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Existence and uniqueness of solution for a nonhomogeneous nonlocal problem

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Abstract. In this paper we analyze a class of elliptic problems involving a nonlocal Kirchhoff-type operator with variable coefficients and with a sign-changing datum. Under appropriated conditions on the coefficients, we have shown existence and uniqueness of solution.

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1. Introduction

In this paper we are concerned with the uniqueness of nontrivial classical solution for the following class of nonlocal elliptic equations

$$\begin{cases} -\left(a(x)+b(x)\int\limits_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = h(x) \text{ in } \Omega, \\ u=0 \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with smooth boundary, $a, b \in C^{0,\gamma}(\overline{\Omega})$, $\gamma \in (0,1)$, are positive functions with $a(x) \geq a_0 > 0$, $b(x) \geq b_0 > 0$ and $h \in C^{0,\gamma}(\overline{\Omega})$ is given.

In the case that functions a, b are positive constants, problem (P) is the N-dimensional stationary version of a hyperbolic problem proposed in [5]. Kirchhoff's equation models small transversal vibrations of an elastic string with fixed ends which is composed by a homogeneous material. Such an equation is a more realistic model than that provided by the classic D'Alembert's wave equation because it takes into account the change in the length of the string produced by transverse vibrations. The hyperbolic Kirchhoff problem (with a, b constants) receives special attention mainly after the apparition of [6]. In [6] the author proposes an abstract framework to the problem.

To our best knowledge, it was [2] the first work to study uniqueness of solution for problem (P) with a, b constants. It is an immediate consequence of Theorem 1 in [2] that if h is a Hölder continuous nonnegative (nonzero) function, then problem (P), with a, b constants, has a unique positive solution. In [2], sign-changing functions h were not considered. When a, b are not constants, problem (P) is even more relevant in terms of applications, because its unidimensional version models small transversal vibrations of an elastic string composed by nonhomogeneous materials (see [4], section 2). In [4] (see Theorem 1) the authors proved that for each $h \in L^\infty(\Omega)$ ($h \not\equiv 0$), problem (P) admits at least a nontrivial solution. Moreover, if h has defined sign ($h \leq 0$ or $h \geq 0$) such a solution is unique. Unfortunately, their approach

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does not work when h is a sign-changing function. To our best knowledge, the uniqueness of solution to problem (P) in the general case is actually an open problem.

In this article we obtain sufficient conditions on the quotient a/b to ensure uniqueness of solution when h is a sign-changing function. The main results of this paper are as follows:

Theorem 1.1. *If there exists $\theta > 0$ such that $a/b = \theta$ in Ω, then for each $h \in C^{0,\gamma}(\overline{\Omega})$ problem (P) has a unique solution.*

Theorem 1.2. *Let $a, b \in C^{2,\gamma}(\overline{\Omega})$, $h \in C^{0,\gamma}(\overline{\Omega})$ is sign-changing and suppose $c = a/b$ is not constant.*

(i) *If $\Delta c \geq 2|\nabla c|^2/c$ in Ω, then, for each $h \in C^{0,\gamma}(\overline{\Omega})$ given, problem (P) has a unique nontrivial classical solution.*

(ii) *If $\Delta c < 2|\nabla c|^2/c$ in some open $\Omega_0 \subset \Omega$, then, for each $h \in C^{0,\gamma}(\overline{\Omega})$ given, problem (P) has a unique nontrivial classical solution, provided that*

$$\frac{|\nabla c|_\infty c_M}{\sqrt{\lambda_1} c_L^2} \leq 3/2,$$

where λ_1 is the first eigenvalue of Laplacian operator with homogeneous Dirichlet boundary condition, $c_L = \min_{x \in \overline{\Omega}} c(x)$, $c_M = \max_{x \in \overline{\Omega}} c(x)$ and $|\nabla c|_\infty = \max_{x \in \overline{\Omega}} |\nabla c(x)|$.

Theorem 1.1 generalizes Theorem 1 in [2] because it holds for sign-changing or signed functions h. On the other hand, Theorem 1.2 complements Theorem 1 in [4].

The paper is organized as follows: In Sect. 2 we present some abstracts results, notations and definitions. In Sect. 3 we investigate a nonlocal eigenvalue problem which seems to be closely related to the uniqueness of solution to problem (P). In Sect. 4 we prove Theorems 1.1 and 1.2. Moreover, an alternative proof for the existence and uniqueness of solution in [4] is supplied.

2. Preliminaries

In this section we state some results and we define some notations which will be used throughout the paper.

Definition 2.1. We say that a function h is signed in Ω if $h \geq 0$ in Ω or $h \leq 0$ in Ω.

Definition 2.2. An application $\Psi : E \to F$ defined in Banach spaces is locally invertible in $u \in E$ if there are open sets $A \ni u$ in E and $B \ni \Psi(u)$ in F such that $\Psi : A \to B$ is a bijection. If Ψ is locally invertible in any point $u \in E$, it is said that $\Psi : E \to F$ is locally invertible.

Definition 2.3. Let M, N be metric spaces. We say that a map $\Psi : M \to N$ is proper if $\Psi^{-1}(K) = \{u \in M : \Psi(u) \in K\}$ is compact in M for all compact set $K \subset N$.

Now, we enunciate the classic local and global inverse function theorems, whose proofs can be found, for example, in [1].

Theorem 2.4. (Local Inverse Theorem) *Let E, F be two Banach spaces. Suppose $\Psi \in C^1(E, F)$ and $\Psi'(u) : E \to F$ is an isomorphism. Then, Ψ is locally invertible at u and its local inverse, Ψ^{-1}, is also a C^1-function.*

Theorem 2.5. (Global Inverse Theorem) *Let M, N be two metric spaces and $\Psi \in C(M, N)$ a proper and locally invertible function in M. Suppose that M is arcwise connected and N is simply connected. Then, Ψ is a homeomorphism from M onto N.*

Next, we state another classic result which will be used in our arguments and whose proof can be found, for example, in [3].

Proposition 2.6. *Suppose $m \in L^\infty(\Omega)$, $m(x) > 0$ in a set of positive measure and $A \in L^\infty(\Omega)$, $A(x) \geq \mathfrak{m}$ for some positive constant $\mathfrak{m}$. Then, problem*

$$\begin{cases} -div(A(x)\nabla u) = \lambda m(x) u \text{ in } \Omega, \\ u = 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

has a smallest positive eigenvalue $\lambda_1(m)$, which is simple and its corresponding eigenfunctions do not change sign in Ω.

Throughout this paper X is the Banach space

$$X = \left\{u \in C^{2,\gamma}(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\right\}$$

with norm

$$\|u\|_X = \|u\|_{C^2(\overline{\Omega})} + \max_{|\beta|=2} \left[D^\beta u\right]_\gamma,$$

where $\gamma \in (0,1)$, $\beta = (\beta_1, \ldots, \beta_N) \in \mathbb{N}^N$, $|\beta| = \beta_1 + \cdots + \beta_N$,

$$\|u\|_{C^2(\overline{\Omega})} = \sum_{0 \leq |\beta| \leq 2} \|D^\beta u\|_{C(\overline{\Omega})} \text{ and } [D^\beta u]_\gamma = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\gamma}.$$

Moreover, Y will denote the Banach space $C^{0,\gamma}(\overline{\Omega})$ with norm

$$\|f\|_Y = \|f\|_{C(\overline{\Omega})} + [f]_\gamma,$$

where $\|f\|_{C(\overline{\Omega})} = \max_{x\in\overline{\Omega}} |f(x)|$.

Hereafter same symbol C denotes different positive constants.

3. A nonlocal eigenvalue problem

In this section we are interested in studying the following nonlocal eigenvalue problem

$$\begin{cases} -\text{div}\left(\dfrac{\nabla u}{c + |\nabla u|_2^2}\right) = \lambda \left\{-\text{div}\left[\dfrac{\nabla c}{(c + |\nabla u|_2^2)^2}\right]\right\} u \text{ in } \Omega, \\ u = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{EP}$$

where $\Omega \subset \mathbb{R}^N$ is bounded smooth domain, λ is a positive parameter, and $c \in C^2(\overline{\Omega})$ is a positive (not constant) function. As we will see in the next section, problem (EP) arises naturally when one studies questions of uniqueness for the problem (P).

Before stating the main results of this section, we observe that

Lemma 3.1. *The set*

$$\mathcal{A} := \left\{\alpha > 0 : -\text{div}\left[\frac{\nabla \text{c}}{(\text{c} + \alpha)^2}\right] > 0 \text{ in some open set } \Omega_0 \subset \Omega\right\}$$

is not empty if and only if there is an open set $\hat{\Omega} \subset \mathbb{R}^N$ such that

$$\Delta c < 2\frac{|\nabla c|^2}{c} \text{ in } \hat{\Omega}. \tag{3.1}$$

Proof. Differentiating we get

$$-\text{div}\left[\frac{\nabla c}{(c+\alpha)^2}\right] = -\frac{1}{(c+\alpha)^2}\Delta c + \frac{2}{(c+\alpha)^3}|\nabla c|^2. \tag{3.2}$$

Now, note that

$$-\text{div}\left[\frac{\nabla c}{(c+\alpha)^2}\right] > 0 \text{ in some open set } \Omega_0 \tag{3.3}$$

if and only if

$$\Delta c < 2\frac{|\nabla c|^2}{(c+\alpha)} \text{ in } \Omega_0. \tag{3.4}$$

It is clear that the existence of a positive number α satisfying (3.4) is equivalent to inequality (3.1). □

Remark 1. In Lemma 3.1 we have shown also that $\mathcal{A} = \emptyset$ if and only if

$$\Delta c \geq 2\frac{|\nabla c|^2}{c} \text{ in } \Omega. \tag{3.5}$$

Certainly, there are many positive functions $c \in C^2(\overline{\Omega})$ verifying (3.5). For instance, setting $c = \delta e + 1$, where $0 < \delta \leq \min\{1/(4|\nabla e|_\infty^2), 1/(2|e|_\infty)\}$ and

$$\begin{cases} \Delta e = 1 \text{ in } \Omega, \\ e = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

we conclude that $c > 0$ and it satisfies (3.5).

Remark 2. When (3.1) holds, an interesting question is related to the topology of set $\mathcal{A}$. In this direction, the proof of Lemma 3.1 allows us to say that $\mathcal{A}$ contains even a neighborhood $(0, \alpha_0)$.

Now we are ready to claim the following result.

Theorem 3.2. *Suppose that* (3.1) *holds. For each* $\alpha \in \mathcal{A}$, *problem* (*EP*) *has a unique solution* $(\lambda_\alpha, u_\alpha)$ *such that* $\lambda_\alpha > 0$, $u_\alpha > 0$ *and* $|\nabla u_\alpha|_2^2 = \alpha$.

Proof. From Lemma 3.1 we get that $\mathcal{A} \neq \emptyset$. Since $c \in C^2(\overline{\Omega})$ and $b > 0$ in $\overline{\Omega}$, it follows from Proposition 2.6 that, for each $\alpha \in \mathcal{A}$, the eigenvalue problem

$$\begin{cases} -\text{div}\left(\dfrac{\nabla u}{c+\alpha}\right) = \lambda\left\{-\text{div}\left[\dfrac{\nabla c}{(c+\alpha)^2}\right]\right\} u \text{ in } \Omega, \\ u = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{P_α}$$

has a positive smallest eigenvalue λ_α whose associated eigenspace V_α is unidimensional and its eigenfunctions have defined sign. Choosing $u \in V_\alpha$ such that $u > 0$ and $|\nabla u|_2^2 = \alpha$, the result follows. □

Remark 3. In particular, if (3.1) holds, then

$$\int_\Omega u_\alpha^2 \left\{-\text{div}\left[\frac{\nabla c}{(c+\alpha)^2}\right]\right\} \mathrm{d}x = \frac{1}{\lambda_\alpha}\int_\Omega \frac{|\nabla u_\alpha|^2}{c+\alpha}\mathrm{d}x, \ \forall\, \alpha \in \mathcal{A}. \tag{3.7}$$

Corollary 3.3. *Suppose* (3.1). *For each* $\alpha \in \mathcal{A}$, *the following inequality holds*

$$\lambda_\alpha \geq \frac{\sqrt{\lambda_1}(c_L+\alpha)^2}{2|\nabla c|_\infty (c_M+\alpha)},$$

where λ_1 *is the first eigenvalue of Laplacian operator with homogeneous Dirichlet boundary condition,* $c_L = \min_{x\in\overline{\Omega}} c(x)$, $c_M = \max_{x\in\overline{\Omega}} c(x)$ *and* $|\nabla c|_\infty = \max_{x\in\overline{\Omega}} |\nabla c(x)|$.

Proof. From Remark 3, we get

$$\lambda_\alpha = \frac{\displaystyle\int_\Omega \frac{|\nabla u_\alpha|^2}{c+\alpha}\mathrm{d}x}{\displaystyle\int_\Omega u_\alpha^2\left\{-\text{div}\left[\frac{\nabla c}{(c+\alpha)^2}\right]\right\}\mathrm{d}x}. \tag{3.8}$$

Observe that

$$\int_\Omega \frac{|\nabla u_\alpha|^2}{c+\alpha}\mathrm{d}x \geq \frac{\alpha}{c_M+\alpha}. \tag{3.9}$$

Moreover, by using the divergence theorem,

$$\int_\Omega u_\alpha^2 \left\{-\mathrm{div}\left[\frac{\nabla c}{(c+\alpha)^2}\right]\right\}\mathrm{d}x = 2\int_\Omega \frac{u_\alpha \nabla u_\alpha \nabla c}{(c+\alpha)^2}\mathrm{d}x \le \frac{2|\nabla c|_\infty \int_\Omega u_\alpha |\nabla u_\alpha|\mathrm{d}x}{(c_L+\alpha)^2}.$$

From Hölder and Poincaré inequalities, we conclude that

$$\int_\Omega u_\alpha^2 \left\{-\mathrm{div}\left[\frac{\nabla c}{(c+\alpha)^2}\right]\right\}\mathrm{d}x \le \frac{2|\nabla c|_\infty \alpha}{\sqrt{\lambda_1}(c_L+\alpha)^2}. \tag{3.10}$$

From (3.8), (3.9) and (3.10) we have

$$\lambda_\alpha \ge \frac{\sqrt{\lambda_1}(c_L+\alpha)^2}{2|\nabla c|_\infty (c_M+\alpha)},$$

for all $\alpha \in \mathcal{A}$. □

4. Uniqueness results

In order to apply Theorem 2.5 we define operator $\Psi : X \to Y$ by

$$\Psi(u) = \left(a(x) + b(x)\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u.$$

In what follows, we will denote $M\left(x, |\nabla u|_2^2\right) = a(x) + b(x)\int_\Omega |\nabla u|^2 \mathrm{d}x$ for short, where $|\nabla u|_2^2 = \int_\Omega |\nabla u|^2 \mathrm{d}x$. The proof of main results of this paper will be divided in various propositions.

Proposition 4.1. *Operator $\Psi : X \to Y$ is proper.*

Proof. It is enough to prove that if $\{h_n\} \subset Y$ is a sequence converging to $h \in Y$ and $\{u_n\} \subset X$ is another sequence with $\Psi(u_n) = -\, h_n$, then $\{u_n\}$ possesses a convergent subsequence in X. For this, note that the equality $\Psi(u_n) = -\, h_n$ is equivalent to

$$-\ \Delta u_n = \frac{h_n}{M\left(x, |\nabla u_n|_2^2\right)}. \tag{4.1}$$

Observe that $h_n / M\left(., |\nabla u_n|_2^2\right) \in Y$ because $h_n \in Y$, $M\left(., |\nabla u_n|_2^2\right) \in Y$ and $M\left(x, |\nabla u_n|_2^2\right) \ge a_0$.

Moreover,

$$\left\| \frac{h_n(x)}{M\left(x, |\nabla u_n|_2^2\right)} \right\|_{C(\overline{\Omega})} \le \|h_n\|_{C(\overline{\Omega})}/a_0,\ \forall\ n \in \mathbb{N}. \tag{4.2}$$

From $\|h_n\|_{C(\overline{\Omega})} \le \|h_n\|_Y$, (4.2) and the boundedness of $\{h_n\}$ in Y, it follows that $\left\{h_n / M\left(x, |\nabla u_n|_2^2\right)\right\}$ is bounded in $C(\overline{\Omega})$. Thus, the continuous embedding from $C^{1,\gamma}(\overline{\Omega})$ into $C(\overline{\Omega})$ and equality in (4.1) tell us that $\{u_n\}$ is bounded in $C^{1,\gamma}(\overline{\Omega})$ (see Theorem 0.5 in [1]). Finally, by compact embedding from $C^{1,\gamma}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, we conclude that there exists $u \in C^1(\overline{\Omega})$ such that, passing to a subsequence,

$$u_n \to u \text{ in } C^1(\overline{\Omega}). \tag{4.3}$$

This convergence leads to

$$|\nabla u_n(x)|^2 \to |\nabla u(x)|^2 \text{ uniformly in } x \in \Omega. \tag{4.4}$$

Whence

$$|\nabla u_n|_2^2 \to |\nabla u|_2^2. \tag{4.5}$$

In what follows, we show that

$$\left\| \frac{h_n}{M\left(.,|\nabla u_n|_2^2\right)} \right\|_Y \leq C, \tag{4.6}$$

for some positive constant C. In fact, since $\{h_n\} \subset Y$ and $\left\{M\left(.,|\nabla u_n|_2^2\right)\right\} \subset Y$, with $M(x,t) \geq a_0 > 0$ for all $t \geq 0$, a straightforward manipulation shows us that

$$\left[\frac{h_n}{M\left(.,|\nabla u_n|_2^2\right)}\right]_\gamma \leq \frac{1}{a_0^2}\left(\|h_n\|_{C(\overline{\Omega})}\left[M\left(.,|\nabla u_n|_2^2\right)\right]_\gamma + \left\|M\left(.,|\nabla u_n|_2^2\right)\right\|_{C(\overline{\Omega})}[h_n]_\gamma\right).$$

From $\|h_n\|_{C(\overline{\Omega})}, [h_n]_\gamma \leq C$,

$$\left[M\left(.,|\nabla u_n|_2^2\right)\right]_\gamma \leq [a]_\gamma + [b]_\gamma |\nabla u_n|_2^2 \leq [a]_\gamma + C[b]_\gamma \tag{4.7}$$

and

$$\left\|M\left(.,|\nabla u_n|_2^2\right)\right\|_{C(\overline{\Omega})} \leq \|a\|_{C(\overline{\Omega})} + \|b\|_{C(\overline{\Omega})}|\nabla u_n|_2^2 \leq \|a\|_{C(\overline{\Omega})} + C\|b\|_{C(\overline{\Omega})}, \tag{4.8}$$

it follows that

$$\left[\frac{h_n}{M\left(.,|\nabla u_n|_2^2\right)}\right]_\gamma \leq \frac{C}{a_0^2}\left([a]_\gamma + C[b]_\gamma + \|a\|_{C(\overline{\Omega})} + C\|b\|_{C(\overline{\Omega})}\right) = \frac{C}{a_0^2}\|a\|_Y + \frac{C^2}{a_0^2}\|b\|_Y.$$

Since $\left\{h_n/M\left(x,|\nabla u_n|_2^2\right)\right\}$ is bounded in $C(\overline{\Omega})$, the last inequality proves (4.6).

By (4.1), (4.6) and Theorem 0.5 in [1], sequence $\{u_n\}$ is bounded in X. By compact embedding from X in $C^2(\overline{\Omega})$, passing to a subsequence, we get

$$u_n \to u \text{ in } C^2(\overline{\Omega}). \tag{4.9}$$

By (4.9), passing to the limit in $n \to \infty$ in (4.1) we have

$$-\Delta u = \frac{h}{M\left(x,|\nabla u|_2^2\right)}. \tag{4.10}$$

Last equality and Theorem 0.5 in [1] allow us to conclude that $u \in X$.

Finally, by linearity of Laplacian, we have

$$-\Delta(u_n - u) = \frac{h_n}{M\left(x,|\nabla u_n|_2^2\right)} - \frac{h}{M\left(x,|\nabla u|_2^2\right)}. \tag{4.11}$$

From (4.11) and Theorem 0.5 in [1] we conclude that $u_n \to u$ in X. □

Proposition 4.2. *Let $a, b \in C^{0,\gamma}(\overline{\Omega})$ and $u \in X$. If*

$$\int_\Omega \frac{b(x)u\Delta u}{M(x,|\nabla u|_2^2)}\mathrm{dx} \neq 1/2 \tag{4.12}$$

holds, then Ψ is locally invertible in u.

Proof. We use Theorem 2.4 to prove this lemma. It is standard to show that $\Psi \in C^1(X,Y)$ and that

$$\Psi'(u)v = 2b(x)\Delta u \int_\Omega \nabla u \nabla v \mathrm{d}x + M(x,|\nabla u|_2^2)\Delta v.$$

It remains to prove that $\Psi'(u) : X \to Y$ is an isomorphism. If $u = 0$, there is nothing to prove. Now, if $u \neq 0$, observe that $\Psi'(u)$ is an isomorphism if and only if, for each $g \in Y$ given, there is a unique $v \in X$ such that $\Psi'(u)v = -\, g$, this is

$$-\ M\big(x,|\nabla u|_2^2\big)\Delta v = g(x) + 2b(x)\Delta u \int_\Omega \nabla u \nabla v \mathrm{d}x. \tag{4.13}$$

From divergence theorem, (4.13) is equivalent to

$$-M\big(x,|\nabla u|_2^2\big)\Delta v = g(x) - 2b(x)\Delta u \int_\Omega u\Delta v \mathrm{d}x. \tag{4.14}$$

Consequently, $\Psi'(u)$ is an isomorphism if and only if, for each $g \in Y$ given, there is a unique $v \in X$ such that

$$\Delta v = \frac{2b(x)\Delta u \int_\Omega u\Delta v \mathrm{d}x}{M(x,|\nabla u|_2^2)} - \frac{g(x)}{M(x,|\nabla u|_2^2)}. \tag{4.15}$$

To study equation (4.15), we define the mapping $T : Y \to Y$ by

$$T(w) = \frac{2b(x)\Delta u \int_\Omega uw \mathrm{d}x}{M(x,|\nabla u|_2^2)} - \frac{g(x)}{M(x,|\nabla u|_2^2)} \tag{4.16}$$

and we note that, since for each $w \in Y$ problem

$$\begin{cases} \Delta z = w(x) \text{ in } \Omega, \\ z = 0 \qquad\quad \text{on } \partial\Omega, \end{cases} \tag{LP}$$

has a unique solution $z \in X$, looking for solutions of (4.15) is equivalent to find fixed points of T. Denoting $t = \int\limits_\Omega uwdx$, it follows that w is a fixed point of T if and only if

$$w = T(w) = t\frac{2b(x)\Delta u}{M(x,|\nabla u|_2^2)} - \frac{g(x)}{M(x,|\nabla u|_2^2)}. \tag{4.17}$$

Therefore, w is a fixed point of T if and only if

$$T\left(t\frac{2b(x)\Delta u}{M(x,|\nabla u|_2^2)} - \frac{g(x)}{M(x,|\nabla u|_2^2)}\right) = t\frac{2b(x)\Delta u}{M(x,|\nabla u|_2^2)} - \frac{g(x)}{M(x,|\nabla u|_2^2)}.$$

From (4.16), we get

$$\frac{2b(x)\Delta u}{M(x,|\nabla u|_2^2)}\int_\Omega u\left[t\frac{2b(x)\Delta u}{M(x,|\nabla u|_2^2)} - \frac{g(x)}{M(x,|\nabla u|_2^2)}\right]\mathrm{d}x = t\frac{2b(x)\Delta u}{M(x,|\nabla u|_2^2)}.$$

Since $b > 0$ and $\Delta u \not\equiv 0$ (because $u \neq 0$), T admits a fixed point if and only if

$$2\int_\Omega u\left[t\frac{2b(x)\Delta u}{M(x,|\nabla u|_2^2)} - \frac{g(x)}{M(x,|\nabla u|_2^2)}\right]\mathrm{d}x = t,$$

namely

$$t\left[\int_\Omega \frac{2b(x)u\Delta u}{M(x,|\nabla u|_2^2)}\mathrm{d}x - 1\right] = 2\int_\Omega \frac{g(x)u}{M(x,|\nabla u|_2^2)}\mathrm{d}x. \tag{4.18}$$

Equality (4.18) says us that if (4.12) occurs, then T has a unique fixed point w given by

$$w = t\frac{2b(x)\Delta u}{M(x,|\nabla u|_2^2)} - \frac{g(x)}{M(x,|\nabla u|_2^2)},$$

with

$$t = 2\int_\Omega \frac{g(x)u}{M(x,|\nabla u|_2^2)}\mathrm{d}x \Big/ \left[\int_\Omega \frac{2b(x)u\Delta u}{M(x,|\nabla u|_2^2)}\mathrm{d}x - 1\right].$$

□

Remark 4. Equality (4.18) shows us that $\Psi'(u) : X \to Y$ is not surjective if

$$\int_\Omega \frac{b(x)u\Delta u}{M(x, |\nabla u|_2^2)}\mathrm{d}x = 1/2.$$

In fact, in this case, functions $g \in Y$ such that

$$\int_\Omega \frac{g(x)u}{M(x, |\nabla u|_2^2)}\mathrm{d}x \neq 0$$

do not belong to the range of $\Psi'(u)$.

Actually, it is possible to get the same result of (existence and) uniqueness provided in [4] for signed functions as a consequence of global inverse theorem and previous proposition. This is exactly the content of next corollary.

Corollary 4.3. *For each signed function $h \in Y$ given, problem (P) has a unique solution.*

Proof. First of all, we define the sets

$$P_1 = \{u \in X : \Delta u \geq 0\} \subset X$$

and

$$P_2 = \{h \in Y : h \geq 0\} \subset Y.$$

Consider $P_1 \cup (-P_1)$ and $P_2 \cup (-P_2)$ as metric spaces whose metrics are induced from X and Y, respectively.

It is clear that $P_1 \cup (-P_1)$ is arcwise connected (because P_1 and $-P_1$ are convex sets and $P_1 \cap (-P_1) = \{0\}$) closed in X. On the other hand, since $P_2 \cup (-P_2)$ is the union of the closed cone of nonnegative functions of Y with the closed cone of nonpositive functions of Y, it follows that $P_2 \cup (-P_2)$ is simply connected.

From $\Psi(P_1) \subset P_2$ and $\Psi(-P_1) \subset (-P_2)$, it follows that Ψ is well defined from $P_1 \cup (-P_1)$ to $P_2 \cup (-P_2)$.

Moreover, being Ψ proper from X to Y (see Proposition 4.1) and $P_1 \cup (-P_1)$ and $P_2 \cup (-P_2)$ are closed metric spaces in X and Y, respectively, we conclude that Ψ is proper from $P_1 \cup (-P_1)$ to $P_2 \cup (-P_2)$.

Note that if $u \in P_1$ (resp. $-P_1$), then, as u is (the unique) solution to problem

$$\begin{cases} \Delta u = \Delta u \text{ in } \Omega, \\ u = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{4.19}$$

From the maximum principle, it follows that $u \leq 0$ (resp. $u \geq 0$). Whence, we have

$$\int_\Omega \frac{b(x)u\Delta u}{M(x, |\nabla u|_2^2)}\mathrm{d}x \leq 0, \qquad \forall\, u \in P_1 \cup (-P_1).$$

Therefore, from Proposition 4.2, $\Psi : P_1 \cup (-P_1) \to P_2 \cup (-P_2)$ is locally invertible. The result follows now from the global inverse theorem. □

Next result tells us that if $h \in Y$ given (signed or not) is "small," then there is a unique solution with "little variation."

Corollary 4.4. *There are positive constants ε, δ such that for each $h \in Y$ with $\|h\|_Y < \varepsilon$, problem (P) has a unique solution u with $\|u\|_X < \delta$.*

Proof. It is sufficient to note that when $u = 0$ the integral (4.12), in Proposition 4.2, is null. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Since X and Y are Banach spaces, then X is arcwise connected and Y is simply connected. Moreover, from Proposition 4.1, operator Ψ is proper and by divergence theorem we obtain

$$\int_\Omega \frac{b(x)u\Delta u}{M(x,|\nabla u|_2^2)}\mathrm{d}x = \frac{1}{\theta+|\nabla u|_2^2}\int_\Omega u\Delta u \mathrm{d}x = -\frac{|\nabla u|_2^2}{\theta+|\nabla u|_2^2} < 0,\ \forall\ u\in X.$$

The result follows directly from Proposition 4.2 and global inverse theorem. □

Next proposition provides us a sufficient condition on functions a and b for that (4.12) occurs when a/b is not constant.

Proposition 4.5. *Let $a,b\in C^{2,\gamma}(\overline{\Omega})$ and $c=a/b$.*

(i) *If $\Delta c\geq 2|\nabla c|^2/c$ in Ω, then*

$$\int_\Omega \frac{b(x)u\Delta u}{M(x,|\nabla u|_2^2)}dx \leq 0,\ \forall\ u\in X. \tag{4.20}$$

(ii) *If $\Delta c< 2|\nabla c|^2/c$ in some open $\Omega_0\subset\Omega$, then*

$$\int_\Omega \frac{b(x)u\Delta u}{M(x,|\nabla u|_2^2)}dx < 1/2,\ \forall\ u\in X, \tag{4.21}$$

provided that

$$\frac{|\nabla c|_\infty c_M}{\sqrt{\lambda_1}c_L^2}\leq 3/2.$$

Proof. Putting b in evidence in the integral (4.12), we get

$$\int_\Omega \frac{b(x)u\Delta u}{M(x,|\nabla u|_2^2)}\mathrm{d}x = \int_\Omega \frac{u\Delta u}{c+|\nabla u|_2^2}\mathrm{d}x,$$

where $c=c(x)=a(x)/b(x)$. From divergence theorem, we have

$$\int_\Omega \frac{u\Delta u}{c+|\nabla u|_2^2}\mathrm{d}x = -\int_\Omega \nabla\left(\frac{u}{c+|\nabla u|_2^2}\right)\nabla u\mathrm{d}x.$$

Since

$$\nabla\left(\frac{u}{c+|\nabla u|_2^2}\right) = \frac{1}{c+|\nabla u|_2^2}\nabla u - \frac{u}{(c+|\nabla u|_2^2)^2}\nabla c,$$

we conclude that

$$\begin{aligned}\int_\Omega \frac{b(x)u\Delta u}{M(x,|\nabla u|_2^2)}\mathrm{d}x &= -\int_\Omega \frac{|\nabla u|^2}{c+|\nabla u|_2^2}\mathrm{d}x + \int_\Omega \frac{u\nabla u\nabla c}{(c+|\nabla u|_2^2)^2}dx\\ &= -\int_\Omega \frac{|\nabla u|^2}{c+|\nabla u|_2^2}\mathrm{d}x + \frac{1}{2}\int_\Omega \frac{\nabla(u^2)\nabla c}{(c+|\nabla u|_2^2)^2}\mathrm{d}x.\end{aligned}$$

Using again the divergence theorem

$$\int_\Omega \frac{b(x)u\Delta u}{M(x,|\nabla u|_2^2)}\mathrm{d}x = -\int_\Omega \frac{|\nabla u|^2}{c+|\nabla u|_2^2}\mathrm{d}x + \frac{1}{2}\int_\Omega u^2\left\{-\mathrm{div}\left[\frac{\nabla c}{(c+|\nabla u|_2^2)^2}\right]\right\}\mathrm{d}x. \tag{4.22}$$

(i) In this case, from Lemma 3.1 (see also Remark 1) we get that $\mathcal{A} = \emptyset$. Consequently, for each $u \in X$ we have

$$\int_\Omega u^2 \left\{-\mathrm{div}\left[\frac{\nabla c}{(c+|\nabla u|_2^2)^2}\right]\right\} \mathrm{d}x \leq 0.$$

Whence, by (4.22),

$$\int_\Omega \frac{b(x)u\Delta u}{M(x,|\nabla u|_2^2)} \mathrm{d}x \leq 0, \ \forall\ u \in X.$$

(ii) In this case $\mathcal{A} \neq \emptyset$. If $u \in X$ is such that $|\nabla u|_2^2 \notin \mathcal{A}$, we already know that

$$\int_\Omega \frac{b(x)u\Delta u}{M(x,|\nabla u|_2^2)} \mathrm{d}x \leq 0.$$

Now, if $u \in X$ is such that $|\nabla u|_2^2 \in \mathcal{A}$, then, from (4.22) and Proposition 3.2 (see also Remark 3), we obtain

$$\int_\Omega \frac{b(x)u\Delta u}{M(x,|\nabla u|_2^2)} \mathrm{d}x \leq \left(\frac{1}{2\lambda_\alpha} - 1\right) \int_\Omega \frac{|\nabla u|^2}{c+\alpha} \mathrm{d}x, \tag{4.23}$$

where $\alpha := |\nabla u|_2^2$. If α is such that $1/2 \leq \lambda_\alpha$, then, by (4.23), $\int_\Omega b(x)u\Delta u/M(x,|\nabla u|_2^2)\mathrm{d}x \leq 0$. Finally, if $0 < \lambda_\alpha < 1/2$, it follows from $\alpha = |\nabla u|_2^2$ and from Corollary 3.3 that,

$$\int_\Omega \frac{b(x)u\Delta u}{M(x,|\nabla u|_2^2)} \mathrm{d}x < \frac{|\nabla c|_\infty (c_M+\alpha)}{\sqrt{\lambda_1}(c_L+\alpha)^2} - 1 =: g(\alpha). \tag{4.24}$$

We have that $g(0) = |\nabla c|_\infty c_M/\sqrt{\lambda_1}c_L^2 - 1$ and

$$g'(\alpha) = \frac{|\nabla c|_\infty (c_L - 2c_M - \alpha)}{\sqrt{\lambda_1}(c_L+\alpha)^3} < 0, \qquad \forall \alpha > 0.$$

Therefore, g is decreasing and, from (4.24), we conclude that if

$$\frac{|\nabla c|_\infty c_M}{\sqrt{\lambda_1}c_L^2} \leq \frac{3}{2},$$

then (4.21) holds. □

Now, we give the proof of our main uniqueness result to problem (P) which covers sign-changing functions.

Proof of Theorem 1.2. It follows directly from Propositions 4.1, 4.2, 4.5 and the global inverse theorem. □

Theorems 1.1 and 1.2 seem to indicate that in the case that h is sign-changing the uniqueness of solution to the problem (P) is, in some way, related to the variation of a/b. In any way, it remains open the question to know what happens with the number of solutions of (P) in the case that h is sign-changing, $\Delta c < 2|\nabla c|^2/c$ in some open $\Omega_0 \subset \Omega$ and $|\nabla c|_\infty c_M/\sqrt{\lambda_1}c_L^2$ is large.

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