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Positive solutions for nonlocal dispersal equation with spatial degeneracy

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Abstract. In this paper, we consider the positive solutions of the nonlocal dispersal equation

$$\int_{\Omega} J(x,y)[u(y)-u(x)]\mathrm{d}y = -\lambda m(x)u(x) + [c(x)+\varepsilon]u^p(x) \quad \text{in } \bar{\Omega},$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, λ, ε and $p > 1$ are positive constants. The dispersal kernel J and the coefficient $c(x)$ are nonnegative, but $c(x)$ has a degeneracy in some subdomain of Ω. In order to study the influence of heterogeneous environment on the nonlocal system, we study the sharp spatial patterns of positive solutions as $\varepsilon \to 0$. We obtain that the positive solutions always have blow-up asymptotic profiles in $\bar{\Omega}$. Meanwhile, we find that the profiles in degeneracy domain are different from the domain without degeneracy.

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1. Introduction and main results

Let $J : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a nonnegative continuous function. Recently, there has been considerable interest in the study of the nonlocal dispersal equation

$$u_t(x,t) = \int_{\Omega} J(x,y)[u(y,t)-u(x,t)]\mathrm{d}y + f(x,u),$$

see among other references [1,2,9], here Ω is a subdomain of $\mathbb{R}^N$. In fact, nonlocal dispersal equations have been widely used to model different dispersal phenomena in material science and ecology, see, e.g., [3–5,14–16]. As stated in [11], if $u(y,t)$ is thought of as the density at location y at time t, and $J(x,y)$ is thought of as the probability distribution of jumping from y to x, then $\int_{\Omega} J(x,y)u(y,t)\mathrm{d}y$ denotes the rate at which individuals are arriving to location x from all other places and $-\int_{\Omega} J(y,x)u(x,t)\mathrm{d}y$ is the rate at which they are leaving location x to all other places. Nonlocal dispersal operator also characterizes the diffusion of species which may occur between nonadjacent locations.

In this paper, we consider the nonlocal dispersal equation

$$\int_{\Omega} J(x,y)[u(y)-u(x)]\mathrm{d}y = -\lambda m(x)u(x) + c(x)u^p(x) \quad \text{in } \bar{\Omega}, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, λ and $p > 1$ are positive constants. The coefficient $c(x)$ is nonnegative, and $m(x)$ may change sign in Ω. Thus, (1.1) describes the change in the population density $u(x)$ in heterogeneous environment with nonlocal dispersal and Logistic type growth rate. In (1.1), the dispersal takes place only in Ω. The individuals may not enter nor leave Ω. This is called homogeneous Neumann boundary condition or nonlocal impassable boundary condition [10,13].

Throughout this paper, we make the following assumptions on J, $c(x)$ and $m(x)$:

(**A1**) $J \in C(\bar{\Omega} \times \bar{\Omega})$ verifies $J(x,y) \geq 0$ and $J(x,y) = J(y,x)$ in $\bar{\Omega} \times \bar{\Omega}$; there exist $\alpha > 0$ and $l > 0$ such that $J(x,y) > \alpha$ if $x, y \in \bar{\Omega}$ and $|x-y| < l$.

(**A2**) $c(x) \in C(\bar{\Omega})$ is nontrivial,

$$c(x) = 0 \text{ in } \Omega_0 \text{ and } c(x) > 0 \text{ in } \bar{\Omega} \setminus \bar{\Omega}_0,$$

where Ω_0 is a subdomain of Ω with a positive measure.

(**A3**) $m(x) \in C(\bar{\Omega})$ and $\{x \in \Omega_0 : m(x) > 0\}$ has a positive measure.

Since the dispersal kernel function J is symmetric, the existence, uniqueness and stability of positive solutions of (1.1) are followed by the resent works of Coville [6] and Sun et al. [13,15]. More precisely, we know that there exist two constants λ_* and λ^* satisfying $0 \leq \lambda_* \leq \lambda^* < \infty$ such that (1.1) admits a unique positive solution $u \in C(\bar{\Omega})$ if $\lambda_* < \lambda < \lambda^*$. Meanwhile, if $\lambda_* = \lambda^*$ or $\lambda \geq \lambda^*$ or $\lambda \leq \lambda_*$ with $\int_\Omega m(x)\mathrm{d}x < 0$, then (1.1) admits no positive solution [13]. The constants λ_* and $\lambda^* > 0$ can be unique determined as follows:

$$\lambda_* = \inf_{u \in \mathcal{H}(\Omega)} \frac{\int_\Omega \int_\Omega J(x,y)(u(x)-u(y))^2 \mathrm{d}x\mathrm{d}y}{2\int_\Omega m(x)u^2(x)\mathrm{d}x},$$

$$\lambda^* = \inf_{u \in \mathcal{H}(\Omega_0)} \frac{\int_{\Omega_0} d(x)u^2(x)\mathrm{d}x - \int_{\Omega_0}\int_{\Omega_0} J(x,y)u(x)u(y)\mathrm{d}x\mathrm{d}y}{\int_{\Omega_0} m(x)u^2(x)\mathrm{d}x},$$

where $d(x) = \int_\Omega J(x,y)\mathrm{d}y$ and

$$\mathcal{H}(\Omega) = \left\{ v \in L^2(\Omega) \middle| \int_\Omega m(x)v^2(x)\mathrm{d}x > 0 \right\}.$$

We know from [13] that the constant $\lambda^* > 0$, but λ_* may be zero. In fact, the integral $\int_\Omega m(x)\mathrm{d}x$ plays important roles on the sign of λ_* and $\lambda_* > 0$ ($\lambda_* = 0$) if $\int_\Omega m(x)\mathrm{d}x < 0$ ($\int_\Omega m(x)\mathrm{d}x \geq 0$). We are interested in the positive solutions of (1.1), so we always assume that $\lambda_* < \lambda^*$ in the rest of the paper.

In (1.1), the coefficient $c(x)$ has a spatial degeneracy and the sign of $m(x)$ may change. We want to know the influence of heterogeneous environment on the nonlocal dispersal system (1.1). To this end, we shall consider the sharp spatial patterns of positive solutions. We study the perturbed nonlocal dispersal equation

$$\int_\Omega J(x,y)[u(y)-u(x)]\mathrm{d}y = -\lambda m(x)u(x) + [c(x)+\varepsilon]u^p(x) \text{ in } \bar{\Omega}, \tag{1.2}$$

where $\varepsilon > 0$ is a positive parameter. In this case, the sufficiently large constant is an upper solution of (1.2). On the other hand, we can construct a lower solution by the argument in [6,14]. Thus, we know that (1.2) admits a unique positive solution $u_\varepsilon \in C(\bar{\Omega})$ for every $\lambda > \lambda_*$, see [13].

Now we are ready to state the main results. We first consider the asymptotic profiles for positive solutions of (1.2) as $\varepsilon \to 0$.

Theorem 1.1. *Assume that $\lambda > \lambda_*$ and $\varepsilon > 0$. Let $u_\varepsilon(x)$ be the positive solution of (1.2), then we know that*

$$u_{\varepsilon_1}(x) \geq u_{\varepsilon_2}(x) \quad \text{in } \bar{\Omega}$$

for $\varepsilon_2 \geq \varepsilon_1 > 0$*. Moreover, we have the following asymptotic behavior.*

(*i*) *If* $\lambda_* < \lambda < \lambda^*$*, then*

$$\lim_{\varepsilon\to 0+} u_\varepsilon(x) = u(x) \quad \text{uniformly in } \bar{\Omega},$$

where $u(x)$ *is the unique positive solution of* (1.1).

(*ii*) *If* $\lambda > \lambda^*$*, then*

$$\lim_{\varepsilon\to 0+} u_\varepsilon(x) = \infty \text{ uniformly in } \bar{\Omega}.$$

Theorem 1.1 gives the asymptotic profiles of positive solutions to (1.2). As $\varepsilon \to 0+$, the unique positive solution $u_\varepsilon(x) \to \infty$ in $\bar{\Omega}$. This is different from the classical problems. Let $v_\varepsilon(x)$ be the positive solution of reaction-diffusion equation

$$\begin{cases} \Delta u = -\lambda m(x)u + [c(x) + \varepsilon]u^p(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

here we assume further that Ω is smooth and ν is the unit outward normal to $\partial\Omega$. It follows from [7,8] that under some assumptions on λ, $v_\varepsilon(x)$ tends to infinity only in Ω_0 and is still bounded in $\bar{\Omega} \setminus \bar{\Omega}_0$ as $\varepsilon \to 0+$.

To reveal the complex influence of heterogeneous environment on the nonlocal dispersal systems, we will further investigate the sharp spatial patterns of positive solutions.

Theorem 1.2. *Assume that* $u_\varepsilon(x)$ *is the positive solution of* (1.2) *for* $\lambda > \lambda_*$ *and* $\varepsilon > 0$*. Set* $v_\varepsilon(x) = \varepsilon^{\frac{1}{p-1}} u_\varepsilon(x)$ *and* $\omega_\varepsilon(x) = \varepsilon^{\frac{1}{p(p-1)}} u_\varepsilon(x)$*, we have the following results.*

(*i*) *If* $\lambda_* < \lambda < \lambda^*$*, then*

$$\lim_{\varepsilon\to 0+} v_\varepsilon(x) = \lim_{\varepsilon\to 0+} \omega_\varepsilon(x) = 0 \text{ uniformly in } \bar{\Omega}.$$

(*ii*) *If* $\lambda > \lambda^*$*, then*

$$\lim_{\varepsilon\to 0+} v_\varepsilon(x) = \theta(x) \text{uniformly in} \bar{\Omega}_0 \tag{1.3}$$

and

$$\lim_{\varepsilon\to 0+} \omega_\varepsilon(x) = \left[\frac{\int_{\Omega_0} J(x,y)\theta(y)\mathrm{d}y}{c(x)} \right]^{\frac{1}{p}} \text{ locally uniformly in } \bar{\Omega} \setminus \bar{\Omega}_0,$$

where $\theta \in C(\bar{\Omega}_0)$ *satisfies* $\theta(x) > 0$ *in* $\bar{\Omega}_0$ *and*

$$\int_{\Omega_0} J(x,y)\theta(y)\mathrm{d}y - d(x)\theta(x) = -\lambda m(x)\theta(x) + \theta^p(x) \text{ in } \bar{\Omega}_0. \tag{1.4}$$

We know that the positive solutions of (1.2) always tend to ∞ in $\bar{\Omega}$ as $\varepsilon \to 0$. From Theorem 1.2, we obtain that the asymptotic profiles in degeneracy domain are different from the domain without degeneracy. The results are different from the case of classical diffusion problems, see [7].

The rest of this paper is organized as follows. In Sect. 2, we give some preliminaries. In Sect. 3, we prove the main results.

2. Preliminaries

In this section, we give some preliminaries, which will be used in the proof of the main theorems. To do this, we consider the nonlocal dispersal equation

$$\int_{\Omega} J(x,y)[u(y)-u(x)]\mathrm{d}y = -\lambda m(x)u(x) + u^p(x) \text{ in } \bar{\Omega}. \tag{2.1}$$

In (2.1), there is no spatial degeneracy appears. It follows from [13] that there exists a unique continuous positive solution $\hat{u}(x)$ for $\lambda > \lambda_*$.

We first give an upper estimate for $v_\varepsilon(x) = \varepsilon^{\frac{1}{p-1}} u_\varepsilon(x)$. Since $u_\varepsilon \in C(\bar{\Omega})$ is the positive solution of (1.2), we have

$$\int_{\Omega} J(x,y)[v_\varepsilon(y)-v_\varepsilon(x)]\mathrm{d}y = -\lambda m(x)v_\varepsilon(x) + \left[\frac{c(x)}{\varepsilon}+1\right] v_\varepsilon^p(x) \text{ in } \bar{\Omega}. \tag{2.2}$$

Lemma 2.1. *Let $\hat{u}(x)$ be the positive solution of (2.1) for $\lambda > \lambda_*$, then we have*

$$0 < v_\varepsilon(x) \le \hat{u}(x) \text{ in } \bar{\Omega} \tag{2.3}$$

for $\varepsilon > 0$.

Proof. Note that

$$\int_{\Omega} J(x,y)[v_\varepsilon(y)-v_\varepsilon(x)]\mathrm{d}y + \lambda m(x)v_\varepsilon(x) - v_\varepsilon^p(x) = \frac{c(x)}{\varepsilon} v_\varepsilon^p(x) \ge 0 \text{ in } \bar{\Omega},$$

we have $v_\varepsilon(x)$ is a lower solution to (2.1). By the uniqueness of positive solutions and upper-lower solutions argument, we know that (2.3) holds. □

Consider the nonlocal dispersal equation

$$\int_{\Omega_0} J(x,y)u(y)\mathrm{d}y - d(x)u(x) = -\lambda m(x)u(x) + u^p(x) \text{ in } \bar{\Omega}_0, \tag{2.4}$$

where $d(x) = \int_{\Omega} J(x-y)\mathrm{d}y$. We know that (2.4) admits a unique continuous positive solution $\bar{u}(x)$ for $\lambda > \lambda^*$, see [13]. It is similar to Lemma 2.1 that the following result holds.

Lemma 2.2. *Let $\bar{u}(x)$ be the positive solution of (2.4) for $\lambda > \lambda^*$, then we have*

$$v_\varepsilon(x) \ge \bar{u}(x) > 0 \text{ in } \bar{\Omega}_0 \tag{2.5}$$

for $\varepsilon > 0$.

Proposition 2.3. *Assume that $\lambda > \lambda^*$. Then, there exists $\delta > 0$ which is independent of ε such that*

$$d(x) - \lambda m(x) + v_\varepsilon^{p-1}(x) \ge \delta \text{ in } \bar{\Omega}_0$$

for $\varepsilon > 0$.

Proof. We know from (2.2) that

$$\int_{\Omega_0} J(x,y)v_\varepsilon(y)\mathrm{d}y \le \int_{\Omega} J(x,y)v_\varepsilon(y)\mathrm{d}y = [d(x) - \lambda m(x) + v_\varepsilon^{p-1}(x)]v_\varepsilon(x) \text{ in } \bar{\Omega}_0$$

and so

$$d(x) - \lambda m(x) + v_\varepsilon^{p-1}(x) \ge 0 \text{ in } \bar{\Omega}_0.$$

Then, it follows from Lemmas 2.1–2.2 that

$$\int_{\Omega_0} J(x,y)\bar{u}(y)\mathrm{d}y \le [d(x) - \lambda m(x) + v_\varepsilon^{p-1}(x)]\hat{u}(x) \text{ in } \bar{\Omega}_0.$$

Since $\hat{u}(x) > 0$ and $\bar{u}(x) > 0$ in $\bar{\Omega}_0$, we complete the proof. □

3. Proof of Theorems 1.1–1.2

In this section, we will prove the main theorems. We first prove claim (i) of Theorem 1.1. Since there exists a unique positive solution to (1.2), a simple upper-lower solutions argument shows that $u_{\varepsilon_1}(x) \ge u_{\varepsilon_2}(x)$ in $\bar{\Omega}$ provided $\varepsilon_2 \ge \varepsilon_1 > 0$. Thus, there exists $0 < u_1(x) \le u(x)$ such that

$$\lim_{\varepsilon \to 0+} u_\varepsilon(x) = u_1(x)$$

for any given $x \in \bar{\Omega}$. Applying dominated convergence theorem, we know that

$$\int_{\Omega} J(x,y)[u_1(y) - u_1(x)]\mathrm{d}y = -\lambda m(x)u_1(x) + c(x)u_1^p(x). \tag{3.1}$$

But (1.1) admits a unique positive solution $u(x)$ for $\lambda_* < \lambda < \lambda^*$. In view of (1.1) and (3.1), we know that $u_1(x) = u(x)$ in $\bar{\Omega}$. Then, it follows from Dini's theorem that

$$\lim_{\varepsilon \to 0+} u_\varepsilon(x) = u(x) \text{ uniformly in } \bar{\Omega}.$$

Now, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. (i) Since $\lambda_* < \lambda < \lambda^*$, the conclusion is followed by (i) of Theorem 1.1.

(ii) Set

$$h(x) = d(x) - \lambda m(x) + v_\varepsilon^{p-1}(x),$$

we know from Proposition 2.3 that $h(x) \ge \delta$ in $\bar{\Omega}_0$ for $\varepsilon > 0$. Take $x_1, x_2 \in \bar{\Omega}_0$, without loss of generality, we assume that $v_\varepsilon(x_1) \ge v_\varepsilon(x_2)$. Then, by (2.2), we get

$$\begin{aligned}
&\int_{\Omega} [J(x_1,y) - J(x_2,y)]v_\varepsilon(y)\mathrm{d}y \\
&= [d(x_1) - \lambda m(x_1) + p\hat{v}^{p-1}][v_\varepsilon(x_1) - v_\varepsilon(x_2)] \\
&\quad + [(d(x_1) - \lambda m(x_1)) - (d(x_2) - \lambda m(x_2))]v_\varepsilon(x_2) \\
&\ge [d(x_1) - \lambda m(x_1) + v_\varepsilon^{p-1}(x_2)][v_\varepsilon(x_1) - v_\varepsilon(x_2)] \\
&\quad + [(d(x_1) - \lambda m(x_1)) - (d(x_2) - \lambda m(x_2))]v_\varepsilon(x_2) \\
&\ge \delta[v_\varepsilon(x_1) - v_\varepsilon(x_2)] + [(d(x_1) - \lambda m(x_1)) - (d(x_2) - \lambda m(x_2))]v_\varepsilon(x_2),
\end{aligned}$$

where $\hat{v}$ is between $v_\varepsilon(x_1)$ and $v_\varepsilon(x_2)$. In view of Lemma 2.1, there holds

$$\begin{aligned}
&|v_\varepsilon(x_1) - v_\varepsilon(x_2)| \\
&\le \frac{1}{\delta}\int_{\Omega} |J(x_1,y) - J(x_2,y)|\hat{u}(y)\mathrm{d}y \\
&\quad + \frac{\max_{\bar{\Omega}_0} \hat{u}(x)}{\delta}[|d(x_1) - d(x_2)| + \lambda|m(x_1) - m(x_2)|]
\end{aligned}$$

for any $x_1, x_2 \in \bar{\Omega}_0$. It follows from (2.3) and (2.5) that $v_\varepsilon(x)$ is uniformly bounded in $\bar{\Omega}_0$. Subject to a subsequence, a simple compactness argument gives that there exists $v_1 \in C(\bar{\Omega}_0)$ such that $v_1(x) > 0$ in $\bar{\Omega}_0$ and

$$\lim_{\varepsilon\to 0+} v_\varepsilon(x) = v_1(x) \text{ uniformly in } \bar{\Omega}_0. \tag{3.2}$$

On the other hand, since

$$\int_\Omega J(x,y)v_\varepsilon(y)\mathrm{d}y - d(x)v_\varepsilon(x) + \lambda m(x)v_\varepsilon(x) = \left[\frac{c(x)}{\varepsilon} + 1\right] v_\varepsilon^p(x) \text{ in } \bar{\Omega},$$

we know from (2.3) that

$$\begin{aligned}\left[\frac{c(x)}{\varepsilon} + 1\right] v_\varepsilon^p(x) &\le \int_\Omega J(x,y)\hat{u}(y)\mathrm{d}y + d(x)\hat{u}(x) + \lambda|m(x)|\hat{u}(x)\\ &\le \int_\Omega J(x,y)\hat{u}(y)\mathrm{d}y + d(x)\hat{u}(x) + \lambda|m(x)|\hat{u}(x)\\ &\le \left[2\max_{\bar{\Omega}} d(x) + \lambda \max_{\bar{\Omega}} |m(x)|\right] \max_{\bar{\Omega}} \hat{u}(x).\end{aligned}$$

Thus, we have

$$v_\varepsilon(x) \le \left[\frac{[2\max_{\bar{\Omega}} d(x) + \lambda \max_{\bar{\Omega}} |m(x)|] \max_{\bar{\Omega}} \hat{u}(x)}{\frac{c(x)}{\varepsilon} + 1}\right]^{\frac{1}{p}}$$

and

$$\lim_{\varepsilon\to 0+} v_\varepsilon(x) = 0 \text{ locally uniformly in } \bar{\Omega} \setminus \bar{\Omega}_0. \tag{3.3}$$

In view of (3.2) and (3.3), by dominated convergence theorem, we know that

$$\int_{\Omega_0} J(x,y)v_1(y)\mathrm{d}y - d(x)v_1(x) = -\lambda m(x)v_1(x) + v_1^p(x) \text{ in } \bar{\Omega}_0. \tag{3.4}$$

But (3.4) admits a unique continuous positive solution for $\lambda > \lambda^*$, and we necessarily have

$$v_1(x) = \theta(x) \text{ in } \bar{\Omega}_0,$$

where $\theta(x)$ is the unique positive solution of (1.4). This also implies that (1.3) holds for the entire original sequences.

On the other hand, we can see that $\omega_\varepsilon(x) = \varepsilon^{\frac{1}{p(p-1)}} u_\varepsilon(x)$ satisfies

$$\int_\Omega J(x,y)[\omega_\varepsilon(y) - \omega_\varepsilon(x)]\mathrm{d}y = -\lambda m(x)\omega_\varepsilon(x) + [c(x)+\varepsilon]\frac{\omega_\varepsilon^p(x)}{\varepsilon^{1/p}} \quad \text{in } \bar{\Omega}$$

and so

$$\int_\Omega J(x,y)[v_\varepsilon(y) - v_\varepsilon(x)]\mathrm{d}y = -\lambda m(x)v_\varepsilon(x) + [c(x)+\varepsilon]\omega_\varepsilon^p(x) \quad \text{in } \bar{\Omega}.$$

Then, we get

$$\omega_\varepsilon(x) = \left[\frac{\int_\Omega J(x,y)[v_\varepsilon(y) - v_\varepsilon(x)]\mathrm{d}y + \lambda m(x)v_\varepsilon(x)}{c(x)+\varepsilon}\right]^{\frac{1}{p}}.$$

Again by (3.2)–(3.3), we have

$$\lim_{\varepsilon\to 0+}\omega_\varepsilon(x)=\left[\frac{\int\limits_{\Omega_0} J(x-y)\theta(y)\mathrm{d}y}{c(x)}\right]^{\frac{1}{p}} \text{ locally uniformly in } \bar{\Omega}\setminus\bar{\Omega}_0. \tag{3.5}$$

□

If $\lambda>\lambda^*$, by (3.2) and (3.5), we know that

$$\lim_{\varepsilon\to 0+}u_\varepsilon(x)=\infty \text{ uniformly in } \bar{\Omega}_0.$$

and

$$\lim_{\varepsilon\to 0+}u_\varepsilon(x)=\infty \text{ locally uniformly in } \bar{\Omega}\setminus\bar{\Omega}_0.$$

Note that

$$\begin{aligned}\int\limits_{\Omega_0} J(x,y)u_\varepsilon(y)\mathrm{d}y &\le \int\limits_{\Omega} J(x,y)u_\varepsilon(y)\mathrm{d}y\\ &= [d(x)-\lambda m(x)+[c(x)+\varepsilon]u_\varepsilon^{p-1}(x)]u_\varepsilon(x) \text{ in } \bar{\Omega},\end{aligned}$$

we can see that

$$\lim_{\varepsilon\to 0+}u_\varepsilon(x)=\infty \text{ uniformly in } \bar{\Omega}.$$

This completes the proof of Theorem 1.1.

In this paper, we study the sharp spatial patterns of positive solutions for nonlocal dispersal equations if $c(x)$ has a spatial degeneracy. Similarly, if $c(x)>0$ in $\bar{\Omega}$, we have the following result.

Theorem 3.1. *Assume that (A1) and (A3) hold. Assume further that $c(x)>0$ in $\bar{\Omega}$. Let $u_\varepsilon(x)$ be the positive solution of (1.2), then we know that*

$$u_{\varepsilon_1}(x)\ge u_{\varepsilon_2}(x) \text{ in } \bar{\Omega}$$

for $\varepsilon_2\ge\varepsilon_1>0$. Moreover, we have

$$\lim_{\varepsilon\to 0+}u_\varepsilon(x)=u(x) \text{ uniformly in } \bar{\Omega},$$

where $u(x)$ is the unique positive solution of (1.1).

At the end of this section, we consider the nonlocal dispersal equations

$$\begin{cases}\int\limits_{\mathbb{R}^N} J(x,y)u(y)\mathrm{d}y-u(x)=-\lambda m(x)u(x)+c(x)u^p(x), & x\in\bar{\Omega},\\ u(x)=0, & x\in\mathbb{R}^N\setminus\bar{\Omega}.\end{cases} \tag{3.6}$$

and

$$\begin{cases}\int\limits_{\mathbb{R}^N} J(x,y)u(y)\mathrm{d}y-u(x)=-\lambda m(x)u(x)+[c(x)+\varepsilon]u^p(x), & x\in\bar{\Omega},\\ u(x)=0, & x\in\mathbb{R}^N\setminus\bar{\Omega}.\end{cases} \tag{3.7}$$

In (3.6) and (3.7), we assume that the kernel function J satisfies

$$\int\limits_{\Omega} J(x,y)\mathrm{d}y\le 1 \text{and} \int\limits_{\Omega} J(x,y)\mathrm{d}y\not\equiv 1 \text{in} \bar{\Omega}. \tag{3.8}$$

Then, we can see that the dispersal takes place in $\mathbb{R}^N$, but we impose that $u(x)$ vanishes outside $\bar{\Omega}$, which is called homogeneous nonlocal Dirichlet boundary condition [3]. For heterogeneous nonlocal dispersal (1.1) and (3.6), $m(x)$ may change sign and the precisely upper bound of λ^* is unknown. The study of sharp spatial patterns and asymptotic profiles is different from the case that $m(x)$ is constant, see [10,12]. Let us define

$$\hat{\lambda}_* = \inf_{u\in\mathcal{H}(\Omega)} \frac{\int\limits_\Omega u^2(x)\mathrm{d}x - \int\limits_\Omega\int\limits_\Omega J(x,y)u(x)u(y)\mathrm{d}x\mathrm{d}y}{\int\limits_\Omega m(x)u^2(x)\mathrm{d}x},$$

$$\hat{\lambda}^* = \inf_{u\in\mathcal{H}(\Omega_0)} \frac{\int\limits_{\Omega_0} u^2(x)\mathrm{d}x - \int\limits_{\Omega_0}\int\limits_{\Omega_0} J(x,y)u(x)u(y)\mathrm{d}x\mathrm{d}y}{\int\limits_{\Omega_0} m(x)u^2(x)\mathrm{d}x},$$

where

$$\mathcal{H}(\Omega) = \left\{ v\in L^2(\Omega) \middle| \int\limits_\Omega m(x)v^2(x)\mathrm{d}x > 0 \right\}.$$

We know from [14] that $\hat{\lambda}^* \geq \hat{\lambda}_* > 0$.

The techniques and ideas of (1.1)–(1.2) in [13,15] can be modified to treat (3.6). We have the following existence and uniqueness theorems.

Theorem 3.2. *Assume that (3.8) holds. Then, (3.6) admits a unique positive solution if and only if $\hat{\lambda}_* < \lambda < \hat{\lambda}^*$.*

Theorem 3.3. *Assume that (3.8) holds. Then, (3.7) admits a unique positive solution if and only if $\lambda > \hat{\lambda}_*$.*

Similarly to Theorem 1.2, we can obtain the sharp patterns of positive solution to (3.7).

Theorem 3.4. *Assume that (3.8) holds. Let $u_\varepsilon(x)$ be the positive solution of (3.7) for $\lambda > \lambda_*$ and $\varepsilon > 0$. Set $v_\varepsilon(x) = \varepsilon^{\frac{1}{p-1}}u_\varepsilon(x)$ and $\omega_\varepsilon(x) = \varepsilon^{\frac{1}{p(p-1)}}u_\varepsilon(x)$, we have the following results.*

(*i*) *If $\hat{\lambda}_* < \lambda < \hat{\lambda}^*$, then*

$$\lim_{\varepsilon\to 0+} v_\varepsilon(x) = \lim_{\varepsilon\to 0+}\omega_\varepsilon(x) = 0 \text{ uniformly in } \bar{\Omega}.$$

(*ii*) *If $\lambda > \hat{\lambda}^*$, then*

$$\lim_{\varepsilon\to 0+} v_\varepsilon(x) = \hat{\theta}(x) \text{ uniformly in } \bar{\Omega}_0$$

and

$$\lim_{\varepsilon\to 0+}\omega_\varepsilon(x) = \left[\frac{\int\limits_{\Omega_0} J(x,y)\hat{\theta}(y)\mathrm{d}y}{c(x)}\right]^{\frac{1}{p}} \text{ locally uniformly in } \bar{\Omega}\setminus\bar{\Omega}_0,$$

where $\hat{\theta}\in C(\bar{\Omega}_0)$ satisfies $\hat{\theta}(x) > 0$ in $\bar{\Omega}_0$ and

$$\int\limits_{\Omega_0} J(x,y)\hat{\theta}(y)\mathrm{d}y - \hat{\theta}(x) = -\lambda m(x)\hat{\theta}(x) + \hat{\theta}^p(x) \text{ in } \bar{\Omega}_0.$$

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