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Multidimensional stability of traveling fronts in combustion and non-KPP monostable equations

Zhen-Hui Bu and Zhi-Cheng Wang

Abstract. This paper is concerned with the multidimensional stability of traveling fronts for the combustion and non-KPP monostable equations. Our study contains two parts: in the first part, we first show that the two-dimensional V-shaped traveling fronts are asymptotically stable in $\mathbb{R}^{n+2}$ with $n \geq 1$ under any (possibly large) initial perturbations that decay at space infinity, and then, we prove that there exists a solution that oscillates permanently between two V-shaped traveling fronts, which implies that even very small perturbations to the V-shaped traveling front can lead to permanent oscillation. In the second part, we establish the multidimensional stability of planar traveling front in $\mathbb{R}^{n+1}$ with $n \geq 1$.

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Keywords. Planar traveling front, V-shaped traveling front, Combustion nonlinearity, Non-KPP monostable nonlinearity, Multidimensional asymptotic stability.

1. Introduction and main results

In this paper, we investigate the Cauchy problem for the following equation

$$\begin{cases} u_t = \Delta u + f(u), & \mathbf{x} \in \mathbb{R}^m,\ t > 0, \\ u(0, \mathbf{x}) = u_0(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^m, \end{cases} \tag{1.1}$$

where $m \in \mathbb{N}$. Here a given initial function u_0 is continuous and bounded. The following is the standing assumption on the nonlinearity f:

(A1) $f(u)$ is of class $C^{1+\vartheta}([-\epsilon, 1+\epsilon], \mathbb{R})$ for some constants $\vartheta \in (0, 1]$ and $\epsilon \in (0, 1)$ and such that

$$f(0) = f(1) = 0,\ f'(0) \geq 0,\ f'(1) < 0,\ f(u) \geq 0 \text{ for } u \in (0, 1).$$

In one-dimensional space, we write $\mathbf{x} = z$. A traveling front of (1.1) is a special translation invariant solution of the form $u(t, z) = \phi(z - bt)$. ϕ is the wave profile that propagates through the one-dimensional spatial domain at a constant velocity b. Of interest are traveling fronts connecting an equilibrium state 0 and an asymptotically stable equilibrium state 1. Set $\mathcal{P} = z - bt$, then the function $\phi(\mathcal{P})$ satisfies

$$\begin{cases} \phi''(\mathcal{P}) + b\phi'(\mathcal{P}) + f(\phi(\mathcal{P})) = 0, & \phi'(\mathcal{P}) < 0, \quad \mathcal{P} \in \mathbb{R}, \\ \phi(+\infty) = 0, \quad \phi(-\infty) = 1. \end{cases} \tag{1.2}$$

The function ϕ is called the planar traveling front of (1.1) on $\mathbb{R}$. The existence, uniqueness and stability of such planar traveling fronts for (1.2) with various types of nonlinearities f in one-dimensional space have been studied by many works. We can refer to [1,2,9,13,28,30,33,42–44,53,60,61] and the references therein for more details. Throughout this paper, we further assume that the following condition holds.
(A2) There exists a traveling front $\phi(\mathcal{P}) \in C^2(\mathbb{R})$ with speed $c_* > 0$ satisfying (1.2) and

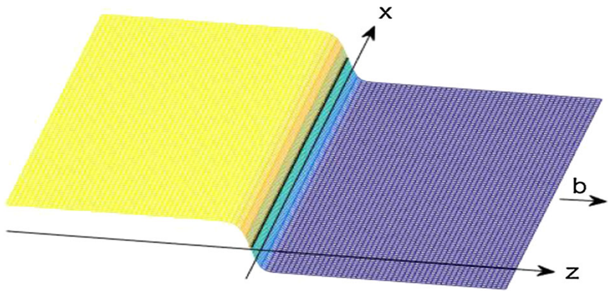

FIG. 1. Planar traveling front $\phi(z-bt)$ on $\mathbb{R}^2$. The black line shows the level line $\phi = 0.5$

$$\lim_{\mathcal{P}\to+\infty} \frac{\phi'(\mathcal{P})}{\phi(\mathcal{P})} = \Lambda < \Lambda_1 \le 0,$$

where Λ and Λ_1 are two real roots of the equation $\mu^2 + c_*\mu + f'(0) = 0$.

Remark 1.1. The assumptions $(A1)$ and $(A2)$ hold in many situations. If the nonlinear term f is of degenerate KPP type ($f'(0) = 0$), Leach et al. [28] proved that there exists a constant $c_* > 0$ such that Eq. (1.2) has solutions (b, ϕ) if and only if $b \ge c_*$. Moreover, the traveling front with minimal speed c_* satisfies

$$\phi(\mathcal{P}) \sim A_1 \mathrm{e}^{-\Lambda\mathcal{P}} \quad \text{as } \mathcal{P} \to +\infty \text{ with } A_1 > 0.$$

For the general non-KPP monostable reactions f with $f'(0) > 0$, Aronson and Weinberger [1] found that there exists a minimal speed $c_* \ge 2\sqrt{f'(0)}$ such that for each $b \ge c_*$, there is a unique traveling front $\phi(\mathcal{P})$ of (1.2). In addition, if the minimal wave speed $c_* > 2\sqrt{f'(0)}$, the traveling front $\phi(\mathcal{P})$ with speed c_* satisfies

$$\phi(\mathcal{P}) \sim A_2 \mathrm{e}^{-\Lambda\mathcal{P}} \quad \text{as } \mathcal{P} \to +\infty \text{ with } A_2 > 0.$$

Thus, in both the cases above, which are called non-KPP monostable type, the assumptions $(A1)$ and $(A2)$ hold and the number c_* is just the minimal wave speed of one-dimensional traveling front ϕ. In addition, the conditions $(A1)$ and $(A2)$ also hold when the nonlinear term f is of combustion type. In this case, the number c_* is the unique wave speed of the traveling front ϕ, see [1].

It is obvious that the function $\phi(z - bt)$ is also a traveling front of Eq. (1.1) with $m \ge 2$ (Fig. 1). In one-dimensional space, an exponential weight in the z-variable (the direction of propagation) allows for the pushing of the essential spectrum for the linearization problem into the open left-half of the complex plane. It is essential that $\Lambda < \Lambda_1$ in this paper. It should be mentioned that this shifting of the essential spectrum into the open left-half of the complex plane is possible only when the absolute spectrum $\left(-\infty, f'(0) - \frac{c_*^2}{4}\right] \bigcup \left(-\infty, f'(1) - \frac{c_*^2}{4}\right]$ does not touch the origin. About the absolute spectrum, see [26, Chapter 3] and [48, Definition 3.5]. In particular, the assumption $\Lambda < \Lambda_1$ rules out the possibility that $c_* = 2\sqrt{f'(0)}$. From the viewpoint of spectrum, linearizing about the front in the z-direction makes that 0 is an eigenvalue in the weighted space. But in the generic case 0 is not an eigenvalue. Regarding the stability of planar traveling fronts in weighted spaces, the readers can see [23,24,60] and the references therein. However, in the multidimensional case, the gap in the spectrum disappears, instead, the continuous spectrum includes zero and cannot be shifted even by introducing spatial weights. Intuitively, this is due to the effects of the transverse diffusion along the planar traveling front [62]. Thus, it is necessary to study the stability of planar traveling fronts in multidimensional spaces.

For Eq. (1.1) with bistable nonlinearity, by decomposing the perturbation into normal and transverse components and using known one-dimensional results, the Fourier transformation and spectral estimates of the heat kernel, Xin [62] showed the multidimensional stability of planar traveling front with perturbations decaying to zero and $m \geq 4$. For the case $m = 2, 3$, Levermore and Xin [29] again studied the problem. Their analysis was based on energy methods and maximum principle arguments. Kapitula [25] extended the results in [62] to the case of $m \geq 2$, using the idea of introducing a drift of perturbations along translate of the front [14]. Notice that, the proofs in [25,29,62] are based on the hypothesis on the spectral gap of the one-dimensional linearized operator. That is, the one-dimensional linearized operator around the traveling front has only a simple eigenvalue 0, with the rest of the spectrum away from the origin. Since the spectral gap appears in a weighted space when $\Lambda < \Lambda_1$, the method of [25,29,62] can apply not only to bistable equation but also to the equation of this paper. Moreover, their results also apply to systems. However, it should be noted that in [25,29,62], the perturbations of the plane traveling front always assumed to be sufficiently small.

Recently, for Allen–Cahn equation, by applying the supersolutions and subsolutions method coupled with the comparison principle, Matano et al. [34] established the asymptotic stability of planar front under any (possibly large) initial perturbations that decay at space infinity and almost periodic perturbations. For Eq. (1.1) with Fisher-KPP nonlinearity, Lv and Wang [31] concerned with the asymptotic stability of planar fronts under general initial perturbations by constructing supersolutions and subsolutions. For more details on the multidimensional stability of planar fronts, we can refer to [32,35,41,63,64] and the references therein.

From the above introduction, all the results discussed in the preceding paragraph about multidimensional stability of planar traveling fronts either assume the initial perturbation is small [25,29,62] or are for reaction–diffusion equations with bistable or KPP-type nonlinearities [31,34]. However, for the equations with combustion and non-KPP monostable nonlinearities, the multidimensional stability of planar traveling fronts by removing the assumption that the perturbation of the planar traveling front is small remains open to the best of our knowledge. It leads to the first question which needs to be solved:

Question 1. *How to establish the multidimensional stability of planar traveling fronts of Eq.* (1.1) *with combustion and non-KPP monostable nonlinearities in* $\mathbb{R}^m (m \geq 2)$*?*

In recent years, nonplanar traveling fronts in the whole space have been investigated. For nonplanar traveling fronts researchers are interested in the shapes of the contour lines or surfaces. Propagating wave fronts evolve to new traveling fronts, such as V-shaped waves in two-dimensional spaces, pyramidal traveling waves in three-dimensional spaces and conical traveling waves in high-dimensional spaces. The mathematical study on these multidimensional traveling fronts will give information for chemists or biochemists to study multidimensional chemical waves or nerve transmission phenomena in future. For the Allen–Cahn equation, the existence and stability of nonplanar traveling fronts have been studied by [15,19,20,27,37,38,49–51]. For the Fisher-KPP equation, Hamel and Nadirashvili [17] and Huang [22] proved the existence and stability of nonplanar traveling fronts. For the equation with combustion nonlinearity, the existence and stability of curved traveling fronts have been established by [3,16,18]. Very recently, by constructing new types subsolution and supersolution, the authors of this paper [6, 7,56] established the existence and stability of two-dimensional V-shaped fronts and three-dimensional pyramidal traveling fronts of Eq. (1.1) with combustion and non-KPP monostable nonlinearities. For more results about multidimensional traveling fronts of reaction–diffusion systems and periodic reaction–diffusion equations, see [5,12,21,36,45,52,54,55,57–59] and the references therein.

Recently, the study on the multidimensional stability of nonplanar traveling fronts in $\mathbb{R}^m$ has attracted attention. For the bistable reaction–diffusion equation, Sheng et al. [46] established the multidimensional stability of V-shaped traveling front to (1.1) in $\mathbb{R}^m$ with $m \geq 3$ under any spatially decaying initial perturbations. Cheng and Yuan [10] further obtained the stability of three-dimensional pyramidal traveling front to (1.1) in $\mathbb{R}^m$ with $m \geq 4$.

It is well known that premixed Bunsen flames are typical examples of nonplanar traveling fronts, see [8,11,47]. In fact, the two-dimensional V-shaped Bunsen flames studied by Bonnet and Hamel [3] can

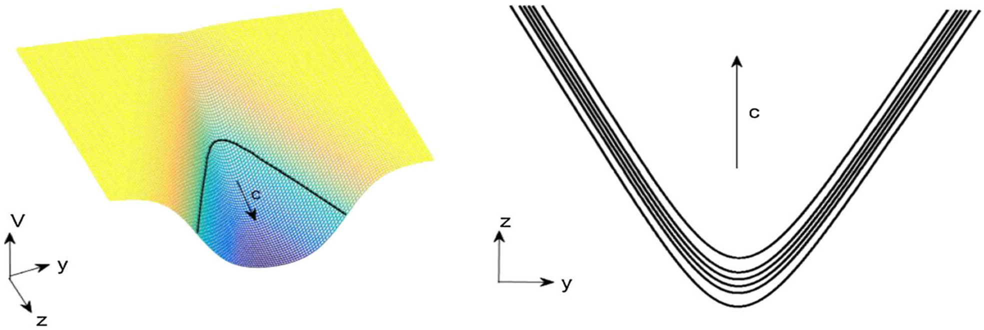

Fig. 2. The profiles (left figure) and the contour lines (right figure) of the V-shaped traveling front. The thick solid curve in the left figure shows the level line $V = 0.5$

be regarded as the stationary states of premixed flames that are invariant by translation in one of the directions orthogonal to the flow. This situation occurs with Bunsen burners that have a thin rectangular cross section, see [3]. This also shows that the two-dimensional V-shaped Bunsen flames occurs in three-dimensional spaces. However, for Eq. (1.1) with combustion and monostable nonlinearities, there are no results devoted to the multidimensional stability of two-dimensional V-shaped traveling fronts in $\mathbb{R}^m$ with $m \geq 3$. This naturally yields the second question:

Question 2. *How to establish the multidimensional stability of V-shaped traveling fronts of Eq.* (1.1) *with the combustion and monostable nonlinearities in* $\mathbb{R}^m (m \geq 3)$*?*

The aim of this paper is to resolve the above two questions. Our method is to use the supersolution and subsolution technique coupled with the comparison principle, which is motivated by [34,46]. However, since the combustion and monostable equations only admit one stable equilibrium, which is different from the bistable type [34,46], we need to construct new types of supersolution and subsolution to overcome this difficulty. It is worth pointing out that the supersolutions and the subsolutions constructed in this paper are more elaborate than those of [34,46] and the results of [34,46] can be improved by using supersolutions and subsolutions similar to those of this paper. In addition, it is worth pointing out that the multidimensional convergence rate we obtain is algebraic but not exponential due to the absence of gap in the spectrum of the linearized operator. This is one of the main differences with [18,42,53,60].

The study of this paper contains two parts. In the first part, we show the multidimensional stability of two-dimensional V-shaped traveling fronts to the Cauchy problem (1.1) in $\mathbb{R}^m$ with $m \geq 3$. The second part is to prove the multidimensional stability of planar traveling fronts of (1.1) in $\mathbb{R}^m$ with $m \geq 2$.

Part 1. *Multidimensional stability of V-shaped traveling fronts in* $\mathbb{R}^{n+2}$ *with* $n \geq 1$. Here, we write $m = n + 2$ with $n \geq 1$ and $\mathbf{x} = (x, y, z)$ with $x \in \mathbb{R}^n$ and $(y, z) \in \mathbb{R}^2$. A V-shaped traveling front (Fig. 2) is referred to $V(y, s) = V(y, z - ct)$ for some positive constant $c > c_*$, where $c_* > 0$ is defined as in (A2). For simplicity, we still denote $V(y, s)$ by $V(y, z)$. We remark here that the profile equation for V is

$$V_{yy} + V_{zz} + cV_z + f(V) = 0. \tag{1.3}$$

Throughout this paper, we always let $\tau_* := \frac{\sqrt{c^2 - c_*^2}}{c_*}$. The existence of such traveling curved front was proved in [56].

Theorem 1.2. [56] *Assume that* (A1) *and* (A2) *hold. For any* $c > c_*$*, there exists* $V(y, z)$ *satisfying* (1.3) *and*

$$\lim_{R\to\infty} \sup_{y^2+z^2>R^2} \frac{\left|V(y,z) - \phi\left(\frac{c_*}{c}\left(z - \tau_*|y|\right)\right)\right|}{\phi^\gamma\left(\frac{c_*}{c}\left(z - \tau_*|y|\right)\right)} = 0, \quad \forall\, \gamma \in \left(\frac{\Lambda_1}{\Lambda}, 1\right). \tag{1.4}$$

Moreover, for any $(y_0, z_0) \in \mathbb{R}^2$ *with* $z_0 \geq \tau_*|y_0|$, *one has*

$$V(y - y_0, z) \leq V(y, z - z_0) \quad \text{for any } (y, z) \in \mathbb{R}^2 \tag{1.5}$$

and for any $M > 0$, *there exists a constant* $\gamma' > 0$ *such that*

$$-V_z(y, z) > 0, \quad (y, z) \in \mathbb{R}^2, \tag{1.6}$$

$$-V_z(y, z) \geq \gamma' \quad if \ -M \leq \frac{c_*}{c}(z - \tau_*|y|) \leq M, \tag{1.7}$$

$$V_y(y, z) > 0, \quad (y, z) \in (0, +\infty) \times \mathbb{R},$$

$$V(y, z) = V(-y, z), \quad (y, z) \in \mathbb{R}^2,$$

$$\lim_{R\to\infty} \sup_{|z-\tau_*|y||\geq R} \frac{|V_z(y, z)|}{\phi^\gamma\left(\frac{c_*}{c}(z - \tau_*|y|)\right)} = 0, \quad \forall\, \gamma \in \left(\frac{\Lambda_1}{\Lambda}, 1\right). \tag{1.8}$$

Without loss of generality, we use the moving coordinate of speed c toward the z direction. We put $\tilde{z} = z - ct$ and $u(t, x, y, z) = v(t, x, y, \tilde{z})$. For simplicity, we still denote $v(t, x, y, \tilde{z})$ by $v(t, x, y, z)$. Then, Eq. (1.1) can be rewritten as

$$\begin{cases} v_t - \Delta v - cv_z - f(v) = 0, \ (x, y, z) \in \ \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, \ t > 0, \\ v(0, x, y, z) = v_0(x, y, z), \ (x, y, z) \in \ \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}. \end{cases} \tag{1.9}$$

We write the solution as $v(t, x, y, z; v_0)$. Our main results of this part are the following theorems.

Theorem 1.3. *Let* (*A*1) *and* (*A*2) *hold. Assume that the initial value* $v_0 \in C\left(\mathbb{R}^{n+2}, [0, 1]\right)$ $(n \geq 1)$ *satisfies*

$$\lim_{R\to+\infty} \sup_{|x|+|y|+|z|\geq R} \frac{|v_0(x, y, z) - V(y, z)|}{\phi^\gamma\left(\frac{c_*}{c}(z - \tau_*|y|)\right)} = 0$$

for some $\gamma \in \left(\frac{\Lambda_1}{\Lambda}, 1\right)$. *Then, the solution* $v(t, x, y, z; v_0)$ *of* (1.9) *satisfies*

$$\lim_{t\to\infty} \sup_{(x,y,z)\in\mathbb{R}^{n+2}} \frac{|v(t, x, y, z; v_0) - V(y, z)|}{\phi^\gamma\left(\frac{c_*}{c}(z - \tau_*|y|)\right)} = 0. \tag{1.10}$$

Theorem 1.3 shows that the V-shaped traveling front is asymptotically stable under the condition that the initial perturbations decay as $|x| + |y| + |z| \to \infty$ in weighted L^∞ spaces. In particular, if the initial perturbation belongs to L^1 in a certain sense, the following theorem gives the convergence rate for (1.10).

Theorem 1.4. *Let* (*A*1) *and* (*A*2) *hold. Assume that the initial value* $v_0(x, y, z)$ *of* (1.9) *is given by*

$$V\left(y, z - w_0^-(x)\right) \leq v_0(x, y, z) \leq V\left(y, z - w_0^+(x)\right) \tag{1.11}$$

for some smooth functions w_0^-, $w_0^+ \in L^1(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$. *Then, for any* $\gamma \in \left(\frac{\Lambda_1}{\Lambda}, 1\right)$, *the solution* $v(t, x, y, z; v_0)$ *to* (1.9) *satisfies*

$$\sup_{(x,y,z)\in\mathbb{R}^{n+2}} \frac{|v(t, x, y, z; v_0) - V(y, z)|}{\phi^\gamma\left(\frac{c_*}{c}(z - \tau_*|y|)\right)} \leq Ct^{-\frac{n}{2}}, \quad t > 0, \tag{1.12}$$

where $C > 0$ *is a constant depending on* γ, f, $\|w_0^-\|_{L^1(\mathbb{R}^n)}$, $\|w_0^-\|_{L^\infty(\mathbb{R}^n)}$, $\|w_0^+\|_{L^1(\mathbb{R}^n)}$ *and* $\|w_0^+\|_{L^\infty(\mathbb{R}^n)}$.

Remark 1.5. The monotonicity of $V(y, z)$ implies that $w_0^-(x) \leq w_0^+(x)$ for all $x \in \mathbb{R}$. The following are special cases to which Theorem 1.4 applies:

(i) $v_0(x, y, z) = V(y, z - w_0(x))$ for some smooth function $w_0 \in L^1(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$.
(ii) $v_0(x, y, z) = V(y - w_1(x), z - w_0(x))$ for some smooth functions w_0, $w_1 \in L^1(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$.

In fact, if the function $w_0(x)$ satisfies $w_0^-(x) \le w_0(x) \le w_0^+(x)$ for all $x \in \mathbb{R}^n$ in Case (i), then the assumption (1.11) holds. In Case (ii), it follows from (1.5) that we have

$$V\left(y, z - w_0(x) + \tau_* \left|w_1(x)\right|\right) \le V\left(y - w_1(x), z - w_0(x)\right) \le V\left(y, z - w_0(x) - \tau_* \left|w_1(x)\right|\right)$$

for all $(x, y, z) \in \mathbb{R}^{n+2}$. Thus, Theorem 1.4 covers Case (ii) with $w_0^-(x) = w_0(x) - \tau_* \left|w_1(x)\right|$ and $w_0^+(x) = w_0(x) + \tau_* \left|w_1(x)\right|$.

In contrast to that in [46] which the convergence rate was obtained only for the special initial data (i), in this paper we establish the convergence rate for more general initial data.

The following theorem proves that the convergence rate (1.12) is optimal in some sense if we further assume the initial perturbations keep the sign.

Theorem 1.6. *Let v_0 be as in (1.11) and assume that either $w_0^- \ge 0$, $w_0^- \not\equiv 0$ or $w_0^+ \le 0$, $w_0^+ \not\equiv 0$. Then, for any $\gamma \in \left(\frac{\Lambda_1}{\Lambda}, 1\right)$, there exist constants $C_1 > 0$ and $C_2 > 0$ such that*

$$C_1(1+t)^{-\frac{n}{2}} \le \sup_{(x,y,z)\in\mathbb{R}^{n+2}} \frac{|v(t,x,y,z;v_0) - V(y,z)|}{\phi^\gamma\left(\frac{c_*}{c}\left(z - \tau_*|y|\right)\right)} \le C_2 t^{-\frac{n}{2}}, \quad t \ge 0.$$

The next result we will show is the existence of a solution of (1.1) which oscillates permanently between two V-shaped traveling fronts.

Theorem 1.7. *Let $n = 1$ and (A1) − −(A2) hold. Then, for any $\gamma \in \left(\frac{\Lambda_1}{\Lambda}, 1\right)$ and $\delta > 0$, there exists a bounded function $\bar{w}_0(x)$ on $\mathbb{R}$ with $\|\bar{w}_0\|_{L^\infty(\mathbb{R})} = \delta$ such that the solution $u(t, x, y, z)$ to (1.1) with $u(0, x, y, z) = V\left(y, z - \bar{w}_0(x)\right)$ satisfies*

$$\lim_{m\to\infty} \sup_{|x|\le m!-1,(y,z)\in\mathbb{R}^2} \frac{|u(t_m,x,y,z) - V(y, z - ct_m + (-1)^m\delta)|}{\phi^\gamma\left(\frac{c_*}{c}\left(z - ct_m - \tau_*|y|\right)\right)} = 0,$$

where $t_m = \frac{m(m!)^2}{4}$.

Remark 1.8. The above result implies that even very small perturbations to the V-shaped traveling front $V(y, z)$ can lead to permanent oscillation. Thus, the V-shaped traveling front is not asymptotically stable under general bounded perturbations. From the viewpoint of dynamical systems, Theorem 1.7 implies that the ω-limit set of the solution u in the weighted $L^\infty_{\text{loc}}(\mathbb{R}^3)$-topology contains at least two distinct points. And each of them is a translation of the V-shaped traveling front.

Part 2. *Multidimensional stability of planar traveling fronts in $\mathbb{R}^{n+1}$ with $n \ge 1$.* In this part, we write $m = n + 1$ with $n \ge 1$ and $\mathbf{x} = (x, z)$ with $x \in \mathbb{R}^n$ and $z \in \mathbb{R}$. Without loss of generality, we use the moving coordinate of speed c_* toward the z direction. Let $u(t, x, z) = v(t, x, z - c_*t)$ and $\hat{z} = z - c_*t$. For simplicity, we still denote $v(t, x, \hat{z})$ by $v(t, x, z)$. Then, we have

$$\begin{cases} v_t - \Delta v - c_* v_z - f(v) = 0, & (x,z) \in \mathbb{R}^n \times \mathbb{R},\ t > 0, \\ v(0,x,z) = v_0(x,z), & (x,z) \in \mathbb{R}^n \times \mathbb{R}. \end{cases} \tag{1.13}$$

We write the solution as $v(t, x, z; v_0)$. The following theorems are the main results of this part. Theorems 1.9–1.11 show that planar traveling fronts are asymptotically stable under any-possible large-initial perturbations that decay at space infinity. Theorem 1.12 proves that there exists a solution that oscillates permanently between two planar traveling fronts, which implies that planar traveling fronts are not asymptotically stable under more general perturbations.

Theorem 1.9. *Let (A1) and (A2) hold. Assume that the initial value $v_0 \in C\left(\mathbb{R}^{n+1}, [0,1]\right)$ $(n \ge 1)$ satisfies*

$$\lim_{R\to+\infty} \sup_{|x|+|z|\ge R} \frac{|v_0(x,z) - \phi(z)|}{\phi^\gamma(z)} = 0$$

for some $\gamma \in \left(\frac{\Lambda_1}{\Lambda}, 1\right)$. *Then, the solution* $v(t,x,z;v_0)$ *of* (1.13) *satisfies*

$$\lim_{t\to\infty} \sup_{(x,z)\in\mathbb{R}^{n+1}} \frac{|v(t,x,z;v_0)-\phi(z)|}{\phi^\gamma(z)} = 0.$$

Theorem 1.10. *Let* (A1) *and* (A2) *hold. Assume that the initial value* $v_0(x,z)$ *of* (1.13) *is given by*

$$\phi\left(z - w_0^{*,-}(x)\right) \le v_0(x,z) \le \phi\left(z - w_0^{*,+}(x)\right) \tag{1.14}$$

for some smooth functions $w_0^{*,-}$, $w_0^{*,+} \in L^1(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$. *Then, for any* $\gamma \in \left(\frac{\Lambda_1}{\Lambda}, 1\right)$, *the solution* $v(t,x,z;v_0)$ *to* (1.13) *satisfies*

$$\sup_{(x,z)\in\mathbb{R}^{n+1}} \frac{|v(t,x,z;v_0)-\phi(z)|}{\phi^\gamma(z)} \le C^* t^{-\frac{n}{2}}, \quad t>0,$$

where $C^* > 0$ *is a constant depending on* γ, f, $\|w_0^{*,-}\|_{L^1(\mathbb{R}^n)}$, $\|w_0^{*,-}\|_{L^\infty(\mathbb{R}^n)}$, $\|w_0^{*,+}\|_{L^1(\mathbb{R}^n)}$ *and* $\|w_0^{*,+}\|_{L^\infty(\mathbb{R}^n)}$.

Theorem 1.11. *Let* v_0 *be as in* (1.14) *and assume that either* $w_0^* \ge 0$, $w_0^* \not\equiv 0$ *or* $w_0^* \le 0$, $w_0^* \not\equiv 0$. *Then, for any* $\gamma \in \left(\frac{\Lambda_1}{\Lambda}, 1\right)$, *there exist constants* $C_1^* > 0$ *and* $C_2^* > 0$ *such that*

$$C_1^*(1+t)^{-\frac{n}{2}} \le \sup_{(x,z)\in\mathbb{R}^{n+1}} \frac{|v(t,x,z;v_0)-\phi(z)|}{\phi^\gamma(z)} \le C_2^* t^{-\frac{n}{2}}, \quad t \ge 0.$$

Theorem 1.12. *Let* $n=1$ *and* $(A1)--(A2)$ *hold. Then, for any* $\gamma \in \left(\frac{\Lambda_1}{\Lambda}, 1\right)$ *and* $\delta > 0$, *there exists a bounded function* $\bar{w}_0^*(x)$ *on* $\mathbb{R}$ *with* $\|\bar{w}_0^*\|_{L^\infty(\mathbb{R})} = \delta$ *such that the solution* $u(t,x,z)$ *to* (1.1) *with* $u(0,x,z) = \phi\left(z - \bar{w}_0^*(x)\right)$ *satisfies*

$$\lim_{m\to\infty} \sup_{|x|\le m!-1, z\in\mathbb{R}} \frac{|u(t_m,x,z)-\phi(z-c_*t_m+(-1)^m\delta)|}{\phi^\gamma(z-c_*t_m)} = 0,$$

where $t_m = \frac{m(m!)^2}{4}$.

Obviously, the above results give (at least partially) positive answers to Questions 1 and 2. Another important question is to establish the multidimensional stability of nonplanar traveling fronts of (1.1) with Fisher-KPP nonlinearity in $\mathbb{R}^{n+2}$ with $n \ge 1$, which will be considered in the future. This paper is organized as follows. In Sect. 2, we summarize some preliminaries including some known results of the curvature flow problem. In Sect. 3, by constructing new types of supersolution and subsolution, we give the proof of Theorems 1.3, 1.4, 1.6 and 1.7. In Sect. 4, we prove Theorems 1.9–1.12. For the sake of convenience, in this paper we always denote

$$\lambda_1 := \sup_{\mathcal{P}\in\mathbb{R}} \left|\frac{\phi'(\mathcal{P})}{\phi(\mathcal{P})}\right|, \quad \lambda_2 := \sup_{\mathcal{P}\in\mathbb{R}} \left|\frac{\phi''(\mathcal{P})}{\phi(\mathcal{P})}\right|, \quad \lambda_3 := \sup_{u\in[-\epsilon,1+\epsilon]} |f'(u)|$$

and fix $\epsilon_1 \in \left(0, \frac{\epsilon}{2}\right)$ such that

$$\frac{3}{2}f'(1) < f'(u) < \frac{1}{2}f'(1), \quad \forall\, u \in (1-2\epsilon_1, 1+2\epsilon_1).$$

2. Preliminaries

In this section we give some known results about the curvature flow problem. For more details, see [34]. The mean curvature flow for a graphical surface $w(t,x)$ on $\mathbb{R}^n$ is given by the following Cauchy problem:

$$\begin{cases} \frac{w_t}{\sqrt{1+|\nabla w|^2}} = \operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right), & x\in\mathbb{R}^n,\ t>0, \\ w(0,x) = w_0(x), & x\in\mathbb{R}^n. \end{cases} \tag{2.1}$$

If the first and second derivatives of w with respect to x are bounded on $\mathbb{R}^n$, then taking some large constant $k > 0$, we have

$$\begin{aligned} 0 &= w_t - \sqrt{1+|\nabla w|^2}\cdot \operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\right) \\ &= w_t - \Delta w + \sum_{i,j=1}^{n} \frac{w_{y_i} w_{y_j} w_{y_i y_j}}{1+|\nabla w|^2} \\ &\geq w_t - \Delta w - k|\nabla w|^2. \end{aligned}$$

That is, $w(t,x)$ is a subsolution of the problem of the form

$$\begin{cases} w_t^+ = \Delta w^+ + k|\nabla w^+|^2, & x\in\mathbb{R}^n,\ t>0, \\ w^+(0,x) = w_0(x), & x\in\mathbb{R}^n. \end{cases}$$

Taking the Cole–Hopf transformation $U^+(t,x) = \exp\left(kw^+(t,x)\right)$, then $U^+(t,x)$ satisfies the following Cauchy problem for the heat equation:

$$\begin{cases} U_t^+ = \Delta U^+, & x\in\mathbb{R}^n,\ t>0, \\ U^+(0,x) = \exp\left(kw_0(x)\right), & x\in\mathbb{R}^n. \end{cases}$$

Thus, an explicit expression for $w^+(t,x)$ is given by

$$w^+(t,x) = \frac{1}{k}\log\left(\int_{\mathbb{R}^n} \Gamma(t,x-\eta)\exp\left(kw_0(\eta)\right)\mathrm{d}\eta\right), \tag{2.2}$$

where

$$\Gamma(t,\eta) = \frac{1}{(4\pi t)^{\frac{n}{2}}}\exp\left(-\frac{|\eta|^2}{4t}\right).$$

Consequently, the expression (2.2) gives an upper estimate for the solution $w(t,x)$ of (2.1). The lower estimate for $w(t,x)$ can be given in a similar way by considering the equation

$$\begin{cases} w_t^- = \Delta w^- - k|\nabla w^-|^2, & x\in\mathbb{R}^n,\ t>0, \\ w^-(0,x) = w_0(x), & x\in\mathbb{R}^n. \end{cases}$$

Now we introduce a lemma which gives the large time behavior of solutions to

$$w_t^\pm = \Delta w^\pm \pm k|\nabla w^\pm|^2, \quad x\in\mathbb{R}^n,\ t>0.$$

Lemma 2.1. [34, Lemma 2.4] *Let $k>0$ be any constant and $w^\pm(t,x)$ be solutions to the following Cauchy problems:*

$$\begin{cases} w_t^\pm = \Delta w^\pm \pm k|\nabla w^\pm|^2, & x\in\mathbb{R}^n,\ t>0, \\ w^\pm(0,x) = w_0(x), & x\in\mathbb{R}^n. \end{cases}$$

If the initial value $w_0(x)$ is bounded and continuous on $\mathbb{R}^n$ and satisfies $\lim_{|x|\to\infty}|w_0(x)| = 0$, then the solutions $w^\pm(t,x)$ satisfy

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}^n}\left|w^\pm(t,x)\right| = 0, \tag{2.3}$$

respectively. Moreover, if we further assume that $w_0\in L^1(\mathbb{R}^n)$, then we have

$$\sup_{x\in\mathbb{R}^n}\left|w^\pm(t,x)\right| \leq \frac{1}{k}\left\|\exp(kw_0)-1\right\|_{L^1(\mathbb{R}^n)}\cdot t^{-\frac{n}{2}}, \quad t>0.$$

3. Multidimensional stability of V-shaped traveling fronts in $\mathbb{R}^{n+2}$ with $n \geq 1$

In this section, we establish the multidimensional stability of V-shaped traveling fronts of Eq. (1.1) in $\mathbb{R}^{n+2}$ with $n \geq 1$. That is, we give the proofs of Theorems 1.4, 1.6 and 1.7 in Sect. 3.1 and Theorem 1.3 in Sect. 3.2. Before this, we give some preliminaries.

Consider

$$W_t = \frac{W_{yy}}{1+W_y^2} + c_*\sqrt{1+W_y^2}, \quad y \in \mathbb{R},\ t > 0. \tag{3.1}$$

For any $c > c_*$, there exists a unique solution $\varphi(y; c)$ of (3.1) with asymptotic lines $z = \tau_*|y|$ satisfying

$$c = \frac{\varphi_{yy}}{1+\varphi_y^2} + c_*\sqrt{1+\varphi_y^2}, \quad y \in \mathbb{R}.$$

Lemma 3.1. Brazhnik [4], Ninomiya and Taniguchi [38–40] *There exist positive constants* $\Theta, k_i (i = 1, 2, 3)$ *and* $\omega_\pm$ *such that*

$$\begin{aligned} \max\{|\varphi''(y)|,\ |\varphi'''(y)|\} &\leq k_1 \mathrm{sech}(\Theta y), \\ k_2 \mathrm{sech}(\Theta y) \leq \frac{c}{\sqrt{1+\varphi'(y)^2}} - c_* &\leq k_3 \mathrm{sech}(\Theta y), \\ \tau_*|y| &\leq \varphi(y), \\ \omega_- &\leq \tilde{\omega}(y) \leq \omega_+ \end{aligned}$$

for any $y \in \mathbb{R}$, *where*

$$\tilde{\omega}(y) = \frac{c(\varphi(y) - \tau_*|y|)}{c - c_*\sqrt{1+\varphi'(y)^2}}.$$

By Lemma 3.1, it is easy to obtain that there exists a constant $a > 0$ such that

$$\tau_*|y| \leq \varphi(y) \leq \tau_*|y| + a \quad \text{for all } y \in \mathbb{R}.$$

The following lemma plays an important role in constructing supersolutions and subsolutions.

Lemma 3.2. [46, Lemma 3.2] *Let* $V(y, z)$ *be defined by Theorem 1.2. Then, there exists a constant* $k_* > 0$ *which depends on* f *such that*

$$k_* V_z(y, z) \leq V_{zz}(y, z) \leq -k_* V_z(y, z), \quad (y, z) \in \mathbb{R}^2. \tag{3.2}$$

By the assumptions $(A1)$ and $(A2)$, similar to the proof of Lemma 2.2 in [34], we can obtain the following auxiliary lemma.

Lemma 3.3. *Let* $(A1)$ *and* $(A2)$ *hold. There exists a constant* $k^* > 0$ *depending only on* f *such that the following inequalities*

$$k^*\phi'(z) \leq \phi''(z) \leq -k^*\phi'(z), \quad z \in \mathbb{R}$$

hold true.

Let $k = \max\{k_*, k^*\}$. Then, one has

$$\begin{cases} kV_z(y, z) \leq V_{zz}(y, z) \leq -kV_z(y, z), & (y, z) \in \mathbb{R}^2, \\ k\phi'(z) \leq \phi''(z) \leq -k\phi'(z), & z \in \mathbb{R}. \end{cases} \tag{3.3}$$

3.1. Proofs of Theorems 1.4, 1.6 and 1.7

In this subsection, we prove Theorems 1.4, 1.6 and 1.7. Our arguments are similar to the earlier works [34,46]. However, since we are treating reaction–diffusion equations with combustion and non-KPP monostable nonlinearities, many modifications and techniques are needed. In the following, we firstly show that the functions $V\left(y, z-w^{\pm}(t,x)\right)$ are a supersolution and a subsolution to (1.9), respectively. In the sequel, Δ_x and ∇_x denote the n-dimensional Laplacian and the n-dimensional gradient operator, respectively.

Lemma 3.4. *Suppose that the functions $w^+(t,x)$ and $w^-(t,x)$ are solutions to the following problem:*

$$w_t^+ = \Delta_x w^+ + k_*|\nabla_x w^+|^2, \quad x\in\mathbb{R}^n,\ t>0, \tag{3.4}$$

$$w_t^- = \Delta_x w^- - k_*|\nabla_x w^-|^2, \quad x\in\mathbb{R}^n,\ t>0 \tag{3.5}$$

with initial values $w^+(0,x)$ and $w^-(0,x)$, respectively, where $k_*>0$ is the constant defined in Lemma 3.2. Let $v(t,x,y,z;v_0)$ be the solution to (1.9) with the initial value $v_0(x,y,z)$ satisfying

$$V\left(y, z-w^-(0,x)\right) \le v_0(x,y,z) \le V\left(y, z-w^+(0,x)\right), \quad (x,y,z)\in\mathbb{R}^{n+2}.$$

Then, we have

$$V\left(y, z-w^-(t,x)\right) \le v(t,x,y,z;v_0) \le V\left(y, z-w^+(t,x)\right), \quad (x,y,z)\in\mathbb{R}^{n+2},\ t\ge 0. \tag{3.6}$$

Proof. We only show that the later inequality of (3.6) holds, since the former can be proven in a similar way. Let $v^+(t,x,y,z) := V\left(y, z-w^+(t,x)\right)$. Now we prove that the function $v^+(t,x,y,z)$ is a supersolution of (1.9), that is, it satisfies

$$\mathcal{L}[v^+] := v_t^+ - \Delta v^+ - cv_z^+ - f\left(v^+\right) \ge 0. \tag{3.7}$$

It follows from $V_{yy}+V_{zz}+cV_z+f(V)=0$, Lemma 3.2 and (3.4) that

$$\begin{aligned}
\mathcal{L}[v^+] &= -w_t^+ V_z - \sum_{i=1}^n \left(-w^+_{x_ix_i}V_z + \left(w^+_{x_i}\right)^2 V_{zz}\right) - V_{yy} - V_{zz} - cV_z - f(V)\\
&= -w_t^+V_z + \Delta_x w^+ V_z - \left|\nabla_x w^+\right|^2 V_{zz}\\
&= \left|\nabla_x w^+\right|^2 \left(-k_*V_z - V_{zz}\right) \ge 0.
\end{aligned}$$

Then, by the comparison principle and the assumption of initial values, we complete the proof. □

Proof of Theorem 1.4. For simplicity, we denote $v(t,x,y,z;v_0)$ by $v(t,x,y,z)$. Define the functions $w^-(t,x)$ and $w^+(t,x)$ as in Lemma 3.4 with initial values $w_0^-(x)$ and $w_0^+(x)$, respectively. Lemma 3.4 immediately implies that

$$V\left(y, z-w^+(t,x)\right) \ge v(t,x,y,z) \ge V\left(y, z-w^-(t,x)\right)$$

for any $(t,x,y,z)\in[0,+\infty)\times\mathbb{R}^{n+2}$. For any fixed $\gamma\in\left(\frac{\Lambda_1}{\Lambda},1\right)$, we have

$$\frac{v(t,x,y,z)-V(y,z)}{\phi^\gamma\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)} \ge \frac{V\left(y,z-w^-(t,x)\right)-V(y,z)}{\phi^\gamma\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)} = \frac{-V_z\left(y,z-\tau w^-(t,x)\right)}{\phi^\gamma\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)}w^-(t,x),$$

where $\tau\in(0,1)$. Since the function $w_0^-\in L^1(\mathbb{R}^n)\bigcap L^\infty(\mathbb{R}^n)$, then (2.2) yields that $w^-(t,x)$ is bounded uniformly in $(t,x)\in[0,+\infty)\times\mathbb{R}^n$. It follows from (1.8) that there exists a constant $A>0$ depending on γ such that

$$\begin{aligned}
&\frac{-V_z\left(y,z-\tau w^-(t,x)\right)}{\phi^\gamma\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)}\\
=&\frac{-V_z\left(y,z-\tau w^-(t,x)\right)}{\phi^\gamma\left(\frac{c_*}{c}\left(z-\tau w^-(t,x)-\tau_*|y|\right)\right)}\frac{\phi^\gamma\left(\frac{c_*}{c}\left(z-\tau w^-(t,x)-\tau_*|y|\right)\right)}{\phi^\gamma\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)}
\end{aligned}$$

$$\leq \sup_{(x,y,z)\in\mathbb{R}^{n+2}} \left| \frac{-V_z\left(y,z\right)}{\phi^{\gamma}\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)} \right| \sup_{(t,x,y,z)\in[0,+\infty)\times\mathbb{R}^{n+2}} \left| \frac{\phi^{\gamma}\left(\frac{c_*}{c}\left(z-\tau w^-(t,x)-\tau_*|y|\right)\right)}{\phi^{\gamma}\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)} \right|$$
$$\leq A.$$

By Lemma 2.1, we have that there exists a constant $C^- > 0$ depending on γ, f, $\|w_0^-\|_{L^1(\mathbb{R}^n)}$ and $\|w_0^-\|_{L^\infty(\mathbb{R}^n)}$ such that

$$\frac{v(t,x,y,z)-V(y,z)}{\phi^{\gamma}\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)} \geq -A\sup_{x\in\mathbb{R}^n}\left|w^-(t,x)\right| \geq -C^- t^{-\frac{n}{2}}.$$

Similarly, it follows from Lemma 2.1 that there exists a constant $C^+ > 0$ depending on γ, f, $\|w_0^+\|_{L^1(\mathbb{R}^n)}$ and $\|w_0^+\|_{L^\infty(\mathbb{R}^n)}$ such that

$$\frac{v(t,x,y,z)-V(y,z)}{\phi^{\gamma}\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)} \leq C^+ t^{-\frac{n}{2}}.$$

Let $C := \max\{C^-, C^+\}$. Combining these two inequalities, we complete the proof of Theorem 1.4. □

Proof of Theorem 1.6. We only consider the case where $w_0^- \geq 0$, $w_0^- \not\equiv 0$, since the other case can be discussed in a similar way. By Theorem 1.4 and (3.6), it needs only to show that the solution $w(t,x)$ to the problem

$$\begin{cases} w_t = \Delta w - k_*\left|\nabla_x w\right|^2, & x\in\mathbb{R}^n,\ t>0, \\ w(0,x) = w_0^-(x), & x\in\mathbb{R}^n \end{cases} \tag{3.8}$$

satisfies $w(t,0) \geq C_1(1+t)^{-\frac{n}{2}}$ for some constant $C_1 > 0$. Indeed, the first inequality of (3.6) yields that

$$\begin{aligned}\frac{v(t,0,0,0;v_0)-V(0,0)}{\phi^{\gamma}(0)} \geq & \frac{V(0,-w(t,0))-V(0,0)}{\phi^{\gamma}(0)} \\ \geq & \min_{z\in\left[-\|w\|_{L^\infty(\mathbb{R}^n)},0\right]}\left|\frac{V_z(0,z)}{\phi^{\gamma}(0)}\right| \cdot w(t,0) \\ \geq & C_1(1+t)^{-\frac{n}{2}}, \quad t\geq 0.\end{aligned}$$

Now we show $w(t,0) \geq C_1(1+t)^{-\frac{n}{2}}$ for some constant $C_1 > 0$. Similar to (2.2), we can obtain that the explicit expression of the solution to (3.8) is given by

$$w(t,x) = -\frac{1}{k_*}\ln\left(\int_{\mathbb{R}^n}\Gamma(t,x-\eta)\exp\left(-k_* w_0^-(\eta)\right)\mathrm{d}\eta\right).$$

Since $w_0^- \geq 0$ and $w_0^- \not\equiv 0$, there exist a constant $\delta > 0$ and a nonempty open set $D\subset\mathbb{R}^n$ such that $w_0^- \geq \delta$ for $x \in D$. Thus, we have

$$\begin{aligned} w(t,x) \geq & -\frac{1}{k_*}\ln\left(1-\int_D \Gamma(t,x-\eta)\left(1-\exp\left(-k_*\delta\right)\right)\mathrm{d}\eta\right) \\ \geq & -\frac{1}{k_*}\ln\left(1-|D|\left(1-\exp\left(-k_*\delta\right)\right)\min_{\eta\in D}\Gamma(t,x-\eta)\right) \\ \geq & \frac{|D|}{k_*}\left(1-\exp\left(-k_*\delta\right)\right)\min_{\eta\in D}\Gamma(t,x-\eta), \end{aligned}$$

which implies $w(t,0) \geq C_1(1+t)^{-\frac{n}{2}}$. We complete the proof of Theorem 1.6. □

Now we prove Theorem 1.7. By combining Lemma 3.4 and the next lemma, we construct a sequence of supersolutions and a sequence of subsolutions that push the solution back and forth in the $z-$direction, thus forcing the solution to oscillate permanently with non-decaying amplitude.

Lemma 3.5. [34, Lemmas 3.1 and 3.2] *Let $k_* > 0$ be defined as in Lemma 3.2 and $w^{\pm}(t,x)$ be solutions to the Cauchy problem*

$$\begin{cases} w_t^{\pm} = w_{xx}^{\pm} \pm k_* \left| w_x^{\pm} \right|^2, & x \in \mathbb{R},\ t > 0, \\ w^{\pm}(0,x) = w_0^{\pm}(x), & x \in \mathbb{R}, \end{cases}$$

respectively. Suppose that initial values $w_0^{\pm}(x)$ are all bounded functions on $\mathbb{R}$ and satisfy

$$\begin{aligned} &w_0^+(x) \le \delta, \quad x \in \mathbb{R}, \\ &w_0^+(x) \le -\delta, \quad |x| \in [B!+1, (B+1)!-1] \end{aligned}$$

and

$$\begin{aligned} &w_0^-(x) \ge -\delta, \quad x \in \mathbb{R}, \\ &w_0^-(x) \ge \delta, \quad |x| \in [B!+1, (B+1)!-1] \end{aligned}$$

for some constant $\delta > 0$ and some integer $B \ge 2$, respectively. Then, there exists a constant $C > 0$ depending only on δ and k_* such that

$$\sup_{|x| \le B!-1} w^+(T,x) \le -\delta + C \int\limits_{|\zeta| \in \left[0, 2/\sqrt{B}\right] \cup \left[\sqrt{B}, \infty\right)} \mathrm{e}^{-\zeta^2} \mathrm{d}\zeta$$

and

$$\sup_{|x| \le B!-1} w^-(T,x) \ge \delta - C \int\limits_{|\zeta| \in \left[0, 2/\sqrt{B}\right] \cup \left[\sqrt{B}, \infty\right)} \mathrm{e}^{-\zeta^2} \mathrm{d}\zeta,$$

respectively, where $T = \frac{B(B!)^2}{4}$.

Proof of Theorem 1.7. Let

$$I_B := [B!+1, (B+1)!-1], \quad \tilde{I}_B := [0, B!] \cup [(B+1)!, \infty).$$

Define two sequences of smooth functions $\left\{ w_{0,i}^{\pm}(x) \right\}_{i=1,2,\ldots}$ satisfying

$$\left| w_{0,i}^+(x) \right| \le \delta, \quad x \in \mathbb{R} \text{ and } w_{0,i}^+(x) = \begin{cases} -\delta, & |x| \in I_{2i}, \\ \delta, & |x| \in \tilde{I}_{2i} \end{cases}$$

and

$$\left| w_{0,i}^-(x) \right| \le \delta, \quad x \in \mathbb{R} \text{ and } w_{0,i}^-(x) = \begin{cases} \delta, & |x| \in I_{2i+1}, \\ -\delta, & |x| \in \tilde{I}_{2i+1}, \end{cases}$$

respectively. Choose a function $\bar{w}_0 \in C^{\infty}(\mathbb{R})$ satisfying

$$w_{0,i}^-(x) \le \bar{w}_0(x) \le w_{0,i}^+(x) \quad \text{for all } i \ge 1.$$

Let $v^*(t,x,y,z)$ be the solution to (1.9) with $v^*(0,x,y,z) = V(y, z - \bar{w}_0(x))$ and $w_i^+(t,x)$ be the solution to the following problem

$$\begin{cases} w_{i,t}^+ = w_{i,xx}^+ + k_* |w_{i,x}^+|^2, & x \in \mathbb{R},\ t > 0, \\ w_i^+(0,x) = w_{0,i}^+(x), & x \in \mathbb{R}. \end{cases}$$

It follows from the definition of $\bar{w}_0(x)$ and Lemma 3.4 that we have

$$V(y,z+\delta) \le v^*(t,x,y,z) \le V(y,z-w_i^+(t,x)) \le V(y,z-\delta).$$

Thus, Lemma 3.5 yields that there exists a constant $C>0$ depending only on k_* and δ such that

$$\begin{aligned}
&\sup_{|x|\le (2i)!-1} \frac{v^*(t_{2i},x,y,z)-V(y,z+\delta)}{\phi^\gamma\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)} \\
&\le \sup_{|x|\le (2i)!-1} \frac{V(y,z-w_i^+(t_{2i},x))-V(y,z+\delta)}{\phi^\gamma\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)} \\
&\le \sup_{(y,z)\in\mathbb{R}^2} \frac{\left|V_z\left(y,z+\delta-\tau C\int_{|\zeta|\in[0,2/\sqrt{2i}]\cup[\sqrt{2i},\infty)} \mathrm{e}^{-\zeta^2}\mathrm{d}\zeta\right)\right|}{\phi^\gamma\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)}\cdot C\int_{|\zeta|\in[0,2/\sqrt{2i}]\cup[\sqrt{2i},\infty)} \mathrm{e}^{-\zeta^2}\mathrm{d}\zeta,
\end{aligned}$$

where $\tau\in(0,1)$ and $t_{2i}=\frac{(2i)((2i)!)^2}{4}$. Together with (1.8), the above inequality implies that

$$\lim_{i\to\infty}\sup_{(y,z)\in\mathbb{R}^2}\sup_{|x|\le(2i)!-1}\frac{|v^*(t_{2i},x,y,z)-V(y,z+\delta)|}{\phi^\gamma\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)}=0. \tag{3.9}$$

Similarly, by using Lemma 3.5 and the inequalities $\bar{w}_0(x)\ge w_{0,i}^-(x)$ for $i=1,2,3,\ldots$, we obtain

$$\begin{aligned}
&\sup_{|x|\le (2i+1)!-1} \frac{v^*(t_{2i+1},x,y,z)-V(y,z-\delta)}{\phi^\gamma\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)} \\
&\ge -\sup_{(y,z)\in\mathbb{R}^2} \frac{\left|V_z\left(y,z-\delta+\tau C\int_{|\zeta|\in[0,2/\sqrt{2i}]\cup[\sqrt{2i},\infty)} \mathrm{e}^{-\zeta^2}\mathrm{d}\zeta\right)\right|}{\phi^\gamma\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)} \\
&\quad\times C\int_{|\zeta|\in[0,2/\sqrt{2i+1}]\cup[\sqrt{2i+1},\infty)} \mathrm{e}^{-\zeta^2}\mathrm{d}\zeta,
\end{aligned}$$

where $\tau\in(0,1)$ and $t_{2i+1}=\frac{(2i+1)((2i+1)!)^2}{4}$. Together with (1.8), the above inequality implies that

$$\lim_{i\to\infty}\sup_{(y,z)\in\mathbb{R}^2}\sup_{|x|\le(2i+1)!-1}\frac{|v^*(t_{2i+1},x,y,z)-V(y,z-\delta)|}{\phi^\gamma\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)}=0. \tag{3.10}$$

Combining (3.9) and (3.10), we obtain the desired result. We complete the proof of Theorem 1.7. □

3.2. Proof of Theorem 1.3

In this subsection, by constructing new types of supersolution and subsolution coupled with the comparison principle, we give the proof of Theorem 1.3.

Let $h(\mu):=\mu^2+c_*\mu+f'(0)$. It is obvious that for any $\gamma\in\left(\frac{\Lambda_1}{\Lambda},1\right)$, we have $h(\gamma\Lambda)<0$. Let $\lambda:=\min\left\{-\frac{1}{16}h(\gamma\Lambda),-\frac{f'(1)}{8}\right\}$.

Lemma 3.6. *Let $k>0$ be defined as in (3.3). Then, for $\gamma\in\left(\frac{\Lambda_1}{\Lambda},1\right)$ and $\nu\in(0,1)$ with $\nu<-\frac{h(\gamma\Lambda)c}{16\lambda_1k_1c_*}$, there exist some constants $\delta_0>0$ and $\sigma_0>0$ such that, for any $\delta\in(0,\delta_0]$ and $\sigma\ge\sigma_0$, and any functions $w^+(t,x)$ satisfying*

$$w_t^+=\Delta_x w^+ + k\left|\nabla_x w^+\right|^2,\quad x\in\mathbb{R}^n,\ t>0,$$

the function defined by

$$v^+(t,x,y,z) := V\left(y, z - w^+(t,x) - \sigma\delta\left(1 - \mathrm{e}^{-\lambda t}\right)\right) \\ + \delta \mathrm{e}^{-\lambda t}\phi^\gamma\left(\frac{c_*}{c}\left(z - w^+(t,x) - \sigma\delta\left(1-\mathrm{e}^{-\lambda t}\right) - \frac{\varphi(\nu y)}{\nu}\right)\right) \tag{3.11}$$

is a supersolution of (1.9) *for* $(x,y,z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ *and* $t > 0$, *that is, it satisfies the following inequality*

$$\mathcal{L}[v^+] := v_t^+ - \Delta v^+ - c v_z^+ - f\left(v^+\right) \geq 0.$$

Proof. Let $\xi(t,x,y,z) := \frac{c_*}{c}\left(z - w^+(t,x) - \sigma\delta\left(1-\mathrm{e}^{-\lambda t}\right) - \frac{\varphi(\nu y)}{\nu}\right)$. Using $-V_{yy} - V_{zz} - cV_z - f(V) = 0$, $w_t^+ = \Delta_x w^+ + k\left|\nabla_x w^+\right|^2$ and (3.3), we have

$$\begin{aligned}
\mathcal{L}[v^+] =& v_t^+ - \Delta v^+ - c v_z^+ - f\left(v^+\right) \\
=& - V_z w_t^+ - \sigma\delta\lambda \mathrm{e}^{-\lambda t} V_z - \delta\lambda \mathrm{e}^{-\lambda t}\phi^\gamma(\xi) - \frac{c_*}{c}\delta \mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi'(\xi) w_t^+ \\
&- \frac{c_*}{c}\lambda\sigma\delta^2 \mathrm{e}^{-2\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi'(\xi) - V_{zz}\left|-\nabla_x w^+\right|^2 + V_z \Delta_x w^+ \\
&- \delta \mathrm{e}^{-\lambda t}\gamma(\gamma-1)\phi^{\gamma-2}(\xi)\left(\frac{c_*}{c}\phi'(\xi)\left|-\nabla_x w^+\right|\right)^2 \\
&- \delta \mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi''(\xi)\left(\frac{c_*}{c}\left|-\nabla_x w^+\right|\right)^2 - \delta \mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi'(\xi)\frac{c_*}{c}\left(-\Delta_x w^+\right) \\
&- V_{yy} - \delta \mathrm{e}^{-\lambda t}\gamma(\gamma-1)\phi^{\gamma-2}(\xi)\left(\phi'(\xi)\frac{c_*}{c}\left(-\varphi'(\nu y)\right)\right)^2 \\
&- \delta \mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi''(\xi)\left(\frac{c_*}{c}\left(-\varphi'(\nu y)\right)\right)^2 + \delta \mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi'(\xi)\frac{c_*}{c}\nu\varphi''(\nu y) \\
&- V_{zz} - \delta \mathrm{e}^{-\lambda t}\gamma(\gamma-1)\phi^{\gamma-2}(\xi)\left(\frac{c_*}{c}\phi'(\xi)\right)^2 - \frac{c_*^2}{c^2}\delta \mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi''(\xi) \\
&- cV_z - c_*\delta \mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi'(\xi) - f\left(v^+\right) + f(V) - f(V) \\
=& - \sigma\delta\lambda \mathrm{e}^{-\lambda t} V_z - \delta\lambda \mathrm{e}^{-\lambda t}\phi^\gamma(\xi) - \frac{c_*}{c}\lambda\sigma\delta^2 \mathrm{e}^{-2\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi'(\xi) \\
&+ \left(-w_t^+ + \Delta_x w^+\right) V_z - \left|\nabla_x w^+\right|^2 V_{zz} \\
&+ \delta \mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\xi)\frac{c_*}{c}\left[\left(-w_t^+ + \Delta_x w^+\right)\phi'(\xi) - \frac{c_*}{c}\left|\nabla_x w^+\right|^2\phi''(\xi)\right] \\
&- V_{yy} - V_{zz} - cV_z - f(V) \\
&- \delta \mathrm{e}^{-\lambda t}\gamma(\gamma-1)\phi^{\gamma-2}(\xi)\left(\frac{c_*}{c}\phi'(\xi)\right)^2\left[1 + \varphi'(\nu y)^2\right] \\
&- \delta \mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi''(\xi)\frac{c_*^2}{c^2}\left[1 + \varphi'(\nu y)^2\right] \\
&- \delta \mathrm{e}^{-\lambda t}\gamma(\gamma-1)\phi^{\gamma-2}(\xi)\left(\frac{c_*}{c}\phi'(\xi)\left|\nabla_x w^+\right|\right)^2 \\
&+ \delta \mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi'(\xi)\frac{c_*}{c}\nu\varphi''(\nu y) - c_*\delta \mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi'(\xi) - f\left(v^+\right) + f(V) \\
=& - \sigma\delta\lambda \mathrm{e}^{-\lambda t} V_z - \delta\lambda \mathrm{e}^{-\lambda t}\phi^\gamma(\xi) - \frac{c_*}{c}\lambda\sigma\delta^2 \mathrm{e}^{-2\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi'(\xi) \\
&+ \left(-kV_z - V_{zz}\right)\left|\nabla_x w^+\right|^2 \\
&+ \delta \mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\xi)\frac{c_*}{c}\left(-k\phi'(\xi) - \frac{c_*}{c}\phi''(\xi)\right)\left|\nabla_x w^+\right|^2 \\
&- \delta \mathrm{e}^{-\lambda t}\gamma(\gamma-1)\phi^{\gamma-2}(\xi)\left(\frac{c_*}{c}\phi'(\xi)\right)^2\left[1 + \varphi'(\nu y)^2\right]
\end{aligned}$$

$$
\begin{aligned}
&- \delta \mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi''(\xi)\frac{c_*^2}{c^2}\left[1+\varphi'(\nu y)^2\right]\\
&- \delta \mathrm{e}^{-\lambda t}\gamma(\gamma-1)\phi^{\gamma-2}(\xi)\left(\frac{c_*}{c}\phi'(\xi)\left|\nabla_x w^+\right|\right)^2\\
&+ \delta \mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi'(\xi)\frac{c_*}{c}\nu\varphi''(\nu y) - c_*\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi'(\xi) - f\left(v^+\right)+f(V)\\
\geq &- \sigma\delta\lambda\mathrm{e}^{-\lambda t}V_z - \delta\lambda\mathrm{e}^{-\lambda t}\phi^{\gamma}(\xi) - \frac{c_*}{c}\lambda\sigma\delta^2\mathrm{e}^{-2\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi'(\xi)\\
&- \delta \mathrm{e}^{-\lambda t}\gamma(\gamma-1)\phi^{\gamma-2}(\xi)\left(\frac{c_*}{c}\phi'(\xi)\right)^2\left[1+\varphi'(\nu y)^2\right]\\
&- \delta \mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi''(\xi)\frac{c_*^2}{c^2}\left[1+\varphi'(\nu y)^2\right]\\
&+ \delta \mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi'(\xi)\frac{c_*}{c}\nu\varphi''(\nu y) - c_*\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi'(\xi)\\
&- \delta\mathrm{e}^{-\lambda t}\phi^{\gamma}(\xi)f'\left(V\left(y, z-w^+(t,x)-\sigma\delta\left(1-\mathrm{e}^{-\lambda t}\right)\right)+\tau\delta\mathrm{e}^{-\lambda t}\phi^{\gamma}\left(\xi\right)\right),
\end{aligned}
$$

where $\tau=\tau(t,x,y,z)\in(0,1)$ and $\xi=\xi(t,x,y,z)$.

Since $\lim_{\mathcal{P}\to\infty}\frac{\phi'(\mathcal{P})}{\phi(\mathcal{P})}=\Lambda$, then there exists $R_1>0$ large enough such that

$$
\left(\gamma\frac{\phi'(\mathcal{P})}{\phi(\mathcal{P})}\right)^2 + c_*\gamma\frac{\phi'(\mathcal{P})}{\phi(\mathcal{P})} + f'(0) < \frac{1}{2}h(\gamma\Lambda) \quad \text{for } \mathcal{P}>R_1. \tag{3.12}
$$

The condition $(A2)$ yields that $\lim_{\mathcal{P}\to\infty}\frac{\phi''(\mathcal{P})}{\phi(\mathcal{P})}=\Lambda^2$. Then, by

$$
\lim_{\mathcal{P}\to\infty}\frac{\phi'(\mathcal{P})}{\phi(\mathcal{P})}=\Lambda \quad \text{and} \quad \frac{c_*^2}{c^2}\left[1+\varphi'(\nu y)^2\right]\leq 1,
$$

we can obtain that

$$
\lim_{\mathcal{P}\to\infty}\frac{c_*^2}{c^2}\left[\left(\frac{\phi'(\mathcal{P})}{\phi(\mathcal{P})}\right)^2-\frac{\phi''(\mathcal{P})}{\phi(\mathcal{P})}\right]\left[1+\varphi'(\nu y)^2\right]=0 \quad \text{uniformly in } y\in\mathbb{R}.
$$

Thus, there exists $R_2>0$ large enough such that

$$
\left|\frac{c_*^2}{c^2}\left[\left(\frac{\phi'(\mathcal{P})}{\phi(\mathcal{P})}\right)^2-\frac{\phi''(\mathcal{P})}{\phi(\mathcal{P})}\right]\left[1+\varphi'(\nu y)^2\right]\right| < -\frac{1}{16}h(\gamma\Lambda) \quad \text{for } \mathcal{P}>R_2,\ y\in\mathbb{R}. \tag{3.13}
$$

Under the condition $(A1)$, we can obtain that there exists a positive constant K_1 such that

$$
|f'(u_1)-f'(u_2)|\leq K_1\left|u_1-u_2\right|^{\vartheta}, \quad \forall\ u_1,\ u_2\in[-\epsilon,1+\epsilon].
$$

It follows from (1.4) and $\tau_*|y|\leq\frac{\varphi(\nu y)}{\nu}\leq\tau_*|y|+\frac{a}{\nu}$ that there exists $R_3>0$ large enough such that for all $\delta<\epsilon_1$,

$$
\begin{aligned}
&\left|f'\left(V\left(y,z-w^+(t,x)-\sigma\delta\left(1-\mathrm{e}^{-\lambda t}\right)\right)+\tau\delta\mathrm{e}^{-\lambda t}\phi^{\gamma}\left(\xi\right)\right)-f'(0)\right|\\
\leq &K_1\left|V\left(y,z-w^+(t,x)-\sigma\delta\left(1-\mathrm{e}^{-\lambda t}\right)\right)+\tau\delta\mathrm{e}^{-\lambda t}\phi^{\gamma}\left(\xi\right)\right|^{\vartheta} < -\frac{1}{16}h(\gamma\Lambda)
\end{aligned} \tag{3.14}
$$

for $(x,y,z)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$ and $t\geq 0$ with $\xi(t,x,y,z)>R_3$.

Since $\lim_{\mathcal{P}\to-\infty}\phi(\mathcal{P})=1$, $\lim_{\mathcal{P}\to-\infty}\phi'(\mathcal{P})=0$ and $\lim_{\mathcal{P}\to-\infty}\phi''(\mathcal{P})=0$, then there exists $R_1'>0$ large enough such that

$$
\left|\frac{\phi'(\mathcal{P})}{\phi(\mathcal{P})}\right| < \frac{|f'(1)|}{8k_1}, \quad \left|\frac{\phi''(\mathcal{P})}{\phi(\mathcal{P})}\right| < \frac{|f'(1)|}{8} \quad \text{for } \mathcal{P}<-R_1'. \tag{3.15}
$$

By (1.4) and $\tau_*|y| \le \frac{\varphi(\nu y)}{\nu} \le \tau_*|y| + \frac{a}{\nu}$ that there exists $R_2' > 0$ large enough such that

$$V\left(y, z - w^+(t,x) - \sigma\delta\left(1 - \mathrm{e}^{-\lambda t}\right)\right) > 1 - \epsilon_1 \tag{3.16}$$

for $(x,y,z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ and $t \ge 0$ with $\xi(t,x,y,z) < -R_2'$.

Let $R := \max\{R_1, R_2, R_3, R_1', R_2'\}$ and $M = R + \frac{a}{\nu}$, then (1.7) yields that

$$\min_{-M \le \xi \le M} -V_z\left(y, z - w^+(t,x) - \sigma\delta\left(1 - \mathrm{e}^{-\lambda t}\right)\right) \ge \gamma' > 0. \tag{3.17}$$

Take $\sigma_0 > 0$ large enough such that

$$\sigma\lambda\gamma' - \lambda - \lambda_2 - k_1\lambda_1 - \lambda_3 > 0 \quad \text{for } \sigma \ge \sigma_0. \tag{3.18}$$

Let $\delta_0 := \epsilon_1$.

Note that $\frac{c_*^2}{c^2}\left[1 + \varphi'(\nu y)^2\right] \le 1$ for any $y \in \mathbb{R}$. Let $\sigma \ge \sigma_0$. For $(t,x,y,z) \in [0,+\infty) \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ with $\xi(t,x,y,z) > R$, applying (1.6), (3.12), (3.13) and (3.14) with $0 < \delta < \delta_0$, we have

$$\begin{aligned}
\mathcal{L}[v^+] &= v_t^+ - \Delta v^+ - c v_z^+ - f\left(v^+\right) \\
&\ge \delta \mathrm{e}^{-\lambda t}\phi^\gamma(\xi)\Big\{ -\lambda - \gamma(\gamma-1)\left(\frac{c_*}{c}\frac{\phi'(\xi)}{\phi(\xi)}\right)^2\left[1 + \varphi'(\nu y)^2\right] \\
&\quad - \gamma\frac{\phi''(\xi)}{\phi(\xi)}\frac{c_*^2}{c^2}\left[1 + \varphi'(\nu y)^2\right] + \gamma\frac{\phi'(\xi)}{\phi(\xi)}\frac{c_*}{c}\nu\varphi''(\nu y) - c_*\gamma\frac{\phi'(\xi)}{\phi(\xi)} \\
&\quad - f'\left(V\left(y, z - w^+(t,x) - \sigma\delta\left(1 - \mathrm{e}^{-\lambda t}\right)\right) + \tau\delta\mathrm{e}^{-\lambda t}\phi^\gamma(\xi)\right)\Big\} \\
&= \delta \mathrm{e}^{-\lambda t}\phi^\gamma(\xi)\Big\{ -\lambda + \gamma\frac{c_*^2}{c^2}\left(\left(\frac{\phi'(\xi)}{\phi(\xi)}\right)^2 - \frac{\phi''(\xi)}{\phi(\xi)}\right)\left[1 + \varphi'(\nu y)^2\right] \\
&\quad - \gamma^2\left(\frac{\phi'(\xi)}{\phi(\xi)}\right)^2\frac{c_*^2}{c^2}\left[1 + \varphi'(\nu y)^2\right] + \gamma^2\left(\frac{\phi'(\xi)}{\phi(\xi)}\right)^2 \\
&\quad + \gamma\frac{\phi'(\xi)}{\phi(\xi)}\frac{c_*}{c}\nu\varphi''(\nu y) - \gamma^2\left(\frac{\phi'(\xi)}{\phi(\xi)}\right)^2 - c_*\gamma\frac{\phi'(\xi)}{\phi(\xi)} - f'(0) \\
&\quad - f'\left(V\left(y, z - w^+(t,x) - \sigma\delta\left(1 - \mathrm{e}^{-\lambda t}\right)\right) + \tau\delta\mathrm{e}^{-\lambda t}\phi^\gamma(\xi)\right) + f'(0)\Big\} \\
&\ge \delta \mathrm{e}^{-\lambda t}\phi^\gamma(\xi)\Big\{ -\lambda + \frac{1}{16}h(\gamma\Lambda) + \frac{1}{16}h(\gamma\Lambda) - \frac{1}{2}h(\gamma\Lambda) + \frac{1}{16}h(\gamma\Lambda)\Big\} \\
&\ge 0.
\end{aligned}$$

For $(t,x,y,z) \in [0,+\infty) \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ with $\xi(t,x,y,z) < -R$, applying (1.6), (3.15) and (3.16) with $0 < \delta < \delta_0$, we have

$$\begin{aligned}
\mathcal{L}[v^+] &= v_t^+ - \Delta v^+ - c v_z^+ - f\left(v^+\right) \\
&\ge \delta \mathrm{e}^{-\lambda t}\phi^\gamma(\xi)\Big\{ -\lambda - \gamma\frac{\phi''(\xi)}{\phi(\xi)}\frac{c_*^2}{c^2}\left[1 + \varphi'(\nu y)^2\right] + \gamma\frac{\phi'(\xi)}{\phi(\xi)}\frac{c_*}{c}\nu\varphi''(\nu y) \\
&\quad - f'\left(V\left(y, z - w^+(t,x) - \sigma\delta\left(1 - \mathrm{e}^{-\lambda t}\right)\right) + \tau\delta\mathrm{e}^{-\lambda t}\phi^\gamma(\xi)\right)\Big\} \\
&\ge \delta \mathrm{e}^{-\lambda t}\phi^\gamma(\xi)\Big\{ -\lambda + \frac{f'(1)}{8} + \frac{f'(1)}{8} - \frac{f'(1)}{2}\Big\} \\
&\ge 0.
\end{aligned}$$

For $(t,x,y,z)\in[0,+\infty)\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$ with $|\xi(t,x,y,z)|<R$, using (3.17) and (3.18), we have

$$\begin{aligned}\mathcal{L}[v^+]&=v_t^+-\Delta v^+-cv_z^+-f\left(v^+\right)\\&\geq-\sigma\delta\lambda\mathrm{e}^{-\lambda t}V_z-\delta\lambda\mathrm{e}^{-\lambda t}\phi^\gamma(\xi)-\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi''(\xi)\frac{c_*^2}{c^2}\left[1+\varphi'(\nu y)^2\right]\\&\quad+\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\xi)\phi'(\xi)\frac{c_*}{c}\nu\varphi''(\nu y)\\&\quad-\delta\mathrm{e}^{-\lambda t}\phi^\gamma(\xi)f'\left(V\left(y,z-w^+(t,x)-\sigma\delta\left(1-\mathrm{e}^{-\lambda t}\right)\right)+\tau\delta\mathrm{e}^{-\lambda t}\phi^\gamma\left(\xi\right)\right)\\&\geq\delta\mathrm{e}^{-\lambda t}\left\{-\sigma\lambda V_z-\lambda-\sup_{\xi\in\mathbb{R}}\left|\frac{\phi''(\xi)}{\phi(\xi)}\right|-\sup_{\xi\in\mathbb{R}}\left|\frac{\phi'(\xi)}{\phi(\xi)}\right|\sup_{y\in\mathbb{R}}|\varphi''(\nu y)|-\sup_{u\in[-\epsilon,1+\epsilon]}|f'\left(u\right)|\right\}\\&\geq\delta\mathrm{e}^{-\lambda t}\left\{\sigma\lambda\gamma'-\lambda-\lambda_2-k_1\lambda_1-\lambda_3\right\}\\&\geq0.\end{aligned}$$

This completes the proof. □

Lemma 3.7. *Let $k>0$ be defined as in (3.3). Then, for $\gamma\in\left(\frac{\Lambda_1}{\Lambda},1\right)$ and $\nu\in(0,1)$ with $\nu<-\frac{h(\gamma\Lambda)c}{16\lambda_1k_1c_*}$, there exist some constants $\delta_1>0$ and $\sigma_1>0$ such that, for any $\delta\in(0,\delta_1]$ and $\sigma\geq\sigma_1$, and any bounded functions $w^-(t,x)$ satisfying*

$$w_t^-=\Delta_xw^--k\left|\nabla_xw^-\right|^2,\quad x\in\mathbb{R}^n,\ t>0,$$

the function defined by

$$\begin{aligned}v^-(t,x,y,z):=&\,V\left(y,z-w^-(t,x)+\sigma\delta\left(1-\mathrm{e}^{-\lambda t}\right)\right)\\&-\delta\mathrm{e}^{-\lambda t}\phi^\gamma\left(\frac{c_*}{c}\left(z+w^-(t,x)+\sigma\delta\left(1-\mathrm{e}^{-\lambda t}\right)-\frac{\varphi(\nu y)}{\nu}\right)\right)\end{aligned}\tag{3.19}$$

is a subsolution of (1.9) on $(x,y,z)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$ and $t>0$, that is, it satisfies the following inequality

$$\mathcal{L}[v^-]:=v_t^--\Delta v^--cv_z^--f\left(v^-\right)\leq0.$$

Proof. Let $\eta(t,x,y,z):=\frac{c_*}{c}\left(z+w^-(t,x)+\sigma\delta\left(1-\mathrm{e}^{-\lambda t}\right)-\frac{\varphi(\nu y)}{\nu}\right)$. Applying $-V_{yy}-V_{zz}-cV_z-f(V)=0$, $w_t^-=\Delta_xw^--k\left|\nabla_xw^-\right|^2$ and (3.3), we have

$$\begin{aligned}\mathcal{L}[v^-]&=v_t^--\Delta v^--cv_z^--f\left(v^-\right)\\&=-V_zw_t^-+\sigma\delta\lambda\mathrm{e}^{-\lambda t}V_z+\delta\lambda\mathrm{e}^{-\lambda t}\phi^\gamma(\eta)-\frac{c_*}{c}\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi'(\eta)w_t^-\\&\quad-\frac{c_*}{c}\lambda\sigma\delta^2\mathrm{e}^{-2\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi'(\eta)-V_{zz}\left|-\nabla_xw^-\right|^2+V_z\Delta_xw^-\\&\quad+\delta\mathrm{e}^{-\lambda t}\gamma(\gamma-1)\phi^{\gamma-2}(\eta)\left(\frac{c_*}{c}\phi'(\eta)\left|\nabla_xw^-\right|\right)^2\\&\quad+\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi''(\eta)\left(\frac{c_*}{c}\left|\nabla_xw^-\right|\right)^2+\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi'(\eta)\frac{c_*}{c}\Delta_xw^-\\&\quad-V_{yy}+\delta\mathrm{e}^{-\lambda t}\gamma(\gamma-1)\phi^{\gamma-2}(\eta)\left(\phi'(\eta)\frac{c_*}{c}\left(-\varphi'(\nu y)\right)\right)^2\\&\quad+\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi''(\eta)\left(\frac{c_*}{c}\left(-\varphi'(\nu y)\right)\right)^2-\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi'(\eta)\frac{c_*}{c}\nu\varphi''(\nu y)\\&\quad-V_{zz}+\delta\mathrm{e}^{-\lambda t}\gamma(\gamma-1)\phi^{\gamma-2}(\eta)\left(\frac{c_*}{c}\phi'(\eta)\right)^2+\frac{c_*^2}{c^2}\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi''(\eta)\\&\quad-cV_z+c_*\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi'(\eta)-f\left(v^-\right)+f(V)-f(V)\\&=\sigma\delta\lambda\mathrm{e}^{-\lambda t}V_z+\delta\lambda\mathrm{e}^{-\lambda t}\phi^\gamma(\eta)-\frac{c_*}{c}\lambda\sigma\delta^2\mathrm{e}^{-2\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi'(\eta)\end{aligned}$$

$$
\begin{aligned}
&+\left(-w_t^-+\Delta_x w^-\right)V_z-\left|\nabla_x w^-\right|^2 V_{zz}\\
&+\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\eta)\frac{c_*}{c}\left[\left(-w_t^-+\Delta_x w^-\right)\phi'(\eta)+\frac{c_*}{c}\left|\nabla_x w^-\right|^2\phi''(\eta)\right]\\
&-V_{yy}-V_{zz}-cV_z-f(V)\\
&+\delta\mathrm{e}^{-\lambda t}\gamma(\gamma-1)\phi^{\gamma-2}(\eta)\left(\frac{c_*}{c}\phi'(\eta)\right)^2\left[1+\varphi'(\nu y)^2\right]\\
&+\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi''(\eta)\frac{c_*^2}{c^2}\left[1+\varphi'(\nu y)^2\right]\\
&+\delta\mathrm{e}^{-\lambda t}\gamma(\gamma-1)\phi^{\gamma-2}(\eta)\left(\frac{c_*}{c}\phi'(\eta)\left|\nabla_x w^-\right|\right)^2\\
&-\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi'(\eta)\frac{c_*}{c}\nu\varphi''(\nu y)+c_*\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi'(\eta)-f\left(v^+\right)+f(V)\\
=&\,\sigma\delta\lambda\mathrm{e}^{-\lambda t}V_z+\delta\lambda\mathrm{e}^{-\lambda t}\phi^\gamma(\eta)-\frac{c_*}{c}\lambda\sigma\delta^2\mathrm{e}^{-2\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi'(\eta)\\
&+\left(kV_z-V_{zz}\right)\left|\nabla_x w^-\right|^2\\
&+\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\eta)\frac{c_*}{c}\left(k\phi'(\eta)+\frac{c_*}{c}\phi''(\eta)\right)\left|\nabla_x w^-\right|^2\\
&+\delta\mathrm{e}^{-\lambda t}\gamma(\gamma-1)\phi^{\gamma-2}(\eta)\left(\frac{c_*}{c}\phi'(\eta)\right)^2\left[1+\varphi'(\nu y)^2\right]\\
&+\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi''(\eta)\frac{c_*^2}{c^2}\left[1+\varphi'(\nu y)^2\right]\\
&+\delta\mathrm{e}^{-\lambda t}\gamma(\gamma-1)\phi^{\gamma-2}(\eta)\left(\frac{c_*}{c}\phi'(\eta)\left|\nabla_x w^-\right|\right)^2\\
&-\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi'(\eta)\frac{c_*}{c}\nu\varphi''(\nu y)+c_*\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi'(\eta)-f\left(v^-\right)+f(V)\\
\le&\,\sigma\delta\lambda\mathrm{e}^{-\lambda t}V_z+\delta\lambda\mathrm{e}^{-\lambda t}\phi^\gamma(\eta)-\frac{c_*}{c}\lambda\sigma\delta^2\mathrm{e}^{-2\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi'(\eta)\\
&+\delta\mathrm{e}^{-\lambda t}\gamma(\gamma-1)\phi^{\gamma-2}(\eta)\left(\frac{c_*}{c}\phi'(\eta)\right)^2\left[1+\varphi'(\nu y)^2\right]\\
&+\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi''(\eta)\frac{c_*^2}{c^2}\left[1+\varphi'(\nu y)^2\right]\\
&-\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi'(\eta)\frac{c_*}{c}\nu\varphi''(\nu y)+c_*\delta\mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi'(\eta)\\
&+\delta\mathrm{e}^{-\lambda t}\phi^\gamma(\eta)f'\left(V\left(y,z-w^-(t,x)+\sigma\delta\left(1-\mathrm{e}^{-\lambda t}\right)\right)-\tau\delta\mathrm{e}^{-\lambda t}\phi^\gamma(\eta)\right),
\end{aligned}
$$

where $\tau=\tau(t,x,y,z)\in(0,1)$ and $\eta=\eta(t,x,y,z)$.

The boundedness of functions $w^-(t,x)$ implies that there exists a constant $\widehat{K}>0$ such that $\|w^-\|_{L^\infty((0,+\infty)\times\mathbb{R}^n)}\le\widehat{K}$. Under the condition $(A1)$, there exists a positive constant K_1 such that

$$
\left|f'(u_1)-f'(u_2)\right|\le K_1\left|u_1-u_2\right|^\vartheta,\quad\forall\ u_1,\ u_2\in[-\epsilon,1+\epsilon].
$$

It follows from (1.4) and $\tau_*|y|\le\frac{\varphi(\nu y)}{\nu}\le\tau_*|y|+\frac{a}{\nu}$ that there exists $R_3>0$ large enough such that for all $\delta<\epsilon_1$,

$$
\begin{aligned}
&\left|f'\left(V\left(y,z-w^-(t,x)+\sigma\delta\left(1-\mathrm{e}^{-\lambda t}\right)\right)-\tau\delta\mathrm{e}^{-\lambda t}\phi^\gamma(\eta)\right)-f'(0)\right|\\
\le&K_1\left|V\left(y,z-w^-(t,x)+\sigma\delta\left(1-\mathrm{e}^{-\lambda t}\right)\right)-\tau\delta\mathrm{e}^{-\lambda t}\phi^\gamma(\eta)\right|^\vartheta<-\frac{1}{16}h(\gamma\Lambda)
\end{aligned}
\tag{3.20}
$$

for $(x,y,z)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$ and $t\ge0$ with $\eta(t,x,y,z)>R_3+2\widehat{K}$.

By (1.4) and $\tau_*|y| \le \frac{\varphi(\nu y)}{\nu} \le \tau_*|y| + \frac{a}{\nu}$ that there exists $R_2' > 0$ large enough such that

$$V\left(y, z - w^-(t,x) + \sigma\delta\left(1 - \mathrm{e}^{-\lambda t}\right)\right) > 1 - \epsilon_1 \tag{3.21}$$

for $(x,y,z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ and $t \ge 0$ with $\eta(t,x,y,z) < -R_2' - 2\widehat{K}$.

Let R_1, R_2 and R_1' be defined as in (3.12), (3.13) and (3.15). Let $R := \max\{R_1, R_2, R_3 + 2\widehat{K}, R_1', R_2' + 2\widehat{K}\}$ and $M = R + \frac{a}{\nu}$, then (1.7) yields that

$$\min_{-M \le \eta \le M} -V_z\left(y, z - w^-(t,x) + \sigma\delta\left(1 - \mathrm{e}^{-\lambda t}\right)\right) \ge \gamma' > 0. \tag{3.22}$$

Take $\sigma_1 > 0$ large enough such that

$$-\frac{\sigma\lambda\gamma'}{2} + \lambda + \lambda_2 + k_1\lambda_1 + \lambda_3 < 0 \quad \text{for } \sigma \ge \sigma_1. \tag{3.23}$$

Let $\delta_1 := \min\left\{\epsilon_1, \frac{\gamma'}{2\lambda_1}, -\frac{h(\gamma\Lambda)}{16\sigma\lambda\lambda_1}, \frac{|f'(1)|}{8\sigma\lambda\lambda_1}\right\}$ for $\sigma \ge \sigma_1$.

Note that $\frac{c_*^2}{c^2}\left[1 + \varphi'(\nu y)^2\right] \le 1$ for any $y \in \mathbb{R}$. Let $\sigma \ge \sigma_1$. For $(t,x,y,z) \in [0,+\infty) \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ with $\eta(t,x,y,z) > R$, applying (1.6), (3.12), (3.13) and (3.20) with $0 < \delta < \delta_1$, we have

$$\begin{aligned}
\mathcal{L}[v^-] =& v_t^- - \Delta v^- - c v_z^- - f\left(v^-\right) \\
\le& \delta \mathrm{e}^{-\lambda t}\phi^\gamma(\eta)\Bigg\{\lambda - \lambda\sigma\delta\frac{\phi'(\eta)}{\phi(\eta)} + \gamma(\gamma-1)\left(\frac{c_*}{c}\frac{\phi'(\eta)}{\phi(\eta)}\right)^2\left[1 + \varphi'(\nu y)^2\right] \\
&+ \gamma\frac{\phi''(\eta)}{\phi(\eta)}\frac{c_*^2}{c^2}\left[1 + \varphi'(\nu y)^2\right] - \gamma\frac{\phi'(\eta)}{\phi(\eta)}\frac{c_*}{c}\nu\varphi''(\nu y) + c_*\gamma\frac{\phi'(\eta)}{\phi(\eta)} \\
&+ f'\left(V\left(y, z - w^-(t,x) + \sigma\delta\left(1 - \mathrm{e}^{-\lambda t}\right)\right) - \tau\delta\mathrm{e}^{-\lambda t}\phi^\gamma(\eta)\right)\Bigg\} \\
\le& \delta \mathrm{e}^{-\lambda t}\phi^\gamma(\eta)\Bigg\{\lambda + \lambda\sigma\delta\sup_{\eta\in\mathbb{R}}\left|\frac{\phi'(\eta)}{\phi(\eta)}\right| - \gamma\frac{c_*^2}{c^2}\left(\left(\frac{\phi'(\eta)}{\phi(\eta)}\right)^2 - \frac{\phi''(\eta)}{\phi(\eta)}\right)\left[1 + \varphi'(\nu y)^2\right] \\
&+ \gamma^2\left(\frac{\phi'(\eta)}{\phi(\eta)}\right)^2\frac{c_*^2}{c^2}\left[1 + \varphi'(\nu y)^2\right] - \gamma^2\left(\frac{\phi'(\eta)}{\phi(\eta)}\right)^2 \\
&+ \gamma^2\left(\frac{\phi'(\eta)}{\phi(\eta)}\right)^2 + c_*\gamma\frac{\phi'(\eta)}{\phi(\eta)} + f'(0) - \gamma\frac{\phi'(\eta)}{\phi(\eta)}\frac{c_*}{c}\nu\varphi''(\nu y) \\
&+ f'\left(V\left(y, z - w^-(t,x) + \sigma\delta\left(1 - \mathrm{e}^{-\lambda t}\right)\right) - \tau\delta\mathrm{e}^{-\lambda t}\phi^\gamma(\eta)\right) - f'(0)\Bigg\} \\
\le& \delta \mathrm{e}^{-\lambda t}\phi^\gamma(\eta)\left\{\lambda - \frac{1}{16}h(\gamma\Lambda) - \frac{1}{16}h(\gamma\Lambda) + \frac{1}{2}h(\gamma\Lambda) - \frac{1}{16}h(\gamma\Lambda) - \frac{1}{16}h(\gamma\Lambda)\right\} \\
\le& 0.
\end{aligned}$$

For $(t,x,y,z) \in [0,+\infty) \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ with $\eta(t,x,y,z) < -R$, applying (1.6), (3.15) and (3.21) with $0 < \delta < \delta_1$, we have

$$\begin{aligned}
\mathcal{L}[v^-] =& v_t^- - \Delta v^- - c v_z^- - f\left(v^-\right) \\
\le& \delta \mathrm{e}^{-\lambda t}\phi^\gamma(\eta)\Bigg\{\lambda - \lambda\sigma\delta\frac{\phi'(\eta)}{\phi(\eta)} + \gamma\frac{\phi''(\eta)}{\phi(\eta)}\frac{c_*^2}{c^2}\left[1 + \varphi'(\nu y)^2\right] - \gamma\frac{\phi'(\eta)}{\phi(\eta)}\frac{c_*}{c}\nu\varphi''(\nu y)
\end{aligned}$$

$$
\begin{aligned}
&\quad + f'\left(V\left(y, z - w^-(t,x) + \sigma\delta\left(1 - \mathrm{e}^{-\lambda t}\right)\right) - \tau\delta \mathrm{e}^{-\lambda t}\phi^{\gamma}(\eta)\right)\Big\} \\
&\le \delta \mathrm{e}^{-\lambda t}\phi^{\gamma}(\eta)\left\{\lambda + \frac{|f'(1)|}{8} + \frac{|f'(1)|}{8} + \frac{|f'(1)|}{8} + \frac{f'(1)}{2}\right\} \\
&\le 0.
\end{aligned}
$$

For $(t,x,y,z) \in [0,+\infty) \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ with $|\eta(t,x,y,z)| < R$, using (3.22) and (3.23),

$$
\begin{aligned}
\mathcal{L}[v^-] &= v_t^- - \Delta v^- - c v_z^- - f\left(v^-\right) \\
&\le \sigma\delta\lambda \mathrm{e}^{-\lambda t} V_z + \delta\lambda \mathrm{e}^{-\lambda t}\phi^{\gamma}(\eta) - \frac{c_*}{c}\lambda\sigma\delta^2 \mathrm{e}^{-2\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi'(\eta) \\
&\quad + \delta \mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi''(\eta)\frac{c_*^2}{c^2}\left[1 + \varphi'(\nu y)^2\right] - \delta \mathrm{e}^{-\lambda t}\gamma\phi^{\gamma-1}(\eta)\phi'(\eta)\frac{c_*}{c}\nu\varphi''(\nu y) \\
&\quad + \delta \mathrm{e}^{-\lambda t}\phi^{\gamma}(\eta) f'\left(V\left(y, z - w^-(t,x) + \sigma\delta\left(1 - \mathrm{e}^{-\lambda t}\right)\right) - \tau\delta \mathrm{e}^{-\lambda t}\phi^{\gamma}(\eta)\right) \\
&\le \delta \mathrm{e}^{-\lambda t}\Bigg\{\sigma\lambda V_z + \lambda + \sigma\lambda\delta \sup_{\eta\in\mathbb{R}}\left|\frac{\phi'(\eta)}{\phi(\eta)}\right| + \sup_{\eta\in\mathbb{R}}\left|\frac{\phi''(\eta)}{\phi(\eta)}\right| \\
&\qquad + \sup_{\eta\in\mathbb{R}}\left|\frac{\phi'(\eta)}{\phi(\eta)}\right| \sup_{y\in\mathbb{R}}|\varphi''(\nu y)| + \sup_{u\in[-\epsilon_1, 1+\epsilon_1]}|f'(u)|\Bigg\} \\
&\le \delta \mathrm{e}^{-\lambda t}\left\{-\frac{\sigma\lambda\gamma'}{2} + \lambda + \lambda_2 + k_1\lambda_1 + \lambda_3\right\} \\
&\le 0.
\end{aligned}
$$

This completes the proof. □

To prove Theorem 1.3, we need another auxiliary lemma.

Lemma 3.8. *Suppose that the initial value* $v_0 \in C\left(\mathbb{R}^{n+2}, [0,1]\right)$ $(n \ge 1)$ *satisfies*

$$
\lim_{R\to+\infty} \sup_{|x|+|y|+|z|\ge R} \frac{|v_0(x,y,z) - V(y,z)|}{\phi^{\gamma}\left(\frac{c_*}{c}\left(z - \tau_*|y|\right)\right)} = 0 \tag{3.24}
$$

for some $\gamma \in \left(\frac{\Lambda_1}{\Lambda}, 1\right)$, *then for any fixed* $T > 0$, *we have*

$$
\lim_{R\to+\infty} \sup_{|x|+|y|+|z|\ge R} \frac{|v(T,x,y,z;v_0) - V(y,z)|}{\phi^{\gamma}\left(\frac{c_*}{c}\left(z - \tau_*|y|\right)\right)} = 0.
$$

Proof. For simplicity, we denote $v(t,x,y,z;v_0)$ by $v(t,x,y,z)$. Let $W(t,x,y,z) := v(t,x,y,z) - V(y,z)$ for any $(x,y,z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ and $t \ge 0$. Then, $W(t,x,y,z)$ is a solution to the following Cauchy problem

$$
\begin{cases}
W_t = \Delta W + cW_z + f'(V + \tau W)W, & x \in \mathbb{R}^n,\ y \in \mathbb{R},\ z \in \mathbb{R},\ t > 0, \\
W(0,x,y,z) = v_0(x,y,z) - V(y,z), & x \in \mathbb{R}^n,\ y \in \mathbb{R},\ z \in \mathbb{R},
\end{cases}
$$

where $\tau = \tau(t,x,y,z)$ is a function that satisfies $0 \le \tau(t,x,y,z) \le 1$. By the comparison principle, it suffices to prove the lemma for the case where $W(0,x,y,z) \ge 0$ and the case where $W(0,x,y,z) \le 0$.

Assume $W(0,x,y,z) \ge 0$ for $(x,y,z) \in \mathbb{R}^{n+2}$. By $v_0(x,y,z) \in [0,1]$, the comparison principle implies that $0 \le v(t,x,y,z) \le 1$ for any $(x,y,z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ and $t > 0$. Thus, by $0 < V(y,z) < 1$ for $(y,z) \in \mathbb{R}^2$, we obtain

$$
|V(y,z) + \tau W(t,x,y,z)| = |(1-\tau)V(y,z) + \tau v(t,x,y,z)| \le 1,\ x \in \mathbb{R}^n, y \in \mathbb{R}, z \in \mathbb{R}, t > 0.
$$

It follows from $W(0,x,y,z) \ge 0$ and the maximum principle that we have $W(t,x,y,z) \ge 0$ for $(x,y,z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ and $t > 0$. Thus, the boundedness of $V + \tau W$ yields that

$$
W_t = \Delta W + cW_z + f'(V + \tau W)W \le \Delta W + cW_z + \lambda_3 W.
$$

Here recall that $\lambda_3 := \sup_{u\in[-\epsilon,1+\epsilon]} |f'(u)|$. Let $U(t,x,y,z)$ be the solution of the following initial valued problem

$$\begin{cases} U_t = \Delta U + cU_z + \lambda_3 U, \quad x\in\mathbb{R}^n,\ y\in\mathbb{R},\ z\in\mathbb{R},\ t>0,\\ U(0,x,y,z) = |W(0,x,y,z)| = |v_0(x,y,z) - V(y,z)|, \quad x\in\mathbb{R}^n,\ y\in\mathbb{R},\ z\in\mathbb{R}. \end{cases}$$

Then, we have $W(t,x,y,z)\le U(t,x,y,z)$ and

$$U(t,x,y,z) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \Gamma(t,x,y,z,\xi,\eta,\zeta)U(0,\xi,\eta,\zeta)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta$$

for $(t,x,y,z)\in(0,+\infty)\times\mathbb{R}^{n+2}$, where

$$\Gamma(t,x,y,z,\xi,\eta,\zeta) := \frac{\mathrm{e}^{\lambda_3 t}}{(4\pi t)^{\frac{n+2}{2}}}\exp\left\{-\frac{|x-\xi|^2+|y-\eta|^2+|z-\zeta-ct|^2}{4t}\right\}.$$

Fix $T>0$. Then, there exist constants $B_1>0$ and $B_2>0$ such that

$$\Gamma(t,x,y,z) \le \frac{B_1}{t^{\frac{n+2}{2}}}\mathrm{e}^{-B_2\frac{x^2+y^2+z^2}{t}}, \quad (x,y,z)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R},\ t\in(0,T]. \tag{3.25}$$

It suffices to prove that for any fixed $T>0$,

$$\lim_{R\to+\infty}\sup_{|x|+|y|+|z|\ge R}\frac{U(T,x,y,z)}{\phi^\gamma\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)} = 0, \quad \gamma\in\left(\frac{\Lambda_1}{\Lambda},1\right).$$

For any fixed $T>0$, we have

$$\begin{aligned}
&\frac{U(T,x,y,z)}{\phi^\gamma\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)}\\
=&\frac{1}{\phi^\gamma\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \Gamma(T,x,y,z,\xi,\eta,\zeta)U(0,\xi,\eta,\zeta)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\\
=&\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \Gamma(T,x,y,z,\xi,\eta,\zeta)\frac{U(0,\xi,\eta,\zeta)}{\phi^\gamma\left(\frac{c_*}{c}\left(\zeta-\tau_*|\eta|\right)\right)}\frac{\phi^\gamma\left(\frac{c_*}{c}\left(\zeta-\tau_*|\eta|\right)\right)}{\phi^\gamma\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)}\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\\
\le&\left(\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \Gamma(T,x,y,z,\xi,\eta,\zeta)\frac{U^2(0,\xi,\eta,\zeta)}{\phi^{2\gamma}\left(\frac{c_*}{c}\left(\zeta-\tau_*|\eta|\right)\right)}\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\right)^{\frac{1}{2}}\\
&\times\left(\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \Gamma(T,x,y,z,\xi,\eta,\zeta)\frac{\phi^{2\gamma}\left(\frac{c_*}{c}\left(\zeta-\tau_*|\eta|\right)\right)}{\phi^{2\gamma}\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)}\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\right)^{\frac{1}{2}}\\
:=&I_1^{\frac{1}{2}}(T,x,y,z)I_2^{\frac{1}{2}}(T,x,y,z),
\end{aligned}$$

where

$$I_1(T,x,y,z) := \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \Gamma(T,x,y,z,\xi,\eta,\zeta)\frac{U^2(0,\xi,\eta,\zeta)}{\phi^{2\gamma}\left(\frac{c_*}{c}\left(\zeta-\tau_*|\eta|\right)\right)}\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta,$$

$$I_2(T,x,y,z) := \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \Gamma(T,x,y,z,\xi,\eta,\zeta)\frac{\phi^{2\gamma}\left(\frac{c_*}{c}\left(\zeta-\tau_*|\eta|\right)\right)}{\phi^{2\gamma}\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)}\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta.$$

Now we estimate $I_1(T,x,y,z)$. It follows from (3.24) that, for any fixed $\varepsilon>0$, there exists $\overline{R}>0$ such that

$$\sup_{|\xi|+|\eta|+|\zeta|\geq R}\frac{U(0,\xi,\eta,\zeta)}{\phi^{\gamma}\left(\frac{c_*}{c}\left(\zeta-\tau_*|\eta|\right)\right)}<\frac{\sqrt{2}\varepsilon}{2}\mathrm{e}^{-\frac{\lambda_3 T}{2}}\quad \text{for } R>\overline{R}. \tag{3.26}$$

Direct calculations show that

$$\begin{aligned}I_1(T,x,y,z)=&\int\limits_{|\xi|+|\eta|+|\zeta|\geq\overline{R}}\Gamma(T,x,y,z,\xi,\eta,\zeta)\frac{U^2(0,\xi,\eta,\zeta)}{\phi^{2\gamma}\left(\frac{c_*}{c}\left(\zeta-\tau_*|\eta|\right)\right)}\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\\&+\int\limits_{|\xi|+|\eta|+|\zeta|\leq\overline{R}}\Gamma(T,x,y,z,\xi,\eta,\zeta)\frac{U^2(0,\xi,\eta,\zeta)}{\phi^{2\gamma}\left(\frac{c_*}{c}\left(\zeta-\tau_*|\eta|\right)\right)}\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\\&:=\left(J_1(T,x,y,z)+J_2(T,x,y,z)\right)\mathrm{e}^{\lambda_3 T}.\end{aligned}$$

Obviously, (3.26) yields that $J_1(T,x,y,z)<\left(\frac{\sqrt{2}\varepsilon}{2}\mathrm{e}^{-\frac{\lambda_3 T}{2}}\right)^2$. On the other hand, since

$$\lim_{R\to\infty}\sup_{|x|+|y|+|z|\geq R}\exp\left\{-\frac{|x-\xi|^2+|y-\eta|^2+|z-\zeta-cT|^2}{4T}\right\}=0$$

uniformly for $|\xi|+|\eta|+|\zeta|\leq\overline{R}$ and the function $\frac{U^2(0,\xi,\eta,\zeta)}{\phi^{2\gamma}\left(\frac{c_*}{c}(\zeta-\tau_*|\eta|)\right)}$ is bounded for $(\xi,\eta,\zeta)\in\mathbb{R}^{n+2}$, there exists $\widehat{R}>0$ such that

$$J_2(T,x,y,z)<\left(\frac{\sqrt{2}\varepsilon}{2}\mathrm{e}^{-\frac{\lambda_3 T}{2}}\right)^2\quad \text{for } |x|+|y|+|z|\geq\widehat{R}.$$

Therefore, for $\widetilde{R}\geq\widehat{R}$, we have

$$\sup_{|x|+|y|+|z|\geq\widetilde{R}}I_1^{\frac{1}{2}}(T,x,y,z)\leq\mathrm{e}^{\frac{\lambda_3 T}{2}}\left(J_1(T,x,y,z)+J_2(T,x,y,z)\right)^{\frac{1}{2}}=\varepsilon,$$

which yields that

$$\lim_{R\to\infty}\sup_{|x|+|y|+|z|\geq R}I_1(T,x,y,z)=0.$$

Thus, we can complete the proof by showing that I_2 is bounded on $(x,y,z)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$. In fact, by (3.25), we have

$$\begin{aligned}I_2(T,x,y,z)=&\int\limits_{\mathbb{R}^{n+2}}\Gamma(T,\xi,\eta,\zeta)\frac{\phi^{2\gamma}\left(\frac{c_*}{c}\left(z-\zeta-\tau_*|y-\eta|\right)\right)}{\phi^{2\gamma}\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)}\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\\\leq&\int\limits_{\mathbb{R}^{n+2}}\frac{B_1}{T^{\frac{n+2}{2}}}\mathrm{e}^{-B_2\frac{\xi^2+\eta^2+\zeta^2}{T}}\frac{\phi^{2\gamma}\left(\frac{c_*}{c}\left(z-\zeta-\tau_*|y-\eta|\right)\right)}{\phi^{2\gamma}\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)}\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\\=&B_1\int\limits_{\mathbb{R}^{n+2}}\mathrm{e}^{-B_2(\xi^2+\eta^2+\zeta^2)}\frac{\phi^{2\gamma}\left(\frac{c_*}{c}\left(z-\sqrt{T}\zeta-\tau_*|y-\sqrt{T}\eta|\right)\right)}{\phi^{2\gamma}\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)}\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\\\leq&B_1\int\limits_{\mathbb{R}^{n+2}}\mathrm{e}^{-B_2(\xi^2+\eta^2+\zeta^2)}\frac{\phi^{2\gamma}\left(\frac{c_*}{c}\left(z-\tau_*|y|-\left(\sqrt{T}|\zeta|+\tau_*\sqrt{T}|\eta|\right)\right)\right)}{\phi^{2\gamma}\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)}\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\end{aligned}$$

for $(x,y,z)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$. Recall that $\lambda_1:=\sup_{\mathcal{P}\in\mathbb{R}}\left|\frac{\phi'(\mathcal{P})}{\phi(\mathcal{P})}\right|$. Since the function $\phi(\mathcal{P}+\tilde{\mathcal{P}})\mathrm{e}^{\lambda_1\tilde{\mathcal{P}}}$ is increasing in $\tilde{\mathcal{P}}\in\mathbb{R}$ for each $\mathcal{P}\in\mathbb{R}$, we have

$$I_2(T,x,y,z)\le B_1\int\limits_{\mathbb{R}^{n+2}}\mathrm{e}^{-B_2(\xi^2+\eta^2+\zeta^2)}\exp\left(\frac{2\gamma\lambda_1c_*}{c}\left(\sqrt{T}|\zeta|+\tau_*\sqrt{T}|\eta|\right)\right)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta:=B(T)$$

where $B(T)$ is a positive constant depending on T. This completes the proof. □

Proof of Theorem 1.3. For simplicity, we denote $v\left(t,x,y,z;v_0\right)$ by $v\left(t,x,y,z\right)$. We only show the upper estimate, since the lower estimate can be proved similarly. Take constants $k>0$ as in (3.3) and $\sigma\ge\max\{\sigma_0,\sigma_1,1\}$. It follows from (1.8) that there exists a constant $D_1>0$ such that

$$\sup_{(x,y,z)\in\mathbb{R}^{n+2}}\frac{|V_z\left(y,z\right)|}{\phi^\gamma\left(\frac{c_*}{c}(z-\tau_*|y|)\right)}\le D_1.$$

For any $\varsigma>0$, set $\hat{\varsigma}=\min\left\{\frac{\ln 2}{2\lambda_1},\frac{\varsigma}{8D_1},\frac{\varsigma}{2\sqrt{2}\mathrm{e}^{\lambda_1\frac{\alpha}{\nu}}}\right\}$. Since $v_0\in C\left(\mathbb{R}^{n+2},[0,1]\right)$ satisfies

$$\lim_{R\to+\infty}\sup_{|x|+|y|+|z|\ge R}\frac{|v_0(x,y,z)-V(y,z)|}{\phi^\gamma\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)}=0\quad\text{for some }\gamma\in\left(\frac{\Lambda_1}{\Lambda},1\right),$$

the strong maximum principle yields that

$$0<v(t,x,y,z)<1,\quad (x,y,z)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R},\ t>0.$$

Lemma 3.8 implies that for any fixed $T>0$, there exists a constant $R^*>0$ large enough such that

$$\sup_{|x|+|y|+|z|\ge R^*}\frac{|v(T,x,y,z)-V(y,z)|}{\phi^\gamma\left(\frac{c_*}{c}\left(z-\tau_*|y|\right)\right)}\le\frac{\hat{\varsigma}}{\sigma}.$$

Thus, we can choose a function $w_0(x)\ge 0$ satisfying $\lim_{|x|\to\infty}w_0(x)=0$ and

$$v(T,x,y,z)\le V(y,z-w_0(x))+\frac{\hat{\varsigma}}{\sigma}\phi^\gamma\left(\frac{c_*}{c}\left(z-w_0(x)-\tau_*|y|\right)\right),\quad\forall\ (x,y,z)\in\mathbb{R}^{n+2}.$$

By $\tau_*|y|\le\varphi(y)$ and $\phi'(\mathcal{P})<0$, we have

$$v(T,x,y,z)\le V(y,z-w_0(x))+\frac{\hat{\varsigma}}{\sigma}\phi^\gamma\left(\frac{c_*}{c}\left(z-w_0(x)-\frac{\varphi(\nu y)}{\nu}\right)\right),$$

where ν is defined as in Lemma 3.6. Let $w(t,x)$ be the solution to the following equation

$$\begin{cases}w_t=\Delta_x w+k|\nabla_x w|^2,\\ w(0,x)=w_0(x),\end{cases}$$

where k is defined as in (3.3). Then, Lemma 2.1 yields that there exists $T_1>0$ large enough such that $0\le w(t,x)\le\hat{\varsigma}$ for $x\in\mathbb{R}^n$ and $t\ge T_1$. Therefore, by the comparison principle and the supersolution constructed in Lemma 3.6, we have

$$\begin{aligned}v(t,x,y,z)\le& V\left(y,z-w(t-T,x)-\hat{\varsigma}\left(1-\mathrm{e}^{-\lambda(t-T)}\right)\right)\\&+\frac{\hat{\varsigma}}{\sigma}\mathrm{e}^{-\lambda(t-T)}\phi^\gamma\left(\frac{c_*}{c}\left(z-w(t-T,x)-\frac{\varphi(\nu y)}{\nu}\right)\right)\end{aligned}$$

for $t \geq T+T_1$ and $(x,y,z) \in \mathbb{R}^{n+2}$. Thus,

$$\frac{v(t,x,y,z)-V(y,z)}{\phi^\gamma\left(\frac{c_*}{c}(z-\tau_*|y|)\right)} \leq \frac{V\left(y, z-w(t-T,x)-\hat{\varsigma}\left(1-\mathrm{e}^{-\lambda(t-T)}\right)\right)-V(y,z)}{\phi^\gamma\left(\frac{c_*}{c}(z-\tau_*|y|)\right)} + \frac{\hat{\varsigma}}{\sigma}\mathrm{e}^{-\lambda(t-T)}\frac{\phi^\gamma\left(\frac{c_*}{c}\left(z-w(t-T,x)-\frac{\varphi(\nu y)}{\nu}\right)\right)}{\phi^\gamma\left(\frac{c_*}{c}(z-\tau_*|y|)\right)}.$$

Let $\varpi(t,x) := w(t-T,x)+\hat{\varsigma}\left(1-\mathrm{e}^{-\lambda(t-T)}\right)$. It is easy to obtain that $0 \leq \varpi(t,x) \leq 2\hat{\varsigma}$ for $x \in \mathbb{R}^n$ and $t \geq T+T_1$. Recall that $\lambda_1 := \sup_{\mathcal{P}\in\mathbb{R}}\left|\frac{\phi'(\mathcal{P})}{\phi(\mathcal{P})}\right|$. Then, the function $\phi(\mathcal{P}+\tilde{\mathcal{P}})\mathrm{e}^{\lambda_1\tilde{\mathcal{P}}}$ is increasing in $\tilde{\mathcal{P}} \in \mathbb{R}$ for each $\mathcal{P} \in \mathbb{R}$. Thus, we have

$$\begin{aligned}
&\frac{V(y,z-\varpi(t,x))-V(y,z)}{\phi^\gamma\left(\frac{c_*}{c}(z-\tau_*|y|)\right)} \\
&= \frac{-V_z(y,z-\tau\varpi(t,x))}{\phi^\gamma\left(\frac{c_*}{c}(z-\tau_*|y|)\right)}\varpi(t,x) \\
&= \frac{-V_z(y,z-\tau\varpi(t,x))}{\phi^\gamma\left(\frac{c_*}{c}(z-\tau\varpi(t,x)-\tau_*|y|)\right)}\frac{\phi^\gamma\left(\frac{c_*}{c}(z-\tau\varpi(t,x)-\tau_*|y|)\right)}{\phi^\gamma\left(\frac{c_*}{c}(z-\tau_*|y|)\right)}\varpi(t,x) \\
&\leq 2\hat{\varsigma}\sup_{(x,y,z)\in\mathbb{R}^{n+2}}\frac{-V_z(y,z)}{\phi^\gamma\left(\frac{c_*}{c}(z-\tau_*|y|)\right)}\sup_{(t,x,y,z)\in\Omega}\frac{\phi^\gamma\left(\frac{c_*}{c}(z-\tau\varpi(t,x)-\tau_*|y|)\right)}{\phi^\gamma\left(\frac{c_*}{c}(z-\tau_*|y|)\right)} \\
&\leq 2\hat{\varsigma}D_1\mathrm{e}^{2\lambda_1\hat{\varsigma}},
\end{aligned}$$

and by $\tau_*|y| \leq \varphi(y) \leq \tau_*|y|+a$ for all $y \in \mathbb{R}$, we have

$$\frac{\phi^\gamma\left(\frac{c_*}{c}\left(z-w(t-T,x)-\frac{\varphi(\nu y)}{\nu}\right)\right)}{\phi^\gamma\left(\frac{c_*}{c}(z-\tau_*|y|)\right)} \leq \frac{\phi^\gamma\left(\frac{c_*}{c}\left(z-\tau_*|y|-\hat{\varsigma}-\frac{a}{\nu}\right)\right)}{\phi^\gamma\left(\frac{c_*}{c}(z-\tau_*|y|)\right)} \leq \mathrm{e}^{\lambda_1\frac{a}{\nu}}\mathrm{e}^{\lambda_1\hat{\varsigma}},$$

where $\tau \in (0,1)$ and $\Omega := [T+T_1,\infty)\times\mathbb{R}^{n+2}$.

Combining the above arguments, we obtain

$$\frac{v(t,x,y,z)-V(y,z)}{\phi^\gamma\left(\frac{c_*}{c}(z-\tau_*|y|)\right)} \leq 2\hat{\varsigma}D_1\mathrm{e}^{2\lambda_1\hat{\varsigma}}+\hat{\varsigma}\mathrm{e}^{\lambda_1\frac{a}{\nu}}\mathrm{e}^{\lambda_1\hat{\varsigma}} \leq \frac{\varsigma}{2}+\frac{\varsigma}{2}=\varsigma$$

for $t \geq T+T_1$ and $(x,y,z) \in \mathbb{R}^{n+2}$. We complete the proof of Theorem 1.3. □

4. Multidimensional stability of planar traveling fronts in $\mathbb{R}^{n+1}$ with $n \geq 1$

In this section, we show the multidimensional stability of planar traveling fronts of Eq. (1.1) in $\mathbb{R}^{n+1}$ with $n \geq 1$. That is, we prove Theorems 1.9–1.12.

In the sequel, Δ_x and ∇_x denote the n-dimensional Laplacian and the n-dimensional gradient operator, respectively. Similar to the proof of Lemma 2.3 of [34], we can obtain that $\phi\left(z-w^\pm(t,x)\right)$ are a supersolution and a subsolution to (1.13), respectively, where $w^\pm(t,x)$ are defined as in Sect. 2 with $k=k^*$. Recall that k^* is defined in Lemma 3.3. Then, similar to the proof of Theorems 1.4, 1.6 and 1.7 of this paper, Theorems 1.10–1.12 can be proved.

Similar to those done in Sect. 3.2 (see Lemmas 3.6, 3.7 and 3.8), we have the following lemmas.

Lemma 4.1. *Let $k^*>0$ be defined as in Lemma 3.3. Then, for $\gamma \in \left(\frac{\Lambda_1}{\Lambda},1\right)$, there exist some constants $\delta_0^*>0$ and $\sigma_0^*>0$ such that, for any $\delta \in (0,\delta_0^*]$ and $\sigma \geq \sigma_0^*$, and any functions $w^+(t,x)$ satisfying*

$$w_t^+ = \Delta_x w^+ + k^*\left|\nabla_x w^+\right|^2, \quad x \in \mathbb{R}^n,\ t>0,$$

the function defined by

$$v^+(t,x,z) := \phi\left(z - w^+(t,x) - \sigma\delta(1-\mathrm{e}^{-\lambda t})\right) + \delta \mathrm{e}^{-\lambda t}\phi^\gamma\left(z - w^+(t,x) - \sigma\delta(1-\mathrm{e}^{-\lambda t})\right).$$

is a supersolution of (1.13) *on* $(x,z) \in \mathbb{R}^n \times \mathbb{R}$ *and* $t > 0$*, that is, it satisfies the following inequality*

$$\mathcal{F}[v^+] := v_t^+ - \Delta v^+ - c_* v_z^+ - f\left(v^+\right) \geq 0.$$

Lemma 4.2. *Let* $k^* > 0$ *be defined as in Lemma 3.3. Then, for* $\gamma \in \left(\frac{\Lambda_1}{\Lambda}, 1\right)$*, there exist some constants* $\delta_1^* > 0$ *and* $\sigma_1^* > 0$ *such that, for any* $\delta \in (0, \delta_1^*]$ *and* $\sigma \geq \sigma_1^*$*, and any bounded functions* $w^-(t,x)$ *satisfying*

$$w_t^- = \Delta_x w^- - k^*\left|\nabla_x w^-\right|^2, \quad x \in \mathbb{R}^n,\ t > 0,$$

the function defined by

$$v^-(t,x,z) := \phi\left(z - w^-(t,x) + \sigma\delta(1-\mathrm{e}^{-\lambda t})\right) - \delta \mathrm{e}^{-\lambda t}\phi^\gamma\left(z + w^-(t,x) + \sigma\delta(1-\mathrm{e}^{-\lambda t})\right).$$

is a subsolution of (1.13) *on* $(x,z) \in \mathbb{R}^n \times \mathbb{R}$ *and* $t > 0$*, that is, it satisfies the following inequality*

$$\mathcal{L}[v^-] := v_t^- - \Delta v^- - c_* v_z^- - f\left(v^-\right) \leq 0.$$

Lemma 4.3. *Suppose that the initial value* $v_0 \in C\left(\mathbb{R}^{n+1}, [0,1]\right)$ $(n \geq 1)$ *satisfies*

$$\lim_{R\to+\infty} \sup_{|x|+|z|\geq R} \frac{|v_0(x,z) - \phi(z)|}{\phi^\gamma(z)} = 0$$

for some $\gamma \in \left(\frac{\Lambda_1}{\Lambda}, 1\right)$*, then for any fixed* $T > 0$*, we have*

$$\lim_{R\to+\infty} \sup_{|x|+|z|\geq R} \frac{|v(T,x,z;v_0) - \phi(z)|}{\phi^\gamma(z)} = 0.$$

Hence, similar to the proof of Theorem 1.3 in Sect. 3.2, Theorem 1.9 can be obtained.

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Zhen-Hui Bu and Zhi-Cheng Wang
School of Mathematics and Statistics
Lanzhou University
Lanzhou 730000 Gansu
People's Republic of China
e-mail: wangzhch@lzu.edu.cn

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