}, \quad j = 0, 1, 2, 3, \ldots, n \tag{A.11}$$

Now if we assume the variables, w_j and $\bar{L}_{2j}$ are the components of the w and $\bar{L}_2$, respectively, and define the matrix $\bar{L}_1 = \left[\bar{L}_{1jk}\right]$, then the system of Eq. (A.11) is written as

$$w + \bar{L_1}\, d\, w = \bar{L}_2$$

where $d = [d_j \delta_{ji}]$ is a diagonal matrix containing successive weighting coefficients. Therefore, the above set of equations can be written as

$$\left(I + \bar{L}_1\, d\right) w = \bar{L}_2 \tag{A.12}$$

where, I is the identity matrix of order $n + 1$.

Now we have used the Simpson's rule of integration, for computing the numerical values of all the integration, and we take the following weights:

$$d_0 = d_n = \frac{w}{3};\; d_{2j-1} = \frac{4w}{3},\; j = 1\, 2,\, 3, \ldots, \frac{n}{2};\; d_{2j} = \frac{2w}{3},\; j = 1,\, 2,\, 3, \ldots, n-1.$$

Appendix B

Inversion of Laplace Transform (Bellman et al. [35])

The Laplace transform of a function $u(t)$ can be defined in the following form:

$$\bar{u}(s) = \int_0^\infty u(t)\, e^{-st} \mathrm{d}t, \quad s > 0 \tag{B.1}$$

where $u(t)$ is sufficiently smooth such that it can be approximated.

Now we assume $x = e^{-t}$ in Eq. (B.1), we obtain

$$\bar{u}(s) = \int_0^\infty x^{s-1} v(x)\, \mathrm{d}x, \tag{B.2}$$

where $v(x) = u(-ln(x))$.

By using the Gaussian quadrature formula to Eq. (B.2), we obtain

$$\sum_{i=1}^{N} w_i x_i^{s-1} v(x_i) = \bar{u}(s), \tag{B.3}$$

where, x_i $(i = 1, 2, 3 \ldots N)$ are the roots of the shifted Legendre polynomial $P_N(x) = 0$ and $w_i(i = 1, 2\,, 3 \ldots, N)$ are the respective weights.

Equation (B.3) is written as

$$w_1 x_1^{s-1} v(x_1) + w_2 x_2^{s-1} v(x_2) + w_3 x_3^{s-1} v(x_3) + \cdots + w_N x_N^{s-1} v(x_N) = \bar{u}(s), \tag{B.4}$$

Now putting $s = 1, 2, 3, \ldots, N$ in Eq. (B.4) is written as a system of equations in the following form:

$$w_1 v(x_1) + w_2 v(x_2) + w_3 v(x_3) + \cdots + w_N v(x_N) = \bar{u}(1),$$
$$w_1 x_1 v(x_1) + w_2 x_2 v(x_2) + w_3 x_3 v(x_3) + \cdots + w_N x_N v(x_N) = \bar{u}(2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$w_1 x_1^{N-1} v(x_1) + w_2 x_2^{N-1} v(x_2) + w_3 x_3^{N-1} v(x_3) + \cdots + w_N x_N^{N-1} v(x_N) = \bar{u}(N)$$

Hence, from the system as expressed above, we obtain $v(x_i)$ in the terms as given below

$$\begin{bmatrix} v(x_1) \\ v(x_2) \\ \ldots \\ v(x_N) \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & \ldots & w_N \\ w_1 x_1 & w_2 x_2 & \ldots & w_N x_N \\ \ldots & \ldots & \ldots & \ldots \\ w_1 x_1^{N-1} & w_2 x_2^{N-1} & \ldots & w_N x_N^{N-1} \end{bmatrix}^{-1} \begin{bmatrix} \bar{u}(1) \\ \bar{u}(2) \\ \ldots \\ \bar{u}(N) \end{bmatrix} \tag{B.5}$$

Using Eq. (B.5), we find the discrete values of $v(x_i)$, which is $u(t_i)$, and thereafter, by using interpolation method, we find the function $u(t_i)$.

References

[1] Biot, M.A.: Thermoelasticity and irreversible thermodynamics. J. Appl. Phys. **27**, 240–253 (1956)
[2] Fox, N.: Generalized thermoelasticity. Int. J. Eng. Sci. **7**, 437–445 (1969)
[3] Lord, H.W., Shulman, Y.: A generalized dynamical theory of thermoelasticity. J. Mech. Phys. Solids **15**, 299–309 (1967)
[4] Green, A.E., Lindsay, K.A.: Thermoelasticity. J. Elast. **2**, 1–7 (1972)
[5] Chandrasekharaiah, D.S.: Thermoelasticity with second sound: a review. Appl. Mech. Rev. **39**, 355–376 (1986)
[6] Chandrasekharaiah, D.S.: Hyperbolic thermoelasticity: a review of recent literature. Appl. Mech. Rev. **51**, 705–729 (1998)
[7] Hetnarski, R.B., Ignaczak, J.: Generalized thermoelasticity. J. Therm. Stress. **22**, 451–476 (1999)
[8] Ignaczak, J., Ostoja-Starzewski, M.: Thermoelasticity with Finite Wave Speeds. Oxford University Press, Oxford (2010)
[9] Green, A.E., Naghdi, P.M.: A re-examination of the basic postulates of thermomechanics. Proc. R. Soc. London **A432**, 171–194 (1991)
[10] Green, A.E., Naghdi, P.M.: On undamped heat waves in an elastic solid. J. Therm. Stress. **15**, 253–264 (1992)
[11] Green, A.E., Naghdi, P.M.: Thermoelasticity without energy dissipation. J. Elast. **31**, 189–209 (1993)
[12] Tzou, D.Y.: A unified filed approach for heat conduction from macro to micro scales. ASME J. Heat Transf. **117**, 8–16 (1995)
[13] Quintanilla, R., Racke, R.: A note on stability in dual-phase-lag heat conduction. Int. J. Heat Mass Transf. **49**, 1209–1213 (2006)
[14] Roychoudhuri, S.K.: On a thermoelastic three-phase-lag model. J. Therm. Stress. **30**, 231–238 (2007)
[15] Quintanilla, R., Racke, R.: A note on stability in three-phase-lag heat conduction. Int. J. Heat Mass Transf. **51**, 24–29 (2008)
[16] Dreher, M., Quintanilla, R., Racke, R.: Ill posed problems in thermomechanics. Appl. Math. Lett. **22**, 1374–1379 (2009)
[17] Quintanilla, R.: Some solutions for a family of exact phase phase-lag heat conduction problems. Mech. Res. Commun. **32**, 355–360 (2011)
[18] Leseduarte, M.C., Quintanilla, R.: Phragman-Lindelof alternative for an exact heat conduction equation with delay. Commun. Pure Appl. Anal. **12**(3), 1221–1235 (2013)
[19] Griffith, A.A.: The phenomenon of repute and flow in solids. Philos. Trans. R. Soc. **A221**, 163–198 (1921)
[20] Irwin, G.R.: The crack extension force for a crack at a free surface boundary. Report Naval Research Lab, Washington, DC, USA, 5120 (1958)
[21] Mallik, S.H., Kanoria, M.: A unified generalized thermoelasticity formulation: application to penny-shaped crack analysis. J. Therm. Stress. **32**, 943–965 (2009)
[22] Florence, A.L., Goodier, J.N.: The linear thermoelastic problem of uniform heat flow disturbed by a penny-shaped insulated crack. Int. J. Eng. Sci. **1**, 533–540 (1963)
[23] Sih, G.C.: On the singular character of thermal stress near a crack tip. J. Appl. Mech. **29**, 587–588 (1962)
[24] Kassir, M.K., Bergman, A.M.: Thermal stress in a solid containing parallel circular crack. Appl. Sci. Res. **25**, 262–280 (1971)
[25] Prasad, N.N.V., Aliabadi, M.H.: The dual BEM for transient thermoelastic crack problems. Int. J. Numer. Methods Eng. **35**, 169–187 (1996)

[26] Raveendra, S.T., Banerjee, P.K.: Boundary element analysis of cracks in thermally stressed planar structures. Int. J. Solids Struct. **29**, 2301–2317 (1992)
[27] Elfalaky, A., Abdel-Halim, A.A.: Mode-I crack problem for an infinite space in thermoelasticity. J. Appl. Sci. **6**, 598–606 (2006)
[28] Hosseini-Tehrani, P., Eslami, M.R.: Boundary element analysis of coupled thermoelasticity with relaxation times in finite domain. Am. Inst. Aeronaut. Astronaut. J **38**, 534–541 (2000)
[29] Chaudhuri, P.K., Ray, S.: Thermal stress in a nonhomogeneous transversely isotropic medium containing a penny-shaped crack. Bull. Cal. Math. Soc. **98**, 547–570 (2006)
[30] Sherief, H.H., El-Maghraby, N.M.: An internal penny-shaped crack in an infinite thermoelastic solid. J. Therm. Stress. **26**, 333–352 (2003)
[31] Abdel-Halim, A.A., Elfalaky, A.: An internal penny-shaped crack problem in an infinite thermoelastic solid. J. Appl. Sci. Res. **1**, 325–334 (2005)
[32] Sherief, H.H., El-Maghraby, N.M.: A mode-I crack problem for an infinite space in generalized thermoelasticity. J. Therm. Stress. **28**, 465–484 (2005)
[33] Prasad, R., Mukhopadhyay, S.: A two-dimensional problem of Mode-I crack in a type III thermoelastic medium. Math. Mech. Solids. **18**(6), 506–523 (2013)
[34] Green, A.E., Naghdi, P.M.: A unified procedure for construction of theories of deformable media. I. Classical continuum physics, II. Generalized continua, III. Mixtures of interacting continua. Proc R Soc Lond A **448**, 335–356, 357–377, 379–388 (1995)
[35] Bellman, R., Kalaba, R.E., Lockett, J.A.: Numerical Inversion of Laplace transform. American Elsevier Pub. Co. Inc, New York (1966)
[36] Watson, G.N.: A Treatise on the Theory of Bessel Functions. Cambridge University Press, Cambridge (1996)
[37] Mandal, B.N., Mandal, M.: Advances in Dual Integral Equations. Chapman and Hall/CRC, Boca Raton, FL (1999)
[38] Sneddon, I.N.: Fourier Transforms. McGraw-Hill, Dover, New York (1995)
[39] Churchill, R.V.: Operational Mathematics, 3rd edn. McGraw-Hill, New York (1972)
[40] Delves, L., Mohammed, J.: Computational Methods for Integral Equations. Cambridge University Press, Cambridge (1985)

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