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On magnetoelectric coupling at equilibrium in continua with microstructure

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Abstract. A theory of micromorphic continua, applied to electromagnetic solids, is exploited to study magnetoelectric effects at equilibrium. Microcurrents are modeled by the microgyration tensor of stationary micromotions, compatibly with the balance equations for null microdeformation. The equilibrium of the continuum subject to electric and magnetic fields is reformulated accounting for electric multipoles which are related to microdeformation by evolution equations. Polarization and magnetization are derived for uniform fields under the micropolar reduction in terms of microstrain and octupole structural parameters. Nonlinear dependance on the electromagnetic fields is evidenced, compatibly with known theoretical and experimental results on magnetoelectric coupling.

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1. Introduction

The growing interest in materials and structures characterized by noticeable electromagneto-mechanical coupling has been recently motivated the development of physical models of microscopic interactions in matter. In particular, materials manifesting both ferromagnetic and ferroelectric effects have been investigated from theoretical and experimental point of view to explain structural features involved in these behaviors [1,2].

Concerning with continuum theories of electromagnetic media, beside classical phenomenological models of dielectric and magnetic solids (see for example [3,4]), more recent microcontinuum theories have been introduced in order to account for internal degrees of freedom by which polarization and magnetization can be modeled [5]. These theories are based on the mechanical micromorphic field theory extended to electromagnetic interactions and, differently from the usual approach [6,7], rely on charge microdensity and electric multipoles connected to microdeformation.

Although microdeformation allows for specific contributions to polarization and magnetization, the basic element of magnetic dipoles, i.e., intrinsic microcurrents, is excluded from the domain where rates of microdisplacement apply. In particular, at equilibrium, macro- and microvelocities are zero contextually with nonzero stationary microcurrents. Obviously this argument remains valid for nonequilibrium configurations and, in general, holds for solids as well as for fluids, liquid crystals or composite materials modeled as micromorphic continua.

The previous comment motivates the present work where we look for a description of equilibrium in electromagnetic solids, compatible with stationary micromotions of bound charges. After a summary of the micromorphic electromagnetic model developed in [5,8], we focus on the characterization of equilibrium adopting a lower-scale description which accounts for microcurrents as stationary micromotions. The interaction among different atomic components of the continuum complies with a microdeformation, in order to establish the equilibrium configuration. This approach, introduced in Sect. 3, allows us to obtain an algebraic equation for the microgyration tensor $\hat{\mathbf{N}}$, which involves magnetic induction, electric field and

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its gradient. It is shown that, for uniform fields in the presence of intrinsic polarization, the solution for $\hat{\mathbf{N}}$ cannot be written as a linear map with respect to the electric and magnetic fields. Since magnetization $\boldsymbol{\mathcal{M}}$ is linear in $\hat{\mathbf{N}}$, the previous result implies a nonlinear dependence of $\boldsymbol{\mathcal{M}}$ on the electromagnetic field, as shown in Sect. 5. This can be realized after the solution of the set of equilibrium equations which, beside the balance laws for momentum and angular momentum, encloses the evolution equations for multipoles, here limited to the second order (quadrupole), as functions of microdeformation. The evolution equation for quadrupole can be solved in terms of the successive order parameters (octupoles for null deformation) so that polarization, at first order, can be obtained as function of the microdeformation. The present results are compatible with theoretical and experimental results on the physics of magnetoelectric coupling in solids.

2. Microcontinuum electromagneto-elastic model

Here we resume the essential results of a microcontinuum theory for electromagnetic media which relies on both microdensities of mass and electric charge to account for electromagneto-elastic coupling. Details on the general formulation and derivations of balance equations can be found in previous papers [5,8]. According to the common description of internal degrees of freedom within the continuum particle (see [9]), we introduce the relative position $\boldsymbol{\xi}$ of a point in the particle $\mathcal{P}$ with respect to its center of mass $\mathbf{x}$. Denoting by Ξ and $\mathbf{X}$ the corresponding vectors in the reference (material) configuration, we assume the maps $\mathbf{x} = \tilde{\mathbf{x}}(\mathbf{X}, t)$, $\boldsymbol{\xi} = \tilde{\boldsymbol{\xi}}(\mathbf{X}, \Xi, t)$ to be sufficiently smooth and invertible. In particular, we pose

$$\tilde{\boldsymbol{\xi}}(\mathbf{X}, \Xi, t) = \boldsymbol{\chi}(\mathbf{X}, t)\Xi, \quad \text{or} \quad \xi_i = \chi_{iJ}\Xi_J, \tag{2.1}$$

where $\boldsymbol{\chi}$ is the microdeformation tensor, with inverse $\boldsymbol{\mathfrak{X}}$ $(\chi_{iJ}\mathfrak{X}_{Jk} = \delta_{ik})$. As usual, we denote by $\mathbf{F} = \nabla_{\mathbf{X}}\tilde{\mathbf{x}}$ the deformation tensor and introduce the following strain measures [9]

$$\mathcal{C} = \boldsymbol{\chi}^T\boldsymbol{\chi}, \quad \mathfrak{C} = \mathbf{F}^T\boldsymbol{\mathfrak{X}}^T, \quad \boldsymbol{\Gamma} = \boldsymbol{\mathfrak{X}}(\nabla_{\mathbf{X}}\boldsymbol{\chi})^T, \tag{2.2}$$

known, respectively, as microdeformation strain, deformation strain and wryness. From the last tensor, posing $\gamma_{ijh} = \chi_{iH}\Gamma_{HKL}F^{-1}_{Lj}\Xi_{Kh}$, the gradient of $\boldsymbol{\xi}$ can be expressed as

$$(\nabla\boldsymbol{\xi})^T = \boldsymbol{\gamma}\boldsymbol{\xi}. \tag{2.3}$$

Also, the material time derivative of $\boldsymbol{\xi}$ is expressed by the microgyration tensor $\mathbf{N}(\mathbf{x}, t)$ as

$$\dot{\boldsymbol{\xi}} = \mathbf{N}\boldsymbol{\xi}. \tag{2.4}$$

Then, absolute position and velocity of a point in the particle are $\mathbf{x} + \boldsymbol{\xi}$ and $\mathbf{v} + \dot{\boldsymbol{\xi}}$, respectively, where $\mathbf{v} = \dot{\mathbf{x}}$. Denoting by $\mathcal{P}_t$ the current configuration of the particle $\mathcal{P}$, we define mass and charge densities as

$$\rho(\mathbf{x}, t) = \frac{1}{\Delta v'}\int_{\mathcal{P}_t}\rho'(\mathbf{x} + \boldsymbol{\xi}, t)\,\mathrm{d}v', \quad q(\mathbf{x}, t) = \frac{1}{\Delta v'}\int_{\mathcal{P}_t}\sigma'(\mathbf{x} + \boldsymbol{\xi}, t)\,\mathrm{d}v' \tag{2.5}$$

where $\Delta v' = \mathrm{vol}(\mathcal{P}_t)$ and ρ' and σ' are, respectively, the microdensities of mass and bound charge. Since we are not interested into conducting media, density of free charges is neglected here. Beside the microinertia tensor

$$\boldsymbol{\mathcal{I}}(\mathbf{x}, t) = \frac{1}{\rho\Delta v'}\int_{\mathcal{P}_t}\rho'(\mathbf{x} + \boldsymbol{\xi}, t)\,\boldsymbol{\xi}\otimes\boldsymbol{\xi}\,\mathrm{d}v', \tag{2.6}$$

we introduce electric dipole and quadrupole densities as

$$\mathbf{p}(\mathbf{x}, t) = \frac{1}{\Delta v'}\int_{\mathcal{P}_t}\sigma'(\mathbf{x} + \boldsymbol{\xi}, t)\,\xi\,\mathrm{d}v', \quad \mathbf{Q}(\mathbf{x}, t) = \frac{1}{\Delta v'}\int_{\mathcal{P}_t}\sigma'(\mathbf{x} + \boldsymbol{\xi}, t)\,\boldsymbol{\xi}\otimes\boldsymbol{\xi}\,\mathrm{d}v'. \tag{2.7}$$

Electric multipoles of successive order are similarly defined. A hierarchy of evolution equations for multipoles can be derived using standard techniques of continuum mechanics [10]. In particular, up to the first successive multipole, $\mathbf{p}$ and $\mathbf{Q}$ satisfy the following evolution equations [8]

$$\begin{aligned} \dot{\mathbf{p}} + \mathbf{p}(\nabla \cdot \mathbf{v}) &= \mathbf{N}\mathbf{p} - (\nabla \cdot \mathbf{N})\mathbf{Q} - \mathbf{N} : \boldsymbol{\gamma}\,\mathbf{Q} \\ \dot{\mathbf{Q}} + \mathbf{Q}(\nabla \cdot \mathbf{v}) &= 2\mathrm{Sym}(\mathbf{N}\mathbf{Q}) - (\nabla \cdot \mathbf{N})\,\boldsymbol{\mathcal{Q}} - \mathbf{N} : \boldsymbol{\gamma}\,\boldsymbol{\mathcal{Q}} \end{aligned} \tag{2.8}$$

where

$$\boldsymbol{\mathcal{Q}} = \frac{1}{\Delta v'} \int_{\mathcal{P}_t} \sigma' \boldsymbol{\xi} \otimes \boldsymbol{\xi} \otimes \boldsymbol{\xi}\, \mathrm{d}v'$$

is the electric octupole density. An analogous derivation yields the following evolution equation for the microinertia tensor

$$\dot{\boldsymbol{\mathcal{I}}} = 2\mathrm{Sym}(\mathbf{N}\boldsymbol{\mathcal{I}}) - (\nabla \cdot \mathbf{N})\boldsymbol{\mathfrak{I}} - \mathbf{N} : \boldsymbol{\gamma}\,\boldsymbol{\mathfrak{I}} \tag{2.9}$$

where

$$\boldsymbol{\mathfrak{I}} = \frac{1}{\rho \Delta v'} \int_{\mathcal{P}_t} \rho' \boldsymbol{\xi} \otimes \boldsymbol{\xi} \otimes \boldsymbol{\xi}\, \mathrm{d}v'.$$

Within the present microcontinuum approach, it has been shown that Maxwell's equations in matter can be obtained by a suitable expansion of charge microdensity in terms of the vector $\boldsymbol{\xi}$ [8]. This derivation exploits the evolution equations (2.8) and allows to write polarization $\mathbf{P}$, magnetization $\mathbf{M}$ and current $\mathbf{J}$ in terms of electric multipoles densities. Up to second-order multipoles, we have

$$\begin{aligned} &\mathbf{P} = \mathbf{p} - \frac{1}{2}\nabla \cdot \mathbf{Q}, \quad \mathbf{J} = q\mathbf{v} + \mathbf{N}\mathbf{p}, \\ &\mathbf{M} = \boldsymbol{\mathcal{M}} - \frac{1}{c}\mathbf{v} \times \mathbf{P}, \quad \mathcal{M}_i = \frac{1}{2c}\epsilon_{ijk}(N_{jh} - L_{jh})Q_{hk} \end{aligned} \tag{2.10}$$

where $\mathbf{L} = (\nabla \mathbf{v})^T$. Here we use Heaviside–Lorentz units and, accordingly, Maxwell's equations for electric and magnetic fields $\mathbf{E}$, $\mathbf{H}$ read

$$\begin{aligned} &\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0, \\ &\nabla \cdot \mathbf{D} = q, \quad \nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \frac{1}{c}\mathbf{J} \end{aligned} \tag{2.11}$$

where $\mathbf{B} = \mathbf{H} + \mathbf{M}$ and $\mathbf{D} = \mathbf{E} + \mathbf{P}$ are, respectively, the magnetic induction and the electric displacement. Balance equations for momentum, moment of momentum and energy can be derived after evaluating electromagnetic forces by an expansion of electric and magnetic fields about $\boldsymbol{\xi} = \mathbf{0}$. For the next analysis, we are here interested into the first two balance laws. Denoting by $\mathbf{T}$ and $\mathbf{m}$, respectively, the Cauchy stress tensor and the couple stress tensor, we obtain

$$\rho \dot{\mathbf{v}} = \rho \mathbf{f} + \mathbf{f}^{\mathrm{em}} + \nabla \cdot \mathbf{T}, \tag{2.12}$$

$$\rho \boldsymbol{\sigma} = \rho (\nabla \mathbf{f})^T \boldsymbol{\mathcal{I}} + \mathbf{C}^{\mathrm{em}} + \mathbf{T}^T - \mathbf{S} + \nabla \cdot \mathbf{m}. \tag{2.13}$$

where $\mathbf{f}$ is the bulk mechanical force per unit mass and $\boldsymbol{\sigma}$ is the rate of spin inertia, given by

$$\boldsymbol{\sigma} = \dot{\mathbf{N}}\boldsymbol{\mathcal{I}} + \mathbf{N}\mathbf{N}\boldsymbol{\mathcal{I}}. \tag{2.14}$$

The explicit form of electromagnetic force and couple densities is

$$\begin{aligned}\mathbf{f}^{\text{em}} &= q\boldsymbol{\mathcal{E}} + (\mathbf{p}\cdot\nabla)\boldsymbol{\mathcal{E}} + \frac{1}{2}(\mathbf{Q}\cdot\nabla)\nabla\boldsymbol{\mathcal{E}} + \frac{1}{c}[(\mathbf{N}-\mathbf{L})\mathbf{p}]\times\mathbf{B}\\ &\quad + \frac{1}{c}[(\mathbf{N}-\mathbf{L})\mathbf{Q}\nabla]\times\mathbf{B} + \frac{1}{2c}\mathbf{B}\times\left[(\mathbf{Q}\cdot\nabla)\mathbf{L}^T\right].\end{aligned} \tag{2.15}$$

$$C_{ij}^{\text{em}} = p_i\mathcal{E}_j + \left[\mathcal{E}_{i,k} + \frac{1}{c}\epsilon_{ipq}(N_{pk}-L_{pk})B_q\right]Q_{kj} \tag{2.16}$$

where $\boldsymbol{\mathcal{E}} = \mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}$. Tensor equation (2.13) is the dual form of the vector balance on angular momentum [9]. The second-order tensor $\mathbf{S}$, introduced in this case, is symmetric. In general, additional constitutive equations for $\mathbf{T}$, $\mathbf{S}$ and $\mathbf{m}$ are required to obtain explicit forms of balance laws. Common principles of continuum mechanics imply that these tensors depend on the strain measures (2.2) which involve macro- and microdeformations [8].

3. Equilibrium statements for microstructure

In this section, we look for a characterization of equilibrium conditions in the microstructure of continua in connection with their electromagnetic properties. We observe that the micromorphic model introduced in the previous section accounts for a dimensional scale below the macroscopic one. Usually, in the context of solid phase, this scale pertains to molecular structures such as the cell in a crystal lattice where deformations and motions give rise to polarization variation, local current and magnetization, according to Eq. (2.10). At equilibrium, all the microstructure contributions to current and magnetization, which are due to velocity $\mathbf{v}$ and microgyration tensor $\mathbf{N}$, vanish. Nevertheless, microlocal currents or magnetic dipoles pertaining the lower atomic scale exist at equilibrium, irrespective of molecular deformations, and may contribute to magnetization and polarization.

Owing to these facts, we look for a model which account for atomic behavior and introduce a microstructure at a lower scale which, at equilibrium, represents particles endowed with electric multipoles and stationary currents. More precisely, if we denote by $\hat{\boldsymbol{\xi}}$ the relative position of a point within such particle $\hat{\mathcal{P}}$ with respect to $\boldsymbol{\xi}$ and let $\hat{\mathbf{N}}$ to be the corresponding microrotation tensor, according to (2.4) we can write

$$\ddot{\hat{\boldsymbol{\xi}}} = \dot{\hat{\mathbf{N}}}\hat{\boldsymbol{\xi}} + \hat{\mathbf{N}}\hat{\mathbf{N}}\hat{\boldsymbol{\xi}}.$$

Then, the condition $\dot{\hat{\mathbf{N}}} = \mathbf{0}$ describes a stationary micromotion in $\hat{\mathcal{P}}$. The physical meaning of this statement can be easily recognized in the particular case of a linear micropolar model where $\chi_{hH} = \delta_{hH} - \epsilon_{hij}\phi_j\delta_{iH}$ and Φ is the microrotation vector. In this case, the micromotion is a rigid rotation around Φ with angular velocity of modulus $\|\dot{\Phi}\|$ and $\hat{N}_{ij} = -\epsilon_{ijk}\dot{\phi}_k$. The substitution into the previous equation shows that the condition $\dot{\hat{\mathbf{N}}} = \mathbf{0}$ amounts to a stationary microrotation.

The time independent value of $\hat{\mathbf{N}}$ is assumed to be also independent on the microdeformation, but depends on the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ at equilibrium.

We conclude that if we concern with equilibrium in such a structure, the following conditions hold

$$\mathbf{v} = \mathbf{0}, \quad \mathbf{N} = \mathbf{0}, \quad \dot{\hat{\mathbf{N}}} = \mathbf{0}. \tag{3.1}$$

Now we consider a nonconducting electromagnetic medium $\mathcal{B}$ whose molecular structure consists of n atoms, in the absence of mechanical body forces, and pose $q = 0$, $\mathbf{f} = \mathbf{0}$. We model $\mathcal{B}$ as a macroscopically rigid continuum due to the superposition of a set of n micromorphic continua $\hat{\mathcal{B}}^{(\nu)}$ whose particles have an internal structure described by a relative microposition $\hat{\boldsymbol{\xi}}$ and, according to our previous notation, we denote with a superimposed hat the corresponding mechanical quantities. Also, we represent multipoles $\hat{\mathbf{p}}^{(\nu)}$ and $\hat{\mathbf{Q}}^{(\nu)}$, as well as the microinertia tensor $\hat{\boldsymbol{\mathcal{I}}}^{(\nu)}$ for each element of the set, in the form

$$\hat{\mathbf{p}}^{(\nu)} = \bar{\mathbf{p}}^{(\nu)} + \acute{\mathbf{p}}^{(\nu)}, \quad \hat{\mathbf{Q}}^{(\nu)} = \bar{\mathbf{Q}}^{(\nu)} + \acute{\mathbf{Q}}^{(\nu)}, \quad \hat{\boldsymbol{\mathcal{I}}}^{(\nu)} = \bar{\boldsymbol{\mathcal{I}}}^{(\nu)} + \acute{\boldsymbol{\mathcal{I}}}^{(\nu)} \quad (\nu = 1, \ldots, n)$$

where $\bar{\mathbf{p}}$,$\bar{\mathbf{Q}}$, $\bar{\boldsymbol{\mathcal{I}}}$ denote quantities pertaining the atomic structure in the absence of microdeformation and $\acute{\mathbf{p}}$, $\acute{\mathbf{Q}}$, $\acute{\boldsymbol{\mathcal{I}}}$ are perturbed values due to the interaction among the n microcontinua, which is responsible of the molecular microdeformation. The tensor $\hat{\boldsymbol{\mathcal{I}}}$ is assumed to be positive definite. Owing to Eqs. (2.14)–(2.16) and (3.1), applied to each element of the set $\{\hat{\mathcal{B}}^{(\nu)}\}$, Eqs. (2.12) and (2.13), at equilibrium, reduce to

$$\hat{\mathbf{f}}^{\mathrm{em}(\nu)} = \mathbf{0} \tag{3.2}$$

$$\hat{\rho}^{(\nu)}\hat{\mathbf{N}}^{(\nu)}\hat{\mathbf{N}}^{(\nu)}\bar{\boldsymbol{\mathcal{I}}}^{(\nu)} = \hat{\mathbf{C}}^{\mathrm{em}(\nu)} \tag{3.3}$$

where

$$\hat{\mathbf{f}}^{\mathrm{em}(\nu)} = (\bar{\mathbf{p}}^{(\nu)} \cdot \nabla)\mathbf{E} + \frac{1}{2}(\bar{\mathbf{Q}}^{(\nu)} \cdot \nabla)\nabla\mathbf{E} + \frac{1}{c}(\hat{\mathbf{N}}^{(\nu)}\bar{\mathbf{p}}^{(\nu)}) \times \mathbf{B} + \frac{1}{c}(\hat{\mathbf{N}}^{(\nu)}\bar{\mathbf{Q}}^{(\nu)}\nabla) \times \mathbf{B},$$
$$\hat{C}_{ij}^{\mathrm{em}(\nu)} = \bar{p}_i^{(\nu)}E_j + \left(E_{i,k} + \frac{1}{c}\epsilon_{ipq}\hat{N}_{pk}^{(\nu)}B_q\right)\bar{Q}_{kj}^{(\nu)},$$

and where contributions due to macro- and microdeformations have been suppressed. Here we remember that, according to the previous analysis on the stationary conditions at equilibrium, $\hat{\mathbf{N}}$ does not depend on microdeformation. Moreover, according to this model, the evolution equations (2.8), (2.9) at equilibrium imply

$$\begin{aligned}
&\hat{\mathbf{N}}^{(\nu)}\bar{\mathbf{p}}^{(\nu)} - (\nabla \cdot \hat{\mathbf{N}}^{(\nu)})\bar{\mathbf{Q}}^{(\nu)} = \mathbf{0},\\
&2\mathrm{Sym}(\hat{\mathbf{N}}^{(\nu)}\bar{\mathbf{Q}}^{(\nu)}) - (\nabla \cdot \hat{\mathbf{N}}^{(\nu)})\,\bar{\boldsymbol{\mathcal{Q}}}^{(\nu)} = \mathbf{0},\\
&2\mathrm{Sym}(\hat{\mathbf{N}}^{(\nu)}\bar{\boldsymbol{\mathcal{I}}}^{(\nu)}) - (\nabla \cdot \hat{\mathbf{N}}^{(\nu)})\bar{\boldsymbol{\mathfrak{J}}}^{(\nu)} = \mathbf{0}.
\end{aligned} \tag{3.4}$$

Equations (3.2)–(3.4) determine the tensors $\hat{\mathbf{N}}^{(\nu)}$ and the quantities $\bar{\mathbf{p}}^{(\nu)}$, $\bar{\mathbf{Q}}^{(\nu)}$, $\bar{\boldsymbol{\mathcal{I}}}^{(\nu)}$ $(\nu = 1, \ldots, n)$ for a given electromagnetic static configuration. Interactions among the microcontinua are accounted for in the balance equations of the whole microcontinuum which describe this coupling by means of the microdeformation of the molecular structure. Again from Eqs. (2.12) and (2.13), in this case we have

$$(\acute{\mathbf{p}} \cdot \nabla)\mathbf{E} + \frac{1}{2}(\acute{\mathbf{Q}} \cdot \nabla)\nabla\mathbf{E} + \frac{1}{c}\left(\sum_{\nu=1}^{n}\hat{\mathbf{N}}^{(\nu)}\acute{\mathbf{p}}^{(\nu)}\right) \times \mathbf{B} + \frac{1}{c}\left(\sum_{\nu=1}^{n}\hat{\mathbf{N}}^{(\nu)}\acute{\mathbf{Q}}^{(\nu)}\nabla\right) \times \mathbf{B} + \nabla \cdot \mathbf{T} = \mathbf{0}, \tag{3.5}$$

$$\sum_{\nu=1}^{n}\hat{\rho}^{(\nu)}\hat{N}_{ij}^{(\nu)}\hat{N}_{jh}^{(\nu)}\acute{\mathcal{I}}_{hk}^{(\nu)} = \acute{p}_iE_k + E_{i,h}\acute{Q}_{hk} + \frac{1}{c}\epsilon_{ipq}\sum_{\nu=1}^{n}\hat{N}_{ph}^{(\nu)}B_q\acute{Q}_{hk}^{(\nu)} + T_{ki} - S_{ik} + m_{ikp,p}, \tag{3.6}$$

where

$$\acute{\mathbf{p}} = \sum_{\nu=1}^{n}\acute{\mathbf{p}}^{(\nu)}, \quad \acute{\mathbf{Q}} = \sum_{\nu=1}^{n}\acute{\mathbf{Q}}^{(\nu)}.$$

These last equations have to be coupled with the evolution equations (2.8), (2.9) for dipole, quadrupole and microinertia, which in the present formulation become

$$\begin{aligned}
&\sum_{\nu=1}^{n}\left[\hat{\mathbf{N}}^{(\nu)}\acute{\mathbf{p}}^{(\nu)}-(\nabla\cdot\hat{\mathbf{N}}^{(\nu)})\acute{\mathbf{Q}}^{(\nu)}-\hat{\mathbf{N}}^{(\nu)}:\boldsymbol{\gamma}\,\hat{\mathbf{Q}}^{(\nu)}\right]=\mathbf{0},\\
&\sum_{\nu=1}^{n}\left[2\mathrm{Sym}\left(\hat{\mathbf{N}}^{(\nu)}\acute{\mathbf{Q}}^{(\nu)}\right)-(\nabla\cdot\hat{\mathbf{N}}^{(\nu)})\,\acute{\boldsymbol{\mathcal{Q}}}^{(\nu)}-\hat{\mathbf{N}}^{(\nu)}:\boldsymbol{\gamma}\,\hat{\boldsymbol{\mathcal{Q}}}^{(\nu)}\right]=\mathbf{0},\\
&\sum_{\nu=1}^{n}\hat{\rho}^{(\nu)}\left[2\mathrm{Sym}\left(\hat{\mathbf{N}}^{(\nu)}\acute{\boldsymbol{\mathcal{I}}}^{(\nu)}\right)-(\nabla\cdot\hat{\mathbf{N}}^{(\nu)})\acute{\boldsymbol{\mathfrak{I}}}^{(\nu)}-\hat{\mathbf{N}}^{(\nu)}:\boldsymbol{\gamma}\,\hat{\boldsymbol{\mathfrak{I}}}^{(\nu)}\right]=\mathbf{0}.
\end{aligned}\tag{3.7}$$

We observe that the terms due to deformation in these last equations account for the entire values of second- and third-order multipoles $\hat{\mathbf{Q}}^{(\nu)}$ and

$$\hat{\boldsymbol{\mathcal{Q}}}^{(\nu)}=\bar{\boldsymbol{\mathcal{Q}}}^{(\nu)}+\acute{\boldsymbol{\mathcal{Q}}}^{(\nu)},\quad \hat{\boldsymbol{\mathfrak{I}}}^{(\nu)}=\bar{\boldsymbol{\mathfrak{I}}}^{(\nu)}+\acute{\boldsymbol{\mathfrak{I}}}^{(\nu)}.$$

The first step in looking for a solution to the present equilibrium problem is the determination of the tensors $\hat{\mathbf{N}}^{(\nu)}$. In the following, to save writing, we shall omit the superscript (ν). According to the microstructure assumptions at the basis of Eqs. (3.2) and (3.3), we concern with an atomic (or ionic) element whose center of mass is approximately placed at the center of positive charge. Hence, the second-order moments $\hat{\boldsymbol{\mathcal{I}}}$ and $\hat{\mathbf{Q}}$ contain, respectively, mass and charge microdensities $\hat{\rho}'$ and $\hat{\sigma}'$ of only negative charges (electrons). Then, $\hat{\sigma}'$ is assumed to be proportional to $\hat{\rho}'$ and denoting by α_e the (constant, negative) ratio $\hat{\sigma}'/\hat{\rho}'$ we can write

$$\hat{\mathbf{Q}}=\hat{\rho}\alpha_e\hat{\boldsymbol{\mathcal{I}}}\,.\tag{3.8}$$

Exploiting the positive definiteness of $\boldsymbol{\mathcal{I}}$ and Eq. (3.8), right multiplication of both sides of (3.3) by $\bar{\boldsymbol{\mathcal{I}}}^{-1}$ yields

$$\hat{N}_{ij}\hat{N}_{jh}=\alpha_e\bar{p}_iE_l\bar{Q}^{-1}_{lh}+\alpha_e\left(E_{i,h}+\frac{1}{c}\epsilon_{ipq}\hat{N}_{ph}B_q\right).\tag{3.9}$$

Introducing the second-order tensors $\mathbb{E}$ and $\mathbb{B}$ with entries

$$E_{ij}=\alpha_e\bar{p}_iE_l\bar{Q}^{-1}_{lj},\quad B_{ij}=\frac{\alpha_e}{c}\epsilon_{ijk}B_k,$$

we rewrite Eq. (3.9) in the form of the following equation for the unknown $\hat{\mathbf{N}}$,

$$\hat{\mathbf{N}}\hat{\mathbf{N}}=\mathbb{E}+\alpha_e(\nabla\mathbf{E})^T+\mathbb{B}\hat{\mathbf{N}}.\tag{3.10}$$

In addition, Eq. (3.2) implies a restriction on the allowed values of dipole and quadrupole densities $\hat{\mathbf{p}}$, $\hat{\mathbf{Q}}$ and their dependence on the electromagnetic field. Determination of the general solution of (3.10) requires a specific algebraic analysis on quadratic tensor equations. Here we point out some specific results which will be useful in the next analysis. We observe that if $\mathbf{E}=\mathbf{0}$, the nontrivial solution of (3.10) is

$$\hat{\mathbf{N}}=\mathbb{B}.\tag{3.11}$$

In this case, the microgyration tensor is skewsymmetric and describes a pure microrotation around the direction of the magnetic induction. Alternatively, if $\mathbf{B}=\mathbf{0}$, we obtain

$$\hat{\mathbf{N}}=\left[\mathbb{E}+\alpha_e(\nabla\mathbf{E})^T\right]^{1/2}=:\hat{\mathbf{N}}^{(E)},\tag{3.12}$$

provided the square root exists. In the general case, where both electric and magnetic fields are different from zero, it is easy to show that, if $\hat{\mathbf{N}}^{(E)}$ exists, Eq. (3.10) admits the solution

$$\hat{\mathbf{N}}=\hat{\mathbf{N}}^{(E)}+\mathbb{B},\tag{3.13}$$

if and only if $\hat{\mathbf{N}}^{(E)}\mathbb{B} = \mathbf{0}$. This result, which can be verified by direct substitution of (3.13) into (3.10), characterizes the microgyration tensor for possible uncoupling of electric and magnetic fields.
However, concerning with spatially uniform fields, which will be considered in the following sections, solutions in the form (3.13) are not compatible with the set of Eq. (3.2) and $(3.4)_{1,2}$. In "Appendix A", we give a proof of this result which has an interesting consequence on the dependence of magnetization from the electric field, as shown in Sect. 5.

4. Polarization in uniform fields

Our main interest pertains the representation of polarization and magnetization at equilibrium and their dependence on the applied electromagnetic field. Here we deal with polarization in a spatially uniform magnetic field in order to derive the increment of dipole and quadrupole densities at equilibrium, due to microdeformation.
In the absence of dipole density $\bar{\mathbf{p}}$ for unstrained continua, Eq. (3.11) holds also in the presence of electric field and we have

$$\hat{N}^{(\nu)}_{ij} = \hat{N}_{ij} = \frac{\alpha_e}{c}\epsilon_{ijk}B_k. \tag{4.1}$$

Owing to the independence of $\mathbf{B}$ on spatial variables, we have $\nabla \cdot \hat{\mathbf{N}}^{(\nu)} = 0$ and Eqs. (3.4)–(3.6) reduce to

$$\frac{1}{c}(\hat{\mathbf{N}}\acute{\mathbf{p}}) \times \mathbf{B} + \nabla \cdot \mathbf{T} = \mathbf{0}, \tag{4.2}$$

$$\hat{N}_{ij}\hat{N}_{jh}\acute{\mathcal{I}}_{hk} = \frac{1}{c}\epsilon_{ipq}\hat{N}_{ph}B_q\acute{Q}_{hk} + T_{ki} - S_{ik} + m_{ikp,p} + p_iE_k \tag{4.3}$$

$$\begin{aligned} &\hat{\mathbf{N}}\acute{\mathbf{p}} - \hat{\mathbf{N}} : \boldsymbol{\gamma}\,\hat{\mathbf{Q}} = \mathbf{0},\\ &2\mathrm{Sym}(\hat{\mathbf{N}}\acute{\mathbf{Q}}) - \hat{\mathbf{N}} : \boldsymbol{\gamma}\,\hat{\boldsymbol{\mathcal{Q}}} = \mathbf{0},\\ &2\mathrm{Sym}(\hat{\mathbf{N}}\acute{\boldsymbol{\mathcal{I}}}) - \hat{\mathbf{N}} : \boldsymbol{\gamma}\,\hat{\mathfrak{J}} = \mathbf{0}, \end{aligned} \tag{4.4}$$

where

$$\acute{\boldsymbol{\mathcal{I}}} = \frac{1}{\hat{\rho}}\sum_{\nu=1}^{n}\hat{\rho}^{(\nu)}\acute{\boldsymbol{\mathcal{I}}}^{(\nu)}, \quad \hat{\mathbf{Q}} = \sum_{\nu=1}^{n}\bar{\mathbf{Q}}^{(\nu)} + \acute{\mathbf{Q}},$$

and analogue positions for $\hat{\boldsymbol{\mathcal{Q}}}$, $\hat{\mathfrak{J}}$. After linearization, Eqs. (4.2)–(4.4) can be solved to obtain microdeformation and incremental quantities $\acute{\mathbf{p}}$, $\acute{\mathbf{Q}}$, $\acute{\boldsymbol{\mathcal{I}}}$ up to third-order multipoles. This analysis requires the introduction of suitable constitutive equations which specify the dependence of $\mathbf{T}$, $\mathbf{S}$ and $\mathbf{m}$ on microdeformation strain measures. The general case of micromorphic, anisotropic continua implies complex constitutive laws which yield results in terms of a second-order deformation tensor and a large number of constitutive elastic parameters. However, physically meaningful results can be achieved within the simpler case of micropolar deformations and isotropic behavior of the continuum $\mathcal{B}$. As previously noted, the micropolar deformation tensor in its linear form can be expressed in terms of the microrotation vector Φ as $\chi_{hH} = \delta_{hH} - \epsilon_{hij}\phi_j\delta_{iH}$ and the strain measure $\boldsymbol{\gamma}$ turns out to be

$$\gamma_{lmn} = -\epsilon_{lmp}\phi_{p,n}.$$

According to our hypothesis of rigid macrocontinuum, we can express the stress and couple stress tensors for isotropic continua in the form (see [9])

$$T_{ij} = \kappa\epsilon_{jik}\phi_k, \quad m_{ijk} = -\frac{1}{2}\epsilon_{jkh}m_{ih}, \quad m_{ih} = \alpha\phi_{p,p}\delta_{ih} + \beta\phi_{i,h} + \gamma\phi_{h,i}, \tag{4.5}$$

where κ, α, β, γ are real material parameters with $\kappa > 0$, $\gamma > 0$. Under the micropolar assumption, $\mathbf{S} = \mathbf{0}$ and, accounting for (4.1) and (4.5), Eqs. (4.2) and (4.3) can be rewritten as

$$\frac{\alpha_e}{c^2}(\acute{p}_k B_i - \acute{p}_i B_k)B_k + \kappa\epsilon_{ijk}\phi_{k,j} = 0, \tag{4.6}$$

$$\epsilon_{ijk}\frac{\alpha_e}{c^2}(\hat{\rho}\alpha_e\acute{\mathcal{I}}_{kp} - \acute{Q}_{kp})B_pB_j = 2\kappa\phi_i + (\alpha+\beta)\phi_{h,hi} - \gamma\phi_{i,pp} + \epsilon_{ijk}\acute{p}_jE_k. \tag{4.7}$$

We observe that the reasoning used to justify Eq. (3.8) applies to both quantities $\bar{\mathbf{Q}}$ and $\acute{\mathbf{Q}}$ separately. This implies that the left-hand side of Eq. (4.7) vanishes and we get

$$2\kappa\phi_i + (\alpha+\beta)\phi_{h,hi} + \gamma\phi_{i,pp} + \epsilon_{ijk}\acute{p}_jE_k = 0. \tag{4.8}$$

According to the micropolar assumption, equations $(4.4)_{1,2}$ become

$$\epsilon_{ijp}B_p\acute{p}_j + (\phi_{h,h}B_p - \phi_{q,p}B_q)(\bar{Q}_{pi} + \acute{Q}_{pi}) = 0, \tag{4.9}$$

$$(\epsilon_{ijl}\acute{Q}_{jk} + \epsilon_{kjl}\acute{Q}_{ji})B_l + (\phi_{h,h}B_p - \phi_{q,p}B_q)(\bar{\mathcal{Q}}_{pik} + \acute{\mathcal{Q}}_{pik}) = 0 \tag{4.10}$$

These last equations represent the first two evolution equations of a hierarchical set for multipoles. They can be exploited to derive the incremental multipoles up to the second order if we neglect $\acute{\boldsymbol{\mathcal{Q}}}$. Consistently, the entries of the unperturbed quadrupole $\bar{\mathbf{Q}}$ are required to satisfy Eq. $(3.4)_2$ at equilibrium in the absence of microdeformation and for uniform fields, i.e.,

$$2\mathrm{Sym}(\hat{\mathbf{N}}\bar{\mathbf{Q}}) = \mathbf{0}. \tag{4.11}$$

Finally, without loss of generality, we choose $\mathbf{B} = B\mathbf{e}_2$ to obtain explicit results for Φ, $\acute{\mathbf{p}}$ and $\acute{\mathbf{Q}}$. In this case, Eq. (4.6) yields

$$\begin{aligned}
-\frac{\alpha_e}{c^2}\acute{p}_1B^2 - \kappa(\phi_{2,3} - \phi_{3,2}) &= 0,\\
\phi_{3,1} - \phi_{1,3} &= 0,\\
-\frac{\alpha_e}{c^2}\acute{p}_3B^2 - \kappa(\phi_{1,2} - \phi_{2,1}) &= 0.
\end{aligned} \tag{4.12}$$

Owing to (4.11),

$$\bar{Q}_{11} = \bar{Q}_{33}, \quad \bar{Q}_{ij} = 0, \quad i \neq j,$$

and Eq. (4.9) gives

$$\begin{aligned}
-\acute{p}_3 + \acute{Q}_{12}\varphi - \acute{Q}_{13}\varphi_3 - (\bar{Q}_{11} + \acute{Q}_{11})\varphi_1 &= 0,\\
(\bar{Q}_{22} + \acute{Q}_{22})\varphi - \acute{Q}_{12}\varphi_1 - \acute{Q}_{23}\varphi_3 &= 0,\\
\acute{p}_1 + \acute{Q}_{23}\varphi - \acute{Q}_{13}\varphi_1 - (\bar{Q}_{33} + \acute{Q}_{33})\varphi_3 &= 0,
\end{aligned} \tag{4.13}$$

where we posed

$$\varphi = \phi_{1,1} + \phi_{3,3}, \quad \varphi_1 = \phi_{2,1}, \quad \varphi_3 = \phi_{2,3}.$$

Up to second-order multipoles ($\acute{\boldsymbol{\mathcal{Q}}} = \mathbf{0}$), from (4.10) we obtain

$$\begin{aligned}
-2\acute{Q}_{13} + \bar{\mathcal{Q}}_{211}\varphi - \bar{\mathcal{Q}}_{111}\varphi_1 - \bar{\mathcal{Q}}_{311}\varphi_3 &= 0,\\
-\acute{Q}_{23} + \bar{\mathcal{Q}}_{212}\varphi - \bar{\mathcal{Q}}_{211}\varphi_1 - \bar{\mathcal{Q}}_{123}\varphi_3 &= 0,\\
\acute{Q}_{11} - \acute{Q}_{33} + \bar{\mathcal{Q}}_{123}\varphi - \bar{\mathcal{Q}}_{311}\varphi_1 - \bar{\mathcal{Q}}_{313}\varphi_3 &= 0,\\
\bar{\mathcal{Q}}_{222}\varphi - \bar{\mathcal{Q}}_{212}\varphi_1 - \bar{\mathcal{Q}}_{322}\varphi_3 &= 0,\\
\acute{Q}_{12} + \bar{\mathcal{Q}}_{322}\varphi - \bar{\mathcal{Q}}_{123}\varphi_1 - \bar{\mathcal{Q}}_{323}\varphi_3 &= 0,\\
2\acute{Q}_{13} + \bar{\mathcal{Q}}_{323}\varphi - \bar{\mathcal{Q}}_{313}\varphi_1 - \bar{\mathcal{Q}}_{333}\varphi_3 &= 0.
\end{aligned} \tag{4.14}$$

Equation (4.14) can be solved for the set of unknowns

$$\{\varphi, \varphi_1, \varphi_3, \acute{Q}_{12}, \acute{Q}_{23}, \acute{Q}_{13}\} \tag{4.15}$$

if the incremental quantities $\acute{Q}_{11}, \acute{Q}_{33}$ are given. Then, substitution into (4.13) allows us to achieve $\acute{Q}_{22}$ and $\acute{p}_1$, $\acute{p}_3$ in the following form

$$\begin{aligned}
\acute{Q}_{22} &= \frac{1}{u}(u_1u_{12} + u_3u_{23})\acute{Q} - \bar{Q}_{22}, \quad \acute{Q} := \acute{Q}_{11} - \acute{Q}_{33}\\
\acute{p}_1 &= (u_1u_{13} - uu_{23})\acute{Q}^2 + u_3(\bar{Q}_{33} + \acute{Q}_{33})\acute{Q},\\
\acute{p}_3 &= (uu_{12} - u_3u_{13})\acute{Q}^2 - u_1(\bar{Q}_{11} + \acute{Q}_{11})\acute{Q},
\end{aligned} \tag{4.16}$$

where all the coefficients u depend on the entries of the octupole tensor $\bar{\boldsymbol{Q}}$. These are given in "Appendix B" where solution for the set (4.15) is explicitly derived. The results (4.16) show that the first-order contribution $\acute{\mathbf{p}}$ to polarization can be expressed in terms of the incremental values of the principal momenta Q_{11}, Q_{33} along the direction orthogonal to the applied magnetic field. Obviously, these quantities depend on the atomic structure of the n components of the continuum and also on the intensities of electric and magnetic fields. We introduce the fields

$$f_1 = \phi_{1,2} - \varphi_1, \quad f_3 = \phi_{3,2} - \varphi_3$$

and note that, owing to $(4.12)_2$, $f_{1,3} = f_{3,1}$ and in turn, $\acute{p}_{1,1} + \acute{p}_{3,3} = 0$. Retaining first-order contributions to polarization and exploiting the Gauss' law $(2.11)_3$ in the present case of uniform fields, we obtain

$$\acute{p}_{2,2} = 0. \tag{4.17}$$

Going back to Eq. (4.8), we firstly consider $\mathbf{E}$ parallel to the magnetic field $\mathbf{B}$, posing $\mathbf{E} = E\mathbf{e}_2$, and obtain

$$\begin{aligned}
&2\kappa\phi_1 + (\alpha+\beta)\phi_{h,h1} + \gamma\phi_{1,hh} - \acute{p}_3E = 0,\\
&2\kappa\phi_2 + (\alpha+\beta)\phi_{h,h2} + \gamma\phi_{2,hh} = 0,\\
&2\kappa\phi_3 + (\alpha+\beta)\phi_{h,h3} + \gamma\phi_{3,hh} + \acute{p}_1E = 0.
\end{aligned} \tag{4.18}$$

Substituting $\acute{p}_1$ and $\acute{p}_3$ from equations $(4.12)_{1,3}$ into (4.18), it is easy to show that the following differential equation holds

$$2\kappa f + \gamma\Delta f + \frac{\kappa c^2}{\alpha_e}\frac{E}{B^2} f_{,2} = 0, \tag{4.19}$$

for both $f = f_1$ and $f = f_3$. Also, by the same substitution of $\acute{p}_1$ and $\acute{p}_3$ into equations $(4.16)_{2,3}$, we realize that $\acute{Q}^2$ is a linear combination of f_1 and f_3, since

$$[(u_1^2 - u_3^2)u_{13} - u(u_1u_{23} - u_3u_{12}) - u_1u_3]\acute{Q}^2 = \frac{\kappa c^2}{\alpha_e B^2}(u_1f_3 - u_3f_1).$$

This implies that $\acute{Q}^2$ also satisfies Eq. (4.19). A formal rearrangement of Eqs. (4.12) and (4.16) allows to express the dipole increments in the form

$$\begin{aligned}
\acute{p}_1 &= \frac{(\bar{Q}_{33} + \acute{Q}_{33})(f_3 + \varphi_3) + (u_1u_{13} - uu_{23})\acute{Q}^2}{1 + \frac{\alpha_e B^2}{\kappa c^2}(\bar{Q}_{33} + \acute{Q}_{33})},\\
\acute{p}_3 &= -\frac{(\bar{Q}_{11} + \acute{Q}_{11})(f_1 + \varphi_1) + (u_3u_{13} - uu_{12})\acute{Q}^2}{1 + \frac{\alpha_e B^2}{\kappa c^2}(\bar{Q}_{11} + \acute{Q}_{11})}
\end{aligned} \tag{4.20}$$

As expected, in the absence of microdeformation the previous quantities vanish. Moreover, from a dimensional analysis it follows that the order of magnitude of the quantities f_j is the same as that of $u_k\acute{Q}^2$,

$(j,k=1,3)$ while φ_j/u_k has the order of $\acute{Q}$. This implies that retaining terms at the lowest order in $\acute{Q}$, Eq. (4.19) reduces to

$$\acute{p}_1 \approx \frac{\bar{Q}_{33}u_3}{1+\frac{\alpha_e B^2}{\kappa c^2}\bar{Q}_{33}}\acute{Q}, \quad \acute{p}_3 \approx -\frac{\bar{Q}_{11}u_1}{1+\frac{\alpha_e B^2}{\kappa c^2}\bar{Q}_{11}}\acute{Q}. \tag{4.21}$$

Similar results are obtained if the electric field is orthogonal to $\mathbf{B}$. In fact, posing $\mathbf{E}=E\mathbf{e}_1$ from (4.8) we have

$$\begin{aligned}
&2\kappa\phi_1 + (\alpha+\beta)\phi_{h,h1} + \gamma\phi_{1,hh} = 0,\\
&2\kappa\phi_2 + (\alpha+\beta)\phi_{h,h2} + \gamma\phi_{2,hh} + \acute{p}_3 E = 0,\\
&2\kappa\phi_3 + (\alpha+\beta)\phi_{h,h3} + \gamma\phi_{3,hh} - \acute{p}_2 E = 0.
\end{aligned} \tag{4.22}$$

A derivation analogous to the previous case shows that, accounting for Eq. (4.17), the functions f_1 and f_3, and in turn $\acute{Q}^2$, satisfy the equation

$$2\kappa f + \gamma\Delta f + \frac{\kappa c^2}{\alpha_e}\frac{E}{B^2}f_{,1} = 0. \tag{4.23}$$

The results (4.20) and (4.21) hold again in this case, as well as in the absence of electric field.
Owing to the negative definiteness of $\mathbf{Q}$, and since $\alpha_e/\kappa < 0$, from (4.20) or (4.21) we recognize that $\acute{p}_1$ and $\acute{p}_3$ tend to vanish for large values of B. This agrees with known experimental results on magnetizable and polarizable dielectric solids [11]. Also, theoretical quantum approaches to polarization, based on the Berry-phase formalism [12], yield such a dependence on the applied magnetic field for crystals whose structure allows for polarization controlled by magnetic field [13].

5. Magnetization

The previous results allow us to inquire into the dependence of magnetization on the electromagnetic field at equilibrium. Equation $(2.10)_4$ expresses magnetization for a general micromorphic continuum and accounts for rates of macro and microdeformations. At equilibrium, following the model depicted for the particle $\hat{\mathcal{P}}^{(\nu)}$ we can apply the microrotation tensor $\hat{\mathbf{N}}^{(\nu)}$ to represent intrinsic magnetization $\boldsymbol{\mathcal{M}}$ of the continuum particle $\mathcal{P}$ in the static case, as the resultant of the contribution of the n continua,

$$\mathcal{M}_i = \frac{1}{2c}\epsilon_{ijk}\sum_{\nu=1}^{n}\hat{N}_{kl}^{(\nu)}Q_{lj}^{(\nu)}. \tag{5.1}$$

This result accounts for microdeformation through the quadrupole density as shown in the previous section. From Eq. (3.11) and taking into account the independence of $\hat{\mathbf{N}}$ on ν, we obtain

$$\boldsymbol{\mathcal{M}} = \frac{\alpha_e}{2c^2}[\hat{\mathbf{Q}} - (\mathrm{tr}\hat{\mathbf{Q}})\mathbf{I}]\mathbf{B}. \tag{5.2}$$

Equation (5.2) generalizes the common linear constitutive law for $\boldsymbol{\mathcal{M}}$ in absence of electric field. Here, linearity is expressed in terms of a field-independent electric quadrupole density if microdeformation effects are discarded.
Posing again $\mathbf{B}=B\mathbf{e}_2$, the results of the previous section hold also for nonnull electric field. In particular, from $(4.12)_{1,3}$ and $(4.16)_{2,3}$,

$$\hat{Q}_{11} + \hat{Q}_{33} = \left(1 + 2\frac{uu_{23} - u_1u_{13}}{u_3}\right)\acute{Q}.$$

Then, Eq. (5.2) yields

$$\begin{aligned}\mathcal{M}_1 &= \frac{\alpha_e}{2c^2} u_{12} \acute{Q} B,\\ \mathcal{M}_2 &= -\frac{\alpha_e}{2c^2}\left(1 + 2\frac{u u_{23} - u_1 u_{13}}{u_3}\right)\acute{Q} B,\\ \mathcal{M}_3 &= \frac{\alpha_e}{2c^2} u_{23} \acute{Q} B.\end{aligned} \tag{5.3}$$

As to the dependence on the electric field, we observe that, owing to Eqs. (4.19) or (4.23), admitted values of $\acute{Q}$ depend on the fraction E/B^2. This implies a nonlinear dependence of magnetization on the magnetic field and, in particular, the effect of the electric field on magnetization turns out to be more relevant for lower values of B. This behavior agrees with experimental results on the converse magnetoelectric effect, observed in some magnetic materials [11]. On the other hand, Eq. (5.3) represents a constitutive law for magnetization where a nonlinear coupling between electric and magnetic fields is expressed by microdeformation through $\acute{Q}$. As we showed at the end of Sect. 3, a linear constitutive law obtained by substituting possible solutions (3.13) into (5.1) cannot apply. We remark that in the more general case of uniform or nonuniform fields with $\bar{\mathbf{p}} \neq \mathbf{0}$, a solution of the full equation (3.10) is required to obtain the pertinent constitutive laws for both polarization and magnetization.

6. Concluding remarks

The analysis of equilibrium for a polarizable and magnetizable electromagnetic continuum, developed in Sect. 3, shows that atomic microcurrents can be modeled by the microgyration tensor of a micromorphic description at a lower dimensional order, satisfying the micromorphic balance laws of momentum and angular momentum, together with the evolution equations for electric multipoles. Here we restricted our analysis to the micropolar case although a more general approach could be exploited where microdeformation includes internal degrees of freedom due to stretch. Nevertheless, the microcurrent phenomena can be phenomenologically modeled by simple microrotations when we deal with nonconducting media, as in the present work. Moreover, the isotropic assumption could be replaced by considering anisotropic micromorphic structures, substantially without affecting the derivation of Eq. (3.10). In this respect, we observe that classical isotropic linear models of microcontinua require magnetization be parallel to $\mathbf{B}$ and polarization parallel to $\mathbf{E}$ [6]. In the present electromagneto-elastic model, mechanical isotropy does not prevent to obtain nonzero magnetization or polarization components along directions different from those of $\mathbf{B}$ or $\mathbf{E}$.

Finally, it is worth remarking that, according to the results of Sects. 4 and 5, it does not seem so natural to state linear constitutive laws for polarization and magnetization when both electric and magnetic fields are applied to electromagnetic media. This linearity is a common view within various approaches based on statistical or microcontinuum models, when a dependence on both electric and magnetic fields is assumed (see for example [7,14]). On the other hand, as remarked at the end of Sect. 4, both theoretical quantum approaches and experimental results support the occurrence of nonlinear laws with noticeable magnetoelectric coupling effects.

A Equilibrium compatibility of $\hat{\mathbf{N}} = \hat{\mathbf{N}}^{(E)} + \mathbb{B}$

We consider Eq. (3.10), for spatially uniform fields $\mathbf{E}$, $\mathbf{B}$, i.e.,

$$\hat{\mathbf{N}}\hat{\mathbf{N}} = \mathbb{E} + \hat{\mathbf{N}}\mathbb{B},$$

and inquire into the compatibility of solutions in the form (3.13), where $\bar{\mathbf{N}}^{(E)} \neq \mathbf{0}$, with the equilibrium conditions (3,2), $(3.4)_{1,2}$. Without loss of generality, for a given ν, we choose a basis in $\mathbb{R}^3$ as the triad $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of unit vectors along the principal axes of $\bar{\mathbf{Q}}$ and pose $\mathbf{E} = E\mathbf{e}_\beta$ with $E > 0$, $(\beta = 1, 2, 3)$. Denoting by λ_β the (negative) eigenvalue of $\bar{\mathbf{Q}}$ with respect to $\mathbf{e}_\beta$, the entries of the tensor $\mathbb{E}$ are

$$E_{ij} = \alpha_e \frac{E}{\lambda_\beta} A_{ij}^{(\beta)}, \quad A_{ij}^{(\beta)} = \begin{cases} \bar{p}_i & j = \beta \\ 0 & j \neq \beta \end{cases}.$$

A square root of $\mathbf{A}^{(\beta)}$ exists in the form

$$[A^{(\beta)}]_{ij}^{1/2} = \pi_i^{(\beta)} \delta_{\beta j}, \qquad \boldsymbol{\pi}^{(\beta)} = \frac{1}{\sqrt{\bar{p}_\beta}} (\bar{p}_1, \bar{p}_2, \bar{p}_3), \tag{A.1}$$

if $\bar{p}_\beta > 0$. From (A.1) and (3.12), we obtain

$$\hat{N}_{ij}^{(E)} = \sqrt{\frac{\alpha_e E}{\lambda_\beta}} \pi_i^{(\beta)} \delta_{\beta i}. \tag{A.2}$$

Applying the necessary and sufficient condition $\hat{\mathbf{N}}\mathbb{B} = \mathbf{0}$ for (3.13) to hold, we obtain $\bar{\mathbf{p}} = \mathbf{0}$ or $\mathbf{B} = B\mathbf{e}_\beta$. In the first case, we obtain the trivial result $\hat{\mathbf{N}}^{(E)} = \mathbf{0}$. Allowing for the second alternative, electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ are required to be parallel. Now we exploit Eq. (3.2) which reduces to

$$[(\hat{\mathbf{N}}^{(E)} + \mathbb{B})\bar{\mathbf{p}}] \times \mathbf{B} = \mathbf{0}.$$

By requiring that this equation holds for arbitrary values of E and B, we find the necessary condition $\bar{\mathbf{p}} = \bar{p}_\beta \mathbf{e}_\beta$. Finally, if we use Eq. $(3.4)_1$, which reduces to the more restrictive condition $(\hat{\mathbf{N}}^{(E)} + \mathbb{B})\bar{\mathbf{p}} = \mathbf{0}$, we arrive at the trivial result $\bar{\mathbf{p}} = \mathbf{0}$. We conclude that, at least for uniform fields, (3.13) does not hold, in nontrivial form, under the equilibrium conditions modeled in Sect. 3.

B Solution to algebraic system (4.14)

Owing to the symmetry of octupole third-order tensor $\boldsymbol{\mathcal{Q}}$, its independent entries amount to ten quantities. We pose

$$\begin{aligned} &\mathcal{Q}_{111} = \mathcal{Q}_1, \quad \mathcal{Q}_{222} = \mathcal{Q}_2, \quad \mathcal{Q}_{333} = \mathcal{Q}_3, \quad \mathcal{Q}_{112} = \mathcal{Q}_4, \quad \mathcal{Q}_{113} = \mathcal{Q}_5, \\ &\mathcal{Q}_{122} = \mathcal{Q}_6, \quad \mathcal{Q}_{123} = \mathcal{Q}_7, \quad \mathcal{Q}_{223} = \mathcal{Q}_8, \quad \mathcal{Q}_{133} = \mathcal{Q}_9, \quad \mathcal{Q}_{233} = \mathcal{Q}_{10} \end{aligned}$$

and rewrite system (4.14) in the equivalent form, where, to save writing, we omit the superimposed bar

$$\begin{aligned} -2\acute{Q}_{13} + \mathcal{Q}_4\varphi - \mathcal{Q}_1\varphi_1 - \mathcal{Q}_5\varphi_3 &= 0, \\ -\acute{Q}_{23} + \mathcal{Q}_6\varphi - \mathcal{Q}_4\varphi_1 - \mathcal{Q}_7\varphi_3 &= 0, \\ \acute{Q}_{12} + \mathcal{Q}_8\varphi - \mathcal{Q}_7\varphi_1 - \mathcal{Q}_{10}\varphi_3 &= 0, \\ \mathcal{Q}_7\varphi - \mathcal{Q}_5\varphi_1 - \mathcal{Q}_9\varphi_3 &= -\acute{Q}, \\ \mathcal{Q}_2\varphi - \mathcal{Q}_6\varphi_1 - \mathcal{Q}_8\varphi_3 &= 0, \\ (\mathcal{Q}_4 + \mathcal{Q}_{10})\varphi - (\mathcal{Q}_1 + \mathcal{Q}_9)\varphi_1 - (\mathcal{Q}_3 + \mathcal{Q}_5)\varphi_3 &= 0. \end{aligned} \tag{B.1}$$

The determinant of the coefficient matrix of this system is

$$(\mathcal{Q}_7\mathcal{Q}_6 - \mathcal{Q}_2\mathcal{Q}_5)(\mathcal{Q}_3 + \mathcal{Q}_5) + (\mathcal{Q}_5\mathcal{Q}_8 - \mathcal{Q}_6\mathcal{Q}_9)(\mathcal{Q}_4 + \mathcal{Q}_{10}) + (\mathcal{Q}_2\mathcal{Q}_9 - \mathcal{Q}_7\mathcal{Q}_8)(\mathcal{Q}_1 + \mathcal{Q}_9) := 2D$$

It can be easily verified that the subsystem $(A.1)_{4,5,6}$ for the unknowns φ, φ_1, φ_3 has determinant D. Solving this subsystem, we obtain

$$\varphi = u\acute{Q}, \quad \varphi_1 = u_1\acute{Q}, \quad \varphi_3 = u_3\acute{Q}, \tag{B.2}$$

where

$$u = \frac{1}{D}[\mathcal{Q}_8(\mathcal{Q}_1 + \mathcal{Q}_9) - \mathcal{Q}_6(\mathcal{Q}_3 + \mathcal{Q}_5)],$$

$$u_1 = \frac{1}{D}[\mathcal{Q}_8(\mathcal{Q}_4 + \mathcal{Q}_{10}) - \mathcal{Q}_2(\mathcal{Q}_3 + \mathcal{Q}_5)], \quad u_3 = \frac{1}{D}[\mathcal{Q}_6(\mathcal{Q}_4 + \mathcal{Q}_{10}) - \mathcal{Q}_2(\mathcal{Q}_1 + \mathcal{Q}_9)]$$

Substituting (A.2) into $(A.1)_{1,2,3}$, we get the following result

$$\acute{Q}_{12} = u_{12}\acute{Q}, \quad \acute{Q}_{13} = u_{13}\acute{Q}, \quad \acute{Q}_{23} = u_{23}\acute{Q},$$

where

$$u_{12} = -\mathcal{Q}_8 u + \mathcal{Q}_7 u_1 + \mathcal{Q}_{10} u_3, \quad u_{13} = \frac{1}{2}(\mathcal{Q}_4 u - \mathcal{Q}_1 u_1 - \mathcal{Q}_5 u_3), \quad u_{23} = \mathcal{Q}_6 u - \mathcal{Q}_4 u_1 - \mathcal{Q}_7 u_3.$$

Thus, microstrain measures and off diagonal entries of $\mathbf{Q}$ due to microdeformation in a field $\mathbf{B} = B\mathbf{e}_2$ can be expressed by means of $\acute{Q}_{11} - \acute{Q}_{33}$ when the octupole density $\bar{\boldsymbol{\mathcal{Q}}}$ of the undeformed continuum is given. Analogous results hold for a magnetic field along a different direction.

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