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Global well-posedness and asymptotic behavior of solutions for the three-dimensional MHD equations with Hall and ion-slip effects

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Abstract. In this paper, we consider the small initial data global well-posedness of solutions for the magnetohydrodynamics with Hall and ion-slip effects in $\mathbb{R}^3$. In addition, we also establish the temporal decay estimates for the weak solutions. With these estimates in hand, we study the algebraic time decay for higher-order Sobolev norms of small initial data solutions.

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Keywords. Hall-MHD system, Ion slip, Global existence, Decay rate.

1. Introduction

In this paper, we consider the following three-dimensional incompressible magnetohydrodynamics equations with Hall and ion-slip effects in $\mathbb{R}^3$:

$$u_t + u\cdot\nabla u - \mu\Delta u + \nabla\pi = (\nabla\times b)\times b, \tag{1.1}$$

$$b_t - \nabla\times(u\times b) + \delta\nabla\times((\nabla\times b)\times b) - \nu\Delta b = \kappa\nabla\times[b\times(b\times(\nabla\times b))], \tag{1.2}$$

$$\operatorname{div} u = \operatorname{div} b = 0, \tag{1.3}$$

$$(u,b)(\cdot,0) = (u_0,b_0)(\cdot), \tag{1.4}$$

where $u=(u_1,u_2,u_3)$ is the fluid velocity field, $b=(b_1,b_2,b_3)$ is the magnetic field, and π is the pressure, respectively. The Hall term $\nabla\times((\nabla\times b)\times b)$ is for the Hall effect, and $\nabla\times[b\times(b\times(\nabla\times b))]$ is for the ion-slip effect. The parameters μ, ν, δ and κ denote the viscous coefficient, the resistivity coefficient, the Hall effect coefficient and the ion-slip effect coefficient, respectively. For simplicity, we set $\mu=\nu=\delta=\kappa=1$.

When $\delta=0$ and $\kappa=0$, the system (1.1)–(1.4) reduces to the classical magnetohydrodynamics system, which describes the plasma form a nonlinear system that couples Navier–Stokes equations with Maxwell's equations. In addition, if $\kappa=0$ and $\delta\neq 0$, the system (1.1)–(1.4) reduces to the Hall-MHD system, which represents the momentum conservation equation for the plasma fluid. Both magnetohydrodynamics system and Hall-MHD system have received many studies [1,5,7,9–12,15,16,26,27].

System (1.1)–(1.4) has been studied in [20–22]. Latterly, Fan, Jia, Nakamura and Zhou[13] proved some blow-up criteria, the local well-posedness of strong solutions, global existence of solutions and time decay rate of small data for system (1.1)–(1.4). The authors proved that if the H^2-norm of the initial datum is sufficiently small, system (1.1)–(1.4) has a small unique global-in-time strong solution $(u,b)\in L^\infty(0,\infty;H^2)$. Gala and Ragusa[14] improved Fan et al.'s local well-posedness result to the critical Besov space. They also established a new blow-up criterion of strong solutions in terms of the critical Besov space $\dot{B}^{-1}_{\infty;\infty}$ and multiplier spaces.

The first aim of this paper is to improve Fan et. al.[13]'s global well-posedness result, prove the existence of global solutions for small initial data to system (1.1)–(1.4). More precisely, the result can be stated as follows.

Theorem 1.1. *Suppose* $(u_0, b_0) \in H^s(\mathbb{R}^3)$ *with* $s > 2$ *and* $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$*. There exists a sufficiently small constant* $K > 0$ *and any* $\varepsilon > 0$ *such that if* $\|u_0\|_{H^{\frac{1}{2}+\varepsilon}} + \|b_0\|_{H^{\frac{3}{2}+\varepsilon}} \le K$*, then there exists a unique global strong solution* (u, b) *and satisfy*

$$(u, b) \in C(0, T; H^s(\mathbb{R}^3)) \cap L^2(0, T; H^{s+1}(\mathbb{R}^3)).$$

Remark 1.1. It seems that ε in Theorem 1.1 cannot reduce to 0 because of the following Sobolev's embedding

$$\|b\|_{L^\infty} \le \|b\|_{L^3}^{\frac{\varepsilon}{1+\varepsilon}} \|\Lambda^{\frac{3}{2}+\varepsilon} b\|_{L^2}^{\frac{1}{1+\varepsilon}} \le \|\Lambda^{\frac{1}{2}} b\|_{L^2}^{\frac{\varepsilon}{1+\varepsilon}} \|\Lambda^{\frac{3}{2}+\varepsilon} b\|_{L^2}^{\frac{1}{1+\varepsilon}} \le \|\Lambda^{\frac{1}{2}} b\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon} b\|_{L^2}.$$

Remark 1.2. We suspect that Theorem 1.1 may be hold if $\|u_0\|_{\dot{H}^{\frac{1}{2}}} + \|b_0\|_{\dot{H}^{\frac{3}{2}}}$ is sufficiently small. But, until now, we cannot find any valid method to get ride of the parameter ε. Therefore, we have to leave it as an open problem to be carried out later.

Remark 1.3. In [13], suppose that $(u_0, b_0) \in L^1 \bigcap H^2$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ and there exists a small constant K such that

$$\|u_0\|_{H^2} + \|b_0\|_{H^2} \le K,$$

Fan, Jia, Nakamura and Zhou obtained the existence of unique global-in-time strong solution for system (1.1)–(1.4). Comparing with [13], our theorem can be seen as an improvement of Fan et. al.'s work.

Another purpose of this paper is to obtain the time decay rates in $L^2 \times L^2$ and $H^m \times H^m$ by using Fourier splitting method and the properties of decay character r^*[2], which measures the "order" of $\widehat{v_0}(\xi)$ at $\xi = 0$ in frequency space.

Fourier splitting method was introduced by Schonbek in [24,25] to study the algebraic rates for the asymptotic behavior of solutions to the Navier–Stokes equations. Later, this method was well extended to investigate the decay for the solutions of PDE from mathematical physics, see, e.g., Brandolese and Schonbek[4], Dai et. al.[8], Jiu and Yu[17], Weng[28], etc.

Our first theorem on the L^2-norm decay rate of solutions read as follows:

Theorem 1.2. (L^2-decay) *Suppose that* $(u_0, b_0) \in L^2(\mathbb{R}^3)$, $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$*. Let* $r^* = r^*(u_0) = r^*(b_0) \in (-\frac{3}{2}, +\infty)$ *be the decay character. Then, there exist a positive constant* $C_0 = C_0(\|u_0\|_{L^2}, \|b_0\|_{L^2})$*, such that*

$$\|u\|_{L^2}^2 + \|b\|_{L^2}^2 \le C_0(1+t)^{-\min\{\frac{3}{2}+r^*, \frac{3}{2}\}}, \text{ for large } t. \tag{1.5}$$

The following is a decay estimate for the higher-order Sobolev norms, whose global-in-time existence is guaranteed for sufficiently small initial data.

Theorem 1.3. *Suppose that* $m \in \mathbb{N}$*,* $K_0 = \max\{3, m\}$*,* $(u_0, b_0) \in H^{K_0}(\mathbb{R}^3)$ *and* $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$*. Let* $r^* = r^*(u_0) = r^*(b_0) \in [-\frac{1}{2}, +\infty)$ *be the decay character. Then, there exists a positive constant* $C_m = C_m(\|u_0\|_{H^{K_0}}, \|b_0\|_{H^{K_0}})$*, such that*

$$\|D^m u\|_{L^2}^2 + \|D^m b\|_{L^2}^2 \le C_m(1+t)^{-m-\min\{\frac{3}{2}+r^*, \frac{3}{2}\}}, \text{ for large } t. \tag{1.6}$$

Remark 1.4. When $r^* = 0$, we recover the result in Remark 1.1 of [13]. Compared with the classical decay results of 3D Hall-MHD equations (see[29]), the decay estimates on $\|u\|_{L^2}^2 + \|b\|_{L^2}^2$ cannot reach $O(t^{-\frac{5}{2}})$ because of the ion-slip term.

Remark 1.5. At present, we are not able to show that (1.6) still holds true for $r^* \in (-\frac{3}{2}, -\frac{1}{2})$. The key reason is that the proof heavily relies on Lemma 3.5. Therefore, we leave it as an open problem to be carried out later.

The rest of this paper is organized as follows. In the next section, we study the well-posedness of solutions for system (1.1)–(1.4), i.e., we prove Theorem 1.1. In Sect. 3, we study the time decay rate of solutions.

2. Proof of Theorem 1.1

First of all, we introduce the Kato–Ponce inequality which is of great importance in the proof of Theorem 1.1.

Lemma 2.1. ([18,19]) *Let $1 < p < \infty$, $s > 0$. There exists a positive constant C such that*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{q_1}}\|g\|_{L^{q_2}}) \tag{2.1}$$

and

$$\|\Lambda^s(fg)\|_{L^p} \le C(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{q_1}}\|g\|_{L^{q_2}}, \tag{2.2}$$

where $p_1, q_1, p_2, q_2 \in (1,\infty)$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$.

Now, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. Multiplying(1.1) and (1.2) by u and b, one can get the fundamental energy estimate, for any $t \ge 0$,

$$\|u\|_{L^2}^2 + \|u\|_{L^2}^2 + \int_0^t \left(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \|b\times(\nabla\times b)\|_{L^2}^2\right) \mathrm{d}\tau = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 \tag{2.3}$$

Taking $\Lambda^{\frac12}$ to (1.1) and (1.2), multiplying by $\Lambda^{\frac12}u$ and $\Lambda^{\frac12}b$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Lambda^{\frac12}u\|_{L^2}^2 + \|\Lambda^{\frac12}b\|_{L^2}^2\right) + \left(\|\Lambda^{\frac32}u\|_{L^2}^2 + \|\Lambda^{\frac32}b\|_{L^2}^2\right)\\
&= -\int_{\mathbb{R}^3}(\Lambda^{\frac12}(u\cdot\nabla u) - u\cdot\nabla\Lambda^{\frac12}u)\Lambda^{\frac12}u\,\mathrm{d}x + \int_{\mathbb{R}^3}(\Lambda^{\frac12}(b\cdot\nabla b) - b\cdot\nabla\Lambda^{\frac12}b)\Lambda^{\frac12}u\,\mathrm{d}x\\
&\quad -\int_{\mathbb{R}^3}(\Lambda^{\frac12}(u\cdot\nabla b) - u\cdot\nabla\Lambda^{\frac12}b)\Lambda^{\frac12}b\,\mathrm{d}x + \int_{\mathbb{R}^3}(\Lambda^{\frac12}(b\cdot\nabla u) - b\cdot\nabla\Lambda^{\frac12}u)\Lambda^{\frac12}b\,\mathrm{d}x\\
&\quad -\int_{\mathbb{R}^3}\Lambda^{\frac12}((\nabla\times b)\times b)\Lambda^{\frac12}(\nabla\times b)\,\mathrm{d}x + \int_{\mathbb{R}^3}\Lambda^{\frac12}[b\times(b\times(\nabla\times b))]\Lambda^{\frac12}(\nabla\times b)\,\mathrm{d}x\\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned} \tag{2.4}$$

By using Lemma 2.1, the six terms in the right-hand side of (2.4) can be estimated as

$$|I_5| \le C\|\nabla b\|_{L^3}\|\Lambda^{\frac12}b\|_{L^6}\|\Lambda^{\frac32}b\|_{L^2} \le C\|\Lambda^{\frac32}b\|_{L^2}^3,$$

$$\begin{aligned}
|I_6| &\le C\left(\|b\|_{L^{\frac{3}{\varepsilon}}}^2\|\Lambda^{\frac32}b\|_{L^{\frac{6}{3-2\varepsilon}}}^2 + \|b\|_{L^{\frac{3}{\varepsilon}}}\|\Lambda^{\frac12}b\|_{L^{\frac{6}{1+2\varepsilon}}}\|\nabla b\|_{L^{\frac{3}{1-\varepsilon}}}\|\Lambda^{\frac32}b\|_{L^{\frac{6}{3-2\varepsilon}}}\right)\\
&\le C\|b\|_{\dot H^{\frac32-\varepsilon}}^2\|b\|_{\dot H^{\frac32+\varepsilon}}^2
\end{aligned}$$

and

$$\begin{aligned}
&|I_1| + |I_2| + |I_3| + |I_4|\\
&\le C\|\Lambda^{\frac12}(u\cdot\nabla u) - u\cdot\nabla\Lambda^{\frac12}u\|_{L^2}\|\Lambda^{\frac12}u\|_{L^2} + C\|\Lambda^{\frac12}(b\cdot\nabla b) - b\cdot\nabla\Lambda^{\frac12}b\|_{L^2}\|\Lambda^{\frac12}u\|_{L^2}\\
&\quad + C\|\Lambda^{\frac12}(u\cdot\nabla b) - u\cdot\nabla\Lambda^{\frac12}b\|_{L^2}\|\Lambda^{\frac12}b\|_{L^2} + C\|\Lambda^{\frac12}(b\cdot\nabla u) - u\cdot\nabla\Lambda^{\frac12}b\|_{L^2}\|\Lambda^{\frac12}b\|_{L^2}\\
&\le C(\|\nabla u\|_{L^3} + \|\nabla b\|_{L^3})(\|\Lambda^{\frac12}u\|_{L^6} + \|\Lambda^{\frac12}b\|_{L^6})(\|\Lambda^{\frac12}u\|_{L^2} + \|\Lambda^{\frac12}b\|_{L^2})\\
&\le C(\|\Lambda^{\frac12}u\|_{L^2} + \|\Lambda^{\frac12}b\|_{L^2})(\|\Lambda^{\frac32}u\|_{L^2}^2 + \|\Lambda^{\frac32}b\|_{L^2}^2).
\end{aligned}$$

Taking $\Lambda^{\frac{1}{2}+\varepsilon}$ to (1.1) and (1.2), multiplying by $\Lambda^{\frac{1}{2}+\varepsilon}u$ and $\Lambda^{\frac{1}{2}+\varepsilon}b$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^{\frac{1}{2}+\varepsilon}u\|_{L^2}^2+\|\Lambda^{\frac{1}{2}+\varepsilon}b\|_{L^2}^2)+(\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^2)\\
&=-\int_{\mathbb{R}^3}(\Lambda^{\frac{1}{2}+\varepsilon}(u\cdot\nabla u)-u\cdot\nabla\Lambda^{\frac{1}{2}+\varepsilon}u)\Lambda^{\frac{1}{2}+\varepsilon}u\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}(\Lambda^{\frac{1}{2}+\varepsilon}(b\cdot\nabla b)-b\cdot\nabla\Lambda^{\frac{1}{2}+\varepsilon}b)\Lambda^{\frac{1}{2}+\varepsilon}u\,\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^3}(\Lambda^{\frac{1}{2}+\varepsilon}(u\cdot\nabla b)-u\cdot\nabla\Lambda^{\frac{1}{2}+\varepsilon}b)\Lambda^{\frac{1}{2}+\varepsilon}b\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}(\Lambda^{\frac{1}{2}+\varepsilon}(b\cdot\nabla u)-b\cdot\nabla\Lambda^{\frac{1}{2}+\varepsilon}u)\Lambda^{\frac{1}{2}+\varepsilon}b\,\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^3}\Lambda^{\frac{1}{2}+\varepsilon}((\nabla\times b)\times b)\Lambda^{\frac{1}{2}+\varepsilon}(\nabla\times b)\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}\Lambda^{\frac{1}{2}+\varepsilon}[b\times(b\times(\nabla\times b))]\Lambda^{\frac{1}{2}+\varepsilon}(\nabla\times b)\,\mathrm{d}x\\
&=J_1+J_2+J_3+J_4+J_5+J_6.
\end{aligned}
\tag{2.5}
$$

Now, we estimate J_1-J_6 as follows

$$
\begin{aligned}
&|J_1|+|J_2|+|J_3|+|J_4|\\
&\quad\le C\|\Lambda^{\frac{1}{2}+\varepsilon}(u\cdot\nabla u)-u\cdot\nabla\Lambda^{\frac{1}{2}+\varepsilon}u\|_{L^2}\|\Lambda^{\frac{1}{2}+\varepsilon}u\|_{L^2}\\
&\qquad+C\|\Lambda^{\frac{1}{2}+\varepsilon}(b\cdot\nabla b)-b\cdot\nabla\Lambda^{\frac{1}{2}+\varepsilon}b\|_{L^2}\|\Lambda^{\frac{1}{2}+\varepsilon}u\|_{L^2}\\
&\qquad+C\|\Lambda^{\frac{1}{2}+\varepsilon}(u\cdot\nabla b)-u\cdot\nabla\Lambda^{\frac{1}{2}+\varepsilon}b\|_{L^2}\|\Lambda^{\frac{1}{2}+\varepsilon}b\|_{L^2}\\
&\qquad+C\|\Lambda^{\frac{1}{2}+\varepsilon}(b\cdot\nabla u)-u\cdot\nabla\Lambda^{\frac{1}{2}+\varepsilon}b\|_{L^2}\|\Lambda^{\frac{1}{2}+\varepsilon}b\|_{L^2}\\
&\quad\le C(\|\nabla u\|_{L^3}+\|\nabla b\|_{L^3})(\|\Lambda^{\frac{1}{2}+\varepsilon}u\|_{L^6}+\|\Lambda^{\frac{1}{2}+\varepsilon}b\|_{L^6})(\|\Lambda^{\frac{1}{2}+\varepsilon}u\|_{L^2}+\|\Lambda^{\frac{1}{2}+\varepsilon}b\|_{L^2})\\
&\quad\le C(\|\Lambda^{\frac{1}{2}}u\|_{L^2}+\|\Lambda^{\frac{1}{2}}b\|_{L^2})\left(\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^2\right),\\
&|J_5|\le C\|\nabla b\|_{L^3}\|\Lambda^{\frac{1}{2}+\varepsilon}b\|_{L^6}\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}\le C\|\Lambda^{\frac{1}{2}}b\|_{L^2}\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^2
\end{aligned}
$$

and

$$
\begin{aligned}
|J_6|&\le C\left(\|b\|_{L^\infty}^2\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^2+\|b\|_{L^\infty}\|\nabla b\|_{L^3}\|\Lambda^{\frac{1}{2}+\varepsilon}b\|_{L^6}\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}\right)\\
&\le C\left(\|b\|_{L^\infty}^2\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^2+\|b\|_{L^\infty}\|\Lambda^{\frac{3}{2}}b\|_{L^2}\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^2\right).
\end{aligned}
$$

Taking $\Lambda^{\frac{3}{2}+\varepsilon}$ to (1.2), multiplying by $\Lambda^{\frac{3}{2}+\varepsilon}b$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^2+\|\Lambda^{\frac{5}{2}+\varepsilon}b\|_{L^2}^2\\
&\quad=-\int_0^T\Lambda^{\frac{3}{2}+\varepsilon}(u\cdot\nabla b)\Lambda^{\frac{3}{2}+\varepsilon}b\,\mathrm{d}x+\int_0^T\Lambda^{\frac{3}{2}+\varepsilon}(b\cdot\nabla u)\Lambda^{\frac{3}{2}+\varepsilon}b\,\mathrm{d}x\\
&\qquad-\int_0^T\Lambda^{\frac{3}{2}+\varepsilon}((\nabla\times b)\times b)\Lambda^{\frac{3}{2}+\varepsilon}(\nabla\times b)\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^T \Lambda^{\frac{3}{2}+\varepsilon}[b\times(b\times(\nabla\times b))]\Lambda^{\frac{3}{2}+\varepsilon}(\nabla\times b)\,\mathrm{d}x\\
&=K_1+K_2+K_3+K_4. \qquad (2.6)
\end{aligned}
$$

Applying Lemma 2.1, the four terms in the right-hand side of (2.6) can be estimated as

$$
\begin{aligned}
|K_1| &\le C(\|\nabla b\|_{L^3}\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^6}+\|\nabla u\|_{L^3}\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^6})\\
&\le C(\|\Lambda^{\frac{3}{2}}b\|_{L^2}+\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2})\left(\|\Lambda^{\frac{3}{2}}u\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}^2+\|\Lambda^{\frac{5}{2}+\varepsilon}b\|_{L^2}^2\right),\\
|K_2| &= \int_0^T \Lambda^{\frac{3}{2}+\varepsilon}(b\cdot u)\Lambda^{\frac{3}{2}+\varepsilon}\nabla b\,\mathrm{d}x\\
&\le C(\|b\|_{L^\infty}\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}\|\Lambda^{\frac{5}{2}+\varepsilon}b\|_{L^2}+\|u\|_{L^3}\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^6}\|\Lambda^{\frac{5}{2}+\varepsilon}b\|_{L^2})\\
&\le C(\|\Lambda^{\frac{1}{2}}u\|_{L^2}+\|b\|_{L^\infty})\left(\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^2+\|\Lambda^{\frac{5}{2}+\varepsilon}b\|_{L^2}^2\right),\\
|K_3| &\le C\|\nabla b\|_{L^3}\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^6}\|\Lambda^{\frac{5}{2}+\varepsilon}b\|_{L^2}\le C\|\Lambda^{\frac{3}{2}}b\|_{L^2}\|\Lambda^{\frac{5}{2}+\varepsilon}b\|_{L^2}^2
\end{aligned}
$$

and

$$
\begin{aligned}
|K_4| &\le C(\|b\|_{L^\infty}^2\|\Lambda^{\frac{5}{2}+\varepsilon}b\|_{L^2}^2+\|b\|_{L^\infty}\|\nabla b\|_{L^3}\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^6}\|\Lambda^{\frac{5}{2}+\varepsilon}b\|_{L^2})\\
&\le C\left(\|b\|_{L^\infty}^2\|\Lambda^{\frac{5}{2}+\varepsilon}b\|_{L^2}^2+\|b\|_{L^\infty}\|\Lambda^{\frac{3}{2}}b\|_{L^2}\|\Lambda^{\frac{5}{2}+\varepsilon}b\|_{L^2}^2\right).
\end{aligned}
$$

Summing up, we derive that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Lambda^{\frac{1}{2}}u\|_{L^2}^2+\|\Lambda^{\frac{1}{2}}b\|_{L^2}^2+\|\Lambda^{\frac{1}{2}+\varepsilon}u\|_{L^2}^2+\|\Lambda^{\frac{1}{2}+\varepsilon}b\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^2\right)\\
&+\|\Lambda^{\frac{3}{2}}u\|_{L^2}^2+\|\Lambda^{\frac{3}{2}}b\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^2+\|\Lambda^{\frac{5}{2}+\varepsilon}b\|_{L^2}^2\\
&\le C\left(\|b\|_{L^\infty}+\|b\|_{L^\infty}^2+\|\Lambda^{\frac{1}{2}}u\|_{L^2}+\|\Lambda^{\frac{1}{2}}b\|_{L^2}+\|\Lambda^{\frac{3}{2}}b\|_{L^2}+\|\Lambda^{\frac{3}{2}-\varepsilon}b\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^2\right)\\
&\quad\times\left(\|\Lambda^{\frac{3}{2}}u\|_{L^2}^2+\|\Lambda^{\frac{3}{2}}b\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^2+\|\Lambda^{\frac{5}{2}+\varepsilon}b\|_{L^2}^2\right)\\
&\le C\left(\|\Lambda^{\frac{1}{2}}b\|_{L^2}+\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}+\|\Lambda^{\frac{1}{2}}u\|_{L^2}+\|\Lambda^{\frac{1}{2}}b\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^2\right)\\
&\quad\times\left(\|\Lambda^{\frac{3}{2}}u\|_{L^2}^2+\|\Lambda^{\frac{3}{2}}b\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^2+\|\Lambda^{\frac{5}{2}+\varepsilon}b\|_{L^2}^2\right),
\end{aligned}
$$

where we have used the Gagliardo–Nirenberg inequality:

$$
\begin{aligned}
\|b\|_{L^\infty} &\le \|b\|_{L^3}^{\frac{\varepsilon}{1+\varepsilon}}\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^{\frac{1}{1+\varepsilon}}\le\|\Lambda^{\frac{1}{2}}b\|_{L^2}^{\frac{\varepsilon}{1+\varepsilon}}\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^{\frac{1}{1+\varepsilon}}\le\|\Lambda^{\frac{1}{2}}b\|_{L^2}+\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2},\\
\|\Lambda^{\frac{3}{2}}b\|_{L^2} &\le \|\Lambda^{\frac{1}{2}}b\|_{L^2}^{\frac{\varepsilon}{1+\varepsilon}}\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^{\frac{1}{1+\varepsilon}}\le\|\Lambda^{\frac{1}{2}}b\|_{L^2}+\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}
\end{aligned}
$$

and

$$
\|\Lambda^{\frac{3}{2}-\varepsilon}b\|_{L^2}^2\le\|\Lambda^{\frac{1}{2}}b\|_{L^2}^{\frac{4\varepsilon}{1+\varepsilon}}\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^{\frac{2-2\varepsilon}{1+\varepsilon}}\le\|\Lambda^{\frac{1}{2}}b\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^2.
$$

Choosing K so small that

$$
C\left(\|\Lambda^{\frac{1}{2}}b_0\|_{L^2}+\|\Lambda^{\frac{3}{2}+\varepsilon}b_0\|_{L^2}+\|\Lambda^{\frac{1}{2}}u_0\|_{L^2}+\|\Lambda^{\frac{1}{2}}b_0\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}b_0\|_{L^2}^2\right)\le\frac{1}{2},
$$

then, $\|\Lambda^{\frac{1}{2}}u\|_{L^2}^2+\|\Lambda^{\frac{1}{2}}b\|_{L^2}^2+\|\Lambda^{\frac{1}{2}+\varepsilon}u\|_{L^2}^2+\|\Lambda^{\frac{1}{2}+\varepsilon}b\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^2$ is decreasing. So, for any $0<T<\infty$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Lambda^{\frac{1}{2}}u\|_{L^2}^2+\|\Lambda^{\frac{1}{2}}b\|_{L^2}^2+\|\Lambda^{\frac{1}{2}+\varepsilon}u\|_{L^2}^2+\|\Lambda^{\frac{1}{2}+\varepsilon}b\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^2\right)\\&+\|\Lambda^{\frac{3}{2}}u\|_{L^2}^2+\|\Lambda^{\frac{3}{2}}b\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}b\|_{L^2}^2+\|\Lambda^{\frac{5}{2}+\varepsilon}b\|_{L^2}^2\le 0,\end{aligned}$$

which implies

$$u\in L^\infty(0,T;H^{\frac{1}{2}+\varepsilon})\cap L^2(0,T;H^{\frac{3}{2}+\varepsilon}); \tag{2.7}$$

$$b\in L^\infty(0,T;H^{\frac{3}{2}+\varepsilon})\cap L^2(0,T;H^{\frac{5}{2}+\varepsilon}). \tag{2.8}$$

The high-order estimations can be obtained by the blow-up criteria [13,14] and (2.7), (2.8). So, we omit the details. □

3. Proof of Theorems 1.2 and 1.3

The first definition of decay character r^* can be traced back to Bjorland and Schonbek[2]. In [3,23], the authors found the sharp decay estimates for solutions to the heat equation

$$\partial_t\bar{u}-\Delta\bar{u}=0,\ \ \bar{v}(x,0)=v_0(x), \tag{3.1}$$

in terms of r^*.

Definition 3.1. ([3,23]) Suppose that $v_0\in L^2(\mathbb{R}^n)$, $\Lambda=(-\Delta)^{\frac{1}{2}}$ and

$$P_r^s(v_0)=\lim_{\rho\to 0}\rho^{-2r-n}\int\limits_{B(\rho)}|\xi|^{2s}|\widehat{v_0}(\xi)|^2\mathrm{d}\xi,\ \ s\ge 0,$$

exists, for $r\in(-\frac{n}{2}+s,\infty)$ and $B(\rho)$ the ball at the origin with radius ρ. Then, $P_r^s(v_0)$ is the s-decay indicator corresponding to $\Lambda^s v_0$.

Definition 3.2. ([23]) The decay character of $\Lambda^s v_0$ denoted by $r_s^*=r_s^*(v_0)$ is the unique $r\in(-\frac{n}{2}+s,\infty)$ such that $0<P_r^s(v_0)<\infty$, provided that this number exists. If such $P_r^s(v_0)$ does not exist, set $r_s^*=-\frac{n}{2}+s$, when $P_r^s(v_0)=\infty$ for all $r\in(-\frac{n}{2}+s,\infty)$ or $r_s^*=\infty$, if $P_r^s(v_0)=0$ for all $r\in(-\frac{n}{2}+s,\infty)$.

The following lemma describes the L^2 decay characterization of solutions to the heat equation (3.1) in terms of the decay character $r^*=r^*(v_0)$.

Lemma 3.1. ([23]) *Suppose that $v_0\in L^2(\mathbb{R}^n)$ have decay character $r^*=r^*(v_0)$. Let $\bar{v}(t)$ be a solution to (3.1) with data v_0. Then*

- *if $-\frac{n}{2}<r^*<\infty$, then there exist two positive constants C_1 and C_2 such that*

$$C_1(1+t)^{-\left(\frac{n}{2}+r^*\right)}\le\|\bar{v}(t)\|_{L^2}^2\le C_2(1+t)^{-\left(\frac{n}{2}+r^*\right)};$$

- *if $r^*=-\frac{n}{2}$, then there exists a positive constant $C=C(\varepsilon)$ such that*

$$\|\bar{v}(t)\|_{L^2}^2\ge C(1+t)^{-\varepsilon},\quad\forall\varepsilon>0,$$

  which means the decay of $\|\bar{v}(t)\|_{L^2}^2$ is slower than any uniform algebraic rate;
- *if $r^*=\infty$, then there exists a positive constant C such that*

$$\|\bar{v}(t)\|_{L^2}^2\le C(1+t)^{-m},\quad\forall m>0,$$

  which means the decay of $\bar{v}(t)\|_{L^2}^2$ is faster than any algebraic rate.

In order to prove the decay estimate (1.5), we prepare

Lemma 3.2. *Let $(u_0, b_0) \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ with $div\, u_0 = div\, b_0 = 0$. Suppose that (u, b) is the solution of system (1.1)–(1.4) with initial value (u_0, b_0). Then*

$$|\widehat{u}(\xi,t)|^2 \le C\left[e^{-2|\xi|^2 t}|\widehat{u_0}(\xi) + |\xi|^2\left(\int_0^t \left(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2\right)\mathrm{d}s\right)^2\right], \tag{3.2}$$

and

$$|\widehat{b}(\xi,t)|^2 \le C\left[e^{-2|\xi|^2 t}|\widehat{b_0}(\xi)|^2 + 1 + |\xi|^2\left(\int_0^t \left(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2\right)\mathrm{d}s\right)^2\right]. \tag{3.3}$$

Proof. Taking the Fourier transform for (1.1), we derive that

$$\partial_t\widehat{u}(\xi,t) + |\xi|^2\widehat{u}(\xi,t) = H(\xi,t) := -\widehat{u\cdot\nabla u}(\xi,t) - \widehat{\nabla\pi}(\xi,t) - \widehat{(\nabla\times b)\times b}(\xi,t). \tag{3.4}$$

Integrating in time from 0 to t, we get

$$\partial_t\widehat{u}(\xi,t) = e^{-|\xi|^2 t}\widehat{u_0}(\xi) + \int_0^t e^{-|\xi|^2(t-s)}H(\xi,s)\mathrm{d}s. \tag{3.5}$$

Therefore,

$$|\widehat{u}(\xi,t)| \le |e^{-|\xi|^2 t}\widehat{u_0}(\xi)| + \int_0^t e^{-|\xi|^2(t-s)}|H(\xi,s)|\mathrm{d}s. \tag{3.6}$$

Applying the same calculations as the proof of Lemma 7 in [29], we obtain

$$|H(\xi,t)| \le C|\xi|\left(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2\right). \tag{3.7}$$

By (3.6) and (3.7), we obtain (3.2). Taking the Fourier transform for (1.2), we get

$$\begin{aligned}\partial_t\widehat{b}(\xi,t)] + |\xi|^2\widehat{b}(\xi,t) &= G(\xi,t)\\ &:= \widehat{b\cdot\nabla u}(\xi,t) - \widehat{u\cdot\nabla b}(\xi,t) - \nabla\times(\widehat{(\nabla\times b)\times b})(\xi,s)\\ &\quad + \nabla\times(b\times\widehat{(b\times(\nabla\times b))})(\xi,s).\end{aligned} \tag{3.8}$$

Integrating in time from 0 to t, we deduce that

$$|\widehat{b}(\xi,t)| \le e^{-|\xi|^2 t}|\widehat{b_0}(\xi)| + \int_0^t e^{-|\xi|^2(t-s)}|G(\xi,s)|\mathrm{d}s. \tag{3.9}$$

Note that

$$|\widehat{b\cdot\nabla u}(\xi,t)| + |\widehat{u\cdot\nabla b}(\xi,t)| \le C|\xi|\|u\|_{L^2}\|b\|_{L^2},$$

and

$$|\nabla\times(\widehat{b\times(\nabla\times b)})(\xi,t)| \le |\xi\times\sum_{i=1}^3\xi_i\widehat{b_i b}(\xi,t)| \le C|\xi|^2\|b\|_{L^2}^2.$$

We also have

$$|\nabla\times(b\times\widehat{(b\times(\nabla\times b))})(\xi,t)| \le |\xi|\|b\|_{L^2}\|b\times(\nabla\times b)\|_{L^2} \le C|\xi|\left(\|b\|_{L^2}^2 + \|b\times(\nabla\times b)\|_{L^2}^2\right).$$

Summing up, we have

$$G(\xi,t) \le C|\xi|\left(\|u\|_{L^2}^2 + \|b\|_{L^2}^2 + \|b\times(\nabla\times b)\|_{L^2}^2 + |\xi|\|b\|_{L^2}^2\right). \tag{3.10}$$

By the energy inequality, we obtain $\int_0^t\|b\times(\nabla\times b)\|_{L^2}^2\mathrm{d}s \le C$. Combining (3.9)–(3.10) together, we obtain (3.3). This completes the proof. □

Proof of Theorem 1.2. Testing (1.1) by u, and (1.2) by b, respectively, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(|u|^2+|b|^2)\mathrm{d}x+\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\|(\nabla\times b)\times b\|_{L^2}^2=0. \tag{3.11}$$

Applying the Plancherel's theorem to (3.11), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}[|\hat{u}(\xi,t)|^2+|\hat{b}(\xi,t)|^2]\mathrm{d}\xi+2\int_{\mathbb{R}^3}|\xi|^2(|\hat{u}(\xi,t)|^2+|\hat{b}(\xi,t)|^2)\mathrm{d}\xi\le 0,\quad \forall t>0. \tag{3.12}$$

Set

$$B(t):=\left\{\xi\in\mathbb{R}^3||\xi|^2\le\rho^2=\frac{g'(t)}{2g(t)}\right\},\quad B^c(t):=\mathbb{R}^3\setminus B(t),$$

where $g(t)$ is a differentiable function of t satisfying

$$g(0)=1,\ g'(t)>0 \text{ and } 2g(t)>g'(t),\ \forall t>0.$$

Multiplying (3.12) by $g(t)$, on the basis of the definitions of $B(t)$ and $B^c(t)$, we derive that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(g(t)\int_{\mathbb{R}^3}[|\hat{u}(\xi,t)|^2+|\hat{b}(\xi,t)|^2]\mathrm{d}\xi\right)\le g'(t)\int_{B(t)}\left(|\hat{u}(\xi,t)|^2+|\hat{b}(\xi,t)|^2\right)\mathrm{d}\xi. \tag{3.13}$$

Hence

$$\begin{aligned}
&g(t)\int_{\mathbb{R}^3}[|\hat{u}(\xi,t)|^2+|\hat{b}(\xi,t)|^2]\mathrm{d}\xi\\
&\le C+C\int_0^t g'(s)\int_{B(s)}e^{-2|\xi|^2t}(|\hat{u_0}(\xi)|^2+|\hat{b_0}(\xi)|^2)\mathrm{d}\xi\mathrm{d}s+C\int_0^t g'(s)\int_{B(s)}\mathrm{d}\xi\mathrm{d}s\\
&\quad+C\int_0^t g'(s)\int_{B(s)}|\xi|^2\left(\int_0^t(\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2)\mathrm{d}s\right)^2\mathrm{d}\xi\mathrm{d}t.
\end{aligned} \tag{3.14}$$

Note that

$$\begin{aligned}
&C\int_0^t g'(s)\int_{B(s)}e^{-2|\xi|^2t}(|\hat{u_0}(\xi)|^2+|\hat{b_0}(\xi)|^2)\mathrm{d}\xi\mathrm{d}s\\
&\le C\int_0^t g'(s)(\|\bar{u}(s)\|_{L^2}^2+\|\bar{b}\|_{L^2}^2)\mathrm{d}s\le C\int_0^t g'(s)(1+s)^{-\left(\frac{3}{2}+r^*\right)}\mathrm{d}s,
\end{aligned} \tag{3.15}$$

where $\bar{u}$ and $\bar{b}$ are the solutions of the heat equation, which is the linear part of (1.1)and (1.3). We also have

$$C\int_0^t g'(s)\int_{B(s)}\mathrm{d}\xi\mathrm{d}s\le C\int_0^t g'(s)(1+s)^{-\frac{3}{2}}\mathrm{d}s. \tag{3.16}$$

For the last term of the right hand of (3.14), after integrating in polar coordinates in $B(t)$, we get

$$\begin{aligned} &C\int_0^t g'(t)\int_{B(t)}|\xi|^2\left(\int_0^t(\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2)\mathrm{d}s\right)^2\mathrm{d}\xi\mathrm{d}t \\ &\le C\left(\int_0^t g'(s)\rho^5\mathrm{d}s\right)\left(\int_0^t\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2)\mathrm{d}s\right)^2. \end{aligned} \tag{3.17}$$

For a fixed r^*, we can choose $g(t)=(1+t)^m$ with $m>\max\{\frac{1}{2},\frac{3}{2}+r^*\}$. It is easy to see that $\rho(t)=(1+t)^{-\frac{1}{2}}$. It then follows from (3.14)–(3.17) and the a priori estimate $\|u\|_{L^2}^2+\|b\|_{L^2}^2\le C$ that

$$\begin{aligned} &\|u\|_{L^2}^2+\|b\|_{L^2}^2 \\ &\le C\left((1+t)^{-m}+(1+t)^{-\left(\frac{3}{2}+r^*\right)}+(1+t)^{-\frac{1}{2}}+(1+t)^{-\frac{3}{2}}\right) \\ &\le C(1+t)^{-\min\left\{\frac{3}{2}+r^*,\frac{1}{2}\right\}}. \end{aligned} \tag{3.18}$$

Using this first preliminary decay, we bootstrap to find sharper estimates. Assume that $\min\{\frac{3}{2}+r^*,\frac{1}{2}\}=\frac{3}{2}+r^*$, for $g(t)=(1+t)^{-m}$ with $m>\max\{\frac{3}{2}+r^*,\frac{3}{2}\}$, we get $\rho(t)=C(1+t)^{-\frac{1}{2}}$ and

$$\begin{aligned} &C\int_0^t g'(t)\int_{B(t)}|\xi|^2\left(\int_0^t\left(\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2\right)\mathrm{d}s\right)^2\mathrm{d}\xi\mathrm{d}t \\ &\le C\left(\int_0^t g'(s)\rho^5\mathrm{d}s\right)\left(\int_0^t(1+s)^{-\left(\frac{3}{2}+r^*\right)}\mathrm{d}s\right)^2 \\ &\le C\int_0^t g'(s)\left((1+s)^{-\left(\frac{7}{2}+2r^*\right)}\right)\mathrm{d}s. \end{aligned} \tag{3.19}$$

It then follows from (3.14)–(3.16) and (3.19) that

$$\begin{aligned} &\|u\|_{L^2}^2+\|b\|_{L^2}^2 \\ &\le C\left((1+t)^{-m}+(1+t)^{-\left(\frac{3}{2}+r^*\right)}+(1+t)^{-\frac{7}{2}-2r^*}+(1+t)^{-\frac{3}{2}}\right) \\ &\le C(1+t)^{-\left(\frac{3}{2}+r^*\right)}, \end{aligned} \tag{3.20}$$

the decay is still the slower one, there is no improvement for the decay rate. Suppose that $\min\{\frac{3}{2}+r^*,\frac{1}{2}\}=\frac{1}{2}$, we have

$$C\left(\int_0^t g'(s)\rho^5\mathrm{d}s\right)\left(\int_0^t(1+s)^{-\frac{1}{2}}\mathrm{d}s\right)^2\le C\int_0^t g'(s)(1+s)^{-\frac{3}{2}}\mathrm{d}s. \tag{3.21}$$

By (3.14)–(3.16) and (3.21), we derive that

$$\begin{aligned} \|u\|_{L^2}^2+\|b\|_{L^2}^2 &\le C\left((1+t)^{-m}+(1+t)^{-\left(\frac{3}{2}+r^*\right)}+(1+t)^{-\frac{3}{2}}+(1+t)^{-\frac{3}{2}}\right) \\ &\le C(1+t)^{-\min\left\{\frac{3}{2}+r^*,\frac{3}{2}\right\}}, \end{aligned} \tag{3.22}$$

If we bootstrap once again, the decay rate is also the same as before, there is no improvement. Hence, we complete the proof. □

We observe the following fact.

Lemma 3.3. *Let* $(u_0, b_0) \in H^m(\mathbb{R}^3) \times H^m(\mathbb{R}^3)$ *with* $div\, u_0 = div\, b_0 = 0$. *Then, for all* $|\xi| \le 1$ *and for all* $j \in \mathbb{N}$,

$$|\widehat{D^j u}(\xi, t)| + |\widehat{D^j b}(\xi, t)| \le |\widehat{b}(\xi, t)| + |\widehat{b}(\xi, t)|, \tag{3.23}$$

where $C = C(\|u_0\|_{L^2}, \|b_0\|_{L^2})$ *is a positive constant.*

Proof. Since $|\xi| \le 1$, we have

$$|\widehat{D^j u}(\xi, t)| + |\widehat{D^j b}(\xi, t)| \le |\xi|^j (|\widehat{u}(\xi, t)| + |\widehat{b}(\xi, t)|) \le |\widehat{u}(\xi, t)| + |\widehat{b}(\xi, t)|.$$

Then, we complete the proof. □

The following are decay estimates for high-order Sobolev norms.

Lemma 3.4. *Let* $(u_0, b_0) \in H^m(\mathbb{R}^3) \times H^m(\mathbb{R}^3)$ *with* $div\, u_0 = div\, b_0 = 0$. *Suppose that* $m \in \mathbb{N}$ *and* $m \ge 3$. *Then, that*

$$\|u\|^2_{H^m} + \|b\|^2_{H^m} \le C(1+t)^{-\min\left\{\frac{3}{2}+r^*, \frac{3}{2}\right\}}, \text{ for large } t, \tag{3.24}$$

where $C = C(\|u_0\|_{H^m}, \|b_0\|_{H^m})$ *is a positive constant.*

Proof. On the basis of Theorem 1.1, we easily obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2_{H^s} + \|b\|^2_{H^s}\right) + \|u\|^2_{H^{s+1}} + \|b\|^2_{H^{s+1}} \le 0, \tag{3.25}$$

provided that $\|u_0\|_{\dot{H}^{\frac{1}{2}+\varepsilon}} + \|b_0\|_{\dot{H}^{\frac{3}{2}+\varepsilon}}$ is sufficiently small. The Fourier transform of (3.25) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\left[|\widehat{u}(\xi,t)|^2 + |D^m\widehat{u}(\xi,t)|^2 + |\widehat{b}(\xi,t)|^2 + |\widehat{D^m b}(\xi,t)|^2\right]\mathrm{d}\xi$$
$$\le -\int_{\mathbb{R}^3}|\xi|^2\left[|\widehat{u}(\xi,t)|^2 + |D^m\widehat{u}(\xi,t)|^2 + |\widehat{b}(\xi,t)|^2 + |\widehat{D^m b}(\xi,t)|^2\right]\mathrm{d}\xi.$$

In a similar fashion as the proof of Theorem 1.2, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{g(t)\int_{\mathbb{R}^3}\left[|\widehat{u}(\xi,t)|^2 + |D^m\widehat{u}(\xi,t)|^2 + |\widehat{b}(\xi,t)|^2 + |\widehat{D^m b}(\xi,t)|^2\right]\mathrm{d}\xi\right\}$$
$$\le g'(t)\int_{B(t)}\left[|\widehat{u}(\xi,t)|^2 + |D^m\widehat{u}(\xi,t)|^2 + |\widehat{b}(\xi,t)|^2 + |\widehat{D^m b}(\xi,t)|^2\right]\mathrm{d}\xi.$$

Applying the results of Lemma 3.3, there exists a $T_0 > 0$, such that for any $t > T_0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{g(t)\int_{\mathbb{R}^3}\left[|\widehat{u}(\xi,t)|^2 + |D^m\widehat{u}(\xi,t)|^2 + |\widehat{b}(\xi,t)|^2 + |\widehat{D^m b}(\xi,t)|^2\right]\mathrm{d}\xi\right\}$$
$$\le Cg'(t)\int_{B(t)}\left[|\widehat{u}(\xi,t)|^2 + |\widehat{b}(\xi,t)|^2\right]\mathrm{d}\xi.$$

Arguing as for proving Theorem 1.2, we obtain

$$\|u\|^2_{H^m} + \|b\|^2_{H^m} \le C(1+t)^{-\min\left\{\frac{3}{2}+r^*, \frac{3}{2}\right\}}, \text{ for any } t > T_0.$$

Then, the proof is completed. □

Next lemma is a typical case of Lemma 2.4 of [6].

Lemma 3.5. *Suppose that* $m \in \mathbb{N}$ *and*

$$\|D^{m-1}u\|_{L^2}^2 + \|D^{m-1}b\|_{L^2}^2 \le C_{m-1}(1+t)^{-\rho_{m-1}}, \text{ for large } t.$$

Assume that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|D^m u\|_{L^2}^2 + \|D^m b\|_{L^2}^2\right) \le C_0(1+t)^{-1}\|D^m b\|_{L^2}^2 - \left(\|D^{m+1}u\|_{L^2}^2 + \|D^{m+1}b\|_{L^2}^2\right).$$

Then, for $\rho_m = 1 + \rho_{m-1}$*, we have*

$$\|D^m u\|_{L^2}^2 + \|D^m b\|_{L^2}^2 \le C_m(1+t)^{-\rho_m}, \text{ for large } t.$$

Proof of Theorem 1.3. Operating D^m on (1.1) and (1.2), multiplying them by $D^m u$ and $D^m b$, respectively, integrating by part, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|D^m u\|_{L^2}^2 + \|D^m b\|_{L^2}^2\right) + \|D^{m+1}u\|_{L^2}^2 + \|D^{m+1}b\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^3}(u\cdot\nabla u)D^{2m}u\mathrm{d}x + \int_{\mathbb{R}^3}(b\cdot\nabla b)D^{2m}u\mathrm{d}x - \int_{\mathbb{R}^3}(u\cdot\nabla b)D^{2m}b\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3}(b\cdot\nabla u)D^{2m}b\mathrm{d}x - \int_{\mathbb{R}^3}\nabla\times((\nabla\times b)\times b)D^{2m}b\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3}\nabla\times[b\times(b\times(\nabla\times b))]D^{2m}b\mathrm{d}x \\
&=: J_1 + J_2 + J_3 + J_4 + J_5 + J_6.
\end{aligned} \tag{3.26}$$

Note that $\nabla\cdot u = \nabla\cdot b = 0$. We have

$$J_1 \le \sum_{i=1}^{3}\|D^{m+1}u\|_{L^2}\|D^m(u_i u)\|_{L^2} \le \sum_{i=1}^{3}\|D^{m+1}u\|_{L^2}\|D^m u\|_{L^6}\|u\|_{L^3}. \tag{3.27}$$

From Nirenberg's inequality and (3.24) that

$$\|u\|_{L^3} \le C\|u\|_{L^2}^{\frac{5}{6}}\|D^3 u\|_{L^2}^{\frac{1}{6}} \le C(1+t)^{-\min\left\{\frac{3}{4}+\frac{1}{2}r^*,\frac{5}{4}\right\}}. \tag{3.28}$$

Combining (3.27) and (3.28) together gives

$$J_1 \le C(1+t)^{-\min\left\{\frac{3}{4}+\frac{1}{2}r^*,\frac{5}{4}\right\}}\|D^{m+1}u\|_{L^2}\|D^m u\|_{L^6} \le C(1+t)^{-\min\left\{\frac{3}{4}+\frac{1}{2}r^*,\frac{5}{4}\right\}}\|D^{m+1}u\|_{L^2}^2. \tag{3.29}$$

Similarly, we have

$$J_2 + J_3 + J_4 \le C(1+t)^{-\min\left\{\frac{3}{4}+\frac{1}{2}r^*,\frac{5}{4}\right\}}\left(\|D^{m+1}u\|_{L^2}^2 + \|D^{m+1}b\|_{L^2}^2\right). \tag{3.30}$$

We also have

$$\begin{aligned}
J_5 &\le C\|D^{m+1}b\|_{L^2}\|D^m(b\cdot\nabla b)\|_{L^2} \\
&\le C\|b\|_{L^\infty}\|D^{m+1}b\|_{L^2}^2 + C\|\nabla b\|_{L^\infty}\|D^m b\|_{L^2}\|D^{m+1}b\|_{L^2} \\
&\le C\|D^3 b\|_{L^2}^{\frac{1}{2}}\|b\|_{L^2}^{\frac{1}{2}}\|D^{m+1}b\|_{L^2}^2 + C\|D^3 b\|_{L^2}^{\frac{5}{6}}\|b\|_{L^2}^{\frac{1}{6}}\|D^m b\|_{L^2}\|D^{m+1}b\|_{L^2} \\
&\le C(1+t)^{-\min\left\{\frac{3}{4}+\frac{1}{2}r^*,\frac{5}{4}\right\}}\left(\|D^{m+1}b\|_{L^2}^2 + \|D^m b\|_{L^2}\|D^{m+1}b\|_{L^2}\right) \\
&\le \frac{1}{8}\|D^{m+1}b\|_{L^2}^2 + C(1+t)^{-\min\left\{\frac{3}{2}+r^*,\frac{5}{2}\right\}}\|D^m b\|_{L^2}^2, \text{ for large } t.
\end{aligned} \tag{3.31}$$

For J_6, we have the estimate

$$\begin{aligned}
J_6 &\le \left| \int_{\mathbb{R}^3} D^m(b\times(b\times(\nabla\times b)))\cdot D^{m+1}b\mathrm{d}x \right| \\
&\le C(\|D^m b\|_{L^2}\|b\times(\nabla\times b)\|_{L^\infty} + \|D^m(b\times(\nabla\times b)\|_{L^2}\|b\|_{L^\infty})\|D^{m+1}b\|_{L^2} \\
&\le C(\|b\|_{L^\infty}\|\nabla b\|_{L^\infty}\|D^m b\|_{L^2}\|D^{m+1}b\|_{L^2} + \|b\|_{L^\infty}^2\|D^{m+1}b\|_{L^2}^2 \\
&\quad + \|b\|_{L^\infty}\|\nabla b\|_{L^\infty}\|D^m b\|_{L^2}\|D^{m+1}b\|_{L^2}) \\
&\le \frac{1}{8}\|D^{m+1}b\|_{L^2}^2 + C\|b\|_{L^\infty}^2\|\nabla b\|_{L^\infty}^2\|D^m b\|_{L^2}^2 \\
&\le \frac{1}{8}\|D^{m+1}b\|_{L^2}^2 + C(1+t)^{-\min\{3+2r^*,3\}}\|D^m b\|_{L^2}^2.
\end{aligned} \tag{3.32}$$

Combining (3.26)–(3.32) together, we know that there exists a T_1, such that for $t > T_1$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|D^m u\|_{L^2}^2 + \|D^m b\|_{L^2}^2\right) + \|D^{m+1}u\|_{L^2}^2 + \|D^{m+1}b\|_{L^2}^2 \\
&\le C(1+t)^{-\min\left\{\frac{3}{2}+r^*,\frac{5}{2}\right\}}\|D^m b\|_{L^2}^2, \text{ for large } t.
\end{aligned} \tag{3.33}$$

Applying Lemma 3.5 directly, we obtain the conclusion of the theorem for $r^* \ge -\frac{1}{2}$. In addition, the case $m = 1, 2$ can be obtained by Sobolev's embedding theorem. The proof of Theorem 1.3 is completed. □

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