Z. Angew. Math. Phys. (2017) 68:133

0044-2275/17/060001-17
published online November 1, 2017
https://doi.org/10.1007/s00033-017-0877-6

Zeitschrift für angewandte Mathematik und Physik ZAMP

Approximation of entropy solutions to degenerate nonlinear parabolic equations

Eduardo Abreu, Mathilde Colombeau and Evgeny Yu Panov

Abstract. We approximate the unique entropy solutions to general multidimensional degenerate parabolic equations with BV continuous flux and continuous nondecreasing diffusion function (including scalar conservation laws with BV continuous flux) in the periodic case. The approximation procedure reduces, by means of specific formulas, a system of PDEs to a family of systems of the same number of ODEs in the Banach space L^∞, whose solutions constitute a weak asymptotic solution of the original system of PDEs. We establish well posedness, monotonicity and L^1-stability. We prove that the sequence of approximate solutions is strongly L^1-precompact and that it converges to an entropy solution of the original equation in the sense of Carrillo. This result contributes to justify the use of this original method for the Cauchy problem to standard multidimensional systems of fluid dynamics for which a uniqueness result is lacking.

Mathematics Subject Classification. 35K55, 35L65, 65M12.

Keywords. Partial differential equations, Degenerate parabolic equations, Entropy solutions, Approximate solutions, Stability.

1. Introduction

We construct a sequence $(u_\epsilon), (t,x) \longmapsto u(t,x;\epsilon), t \in [0,+\infty), x \in \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, differentiable in t, essentially bounded in x, that tends to satisfy a periodic PDE in the strong sense in time and in the weak sense in space. The construction consists in resolving, for each value $\epsilon > 0$, a system of ordinary differential equations in the Banach space $L^\infty(\mathbb{T}^n)$, one ODE per equation of the system, whose solutions constitute a weak asymptotic solution of the original PDE. We prove that the approximate solutions so constructed tend to the unique entropy solution in the sense of Carrillo in the case of degenerate parabolic scalar equations, a fortiori to the unique entropy solution in the sense of Kruzhkov for scalar conservation laws. The result in this paper gives an approximation process for these equations, satisfying well posedness, monotonicity and L^1-stability.

In the sequel of this section we recall the definitions of Carrillo and Kruzhkov entropy solutions and we prove uniqueness in the periodic case when the flux vector and the diffusion functions are merely continuous. At the beginning of the next section we give the details of the scheme. The final result of convergence of the approximations is Theorem 4 in Sect. 5.

Theorem 4 (main result) The approximate solutions $u(t,x;\epsilon)$ converge to the unique entropy solution $u(t,x)$ in the space $\mathcal{C}([0,+\infty); L^1(\mathbb{T}^n))$ when $\epsilon \to 0$.

The assumptions are: continuity and bounded variation of the flux function, continuity of the diffusion function which is increasing in a nonstrict sense.

In the half-space $\Pi = \mathbb{R}_+ \times \mathbb{R}^n$, $\mathbb{R}_+ = (0,+\infty)$, we consider the scalar conservation law

$$u_t + \text{div}_x f(u) = \mu \Delta g(u), \quad t > 0,\ x \in \mathbb{R}^n, \tag{1}$$

with continuous flux vector $f(u) = (f_1(u), \ldots, f_n(u)) \in C(\mathbb{R}, \mathbb{R}^n)$ and with a continuous diffusion function $g(u) \in C(\mathbb{R})$ increasing in a nonstrict sense. The constant μ is positive or null. The case $g(u) = u$

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corresponds to the usual viscous term $\mu\Delta u$. The case $\mu = 0$ (or $g(u) \equiv \text{const}$) corresponds to a first-order conservation law

$$u_t + \operatorname{div}_x f(u) = 0. \tag{2}$$

Recall the notion of entropy solution to the Cauchy problem for Eq. (1) with initial data

$$u(0, x) = u_0(x) \in L^\infty(\mathbb{R}^n) \tag{3}$$

in the sense of Carrillo [6].

Definition 1. A bounded measurable function $u = u(t, x) \in L^\infty(\Pi)$ is called an entropy solution (e.s. for short) of (1), (3) if the generalized gradient $\nabla_x g(u) \in L^2_{\text{loc}}(\Pi, \mathbb{R}^n)$ and for all $k \in \mathbb{R}$

$$|u - k|_t + \operatorname{div}_x[\operatorname{sign}(u - k)(f(u) - f(k))] - \mu\Delta|g(u) - g(k)| \le 0 \tag{4}$$

in the sense of distributions on Π (in $\mathcal{D}'(\Pi)$), and

$$\operatorname*{ess\,lim}_{t\to 0} u(t, \cdot) = u_0 \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n). \tag{5}$$

Condition (4) means that for all nonnegative test functions $\phi = \phi(t, x) \in C^2_0(\Pi)$

$$\begin{aligned}&\int\limits_\Pi [|u - k|\phi_t + \operatorname{sign}(u - k)(f(u) - f(k)) \cdot \nabla_x\phi - \mu \operatorname{sign}(u - k)\nabla_x g(u) \cdot \nabla_x\phi]\mathrm{d}t\mathrm{d}x \\ &= \int\limits_\Pi [|u - k|\phi_t + \operatorname{sign}(u - k)(f(u) - f(k)) \cdot \nabla_x\phi + \mu|g(u) - g(k)|\Delta_x\phi]\mathrm{d}t\mathrm{d}x \ge 0\end{aligned}$$

(here and below "$\cdot$" denotes the scalar product in $\mathbb{R}^n$).

Taking in (4) $k = \pm M$, where $M \ge \|u\|_\infty$, we obtain that

$$u_t + \operatorname{div}_x f(u) = \mu\Delta g(u) \quad \text{in } \mathcal{D}'(\Pi),$$

that is, an e.s. is a weak solution of (1) as well.

In the case $\mu = 0$ Definition 1 coincides with the known Kruzhkov definition [20] of an entropy solution of the problem (2), (3).

In the case under consideration when the flux vector and the diffusion function are merely continuous, the uniqueness of an e.s. to problem (1), (3) may be violated if $n > 1$ (for the case of conservation laws, see examples in [21,22]). In the case $n = 1$ the uniqueness is known, see [25]. But in the class of spatially periodic e.s. the uniqueness holds in any dimension. For the sake of completeness we will provide the proof of the uniqueness of periodic e.s. Hence, we assume that initial data u_0 and e.s. $u = u(t, x)$ are space periodic: $u_0(x + e_i) = u_0(x)$ for almost all $x \in \mathbb{R}^n$, $u(t, x + e_i) = u(t, x)$ for almost all $(t, x) \in \Pi$, where $i = 1, \dots, n$, and $\{e_i\}_{i=1}^n$ is a basis of periods in $\mathbb{R}^n$. Without loss of generality, we can suppose that $\{e_i\}_{i=1}^n$ is the canonical basis. We denote by $P = [0, 1)^n$ the corresponding fundamental parallelepiped (cube), which will be identified with the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

The uniqueness of a periodic e.s. is an easy consequence of the following property.

Theorem 1. *Let $u_1(t, x)$, $u_2(t, x)$ be two periodic e.s. of problem (1), (3) with initial data $u_{01}(x)$, $u_{02}(x)$, respectively. Then for a.e. $t > 0$*

$$\int\limits_{\mathbb{T}^n} (u_1(t, x) - u_2(t, x))^+\mathrm{d}x \le \int\limits_{\mathbb{T}^n} (u_{01}(x) - u_{02}(x))^+\mathrm{d}x. \tag{6}$$

Here we denote $z^+ = \max(z, 0)$.

Proof. As was proved in [6] (see also [3,25]), the following Kato inequality holds

$$((u_1-u_2)^+)_t+\operatorname{div}_x[\operatorname{sign}^+(u_1-u_2)(f(u_1)-f(u_2))]-\mu\Delta_x(g(u_1)-g(u_2))^+\le 0 \quad (7)$$

in $\mathcal{D}'(\Pi)$. Here $\operatorname{sign}^+ z=(\operatorname{sign} z)^+$ is the Heaviside function.

Let $\alpha(t)\in C_0^\infty((0,+\infty))$, $\beta(y)\in C_0^\infty(\mathbb{R}^n)$, $\alpha(t),\beta(y)\ge 0$, $\int_{\mathbb{R}^n}\beta(y)\mathrm{d}y=1$. Applying (7) to the test function $\alpha(t)\beta(x/r)$, where $r\in\mathbb{N}$, we obtain that

$$\begin{aligned}\int_0^{+\infty}\int_{\mathbb{R}^n}&(u_1(t,x)-u_2(t,x))^+\beta(x/r)\alpha'(t)\mathrm{d}t\\ &+r^{-1}\int_\Pi\alpha(t)\operatorname{sign}^+(u_1-u_2)(f(u_1)-f(u_2))\cdot\nabla_y\beta(x/r)\mathrm{d}t\mathrm{d}x\\ &+\mu r^{-2}\int_\Pi\alpha(t)(g(u_1)-g(u_2))^+\Delta_y\beta(x/r)\mathrm{d}t\mathrm{d}x\ge 0. \end{aligned} \quad (8)$$

As is rather well known (see, for example [27, Lemma 2.1]), for a bounded spatially periodic function $w(t,x)\in L^\infty(\Pi)$ the following relation holds

$$\lim_{r\to\infty}r^{-n}\int_\Pi w(t,x)a(t)b(x/r)\mathrm{d}t\mathrm{d}x=C\int_0^{+\infty}\left(\int_{\mathbb{T}^n}w(t,x)\mathrm{d}x\right)\alpha(t)\mathrm{d}t, \quad (9)$$

where $a(t)\in C_0((0,\infty))$, $b(y)\in C_0(\mathbb{R}^n)$, $C=\int_{\mathbb{R}^n}b(y)\mathrm{d}y$. Multiplying (8) by r^{-n} and passing to the limit as $r\to\infty$ with the help of (9), we arrive at

$$\int_0^{+\infty}I(t)\alpha'(t)\mathrm{d}t\ge 0$$

for any $\alpha(t)\in C_0^\infty((0,+\infty))$, $\alpha(t)\ge 0$, where $I(t)=\int_{\mathbb{T}^n}(u_1(t,x)-u_2(t,x))^+\mathrm{d}x$. This means that $I'(t)\le 0$ in $\mathcal{D}'((0,+\infty))$ and therefore, for almost all $t>0$

$$I(t)\le\operatorname*{ess\,lim}_{t\to 0}I(t)=I(0)\doteq\int_{\mathbb{T}^n}(u_{01}(x)-u_{02}(x))^+\mathrm{d}x, \quad (10)$$

and (6) follows.

We use also the initial condition in the sense of Definition 1 for e.s. u_1,u_2, which implies the indicated in (10) limit relation. Indeed,

$$|I(t)-I(0)|\le\int_{\mathbb{T}^n}|u_1(t,x)-u_{01}(x)|\mathrm{d}x+\int_{\mathbb{T}^n}|u_2(t,x)-u_{02}(x)|\mathrm{d}x\to 0,$$

as time $t\to 0$, running over a set of full Lebesgue measure.

The proof is complete. □

It readily follows from (6) that $u_1(t,x)\le u_2(t,x)$ a.e. in Π whenever $u_{01}\le u_{02}$ (comparison principle). Clearly, this implies the uniqueness of periodic e.s.

The class of partial differential equations of the type (1) encompasses several interesting nonlinear phenomena, coming from fluid mechanics to the theory of porous media flow [33] as applied to petroleum reservoir engineering [14]. We can mention modelling the movement of contaminants in groundwater [4], the sedimentation–consolidation process [32], freeway traffic flow theory and simulation [34], modelling ice sheet dynamics [19], and also in combustion theory [5], just to quote a few examples.

Approximate solutions for various systems of PDEs have been considered in particular by Albeverio and Danilov [1], Albeverio and Shelkovich [2], Danilov et al. [10–13], Joseph and Sahoo [16,18], Joseph et al. [17] and Sahoo [28,29], Kunzinger et al. [23,24], Panov et al. [26], Shelkovich [30,31]. They are usually called weak asymptotic solutions [10–13,26,30,31] and have proved their ability to replace the solutions in the sense of distributions in many nonlinear instances.

2. Approximation scheme: uniqueness of approximate solutions

We suppose in addition that the vector $f(u)$ has bounded variation on any segment. This allows to decompose $f(u)$ into the difference of two vector-functions with increasing components: $f(u) = p(u) - q(u)$, $p(u) = (p_1(u), \dots, p_n(u))$, $q(u) = (q_1(u), \dots, q_n(u))$; $p_i(u)$, $q_i(u)$, $i = 1, \dots, n$, are (nonstrictly) increasing continuous functions.

Now we introduce the approximation procedure, resembling the method proposed in [7–9] which reduces, by means of formulas (13,14), a system of PDEs to a system of the same number of ODEs in the Banach space $L^\infty(\mathbb{T}^n)$, whose solution constitutes a weak asymptotic solution of the original system of PDEs.

For $\varepsilon > 0$ we define $u(t, x; \varepsilon)$ as a solution of the equation obtained by the following finite difference approximation of the operator $-\mathrm{div}_x f(u) + \mu \Delta_x g(u)$:

$$\frac{\mathrm{d}}{\mathrm{d}t} u(x, t; \varepsilon) = \frac{1}{\varepsilon} \sum_{i=1}^{n} [p_i(u(t, x - \varepsilon e_i; \varepsilon)) - (p_i + q_i)(u(t, x; \varepsilon)) + q_i(u(t, x + \varepsilon e_i; \varepsilon))] + \frac{\mu}{\varepsilon^2} \sum_{i=1}^{n} [g(u(t, x + \varepsilon e_i; \varepsilon)) - 2g(u(t, x; \varepsilon)) + g(u(t, x - \varepsilon e_i; \varepsilon))]. \tag{11}$$

This equation can be rewritten in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} u(x, t; \varepsilon) = \frac{1}{\varepsilon} \sum_{i=1}^{n} [\tilde{p}_i(u(t, x - \varepsilon e_i; \varepsilon)) - (\tilde{p}_i + \tilde{q}_i)(u(t, x; \varepsilon)) + \tilde{q}_i(u(t, x + \varepsilon e_i; \varepsilon))], \tag{12}$$

where

$$\tilde{p}_i(u) = p_i(u) + \frac{\mu}{\varepsilon} g(u), \;\; \tilde{q}_i(u) = q_i(u) + \frac{\mu}{\varepsilon} g(u)$$

are nondecreasing functions for all $i = 1, \dots, n$.

Equation (12) is endowed with the initial condition

$$u(0, x; \varepsilon) = u_0(x). \tag{13}$$

We will consider Eq. (12) as an autonomous ODE $\dot{u} = F(u)$ in the Banach space $L^\infty(\mathbb{T}^n)$. Here

$$F(u)(x) = F_\varepsilon(u)(x) \doteq \frac{1}{\varepsilon} \sum_{i=1}^{n} [\tilde{p}_i(u(x - \varepsilon e_i)) - (\tilde{p}_i + \tilde{q}_i)(u(x)) + \tilde{q}_i(u(x + \varepsilon e_i))]$$

is a continuous nonlinear operator on $L^\infty(\mathbb{T}^n)$. It is known that Peano theorem fails in any infinite-dimensional Banach space (cf. [15]). Therefore, we cannot claim even local existence of the Cauchy problem for a general ODE $\dot{u} = F(u)$. Fortunately, for particular problem (12), (13) the existence of a solution (even global one) is actually fulfilled. We will establish this fact in Sect. 4. Let us firstly investigate the uniqueness. We fix $\varepsilon > 0$ and denote by $u(t, x) = u(t, x; \varepsilon) \in C^1((0, T), L^\infty(\mathbb{T}^n))$ a solution of (12) defined on some interval $(0, T)$, $0 < T \le +\infty$. Since for each $h(x) \in L^1(\mathbb{T}^n)$ the functional

$u \mapsto \int\limits_{\mathbb{T}^n} u(x)h(x)\mathrm{d}x$ is linear and continuous on $L^\infty(\mathbb{T}^n)$, the function $I_h(t) = \int\limits_{\mathbb{T}^n} u(t,x)h(x)\mathrm{d}x \in C^1((0,T))$ and $I'_h(t) = \int\limits_{\mathbb{T}^n} F(u(t,\cdot))(x)h(x)\mathrm{d}x$. This readily implies the following identity

$$u_t = F(u(t,\cdot))(x) \tag{14}$$

in the sense of distributions on $(0,T)\times\mathbb{T}^n$ (in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$).

Theorem 2. *Let $u(t,x), v(t,x) \in C^1((0,T), L^\infty(\mathbb{T}^n))$ be solutions of (12), (13) with initial data $u_0(x)$, $v_0(x) \in L^\infty(\mathbb{T}^n)$, respectively. Then $\forall t \in (0,T)$*

$$\int\limits_{\mathbb{T}^n} (u(t,x)-v(t,x))^+ dx \le \int\limits_{\mathbb{T}^n} (u_0(x)-v_0(x))^+ dx. \tag{15}$$

Proof. In view of (14) $(u-v)_t = F(u)-F(v)$ in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$. Since this distribution is regular, the chain rule $(\varphi(u-v))_t = \varphi'(u-v)(F(u)-F(v))$ holds for every $\varphi(z)\in C^1(\mathbb{R})$. We may choose a sequence $\varphi_r(z) \in C^1(\mathbb{R})$ such that $\varphi_r(z) \underset{r\to\infty}{\to} z^+$ uniformly on $\mathbb{R}$, while $(\varphi_r)'(z) \underset{r\to\infty}{\to} \operatorname{sign}^+ z$ pointwise and $0 \le (\varphi_r)'(z) \le 1$. By the above limit relations $\varphi_r(u-v) \to (u-v)^+$, $(\varphi_r)'(u-v)(F(u)-F(v)) \to \operatorname{sign}^+(u-v)(F(u)-F(v))$ as $r\to\infty$ in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$. Therefore, in the limit as $r\to\infty$ we obtain that

$$((u-v)^+)_t = \operatorname{sign}^+(u-v)_t = \operatorname{sign}^+(u-v)(F(u)-F(v)) \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{T}^n) \tag{16}$$

(actually, the equality $(w^+)_t = (\operatorname{sign}^+ w)w_t$ is well known, see for instance [35, p. 302]). Now observe that for each $u, v \in L^\infty(\mathbb{T}^n)$

$$\begin{aligned} &\operatorname{sign}^+(u-v)(F(u)-F(v)) = \frac{1}{\varepsilon}\sum_{i=1}^n [\operatorname{sign}^+(u-v)(\tilde p_i(u(x-\varepsilon e_i)) \\ &-\tilde p_i(v(x-\varepsilon e_i))) - \operatorname{sign}^+(u-v)(\tilde p_i(u)-\tilde p_i(v)+\tilde q_i(u)-\tilde q_i(v)) \\ &+\operatorname{sign}^+(u-v)(\tilde q_i(u(x+\varepsilon e_i))-\tilde q_i(v(x+\varepsilon e_i)))], \quad u=u(x), v=v(x). \end{aligned} \tag{17}$$

Since $0 \le \operatorname{sign}^+(u-v) \le 1$ and the functions $\tilde p_i(u), \tilde q_i(u)$ increase,

$$\begin{aligned} &\operatorname{sign}^+(u-v)(\tilde p_i(u(x-\varepsilon e_i))-\tilde p_i(v(x-\varepsilon e_i))) \\ &\quad \le (\tilde p_i(u(x-\varepsilon e_i))-\tilde p_i(v(x-\varepsilon e_i)))^+, \\ &\operatorname{sign}^+(u-v)(\tilde q_i(u(x+\varepsilon e_i))-\tilde q_i(v(x+\varepsilon e_i))) \\ &\quad \le (\tilde q_i(u(x+\varepsilon e_i))-\tilde q_i(v(x+\varepsilon e_i)))^+, \\ &\operatorname{sign}^+(u-v)(\tilde p_i(u)-\tilde p_i(v)+\tilde q_i(u)-q_i(v)) \\ &\quad = (\tilde p_i(u)-\tilde p_i(v))^+ + (\tilde q_i(u)-\tilde q_i(v))^+, \end{aligned}$$

and it follows from (17) that

$$\begin{aligned} &\operatorname{sign}^+(u-v)(F(u)-F(v)) \\ &\quad \le \frac{1}{\varepsilon}\sum_{i=1}^n [(\tilde p_i(u(x-\varepsilon e_i))-\tilde p_i(v(x-\varepsilon e_i)))^+ - (\tilde p_i(u(x))-\tilde p_i(v(x)))^+ \\ &\qquad +(\tilde q_i(u(x+\varepsilon e_i))-\tilde q_i(v(x+\varepsilon e_i)))^+ - (\tilde q_i(u(x))-\tilde q_i(v(x)))^+]. \end{aligned} \tag{18}$$

Taking into account that for each $i = 1, \dots, n$

$$\int_{\mathbb{T}^n} (\tilde{p}_i(u(x - \varepsilon e_i)) - \tilde{p}_i(v(x - \varepsilon e_i)))^+ \mathrm{d}x = \int_{\mathbb{T}^n} (\tilde{p}_i(u(x)) - \tilde{p}_i(v(x)))^+ \mathrm{d}x,$$

$$\int_{\mathbb{T}^n} (\tilde{q}_i(u(x + \varepsilon e_i)) - \tilde{q}_i(v(x + \varepsilon e_i)))^+ \mathrm{d}x = \int_{\mathbb{T}^n} (\tilde{q}_i(u(x)) - \tilde{q}_i(v(x)))^+ \mathrm{d}x,$$

we deduce from (18) that

$$\int_{\mathbb{T}^n} \operatorname{sign}^+(u(x) - v(x))(F(u)(x) - F(v)(x))\mathrm{d}x \le 0.$$

It follows from this inequality and (16) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^n} (u(t,x) - v(t,x))^+ \mathrm{d}x = \int_{\mathbb{T}^n} ((u(t,x) - v(t,x))^+)_t \mathrm{d}x \le 0 \quad \text{in } \mathcal{D}'((0,T)).$$

From this and continuity of the function $t \to \int_{\mathbb{T}^n} (u(t,x) - v(t,x))^+ \mathrm{d}x$ we readily obtain relation (15). □

Corollary 1. (Comparison principle) *If $u_0(x) \le v_0(x)$ and $u(t,x), v(t,x)$ are solutions of (12), (13) with initial data u_0, v_0, then $u(t,x) \le v(t,x)$ for all $t \in (0,T)$ and almost all $x \in \mathbb{T}^n$.*

Proof. In view of (15) for all $t \in (0,T)$

$$\int_{\mathbb{T}^n} (u(t,x) - v(t,x))^+ \mathrm{d}x \le \int_{\mathbb{T}^n} (u_0(x) - v_0(x))^+ \mathrm{d}x = 0,$$

which implies that $u(t,x) \le v(t,x)$ for a.e. $x \in \mathbb{T}^n$. □

Obviously, Corollary 1 implies the uniqueness of a solution of (12), (13). Another consequence of this corollary is the following.

Corollary 2. (Maximum/minimum principle) *Assume that $a \le u_0(x) \le b$ and $u(t,x)$ is solution of (12), (13) with initial data u_0. Then $a \le u(t,x) \le b$ for all $t \in (0,T)$ and a.e. $x \in \mathbb{T}^n$. In particular, $\|u(t,\cdot)\|_\infty \le \|u_0\|_\infty$.*

Proof. As is directly verified, any constant $u \equiv c$ is a solution of (12) (because $F(c) \equiv 0$). Thus, the desired statement readily follows from the comparison principle. □

Corollary 3. (L^1-stability) *Assume that $u(t,x), v(t,x)$ are solutions of (12), (13) with initial functions $u_0(x)$, $v_0(x)$, respectively. Then for all $t \in (0,T)$*

$$\int_{\mathbb{T}^n} |u(t,x) - v(t,x)|\mathrm{d}x \le \int_{\mathbb{T}^n} |u_0(x) - v_0(x)|\mathrm{d}x.$$

Proof. For the proof one only need to use the identity $|u - v| = (u - v)^+ + (v - u)^+$ and the statement of Theorem 2. □

3. Estimates on the approximate solutions

In this section we obtain uniform estimates of approximate solutions, which in particular guarantee their compactness in $L^1_{\mathrm{loc}}([0,T) \times \mathbb{T}^n)$. First, notice that by Corollary 2 for all $t \in [0,T)$

$$\|u(t,\cdot;\varepsilon)\|_\infty \le M = \|u_0\|_\infty. \tag{19}$$

Second, since $u(t, x+\Delta x; \varepsilon)$ is a solution of (12), (13) with initial function $u_0(x+\Delta x)$ for every $\Delta x \in \mathbb{R}^n$, then by Corollary 3 for all $t \in [0, T)$

$$\int_{\mathbb{T}^n} |u(t, x+\Delta x; \varepsilon) - u(t, x; \varepsilon)| \mathrm{d}x \leq \int_{\mathbb{T}^n} |u_0(x+\Delta x) - u_0(x)| \mathrm{d}x \leq \omega^x(|\Delta x|), \tag{20}$$

where

$$\omega^x(h) = \sup_{\Delta x \in \mathbb{R}^n, |\Delta x| \leq h} \int_{\mathbb{T}^n} |u_0(x+\Delta x) - u_0(x)| \mathrm{d}x$$

is the continuity modulus of $u_0(x)$ in $L^1(\mathbb{T}^n)$. We use here the notation $|z|$ for the Euclidean norm of a finite-dimensional vector z.

Denote $N_1 = \max_{|u| \leq M} \sum_{i=1}^{n} (|p_i(u)| + |q_i(u)|)$, $N_2 = \max_{|u| \leq M} |g(u)|$. Then, directly from (2),

$$\forall t_1, t_2 \in (0, T) \quad \|u(t_2, x; \varepsilon) - u(t_1, x; \varepsilon)\|_\infty \leq \left(\frac{2N_1}{\varepsilon} + \frac{4\mu n N_2}{\varepsilon^2} \right) |t_2 - t_1|. \tag{21}$$

We need an estimate on the continuity modulus with respect to the time variable which is independent of ε. Let us begin with the following weak estimate.

Lemma 1. *Assume that $\phi(x) \in C^2(\mathbb{T}^n)$. Then $\forall t, \Delta t > 0$ such that $t + \Delta t < T$,*

$$\left| \int_{\mathbb{T}^n} (u(t+\Delta t, x; \varepsilon) - u(t, x; \varepsilon)) \phi(x) \mathrm{d}x \right| \leq (N_1 \|\nabla \phi\|_\infty + \mu n N_2 \|\nabla_2 \phi\|_\infty) \Delta t, \tag{22}$$

where we denote by $\nabla_2 \phi$ the vector of second-order derivatives $\phi_{x_i x_i}$, $i = 1, \ldots, n$.

Proof. We denote $I(t) = \int_{\mathbb{T}^n} u(t, x; \varepsilon) \phi(x) \mathrm{d}x$. Then in view of (2)

$$\begin{aligned}
I'(t) = &\int_{\mathbb{T}^n} \frac{1}{\varepsilon} \sum_{i=1}^{n} [p_i(u(t, x - \varepsilon e_i; \varepsilon)) - (p_i + q_i)(u(t, x; \varepsilon)) \\
&+ \quad q_i(u(t, x + \varepsilon e_i; \varepsilon))] \phi(x) \mathrm{d}x \\
&+ \int_{\mathbb{T}^n} \frac{\mu}{\varepsilon^2} \sum_{i=1}^{n} [g(u(t, x + \varepsilon e_i; \varepsilon)) - 2g(u(t, x; \varepsilon)) + g(u(t, x - \varepsilon e_i; \varepsilon))] \phi(x) \mathrm{d}x \\
= &\int_{\mathbb{T}^n} \sum_{i=1}^{n} \left[p_i(u(t, x; \varepsilon)) \frac{\phi(x + \varepsilon e_i) - \phi(x)}{\varepsilon} \right. \\
&- \quad \left. q_i(u(t, x; \varepsilon)) \frac{\phi(x) - \phi(x - \varepsilon e_i)}{\varepsilon} \right] \mathrm{d}x \\
&+ \int_{\mathbb{T}^n} \mu g(u(t, x; \varepsilon)) \sum_{i=1}^{n} \frac{\phi(x - \varepsilon e_i) - 2\phi(x) + \phi(x + \varepsilon e_i)}{\varepsilon^2} \mathrm{d}x.
\end{aligned} \tag{23}$$

Since for each $i = 1, \ldots, n$

$$\left| \frac{\phi(x \pm \varepsilon e_i) - \phi(x)}{\varepsilon} \right| \leq \|\nabla \phi\|_\infty,$$
$$\left| \frac{\phi(x + \varepsilon e_i) - 2\phi(x) + \phi(x - \varepsilon e_i)}{\varepsilon^2} \right| \leq \|\nabla_2 \phi\|_\infty$$

and $|u(t,x;\varepsilon)|\le M$ it follows from (23) that $I'(t)\le N_1\|\nabla\phi\|_\infty+\mu n N_2\|\nabla_2\phi\|_\infty$. Therefore, $|I(t+\Delta t)-I(t)|\le (N_1\|\nabla\phi\|_\infty+\mu n N_2\|\nabla_2\phi\|_\infty)\Delta t$, which yields (22). □

Let $\beta(y)\in C_0^\infty(\mathbb{R}^n)$ be a nonnegative function supported in the unit ball $|x|\le 1$ such that $\int_{\mathbb{R}^n}\beta(y)\mathrm{d}y=1$. We introduce the averaging kernels $\beta_h(x)=h^{-n}\beta(x/h)$, $h>0$. Obviously, the functions $\beta_h(x)$ converge as $h\to 0$ to the Dirac δ-measure in $\mathcal{D}'(\mathbb{R}^n)$. For a function $u(x)\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ we define the corresponding averaged functions

$$u^h(x)=u*\beta^h(x)=\int_{\mathbb{R}^n}u(x-y)\beta^h(y)\mathrm{d}y=\int_{\mathbb{R}^n}u(x-hy)\beta(y)\mathrm{d}y.$$

It is well known that $u^h(x)\in C^\infty(\mathbb{R}^n)$ for all $h>0$ and $u^h(x)\to u(x)$ as $h\to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and almost everywhere in $\mathbb{R}^n$. It is clear that the functions $u^h(x)$ are periodic whenever $u(x)$ is a periodic function.

The following result is the straightforward adaptation of Kruzhkov's lemma [20, Lemma 1] for the periodic case.

Lemma 2. *Suppose that $u(x)\in L^1(\mathbb{T}^n)$, $h>0$. Then*

$$\int_{\mathbb{T}^n}|u(x)(\operatorname{sign}u)^h(x)-|u(x)||\mathrm{d}x\le 2\omega_u(h), \tag{24}$$

where

$$\omega_u(h)=\sup_{\Delta x\in\mathbb{R}^n,|\Delta x|\le h}\int_{\mathbb{T}^n}|u(x+\Delta x)-u(x)|\mathrm{d}x$$

is the continuity modulus of $u(x)$ in $L^1(\mathbb{T}^n)$.

Proof. For all $x,z\in\mathbb{T}^n$ we have

$$\begin{aligned}|u(x)\operatorname{sign}u(z)-|u(x)||&=|(u(x)-u(z))\operatorname{sign}u(z)+|u(z)|-|u(x)||\\&\le|u(x)-u(z)|+||u(z)|-|u(x)||\le 2|u(x)-u(z)|.\end{aligned} \tag{25}$$

Further, by the definition of averaged functions

$$u(x)(\operatorname{sign}u)^h(x)-|u(x)|=\int_{\mathbb{R}^n}(u(x)\operatorname{sign}u(x-y)-|u(x)|)\beta^h(y)\mathrm{d}y,$$

where we use the identity $\int_{\mathbb{R}^n}\beta^h(y)\mathrm{d}y=1$. Therefore,

$$\begin{aligned}&\int_{\mathbb{T}^n}|u(x)(\operatorname{sign}u)^h(x)-|u(x)||\mathrm{d}x\\&\le\int_{\mathbb{T}^n}\left(\int_{\mathbb{R}^n}|u(x)\operatorname{sign}u(x-y)-|u(x)||\beta^h(y)\mathrm{d}y\right)\mathrm{d}x\\&\le 2\int_{\mathbb{T}^n}\left(\int_{\mathbb{R}^n}|u(x)-u(x-y)|\beta^h(y)\mathrm{d}y\right)\mathrm{d}x\\&=2\int_{\mathbb{R}^n}\left(\int_{\mathbb{T}^n}|u(x)-u(x-y)|\mathrm{d}x\right)\beta^h(y)\mathrm{d}y,\end{aligned} \tag{26}$$

where we take into account inequality (25). Since $\operatorname{supp}\beta^h(y)$ lies in the ball $|y| \le h$ while for $|y| \le h$

$$\int_{\mathbb{T}^n} |u(x) - u(x-y)|\mathrm{d}x = \int_{\mathbb{T}^n} |u(x+y) - u(x)|\mathrm{d}x \le \omega_u(h),$$

then inequality (24) follows from (26) and the identity $\int_{\mathbb{R}^n} \beta^h(y)\mathrm{d}y = 1$. □

Now we are ready to prove the following "strong" estimate.

Proposition 1. *For every $t \ge 0$, $\Delta t > 0$, such that $t + \Delta t < T$*

$$\int_{\mathbb{T}^n} |u(t+\Delta t, x; \varepsilon) - u(t, x; \varepsilon)|\mathrm{d}x \le \omega^t(\Delta t), \tag{27}$$

where $\omega^t(\Delta t) \doteq \inf_{h>0}(4\omega^x(h) + c_1 N_1 \Delta t/h + \mu n c_2 N_2 \Delta t/h^2)$, and c_1, c_2 are universal constants.

Proof. Let $0 \le t < t + \Delta t < T$. We take in (22) $\phi(x) = (\operatorname{sign} v)^h(x)$, where $v = v(x) = u(t+\Delta t, x; \varepsilon) - u(t, x; \varepsilon)$. Since $\nabla\phi = (\operatorname{sign} v) * \nabla\beta^h(x)$, $\nabla_2\phi = (\operatorname{sign} v) * \nabla_2\beta^h(x)$, and $|\operatorname{sign} v| \le 1$, then $\|\nabla\phi\|_\infty \le \|\nabla\beta^h\|_1 = c_1/h$, $\|\nabla_2\phi\|_\infty \le \|\nabla_2\beta^h\|_1 = c_2/h$, where $c_1 = \int |\nabla\beta(y)|\mathrm{d}y$, $c_2 = \int |\nabla_2\beta(y)|\mathrm{d}y$, and it follows from (22) that

$$\int_{\mathbb{T}^n} v(x)(\operatorname{sign} v)^h(x)\mathrm{d}x \le c_1 N_1 \Delta t/h + n\mu c_2 N_2 \Delta t/h^2.$$

This estimate and Lemma 2 imply that

$$\begin{aligned}
\int_{\mathbb{T}^n} |v(x)|\mathrm{d}x &= \int_{\mathbb{T}^n} v(x)(\operatorname{sign} v)^h(x)\mathrm{d}x + \int_{\mathbb{T}^n} (|v(x)| - v(x)(\operatorname{sign} v)^h(x))\mathrm{d}x \\
&\le \int_{\mathbb{T}^n} v(x)(\operatorname{sign} v)^h(x)\mathrm{d}x + \int_{\mathbb{T}^n} |v(x)(\operatorname{sign} v)^h(x) - |v(x)||\mathrm{d}x \\
&\le c_1 N_1 \Delta t/h + n\mu c_2 N_2 \Delta t/h^2 + 2\omega_v(h),
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
\omega_v(h) &= \sup_{\Delta x \in \mathbb{R}^n, |\Delta x| \le h} \int_{\mathbb{T}^n} |v(x+\Delta x) - v(x)|\mathrm{d}x \\
&\le \sup_{\Delta x \in \mathbb{R}^n, |\Delta x| \le h} \int_{\mathbb{T}^n} |u(t+\Delta t, x+\Delta x; \varepsilon) - u(t+\Delta t, x; \varepsilon)|\mathrm{d}x \\
&\quad + \sup_{\Delta x \in \mathbb{R}^n, |\Delta x| \le h} \int_{\mathbb{T}^n} |u(t, x+\Delta x; \varepsilon) - u(t, x; \varepsilon)|\mathrm{d}x \le 2\omega^x(h),
\end{aligned} \tag{29}$$

where we take into account estimate (20). In view of (28), (29) we obtain that

$$\int_{\mathbb{T}^n} |u(t+\Delta t, x; \varepsilon) - u(t, x; \varepsilon)|\mathrm{d}x \le 4\omega^x(h) + c_1 N_1 \Delta t/h + n\mu c_2 N_2 \Delta t/h^2$$

and to complete the proof it only remains to notice that $h > 0$ is arbitrary. □

Observe that both the functions $\omega^x(h) \to 0$, $\omega^t(h) \to 0$ as $h \to 0$ and do not depend on ε.

Lemma 3. *Suppose that $u(x) \in L^\infty(\mathbb{T}^n)$ and $a(u), b(u)$ are nondecreasing functions on $\mathbb{R}$. Then for any $y \in \mathbb{R}^n$*

$$\int_{\mathbb{T}^n} a(u(x))(b(u(x+y)) - b(u(x)))\mathrm{d}x \le 0. \tag{30}$$

Proof. We suppose first that $a(u) = \operatorname{sign}^+(u-k)$, where $k \in \mathbb{R}$. Then

$$\begin{aligned}
&\int_{\mathbb{T}^n} a(u(x))(b(u(x+y)) - b(u(x)))\mathrm{d}x \\
&\quad = \int_{\mathbb{T}^n} \operatorname{sign}^+(u(x)-k)(b(u(x+y)) - b(k))\mathrm{d}x - \int_{\mathbb{T}^n} \operatorname{sign}^+(u(x)-k)(b(u(x)) - b(k))\mathrm{d}x \\
&\quad \le \int_{\mathbb{T}^n} (b(u(x+y)) - b(k))^+\mathrm{d}x - \int_{\mathbb{T}^n} (b(u(x)) - b(k))^+\mathrm{d}x = 0,
\end{aligned}$$

where we take into account that for a nondecreasing $b(u)$

$$\operatorname{sign}^+(u(x)-k)(b(u(x+y)) - b(k)) \le (b(u(x+y)) - b(k))^+,$$
$$\operatorname{sign}^+(u(x)-k)(b(u(x)) - b(k)) = (b(u(x)) - b(k))^+.$$

Thus, (30) holds for $a(u) = \operatorname{sign}^+(u-k)$.

In the case of an arbitrary continuous nondecreasing function $a(u)$ we take $M = \|u\|_\infty$ and notice that for $|u| \le M$

$$a(u) = a(-M) + \int_{-M}^{M} \operatorname{sign}^+(u-k)\mathrm{d}a(k),$$

where $\mathrm{d}a(k)$ is a nonnegative Stieltjes measure. Therefore, using the Fubini theorem, we get

$$\begin{aligned}
\int_{\mathbb{T}^n} a(u(x))(b(u(x+y)) - b(u(x)))\mathrm{d}x &= a(-M)\int_{\mathbb{T}^n} (b(u(x+y)) - b(u(x)))\mathrm{d}x \\
&+ \int_{-M}^{M}\int_{\mathbb{T}^n} \operatorname{sign}^+(u(x)-k)(b(u(x+y)) - b(u(x)))\mathrm{d}x\mathrm{d}a(k) \le 0,
\end{aligned}$$

as was to be proved.

Finally, in the general case when $a(u)$ may be discontinuous we can choose the sequence $a_r(u)$, $r \in \mathbb{N}$, of continuous nondecreasing functions, which pointwise converges to $a(u)$ as $r \to \infty$. As we already proved,

$$\int_{\mathbb{T}^n} a_r(u(x))(b(u(x+y)) - b(u(x)))\mathrm{d}x \le 0.$$

Passing in the above relation to the limit as $r \to \infty$, we arrive at (30). □

Corollary 4. *Let $u(t,x;\varepsilon)$ be the approximate solution. Then for any (merely continuous) convex function $\eta(u)$*

$$\int_{\mathbb{T}^n} \eta(u(t,x;\varepsilon))\mathrm{d}x \le \int_{\mathbb{T}^n} \eta(u_0(x))\mathrm{d}x. \tag{31}$$

In particular, $\|u(t,\cdot;\varepsilon)\|_{L^p(\mathbb{T}^n)} \le \|u_0\|_{L^p(\mathbb{T}^n)}$ for any $p \ge 1$.

Proof. Without loss of generality we can assume that $\eta(u) \in C^1(\mathbb{R})$. The general case is treated using an approximation procedure. By the chain rule and (12)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{T^n} \eta(u(t,x;\varepsilon))\mathrm{d}x &= \int\limits_{T^n} \frac{\mathrm{d}}{\mathrm{d}t}\eta(u(t,x;\varepsilon))\mathrm{d}x \\
&= \frac{1}{\varepsilon}\sum_{i=1}^{n}\left(\int\limits_{\mathbb{T}^n} \eta'(u(t,x;\varepsilon))(\tilde{p}_i(u(t,x-\varepsilon e_i;\varepsilon)) - \tilde{p}_i(u(t,x;\varepsilon)))\mathrm{d}x\right. \\
&\quad \left. + \int\limits_{\mathbb{T}^n} \eta'(u(t,x;\varepsilon))(\tilde{q}_i(u(t,x+\varepsilon e_i;\varepsilon)) - \tilde{q}_i(u(t,x;\varepsilon)))\mathrm{d}x\right). \qquad (32)
\end{aligned}
$$

Since the functions $\eta'(u), \tilde{p}_i(u), \tilde{q}_i(u)$ are nondecreasing, then by Lemma 3 for each $i = 1, \ldots, n$

$$
\begin{aligned}
&\int\limits_{\mathbb{T}^n} \eta'(u(t,x;\varepsilon))(\tilde{p}_i(u(t,x-\varepsilon e_i;\varepsilon)) - \tilde{p}_i(u(t,x;\varepsilon)))\mathrm{d}x \le 0, \\
&\int\limits_{\mathbb{T}^n} \eta'(u(t,x;\varepsilon))(\tilde{q}_i(u(t,x+\varepsilon e_i;\varepsilon)) - \tilde{q}_i(u(t,x;\varepsilon)))\mathrm{d}x \le 0
\end{aligned}
$$

and it follows from (32) that $\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{T^n} \eta(u(t,x;\varepsilon))\mathrm{d}x \le 0$. This readily implies (31). □

Another consequence of Lemma 3 is the following a priori estimate (as we will establish in Sect. 4, the approximate solution $u(t,x;\varepsilon)$ exists for all time $t \in \mathbb{R}_+$, and we can assume that $T = +\infty$).

Proposition 2. *Let* $G(u) = \int\limits_0^u g(s)\mathrm{d}s$ *be a primitive for* $g(u)$. *Then*

$$
\frac{\mu}{\varepsilon^2}\sum_{i=1}^{n}\int\limits_{(0,+\infty)\times\mathbb{T}^n} (g(u(t,x+\varepsilon e_i;\varepsilon)) - g(u(t,x;\varepsilon)))^2\mathrm{d}t\mathrm{d}x \le 2\max_{|u|\le M}|G(u)|. \qquad (33)
$$

Proof. Like in the proof of Corollary 4, we have the relation

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{T^n} G(u(t,x;\varepsilon))\mathrm{d}x \\
&= \frac{1}{\varepsilon}\sum_{i=1}^{n}\left(\int\limits_{\mathbb{T}^n} g(u(t,x;\varepsilon))(p_i(u(t,x-\varepsilon e_i;\varepsilon)) - p_i(u(t,x;\varepsilon)))\mathrm{d}x\right. \\
&\quad \left. + \int\limits_{\mathbb{T}^n} g(u(t,x;\varepsilon))(q_i(u(t,x+\varepsilon e_i;\varepsilon)) - q_i(u(t,x;\varepsilon)))\mathrm{d}x\right) \\
&\quad + \frac{\mu}{\varepsilon^2}\sum_{i=1}^{n}\int\limits_{\mathbb{T}^n} g(u(t,x;\varepsilon))[g(u(t,x+\varepsilon e_i;\varepsilon)) + g(u(t,x-\varepsilon e_i;\varepsilon)) - 2g(u(t,x;\varepsilon)]\mathrm{d}x. \qquad (34)
\end{aligned}
$$

In view of Lemma 3 for all $i = 1, \dots, n$

$$\int_{\mathbb{T}^n} g(u(t,x;\varepsilon))(p_i(u(t,x-\varepsilon e_i;\varepsilon)) - p_i(u(t,x;\varepsilon)))\mathrm{d}x \le 0,$$

$$\int_{\mathbb{T}^n} g(u(t,x;\varepsilon))(q_i(u(t,x+\varepsilon e_i;\varepsilon)) - q_i(u(t,x;\varepsilon)))\mathrm{d}x \le 0$$

and it follows from (34) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{T^n} G(u(t,x;\varepsilon))\mathrm{d}x \le \frac{\mu}{\varepsilon^2}\sum_{i=1}^n \int_{\mathbb{T}^n} g(u(t,x;\varepsilon))[g(u(t,x+\varepsilon e_i;\varepsilon)) + g(u(t,x-\varepsilon e_i;\varepsilon)) - 2g(u(t,x;\varepsilon))]\mathrm{d}x. \tag{35}$$

In view of translation invariance of the Lebesgue measure on torus $\mathbb{T}^n$ we have the equalities

$$\int_{\mathbb{T}^n} g(u(t,x;\varepsilon))g(u(t,x+\varepsilon e_i;\varepsilon))\mathrm{d}x = \int_{\mathbb{T}^n} g(u(t,x-\varepsilon e_i;\varepsilon))g(u(t,x;\varepsilon))\mathrm{d}x,$$

$$\int_{\mathbb{T}^n} (g(u(t,x;\varepsilon)))^2\mathrm{d}x = \int_{\mathbb{T}^n} (g(u(t,x+\varepsilon e_i;\varepsilon)))^2\mathrm{d}x.$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{T}^n} g(u(t,x;\varepsilon))[g(u(t,x+\varepsilon e_i;\varepsilon)) + g(u(t,x-\varepsilon e_i;\varepsilon)) - 2g(u(t,x;\varepsilon))]\mathrm{d}x \\
&= \int_{\mathbb{T}^n} [g(u(t,x;\varepsilon))g(u(t,x+\varepsilon e_i;\varepsilon)) \\
&\quad + g(u(t,x-\varepsilon e_i;\varepsilon))g(u(t,x;\varepsilon)) - 2(g(u(t,x;\varepsilon)))^2]\mathrm{d}x \\
&= \int_{\mathbb{T}^n} [2g(u(t,x;\varepsilon))g(u(t,x+\varepsilon e_i;\varepsilon)) - (g(u(t,x;\varepsilon)))^2 - (g(u(t,x+\varepsilon e_i;\varepsilon)))^2]\mathrm{d}x \\
&= -\int_{\mathbb{T}^n} (g(u(t,x+\varepsilon e_i;\varepsilon)) - g(u(t,x;\varepsilon)))^2\mathrm{d}x
\end{aligned}$$

and it follows from (35) that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{T^n} G(u(t,x;\varepsilon))\mathrm{d}x \ge \frac{\mu}{\varepsilon^2}\sum_{i=1}^n \int_{\mathbb{T}^n} (g(u(t,x+\varepsilon e_i;\varepsilon)) - g(u(t,x;\varepsilon)))^2\mathrm{d}x.$$

Integrating this inequality over the interval $(0,T)$, where $T > 0$, we obtain the estimate

$$\begin{aligned}
&\frac{\mu}{\varepsilon^2}\sum_{i=1}^n \int_{(0,T)\times\mathbb{T}^n} (g(u(t,x+\varepsilon e_i;\varepsilon)) - g(u(t,x;\varepsilon)))^2\mathrm{d}t\mathrm{d}x \\
&\le \int_{\mathbb{T}^n} G(u_0(x))\mathrm{d}x - \int_{\mathbb{T}^n} G(u(T,x;\varepsilon))\mathrm{d}x \le 2\max_{|u|\le M}|G(u)|
\end{aligned}$$

(we use estimate (19) in the last inequality), and to complete the proof, one only need to pass to the limit as $T \to +\infty$. □

4. Existence of approximate solutions

In this section we establish existence of a solution $u = u(t, x) \in C^1([0, +\infty), L^\infty(\mathbb{T}^n))$ to approximate problem (12), (13). Suppose first that the vectors $\tilde{p}(u)$, $\tilde{q}(u)$ are Lipschitz:

$$\sum_{i=1}^{n}[|\tilde{p}_i(u) - \tilde{p}_i(v)| + |\tilde{q}_i(u) - \tilde{q}_i(v)|] \le L|u - v| \quad \forall u, v \in \mathbb{R},$$

where L is a positive constant. Then the map F is Lipschitz continuous as well:

$$\|F(u) - F(v)\|_\infty \le \frac{3L}{\varepsilon}\|u - v\|_\infty \quad \forall u, v \in L^\infty(\mathbb{T}^n).$$

By Picard theorem for ODE in the Banach space $L^\infty(\mathbb{T}^n)$ there is a unique solution $u = u(t) \in C^1([0, T), L^\infty(\mathbb{T}^n))$ of Cauchy problem (12), (13) defined on some interval $[0, T)$, $T > 0$. Moreover in view of (19) this solution is bounded in $L^\infty(\mathbb{T}^n)$: $\|u(t)\|_\infty \le M$. This readily implies that our solution exists on the maximal interval $[0, +\infty)$. In the general case we can choose sequences $\tilde{p}_r(u), \tilde{q}_r(u)$, $r \in \mathbb{N}$, of Lipschitz continuous vectors with nondecreasing components $\tilde{p}_{ri}, \tilde{q}_{ri}$, $i = 1, \ldots, n$, such that $\tilde{p}_r(u) \to \tilde{p}(u)$, $\tilde{q}_r(u) \to \tilde{q}(u)$ as $r \to \infty$ uniformly on any segment. Let $u_r(t) \in C^1([0, +\infty), L^\infty(\mathbb{T}^n))$ be a unique solution to the problem

$$\dot{u} = F_r(u), \quad u(0) = u_0, \tag{36}$$

where for $u = u(x) \in L^\infty(\mathbb{T}^n)$

$$F_r(u)(x) = \frac{1}{\varepsilon}\sum_{i=1}^{n}[\tilde{p}_{ri}(u(x - \varepsilon e_i)) - (\tilde{p}_{ri} + \tilde{q}_{ri})(u(x)) + \tilde{q}_{ri}(u(x + \varepsilon e_i))].$$

Theorem 3. *The sequence $u_r(t)$ converges as $r \to \infty$ to a unique solution $u(t) = u(t, x) \in C^1([0, +\infty), L^\infty(\mathbb{T}^n))$ of original approximate problem (12), (13) in $C^1([0, +\infty), L^1(\mathbb{T}^n))$. In particular, a global solution of this problem exists.*

Proof. In view of estimates (19), (21) (or (27)) the sequence $u_r(t)$ is uniformly bounded and equicontinuous in $C([0, +\infty), L^1(\mathbb{T}^n))$. Moreover, estimates (19), (20) imply precompactness of sequences $u_r(t)$, $r \in \mathbb{N}$, in $L^1(\mathbb{T}^n)$ for any fixed $t \ge 0$. By generalized Arzelà–Ascoli theorem the family u_r is precompact in $C([0, +\infty), L^1(\mathbb{T}^n))$, and we can extract a subsequence of $u_m = u_{r_m}(t)$, which converges to some function $u(t) \in C([0, +\infty), L^1(\mathbb{T}^n))$ uniformly on any segment $[0, T]$. By (19), (21) the limit function $u(t) \in C([0, +\infty), L^\infty(\mathbb{T}^n))$, and $\|u(t)\|_\infty \le M$. Passing to the limit as $m \to \infty$ in the equality

$$u_m(t) = u_0 + \int_0^t F_m(u_m(s))\mathrm{d}s, \quad F_m \doteq F_{r_m},$$

and taking into account that as $m \to \infty$

$$\begin{aligned} F_m(u_m(s)) = {} & \frac{1}{\varepsilon}\sum_{i=1}^{n}[\tilde{p}_{mi}(u_m(s, x - \varepsilon e_i)) \\ & -(\tilde{p}_{mi} + \tilde{q}_{mi})(u_m(s, x)) + \tilde{q}_{mi}(u_m(s, x + \varepsilon e_i))] \\ & \to F(u(s)) = \frac{1}{\varepsilon}\sum_{i=1}^{n}[\tilde{p}_i(u(s, x - \varepsilon e_i)) - (\tilde{p}_i + \tilde{q}_i)(u(s, x)) + \tilde{q}_i(u(s, x + \varepsilon e_i))] \end{aligned}$$

in $C([0, t], L^1(\mathbb{T}^n))$ (where $\tilde{p}_{mi} = \tilde{p}_{r_m i}$, $\tilde{q}_{mi} = \tilde{q}_{r_m i}$), we arrive at the identity

$$u(t) = u_0 + \int_0^t F(u(s))\mathrm{d}s \quad \forall t > 0.$$

This implies that $u(t) = u(t,x) \in C^1([0,+\infty), L^\infty(\mathbb{T}^n))$ is a solution of (12), (13). Since this solution is unique, we see that the limit point $u(t)$ of a subsequence u_m does not depend on the choice of this subsequence. The latter readily implies that the original sequence $u_r(t)$ converges to $u(t)$ as $r \to \infty$ in $C([0,+\infty), L^1(\mathbb{T}^n))$. Since $\frac{\mathrm{d}}{\mathrm{d}t}(u_r - u)(t) = F_r(u_r) - F(u) \underset{r\to\infty}{\to} 0$ in $C([0,+\infty), L^1(\mathbb{T}^n))$, we conclude that $u_r(t)$ converges as $r \to \infty$ to $u(t)$ in $C^1([0,+\infty), L^1(\mathbb{T}^n))$. This concludes the proof. □

5. Convergence of approximations

Let, as above, $u = u(t,x;\varepsilon)$ be a unique solution of (12), (13). It follows from estimates (19), (20), (27) that the family $u = u(t,x;\varepsilon)$ is precompact in $C([0,+\infty), L^1(\mathbb{T}^n))$. This allows to extract a sequence $\varepsilon_r \to 0$ such that $u_r(x,t) = u(t,x;\varepsilon_r) \to u(t,x)$ as $r \to \infty$ in $C([0,+\infty), L^1(\mathbb{T}^n))$ and, therefore, in $L^1_{\mathrm{loc}}(\Pi)$ as well.

Let us prove that this limit function $u(t,x)$ is an e.s. of Cauchy problem (1), (3).

Theorem 4. *The approximate solutions $u(t,x;\varepsilon) \to u(t,x)$ as $\varepsilon \to 0$ in $C([0,+\infty), L^1(\mathbb{T}^n))$, where $u(t,x)$ is a unique e.s. of (1), (3).*

Proof. Let $u_\varepsilon = u(t,x;\varepsilon)$ be a solution of approximate problem (12), (13) and $k \in \mathbb{R}$. Since $v \equiv k$ is a solution of (12), then by relations (16), (18)
for all $t > 0$

$$\begin{aligned}
\frac{\partial}{\partial t}|u(t,x;\varepsilon) - k| &= \frac{\partial}{\partial t}[(u_\varepsilon - k)^+ + (k - u_\varepsilon)^+] \\
&\le \frac{1}{\varepsilon}\sum_{i=1}^{n}[|\tilde{p}_i(u(t,x-\varepsilon e_i;\varepsilon)) - \tilde{p}_i(k)| - |\tilde{p}_i(u(t,x;\varepsilon)) - \tilde{p}_i(k)| \\
&\quad + |\tilde{q}_i(u(t,x+\varepsilon e_i;\varepsilon)) - \tilde{q}_i(k)| - |\tilde{q}_i(u(t,x;\varepsilon)) - \tilde{q}_i(k)|] \\
&= \frac{1}{\varepsilon}\sum_{i=1}^{n}[|p_i(u(t,x-\varepsilon e_i;\varepsilon)) - p_i(k)| - |p_i(u_\varepsilon(t,x;\varepsilon)) - p_i(k)| \\
&\quad + |q_i(u(t,x+\varepsilon e_i;\varepsilon)) - q_i(k)| - |q_i(u(t,x;\varepsilon)) - q_i(k)|] \\
&\quad + \frac{\mu}{\varepsilon^2}\sum_{i=1}^{n}[|g(u(t,x-\varepsilon e_i;\varepsilon)) - g(k)| \\
&\quad - 2|g(u(t,x;\varepsilon)) - g(k)| + |g(u(t,x+\varepsilon e_i;\varepsilon)) - g(k)|].
\end{aligned} \tag{37}$$

Multiplying this inequality by a nonnegative test function $\phi = \phi(t,x) \in C_0^2(\Pi)$ and integrating the left-hand side by parts, we arrive at

$$\begin{aligned}
-\int_\Pi |u(t,x;\varepsilon) - k|\phi_t \mathrm{d}t\mathrm{d}x &\le \int_\Pi \sum_{i=1}^{n}\left\{\frac{1}{\varepsilon}[(|p_i(u(t,x-\varepsilon e_i;\varepsilon)) - p_i(k)|\right. \\
&\quad - |p_i(u_\varepsilon) - p_i(k)|) + (|q_i(u(t,x+\varepsilon e_i;\varepsilon)) - q_i(k)| \\
&\quad - |q_i(u_\varepsilon) - q_i(k)|)] + \frac{\mu}{\varepsilon^2}[|g(u(t,x-\varepsilon e_i;\varepsilon)) - g(k)| \\
&\quad \left. - 2|g(u_\varepsilon) - g(k)| + |g(u(t,x+\varepsilon e_i;\varepsilon)) - g(k)|]\right\}\phi \mathrm{d}t\mathrm{d}x \\
&= \int_\Pi \sum_{i=1}^{n}\left\{|p_i(u_\varepsilon) - p_i(k)|\frac{\phi(t,x+\varepsilon e_i) - \phi(t,x)}{\varepsilon}\right. \\
&\quad - |q_i(u_\varepsilon) - q_i(k)|\frac{\phi(t,x) - \phi(t,x-\varepsilon e_i)}{\varepsilon}
\end{aligned}$$

$$+\ \mu|g(u_\varepsilon)-g(k)|\frac{\phi(t,x+\varepsilon e_i)-2\phi(t,x)+\phi(t,x-\varepsilon e_i)}{\varepsilon^2}\Bigg\}\mathrm{d}t\mathrm{d}x$$

$$=\int\limits_\Pi\sum_{i=1}^n[(|p_i(u_\varepsilon)-p_i(k)|-|q_i(u_\varepsilon)-q_i(k)|)\phi_{x_i}+|g(u_\varepsilon)-g(k)|\phi_{x_ix_i}]\mathrm{d}t\mathrm{d}x+I(\varepsilon), \tag{38}$$

where $I(\varepsilon)\to 0$ as $\varepsilon\to 0$. Since the functions $p_i(u)$, $q_i(u)$ increase, then

$$\begin{aligned}&|p_i(u_\varepsilon)-p_i(k)|-|q_i(u_\varepsilon)-q_i(k)|\\&\quad=\operatorname{sign}(u_\varepsilon-k)[(p_i(u_\varepsilon)-p_i(k))-(q_i(u_\varepsilon)-q_i(k))]\\&\quad=\operatorname{sign}(u_\varepsilon-k)(f_i(u)-f_i(k))\end{aligned}$$

for all $i=1,\dots,n$, and it follows from (38) that

$$\int\limits_\Pi\Bigg\{|u_\varepsilon-k|\phi_t+\sum_{i=1}^n[\operatorname{sign}(u_\varepsilon-k)(f_i(u_\varepsilon)-f_i(k))\phi_{x_i}$$

$$+\mu|g(u_\varepsilon)-g(k)|\phi_{x_ix_i}\Bigg\}\mathrm{d}t\mathrm{d}x\ge -I(\varepsilon),\quad u_\varepsilon=u(t,x;\varepsilon). \tag{39}$$

As was shown above, the sequence $u(t,x;\varepsilon_r)\to u(t,x)$ as $r\to\infty$ in $L^1_{\mathrm{loc}}(\Pi)$. Passing to the limit in (39) as $\varepsilon=\varepsilon_r\to 0$, we obtain the entropy relation

$$\int\limits_\Pi\{|u-k|\phi_t+\operatorname{sign}(u-k)(f(u)-f(k))\cdot\nabla_x\phi$$

$$+\mu|g(u)-g(k)|\Delta_x\phi\}\mathrm{d}t\mathrm{d}x\ge 0,\quad u=u(t,x),\ k\in\mathbb{R}. \tag{40}$$

Since a nonnegative test function $\phi=\phi(t,x)\in C_0^1(\bar\Pi)$ is arbitrary, identity (40) means that u satisfies entropy relation (4). Recall that $u(t,x;\varepsilon_r)\to u(t,x)$ as $r\to\infty$ in $C([0,+\infty),L^1(\mathbb{T}^n))$ and $u(0,x;\varepsilon)=u_0(x)$. Therefore, $u(0,x)=u_0(x)$ as well, and in view of continuity of $u(t,\cdot)$ in $L^1(\mathbb{T}^n)$ we obtain that $u(t,\cdot)\to u_0$ in $L^1(\mathbb{T}^n)$ as $t\to 0$, which implies the initial condition (5). It remains only to establish that $\nabla_x g(u)\in L^2_{\mathrm{loc}}(\Pi)$. We will demonstrate that actually $\nabla_x g(u)\in L^2((0,+\infty)\times\mathbb{T}^n)$. For that observe that by Proposition 2 the sequences $v_{ri}=(g(u(t,x+\varepsilon_r e_i;\varepsilon_r))-g(u(t,x;\varepsilon_r)))/\varepsilon_r$ are bounded in $L^2((0,+\infty)\times\mathbb{T}^n)$ for all $i=1,\dots,n$: $\|v_{ri}\|_2\le C/\mu$, where $C=2\max_{|u|\le M}|G(u)|$. Therefore, may be after extraction of a subsequence, $v_{ri}\rightharpoonup v_i$ as $r\to\infty$ weakly in $L^2((0,+\infty)\times\mathbb{T}^n)$. Let $\phi(t,x)\in C_0^1(\Pi)$. Then

$$\int\limits_\Pi g(u(t,x;\varepsilon_r))\frac{\phi(t,x-\varepsilon_r e_i)-\phi(t,x)}{\varepsilon_r}\mathrm{d}t\mathrm{d}x$$

$$=\int\limits_\Pi v_{ri}(t,x)\phi(t,x)\mathrm{d}t\mathrm{d}x\underset{r\to\infty}{\to}\int\limits_\Pi v_i(t,x)\phi(t,x)\mathrm{d}t\mathrm{d}x.$$

On the other hand, as $r\to\infty$ $u(t,x;\varepsilon_r)\to u(t,x)$ in $L^1_{\mathrm{loc}}(\Pi)$ and $\dfrac{\phi(t,x-\varepsilon_r e_i)-\phi(t,x)}{\varepsilon_r}\to -\phi_{x_i}(t,x)$ uniformly on Π, hence

$$\int\limits_\Pi g(u(t,x;\varepsilon_r))\frac{\phi(t,x-\varepsilon_r e_i)-\phi(t,x)}{\varepsilon_r}\mathrm{d}t\mathrm{d}x\underset{r\to\infty}{\to}-\int\limits_\Pi g(u(t,x))\phi_{x_i}(t,x)\mathrm{d}t\mathrm{d}x.$$

Comparing the above limit relation, we conclude that for all $\phi(t,x)\in C_0^1(\Pi)$

$$-\int\limits_\Pi g(u(t,x))\phi_{x_i}(t,x)\mathrm{d}t\mathrm{d}x=\int\limits_\Pi v_i(t,x)\phi(t,x)\mathrm{d}t\mathrm{d}x,$$

that is, the distributions $g(u)_{x_i} = v_i \in L^2((0,+\infty) \times \mathbb{T}^n)$, as was to be proved. □

Acknowledgements

The research of this author (E.A.) was carried out with the financial support of FAPESP, processo 2016/233741 and CNPQ Universal processo 445758/2014-7. The research of this author (M.C.) was carried out with the financial support of FAPESP, processo 2012/15780-9. The research of this author (E.P.) was carried out with the financial support of the Russian Foundation for Basic Research (Grant No. 15-01-07650-a) and the Ministry of Education and Science of Russian Federation (Project No. 1.445.2016/1.4).

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Eduardo Abreu
Universidade Estadual de Campinas
Campinas
Brazil
e-mail: eabreu@ime.unicamp.br

Mathilde Colombeau
Universidade de São Paulo
São Paulo
Brazil
e-mail: mcolombeau@ime.usp.br

Mathilde Colombeau
Universidade Estadual de Campinas
Campinas
Brazil

Evgeny Yu Panov
Novgorod State University
Veliky Novgorod
Russian Federation
e-mail: Eugeny.Panov@novsu.ru

Evgeny Yu Panov
St. Petersburg State University
Saint Petersburg
Russian Federation

(Received: May 9, 2017; revised: September 22, 2017)