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On the spatial behavior in two-temperature generalized thermoelastic theories

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Abstract. This paper investigates the spatial behavior of the solutions of two generalized thermoelastic theories with two temperatures. To be more precise, we focus on the Green–Lindsay theory with two temperatures and the Lord–Shulman theory with two temperatures. We prove that a Phragmén–Lindelöf alternative of exponential type can be obtained in both cases. We also describe how to obtain a bound on the amplitude term by means of the boundary conditions for the Green–Lindsay theory with two temperatures.

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1. Introduction

The study of the spatial behavior of solutions of partial differential equations is a topic related to the Saint-Venant's principle. This is an interesting question to analyze both from a mathematical and from a thermomechanical viewpoints. Such studies describe how the influence of perturbations on a part of the boundary is damped for the points which are far away from the part of the boundary where the perturbations are applied. There exists a long tradition for the study of this question, and many investigations have been developed to understand the spatial damping of the solutions for several thermoelastic situations (see [12] and the references therein). Spatial decay estimates for elliptic [9], parabolic [13,14], hyperbolic [10] and/or combinations of such [28] have been obtained in the last years. It is worth recalling that some contributions have also been proposed in the study of phase field models (see [20–24] and [25]). However, it is worth noting that such a knowledge for nonlinear problems is very limited. What is usual is to consider a semi-infinite cylinder whose finite end is perturbed and to study how the solutions behave when the spatial variable goes to infinity.

The infinite speed of propagation for the Fourier law of heat conduction is an important drawback from a physical point of view. This led many scientists to look for alternative heat conduction models. At the end of the 1960s, Gurtin and several co-authors proposed and studied a thermoelastic theory that they called "thermoelasticity with two temperatures" ([3–5,32]). Several authors have dedicated their attention to this problem (Iesan [15], Quintanilla [29,30] among others). At the same time, other heat conduction theories have been proposed and developed. We can mention the damped hyperbolic heat conduction proposed by Cattaneo and Maxwell or the theories proposed by Green and Naghdi. In particular, two thermoelastic theories based on the Cattaneo–Maxwell heat conduction were proposed by Green and Lindsay and Lord and Shulman ([11,18]). A remarkable fact is that these theories are susceptible to be merged in a way which allows to consider the generalized thermoelastic theories with two temperatures such as the ones proposed in [8,31] and [33]. In this paper, we focus on such theories. It is worth recalling that the combination of the theories proposed by Green and Lindsay and Lord and Shulman with the two temperatures theory was suggested by Youssef [33]. This theory has attracted a lot of attention in the past years ([2,19,26]). The spatial behavior for the classical theory with two temperatures was considered by Awad [1]. Our aim in this paper is to extend these arguments to the theories of Green–Lindsay with two temperatures and Lord–Shulman with two temperatures. Therefore, it is suitable to recall that the spatial behavior of solutions for the Green–Lindsay theory was studied in [6,27]. However, to the best of our knowledge, there is no such contribution for the Lord–Shulman theory. Nevertheless, the arguments to study the usual Lord–Shulman theory would be very different from the ones proposed here for the Lord–Shulman theory with two temperatures.

Here, we do not study the existence of solutions of the problems; in fact, this can be a difficult question in many nonlinear situations (see, e.g., [24]). We note, however, that the existence of solutions can be done by adapting the arguments proposed in [17] to the three-dimensional unbounded case. We thus assume the existence of solutions and then only study the spatial asymptotic behavior in that case. More precisely, we obtain a Phragmén–Lindelöf alternative for the solutions, i.e., either a growth or a decay estimate of exponential type can be shown. An upper bound on the amplitude term, when the solution decays, is also derived, in terms of the boundary conditions.

The plan for the paper is the following: In the next section, we recall the boundary-initial-value problems that we are going to work with. Section 3 is devoted to the study of the Green–Lindsay thermoelasticity with two temperatures. The exponential alternative is obtained, and an upper bound for the amplitude term by means of the boundary conditions is obtained. Section 4 considers the Lord–Shulman thermoelasticity with two temperatures, and we also obtain an exponential alternative for the solutions. Some conclusions end the paper.

2. Preliminaries

The system of equations that governs the thermoelastic deformations of a centrosymmetric material for the Green and Lindsay theory with two temperatures reads

$$(c_{ijkl}u_{k,l} + a_{ij}(\theta + \alpha\dot{\theta}))_{,j} = \rho\ddot{u}_i, \tag{2.1}$$

$$h\ddot{\theta} + d\dot{\theta} - a_{ij}\dot{u}_{i,j} = (k_{ij}\phi_{,i})_{,j}, \tag{2.2}$$

$$\phi - \theta = a(k_{ij}\phi_{,i})_{,j}. \tag{2.3}$$

Here u_i is the displacement, θ is the thermodynamic temperature and ϕ is the conductive temperature. Furthermore, ρ is the mass density, h and d are constitutive functions, a is a positive constant, c_{ijkl} is the elasticity tensor, a_{ij} is the coupling tensor and k_{ij} is the thermal conductivity tensor. Finally, α is a strictly positive constant, which is typical of the Green–Lindsay theory.

Throughout this paper, we assume that the elasticity tensor satisfies

$$c_{ijkl} = c_{klij}, \tag{2.4}$$

and that the thermal conductivity tensor is also symmetric

$$k_{ij} = k_{ji}. \tag{2.5}$$

We also assume that a and α are two positive constants and that

$$\rho \geq \rho_0 > 0,\ h \geq h_0 > 0,\ d\alpha - h \geq m_0 > 0. \tag{2.6}$$

The last inequality is a consequence of the entropy inequality of Green and Lindsay [11], and ρ_0, h_0 and m_0 are positive constants. At the same time, we also assume that all the constitutive functions and tensors are essentially upper bounded in the region in which we consider our study.

The elasticity tensor and the conductivity tensor are also assumed positive. That is, there exists a positive constant c_0 such that

$$c_{ijkl}\xi_{ij}\xi_{kl} \geq c_0\xi_{ij}\xi_{ij}, \tag{2.7}$$

for every tensor ξ_{ij}, and there exists another positive constant k_0 such that

$$k_{ij}\xi_i\xi_j \geq k_0\xi_i\xi_i, \tag{2.8}$$

for every vector ξ_i.

We also assume that there exists c_1 such that the following inequality

$$c_{ijkl}\xi_{ij}\xi_{kl} \leq c_1\xi_{ij}\xi_{ij},$$

is satisfied for every tensor ξ_{ij}. In a similar way, we assume the existence of k_1 such that

$$k_{ij}\xi_i\xi_j \leq k_1\xi_i\xi_i,$$

for every vector ξ_i.

We denote $\beta = \sup a_{ij}a_{ij}$ and introduce the parameters

$$k = \left(\frac{\lambda \sup k_{11}}{k_0}\right)^{1/2}, \; k^* = \frac{\alpha}{2}\left(\frac{\sup k_{11}}{m_0}\right)^{1/2}, \; k^{**} = \left(\frac{a \sup k_{11}}{2}\right)^{1/2},$$

where λ is the Poincaré constant for the domain D (which will be defined below).

The system that governs the deformations of a thermoelastic solid for the Lord and Shulman theory with two temperatures reads

$$\left(c_{ijkl}u_{k,l} + a_{ij}\theta\right)_{,j} = \rho\ddot{u}_i, \tag{2.9}$$

$$h_1\dot{\hat{\theta}} - a_{ij}\dot{\hat{u}}_{i,j} = (k_{ij}\phi_{,i})_{,j}, \tag{2.10}$$

$$\phi - \theta = a(k_{ij}\phi_{,i})_{,j}, \tag{2.11}$$

where $\hat{f} = f + d_1\dot{f}$. In this theory, d_1 is a constitutive constant. When considering this theory, we assume that (2.4), (2.5), (2.7) and (2.8) hold, but we also need to impose that

$$a > 0, \; \rho \geq \rho_0 > 0, \; h_1 \geq h_0^* > 0, \; d_1 > 0. \tag{2.12}$$

In this paper, we study the spatial behavior of the solutions of systems (2.1)–(2.3) and (2.9)–(2.11). Therefore, we study the problems in a semi-infinite cylinder $R = [0, \infty) \times D$, where D is a two-dimensional bounded domain smooth enough to apply the divergence theorem.

We then need to impose the boundary and initial conditions. We thus assume that

$$u_i(x,t) = \phi(x,t) = 0, \quad x \in [0,\infty) \times \partial D, \tag{2.13}$$

and

$$u_i(x,t) = f_i(x_2,x_3,t), \; \phi(x,t) = g(x_2,x_3,t), \; x \in \{0\} \times D, \tag{2.14}$$

where f_i and g are given functions. We also impose null initial conditions

$$u_i(x,0) = 0, \quad \phi(x,0) = 0, \quad x \in R. \tag{2.15}$$

Remark 2.1. Here, we have not defined initial and boundary conditions for the thermodynamic temperature θ. Actually, in the computations below, θ does not appear and can be seen as an auxiliary unknown only, so that such conditions are not needed. Note, however, that θ can be expressed in terms of ϕ, so that initial and boundary conditions on ϕ imply proper conditions on θ, although, strictly speaking, one would also need to impose boundary conditions on the first and second derivatives of ϕ to do so.

It is worth noting that the existence, uniqueness and continuous dependence of the decaying solutions determined by problems (2.1)–(2.3), (2.13)–(2.15) and (2.9)–(2.11), (2.13)–(2.15) can be obtained by means of a semigroup approach. The existence of solutions can be obtained by extending the arguments proposed in [17] and used there for the one-dimensional problem. However, we do not address this here, and we will assume the existence of solutions to study their spatial behavior. We refer the interested reader to [7] or appendix 1 of [16] for more details on this.

3. Green–Lindsay theory with two temperatures

In this section, we study the spatial behavior of the solutions of the problem determined by systems (2.1)–(2.3) subject to the boundary conditions (2.13), (2.14) and the initial conditions (2.15) whenever assumptions (2.4)–(2.8) hold. To be more precise, we will give a Phragmén–Lindelöf alternative for the solutions of this problem, as well as an upper bound for the amplitude term for the decaying solutions. To simplify the notation, we will write

$$K(\phi) = (k_{ij}\phi_{,i})_{,j}.$$

It will be useful to take into account the following identities:

$$\begin{aligned}
&\left((c_{ijkl}u_{k,l} + a_{ij}(\theta + \alpha\dot{\theta}))\dot{u}_i + k_{ij}\phi_{,i}(\phi + \alpha\dot{\phi})\right)_{,j} \\
&= c_{ijkl}u_{k,l}\dot{u}_{i,j} + a_{ij}(\theta + \alpha\dot{\theta})\dot{u}_{i,j} + (c_{ijkl}u_{k,l} + a_{ij}(\theta + \alpha\dot{\theta}))_{,j}\dot{u}_i \\
&\quad + k_{ij}\phi_{,i}\phi_{,j} + \alpha k_{ij}\phi_{,i}\dot{\phi}_{,j} + (k_{ij}\phi_{,i})_{,j}(\phi + \alpha\dot{\phi}) \\
&= c_{ijkl}u_{k,l}\dot{u}_{i,j} + a_{ij}(\theta + \alpha\dot{\theta})\dot{u}_{i,j} + \rho\ddot{u}_i\dot{u}_i + k_{ij}\phi_{,i}\phi_{,j} + \alpha k_{ij}\phi_{,i}\dot{\phi}_{,j} \\
&\quad + (h\ddot{\theta} + d\dot{\theta} - a_{ij}\dot{u}_{i,j})(\phi + \alpha\dot{\phi}) \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}[c_{ijkl}u_{i,j}u_{k.l} + \rho\dot{u}_i\dot{u}_i + \alpha k_{ij}\phi_{,i}\phi_{,j}] + k_{ij}\phi_{,i}\phi_{,j} + (h\ddot{\theta} + d\dot{\theta} - a_{ij}\dot{u}_{i,j})(\theta + \alpha\dot{\theta}) \\
&\quad + a_{ij}(\theta + \alpha\dot{\theta})\dot{u}_{i,j} + (k_{ij}\phi_{,i})_{,j}(a(k_{lm}\phi_{,l})_{,m} + a\alpha(k_{lm}\dot{\phi}_{,l})_{,m}) \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}[\rho\dot{u}_i\dot{u}_i + c_{ijkl}u_{i,j}u_{k.l} + \frac{h}{\alpha}(\theta + \alpha\dot{\theta})^2 \\
&\quad + \left(d - \frac{h}{\alpha}\right)\theta^2 + \alpha k_{ij}\phi_{,i}\phi_{,j} + \alpha a(K(\phi))^2] + (d\alpha - h)(\dot{\theta})^2 + k_{ij}\phi_{,i}\phi_{,j} + a(K(\phi))^2.
\end{aligned} \tag{3.1}$$

We note that the last equality comes from the relation

$$(h\ddot{\theta} + \mathrm{d}\dot{\theta})(\theta + \alpha\dot{\theta}) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{h}{\alpha}(\theta + \alpha\dot{\theta})^2 + \left(d - \frac{h}{\alpha}\right)\theta^2\right) + (\mathrm{d}\alpha - h)(\dot{\theta})^2.$$

3.1. Phragmén–Lindelöf alternative

We start our analysis by considering the function

$$F_\omega(z,t) = \int_0^t \int_{D(z)} \exp(-2\omega s)\left((c_{i1kl}u_{k,l} + a_{i1}(\theta + \alpha\dot{\theta}))\dot{u}_i + k_{i1}\phi_{,i}(\phi + \alpha\dot{\phi})\right) \mathrm{d}a\,\mathrm{d}s, \tag{3.2}$$

where ω is an arbitrary positive constant to be fixed later and $D(z) = \{x \in R,\ x_1 = z\}$.

We have, owing to the boundary and initial conditions, the evolution equations and the divergence theorem,

$$F_\omega(z+h,t) - F_\omega(z,t) = \frac{\exp(-2\omega t)}{2}\int_{R(z,z+h)} W_1^*\mathrm{d}x + \int_0^t \int_{R(z,z+h)} \exp(-2\omega s)W_2^*\,\mathrm{d}x\,\mathrm{d}s, \tag{3.3}$$

where $R(z,z+h) = \{x \in R,\ z < x_1 < z+h\}$,

$$W_1^* = \rho\dot{u}_i\dot{u}_i + c_{ijkl}u_{i,j}u_{k,l} + \frac{h}{\alpha}(\theta + \alpha\dot{\theta})^2 + \left(d - \frac{h}{\alpha}\right)\theta^2 + \alpha k_{ij}\phi_{,i}\phi_{,j} + \alpha a(K(\phi))^2,$$

and

$$W_2^* = \omega W_1^* + (d\alpha - h)(\dot{\theta})^2 + k_{ij}\phi_{,i}\phi_{,j} + a(K(\phi))^2.$$

We then obtain

$$\frac{\partial F_\omega(z,t)}{\partial z} = \frac{\exp(-2\omega t)}{2} \int\limits_{D(z)} W_1^* \, \mathrm{d}a + \int\limits_0^t \int\limits_{D(z)} \exp(-2\omega s) W_2^* \, \mathrm{d}a \, \mathrm{d}s. \tag{3.4}$$

We note that

$$(\dot{\theta})^2 = (\dot{\phi})^2 - 2aK(\dot{\phi})\dot{\phi} + a^2(K(\dot{\phi}))^2.$$

We now consider an auxiliary function to control the expression involving the term $K(\dot{\phi})\dot{\phi}$. We define

$$G(z,t) = 2a \int\limits_0^t \int\limits_{D(z)} \exp(-2\omega s) m_0 k_{i1} \dot{\phi}\dot{\phi}_{,i} \, \mathrm{d}a \, \mathrm{d}s,$$

where m_0 is given in Sect. 2.

We find, proceeding as above,

$$G(z+h,t) - G(z,t) = 2a \int\limits_0^t \int\limits_{R(z,z+h)} \exp(-2\omega s) m_0 (k_{ij}\dot{\phi}_{,i}\dot{\phi}_{,j} + \dot{\phi}(k_{ij}\dot{\phi}_{,i})_{,j}) \, \mathrm{d}x \, \mathrm{d}s. \tag{3.5}$$

Therefore, we see that

$$\frac{\partial G(z,t)}{\partial z} = 2am_0 \int\limits_0^t \int\limits_{D(z)} \exp(-2\omega s)(k_{ij}\dot{\phi}_{,i}\dot{\phi}_{,j} + K(\dot{\phi})\dot{\phi}) \, \mathrm{d}a \, \mathrm{d}s. \tag{3.6}$$

Next, we consider the function $H_\omega = F_\omega + G$. We have

$$H_\omega(z+h,t) - H_\omega(z,t) = \frac{\exp(-2\omega t)}{2} \int\limits_{R(z,z+h)} W_1^* \mathrm{d}x + \int\limits_0^t \int\limits_{R(z,z+h)} \exp(-2\omega s) W_2 \, \mathrm{d}x \, \mathrm{d}s, \tag{3.7}$$

where

$$W_2 = \omega W_1^* + m_0((\dot{\phi})^2 + 2ak_{ij}\dot{\phi}_{,i}\dot{\phi}_{,j} + a^2(K(\dot{\phi}))^2) + (d\alpha - h - m_0)(\dot{\theta})^2 + k_{ij}\phi_{,i}\phi_{,j} + a(K(\phi))^2.$$

We also have

$$\frac{\partial H_\omega(z,t)}{\partial z} = \frac{\exp(-2\omega t)}{2} \int\limits_{D(z)} W_1^* \mathrm{d}a + \int\limits_0^t \int\limits_{D(z)} \exp(-2\omega s) W_2 \, \mathrm{d}a \, \mathrm{d}s. \tag{3.8}$$

The next step consists in obtaining an estimate on $|H_\omega|$ in terms of the spatial derivative of H_ω. First, we note that

$$|H_\omega| \leq |F_\omega| + |G|. \tag{3.9}$$

In fact

$$|F_\omega| \leq |I_1| + |I_2| + |I_3|,$$

where

$$I_1 = \int\limits_0^t \int\limits_{D(z)} \exp(-2\omega s) c_{i1kl} u_{k,l} \dot{u}_i \, \mathrm{d}a \, \mathrm{d}s,$$

$$I_2 = \int_0^t \int_{D(z)} \exp(-2\omega s) a_{i1}(\theta + \alpha\dot{\theta})\dot{u}_i \, \mathrm{d}a \, \mathrm{d}s,$$

and

$$I_3 = \int_0^t \int_{D(z)} \exp(-2\omega s) k_{i1}\phi_{,i}(\phi + \alpha\dot{\phi}) \, \mathrm{d}a \, \mathrm{d}s.$$

We see that

$$|I_1| \leq \left(\int_0^t \int_{D(z)} \exp(-2\omega s) c_{ijkl} u_{i,j} u_{k,l} \, \mathrm{d}a \, \mathrm{d}s \int_0^t \int_{D(z)} \exp(-2\omega s) c_{ijkl} \dot{u}_i n_j \dot{u}_k n_l \, \mathrm{d}a \, \mathrm{d}s \right)^{1/2}$$

$$\leq \left(\frac{c_1}{4\rho_0\omega^2} \right)^{1/2} \left(\omega \int_0^t \int_{D(z)} \exp(-2\omega s) c_{ijkl} u_{i,j} u_{k,l} \, \mathrm{d}a \, \mathrm{d}s + \omega \int_0^t \int_{D(z)} \exp(-2\omega s) \rho \dot{u}_i \dot{u}_i \, \mathrm{d}a \, \mathrm{d}s \right).$$

$$|I_2| \leq \left(\int_0^t \int_{D(z)} \exp(-2\omega s) a_{ij} a_{ij} (\theta + \alpha\dot{\theta})^2 \, \mathrm{d}a \, \mathrm{d}s \int_0^t \int_{D(z)} \exp(-2\omega s) \dot{u}_i \dot{u}_i \, \mathrm{d}a \, \mathrm{d}s \right)^{1/2}$$

$$\leq \left(\frac{\alpha\beta}{4\rho_0\omega^2 h_0} \right)^{1/2} \left(\omega \int_0^t \int_{D(z)} \exp(-2\omega s) \frac{h}{\alpha} (\theta + \alpha\dot{\theta})^2 \, \mathrm{d}a \, \mathrm{d}s + \omega \int_0^t \int_{D(z)} \exp(-2\omega s) \rho \dot{u}_i \dot{u}_i \, \mathrm{d}a \, \mathrm{d}s \right).$$

$$|I_3| \leq \left(\int_0^t \int_{D(z)} \exp(-2\omega s) k_{ij} \phi_{,i} \phi_{,j} \, \mathrm{d}a \, \mathrm{d}s \int_0^t \int_{D(z)} \exp(-2\omega s) k_{11} \phi^2 \, \mathrm{d}a \, \mathrm{d}s \right)^{1/2}$$

$$+ \alpha \left(\int_0^t \int_{D(z)} \exp(-2\omega s) k_{ij} \phi_{,i} \phi_{,j} \, \mathrm{d}a \, \mathrm{d}s \int_0^t \int_{D(z)} \exp(-2\omega s) k_{11} (\dot{\phi})^2 \, \mathrm{d}a \, \mathrm{d}s \right)^{1/2}$$

$$\leq k \left(\int_0^t \int_{D(z)} \exp(-2\omega s) k_{ij} \phi_{,i} \phi_{,j} \, \mathrm{d}a \, \mathrm{d}s \right)$$

$$+ k^* \left(\int_0^t \int_{D(z)} \exp(-2\omega s) k_{ij} \phi_{,i} \phi_{,j} \, \mathrm{d}a \, \mathrm{d}s + \int_0^t \int_{D(z)} \exp(-2\omega s) m_0 (\dot{\phi})^2 \, \mathrm{d}a \, \mathrm{d}s \right),$$

where k and k^* are given in Sect. 2.

In a similar way, we see that

$$|G| \leq k^{**} \left(\int_0^t \int_{D(z)} \exp(-2\omega s) 2am_0 k_{ij} \dot{\phi}_{,i} \dot{\phi}_{,j} \, \mathrm{d}a \, \mathrm{d}s + \int_0^t \int_{D(z)} \exp(-2\omega s) m_0 (\dot{\phi})^2 \, \mathrm{d}a \, \mathrm{d}s \right).$$

From the previous inequalities, we can select

$$C_3 = \max\left(\left(\frac{c_1}{4\rho_0\omega^2}\right)^{1/2} + \left(\frac{\alpha\beta}{4\rho_0\omega^2 h_0}\right)^{1/2}, k+k^*, k^*+k^{**}\right).$$

We obtain

$$|H_\omega(z,t)| \leq C_3 \frac{\partial H_\omega(z,t)}{\partial z}, \tag{3.10}$$

for every t and $z \geq 0$.

Inequality (3.10) is classical in the study of spatial estimates (see [9]) and yields a Phragmén–Lindelöf alternative. If there exists $z_0 \geq 0$ such that $H_\omega(t,z_0) > 0$, then the solution satisfies the estimate

$$H_\omega(t,z) \geq H_\omega(t,z_0)\exp(C_3^{-1}(z-z_0)),\ \ z \geq z_0. \tag{3.11}$$

This estimate gives information in terms of the measure defined in the cylinder. Indeed, it follows that

$$\frac{\exp(-2\omega t)}{2}\int\limits_{R(0,z)} W_1^* \mathrm{d}x + \int\limits_0^t \int\limits_{R(0,z)} \exp(-2\omega s) W_2 \,\mathrm{d}x\,\mathrm{d}s \tag{3.12}$$

tends to infinity exponentially fast, where $R(0,z) = \{x \in R,\ x_1 \leq z\}$. On the contrary, when $H_\omega(z,t) \leq 0$, for every $z \geq 0$, we deduce that $H_\omega(z,t) \leq 0$ for every $z \geq 0$ and the solution decays and we can obtain an estimate of the form

$$-H_\omega(z,t) \leq -H_\omega(0,t)\exp(-C_3^{-1}z),\ \ z \geq 0. \tag{3.13}$$

This inequality implies that $H_\omega(z,t)$ tends to zero as z goes to infinity. Furthermore, in view of (3.13), we see that

$$E_\omega(z,t) \leq E_\omega(0,t)\exp(-C_3^{-1}z),\ \ z \geq 0, \tag{3.14}$$

where

$$E_\omega(z,t) = \frac{\exp(-2\omega t)}{2}\int\limits_{R(z)} W_1^* \,\mathrm{d}x + \int\limits_0^t \int\limits_{R(z)} \exp(-2\omega s) W_2 \,\mathrm{d}x\,\mathrm{d}s \tag{3.15}$$

and $R(z) = \{x \in R,\ x_1 > z\}$.

Setting finally

$$\mathcal{E}(z,t) = \frac{1}{2}\int_{R(z)} W_1^*\, dx + \int\limits_0^t \int\limits_{R(z)} W_2\, dx\, ds, \tag{3.16}$$

we have the

Theorem 3.1. *Let (u_i,ϕ) be a smooth solution of the problem defined by (2.1)–(2.3), the boundary conditions (2.13), (2.14) and the initial conditions (2.15). Then, either this solution satisfies the growth estimate (3.11) or it satisfies the decay estimate*

$$\mathcal{E}(z,t) \leq E_\omega(0,t)\exp(2\omega t - C_3^{-1}z),\ \ z \geq 0, \tag{3.17}$$

where the energy $\mathcal{E}$ is defined in (3.16) and E_ω is given by (3.15).

3.2. The amplitude term

The spatial decay estimate obtained in the previous subsection is of limited use unless we have an upper bound on the amplitude term in terms of the boundary conditions. The aim of this subsection is thus to obtain such a bound.

We denote by v_i, φ functions satisfying the same boundary conditions as u_i and ϕ and such that they tend to zero in a fast way when x_1 increases. We have

$$
\begin{aligned}
-H_\omega(0,t) = & \int_0^t \int_R \exp(-2\omega s)(c_{ijkl}u_{i,j}\dot{v}_{k,l} + a_{ij}(\theta+\alpha\dot{\theta})\dot{v}_{i,j} + \rho\ddot{u}_i\dot{v}_i + k_{ij}\phi_{,i}(\varphi_{,j}+\alpha\dot{\varphi}_{,j}) \\
& + (K(\phi))(\varphi+\alpha\dot{\varphi}))\,\mathrm{d}v\,\mathrm{d}s \\
& + 2am_0 \int_0^t \int_R \exp(-2\omega s)((k_{ij}\dot{\phi}_{,i})_{,j}\dot{\varphi} + k_{ij}\dot{\phi}_{,i}\dot{\varphi}_{,j})\,\mathrm{d}x\,\mathrm{d}s. \qquad (3.18)
\end{aligned}
$$

We note that

$$
\begin{aligned}
\int_0^t \int_R \exp(-2\omega s)\rho\ddot{u}_i\dot{v}_i\,\mathrm{d}v\,\mathrm{d}s = & \exp(-2\omega t)\int_R \rho\dot{u}_i\dot{v}_i\mathrm{d}x - \int_0^t \int_R \exp(-2\omega s)\rho\dot{u}_i\ddot{v}_i\,\mathrm{d}x\,\mathrm{d}s \\
& + 2\omega \int_0^t \int_R \exp(-2\omega s)\rho\dot{u}_i\dot{v}_i\,\mathrm{d}x\,\mathrm{d}s. \qquad (3.19)
\end{aligned}
$$

We further see that

$$
\begin{aligned}
& \int_0^t \int_R \exp(-2\omega s)c_{ijkl}u_{i,j}\dot{v}_{k,l}\,\mathrm{d}x\,\mathrm{d}s \\
& \quad \le \left(\int_0^t \int_R \exp(-2\omega s)c_{ijkl}u_{i,j}u_{k,l}\,\mathrm{d}x\,\mathrm{d}s\right)^{1/2} \times \left(\int_0^t \int_R c_{ijkl}\dot{v}_{i,j}\dot{v}_{k,l}\,\mathrm{d}x\,\mathrm{d}s\right)^{1/2},
\end{aligned}
$$

$$
\begin{aligned}
\int_0^t \int_R \exp(-2\omega s)a_{ij}(\theta+\alpha\dot{\theta})\dot{v}_{i,j}\,\mathrm{d}x\,\mathrm{d}s \le & \left(\int_0^t \int_R \exp(-2\omega s)a_{ij}a_{ij}(\theta+\alpha\dot{\theta})^2\,\mathrm{d}x\,\mathrm{d}s\right)^{1/2} \\
& \times \left(\int_0^t \int_R \dot{v}_{i,j}\dot{v}_{i,j}\,\mathrm{d}x\,\mathrm{d}s\right)^{1/2},
\end{aligned}
$$

$$
\exp(-2\omega t)\int_R \rho\dot{u}_i\dot{v}_i\,\mathrm{d}x \le \exp(-2\omega t)\left(\int_R \rho\dot{u}_i\dot{u}_i\,\mathrm{d}x\right)^{1/2}\left(\int_R \rho\dot{v}_i\dot{v}_i\,\mathrm{d}x\right)^{1/2},
$$

$$
-\int_0^t \int_R \exp(-2\omega s)\rho\dot{u}_i\ddot{v}_i\,\mathrm{d}x\,\mathrm{d}s \le \left(\int_0^t \int_R \exp(-2\omega s)\rho\dot{u}_i\dot{u}_i\,\mathrm{d}x\,\mathrm{d}s\right)^{1/2}\left(\int_0^t \int_R \rho\ddot{v}_i\ddot{v}_i\,\mathrm{d}x\,\mathrm{d}s\right)^{1/2},
$$

$$
2\omega \int_0^t \int_R \exp(-2\omega s)\rho\dot{u}_i\dot{v}_i\,\mathrm{d}x\,\mathrm{d}s
$$

$$\leq 2\omega \left(\int_0^t \int_R \exp(-2\omega s) \rho \dot{u}_i \dot{u}_i \, \mathrm{d}x \, \mathrm{d}s \right)^{1/2} \left(\int_0^t \int_R \rho \dot{v}_i \dot{v}_i \, \mathrm{d}x \, \mathrm{d}s \right)^{1/2},$$

$$\int_0^t \int_R \exp(-2\omega s) k_{ij} \phi_{,i} (\varphi_{,j} + \alpha \dot{\varphi}_{,j}) \, \mathrm{d}x \, \mathrm{d}s \leq \left(\int_0^t \int_R \exp(-2\omega s) k_{ij} \phi_{,i} \phi_{,j} \, \mathrm{d}x \, \mathrm{d}s \right)^{1/2}$$

$$\times \left(\left(\int_0^t \int_R k_{ij} \varphi_{,i} \varphi_{,j} \, \mathrm{d}x \, \mathrm{d}s \right)^{1/2} + \alpha \left(\int_0^t \int_R k_{ij} \dot{\varphi}_{,i} \dot{\varphi}_{,j} \, \mathrm{d}x \, \mathrm{d}s \right)^{1/2} \right),$$

$$\int_0^t \int_R \exp(-2\omega s) (K(\phi)) (\varphi + \alpha \dot{\varphi}) \, \mathrm{d}x \, \mathrm{d}s \leq \left(\int_0^t \int_R \exp(-2\omega s) (K(\phi))^2 \, \mathrm{d}x \right)^{1/2}$$

$$\times \left(\int_0^t \int_R (\varphi + \alpha \dot{\varphi})^2 \, \mathrm{d}x \right)^{1/2},$$

$$2am_0 \int_0^t \int_R \exp(-2\omega s) k_{ij} \dot{\phi}_{,i} \dot{\varphi}_{,j} \, \mathrm{d}x \, \mathrm{d}s \leq 2am_0 \left(\int_0^t \int_R \exp(-2\omega s) k_{ij} \dot{\phi}_{,i} \dot{\phi}_{,j} \, \mathrm{d}x \, \mathrm{d}s \right)^{1/2}$$

$$\times \left(\int_0^t \int_R k_{ij} \dot{\varphi}_{,i} \dot{\varphi}_{,j} \, \mathrm{d}x \, \mathrm{d}s \right)^{1/2},$$

$$2am_0 \int_0^t \int_R \exp(-2\omega s) K(\dot{\phi}) \dot{\varphi} \, \mathrm{d}x \, \mathrm{d}s \leq 2am_0 \left(\int_0^t \int_R \exp(-2\omega s) K(\dot{\phi})^2 \, \mathrm{d}x \, \mathrm{d}s \right)^{1/2}$$

$$\times \left(\int_0^t \int_R (\dot{\varphi})^2 \, \mathrm{d}x \, \mathrm{d}s \right)^{1/2}.$$

Employing the arithmetic–geometric mean inequality, we see that

$$\begin{aligned} -H_\omega(0,t) \leq \epsilon_1 \int_0^t \int_R \exp(-2\omega s) W_2 \, \mathrm{d}x \, \mathrm{d}s + D_1 \int_0^t \int_R c_{ijkl} \dot{v}_{i,j} \dot{v}_{k,l} \, \mathrm{d}x \, \mathrm{d}s \\ + \epsilon_2 \int_0^t \int_R \exp(-2\omega s) W_2 \, \mathrm{d}x \, \mathrm{d}s + D_2 \int_0^t \int_R \dot{v}_{i,j} \dot{v}_{i,j} \, \mathrm{d}x \, \mathrm{d}s \\ + \epsilon_3 \frac{\exp(-2\omega t)}{2} \int_R W_1^* \, \mathrm{d}x + D_3 \int_R \rho \dot{v}_i \dot{v}_i \, \mathrm{d}x \\ + \epsilon_4 \int_0^t \int_R \exp(-2\omega s) W_2 \, \mathrm{d}x \, \mathrm{d}s + D_4 \int_0^t \int_R \rho \ddot{v}_i \ddot{v}_i \, \mathrm{d}x \, \mathrm{d}s \end{aligned}$$

$$+\epsilon_5\int_0^t\int_R \exp(-2\omega s)W_2\,\mathrm{d}x\,\mathrm{d}s + D_5\int_0^t\int_R \rho\dot{v}_i\dot{v}_i\,\mathrm{d}x\,\mathrm{d}s$$
$$+\epsilon_6\int_0^t\int_R \exp(-2\omega s)W_2\,\mathrm{d}x\,\mathrm{d}s + D_6\int_0^t\int_R (\varphi_{,i}\varphi_{,i}+\dot{\varphi}_{,i}\dot{\varphi}_{,i})\,\mathrm{d}x\,\mathrm{d}s$$
$$+\epsilon_7\int_0^t\int_R \exp(-2\omega s)W_2\,\mathrm{d}x\,\mathrm{d}s + D_7\int_0^t\int_R (\varphi^2+(\dot{\varphi})^2)\,\mathrm{d}x\,\mathrm{d}s$$
$$+\epsilon_8\int_0^t\int_R \exp(-2\omega s)W_2\,\mathrm{d}x\,\mathrm{d}s + D_8\int_0^t\int_R (\dot{\varphi})^2\,\mathrm{d}x\,\mathrm{d}s$$
$$+\epsilon_9\int_0^t\int_R \exp(-2\omega s)W_2\,\mathrm{d}x\,\mathrm{d}s + D_9\int_0^t\int_R \dot{\varphi}_{,i}\dot{\varphi}_{,i}\,\mathrm{d}x\,\mathrm{d}s.$$

Here $\epsilon_i, i=1,\dots,9$, are positive constants that are as small as needed and D_i are positive constants depending on the constitutive tensors, the parameters ϵ_i, the parameter ω and time. In fact, we can take

$$D_1=(4\epsilon_1\omega)^{-1},\;\; D_2=\alpha\beta(4\epsilon_2\omega h_0)^{-1},\;\; D_3=(2\epsilon_3)^{-1},\;\; D_4=(4\epsilon_4\omega)^{-1},\;\; D_5=\omega\epsilon_5^{-1},$$
$$D_6=\max(1,\alpha)k_1(4\epsilon_6)^{-1},\;\; D_7=\max(1,\alpha)(2\epsilon_7 a)^{-1},\;\; D_8=m_0\epsilon_8^{-1},\;\; D_9=am_0k_1(2\epsilon_9)^{-1}.$$

We can always select ϵ_i in such a way that $\epsilon_1+\epsilon_2+\epsilon_4+\epsilon_5+\epsilon_6+\epsilon_7+\epsilon_8+\epsilon_9=1/2$ and $\epsilon_3=1/2$. We then obtain

$$-H_\omega(0,t)\le 2D_1\int_0^t\int_R c_{ijkl}\dot{v}_{i,j}\dot{v}_{k,l}\,\mathrm{d}x\,\mathrm{d}s + 2D_2\int_0^t\int_R \dot{v}_{i,j}\dot{v}_{i,j}\,\mathrm{d}x\,\mathrm{d}s$$
$$+2D_3\int_R \rho\dot{v}_i\dot{v}_i\,\mathrm{d}x + 2D_4\int_0^t\int_R \rho\ddot{v}_i\ddot{v}_i\,\mathrm{d}x\,\mathrm{d}s + 2D_5\int_0^t\int_R \rho\dot{v}_i\dot{v}_i\,\mathrm{d}x\,\mathrm{d}s$$
$$+2D_6\int_0^t\int_R (\varphi_{,i}\varphi_{,i}+\dot{\varphi}_{,i}\dot{\varphi}_{,i})\,\mathrm{d}x\,\mathrm{d}s$$
$$+2D_7\int_0^t\int_R (\varphi^2+(\dot{\varphi})^2)\,\mathrm{d}x\,\mathrm{d}s + 2D_8\int_0^t\int_R (\dot{\varphi})^2\,\mathrm{d}x\,\mathrm{d}s$$
$$+2D_9\int_0^t\int_R \dot{\varphi}_{,i}\dot{\varphi}_{,i}\,\mathrm{d}x\,\mathrm{d}s.$$

We now select $v_i(x,t)=f_i(x_2,x_3,t)\exp(-mx_1)$ and $\varphi(x,t)=g(x_2,x_3,t)\exp(-mx_1)$, where m is an arbitrary positive real number. We have

$$\int_0^t\int_R c_{ijkl}\dot{v}_{i,j}\dot{v}_{k,l}\,\mathrm{d}x\,\mathrm{d}s=\frac{1}{2}\int_0^t\int_D (mc_{i1k1}\dot{f}_i\dot{f}_k+c_{i1k\alpha}\dot{f}_i\dot{f}_{k,\alpha}+c_{i\alpha k1}\dot{f}_{i,\alpha}\dot{f}_k+\frac{1}{m}c_{i\alpha k\beta}\dot{f}_{i,\alpha}\dot{f}_{k,\beta})\,\mathrm{d}a\,\mathrm{d}s,$$

$$\int_0^t \int_R \dot{v}_{i,j}\dot{v}_{i,j}\,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}\int_0^t \int_D (m\dot{f}_i\dot{f}_i + \frac{1}{m}\dot{f}_{i,\alpha}\dot{f}_{i,\alpha})\,\mathrm{d}a\,\mathrm{d}s,$$

$$\int_R \rho\dot{v}_i\dot{v}_i\,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2m}\int_D \rho\dot{f}_i\dot{f}_i\,\mathrm{d}a,$$

$$\int_0^t \int_R \rho\ddot{v}_i\ddot{v}_i\,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2m}\int_0^t \int_D \rho\ddot{f}_i\ddot{f}_i\,\mathrm{d}a\,\mathrm{d}s,$$

$$\int_0^t \int_R \rho\dot{v}_i\dot{v}_i\,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2m}\int_0^t \int_D \rho\dot{f}_i\dot{f}_i\,\mathrm{d}a\,\mathrm{d}s,$$

$$\int_0^t \int_R (\varphi_{,i}\varphi_{,i} + \dot{\varphi}_{,i}\dot{\varphi}_{,i})\,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}\int_0^t \int_D \left(m(g^2+(\dot{g})^2) + \frac{1}{m}(g_{,\alpha}g_{,\alpha} + \dot{g}_{,\alpha}\dot{g}_{,\alpha})\right)\mathrm{d}a\,\mathrm{d}s,$$

$$\int_0^t \int_R (\varphi^2 + (\dot{\varphi})^2)\,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2m}\int_0^t \int_D (g^2+(\dot{g})^2)\,\mathrm{d}a\,\mathrm{d}s.$$

We then obtain

$$\begin{aligned}
E_\omega(0,t) \le{}& D_1\int_0^t \int_D \left(mc_{i1k1}\dot{f}_i\dot{f}_k + c_{i1k\alpha}\dot{f}_i\dot{f}_{k,\alpha} + c_{i\alpha k1}\dot{f}_{i,\alpha}\dot{f}_k + \frac{1}{m}c_{i\alpha k\beta}\dot{f}_{i,\alpha}\dot{f}_{k,\beta}\right)\mathrm{d}a\,\mathrm{d}s \\
&+D_2\int_0^t \int_D \left(m\dot{f}_i\dot{f}_i + \frac{1}{m}\dot{f}_{i,\alpha}\dot{f}_{i,\alpha}\right)\mathrm{d}a\,\mathrm{d}s + \frac{D_3}{m}\int_D \rho\dot{f}_i\dot{f}_i\,\mathrm{d}a \\
&+\frac{D_4}{m}\int_0^t \int_D \rho\ddot{f}_i\ddot{f}_i\,\mathrm{d}a\,\mathrm{d}s + \frac{D_5}{m}\int_0^t \int_D \rho\dot{f}_i\dot{f}_i\,\mathrm{d}a\,\mathrm{d}s \\
&+(D_6+D_9)\int_0^t \int_D \left(m(g^2+(\dot{g})^2) + \frac{1}{m}(g_{,\alpha}g_{,\alpha} + \dot{g}_{,\alpha}\dot{g}_{,\alpha})\right)\mathrm{d}a\,\mathrm{d}s \\
&+\frac{D_7+D_8}{m}\int_0^t \int_D (g^2+(\dot{g})^2)\,\mathrm{d}a\,\mathrm{d}s.
\end{aligned}$$

One would like to optimize the value depending on the parameter m. However, it is clear that this is a very cumbersome task.

4. Lord–Shulman theory with two temperatures

In this section, we study the spatial behavior of the solutions of the problem determined by system (2.9)–(2.11) subject to the boundary conditions (2.13), (2.14) and the initial conditions (2.15) whenever assumptions (2.4), (2.5)–(2.7)–(2.8) hold. As in the case of the Green–Lindsay theory with two temperatures, it is possible to obtain the spatial decay of solutions.

We note that from the equation for the displacement in the Lord–Shulman theory we obtain

$$(c_{ijkl}\hat{u}_{k,l}+a_{ij}\hat{\theta})_{,j}=\rho\ddot{\hat{u}}_i.$$

Now, we consider the equalities

$$\begin{aligned}((c_{ijkl}\hat{u}_{k,l}+a_{ij}\hat{\theta})\dot{\hat{u}}_i+k_{ij}\phi_{,i}\hat{\phi})_{,j}&=(c_{ijkl}\hat{u}_{k,l}+a_{ij}\hat{\theta})\dot{\hat{u}}_{i,j}+(c_{ijkl}\hat{u}_{k,l}+a_{ij}\hat{\theta})_{,j}\dot{\hat{u}}_i+(k_{ij}\phi_{,i})_{,j}\hat{\phi}+k_{ij}\phi_{,i}\hat{\phi}_{,j}\\&=(c_{ijkl}\hat{u}_{k,l}+a_{ij}\hat{\theta})\dot{\hat{u}}_{i,j}+\rho\dot{\hat{u}}_i\ddot{\hat{u}}_i+(h_1\dot{\hat{\theta}}-a_{ij}\dot{\hat{u}}_{i,j})\hat{\phi}+k_{ij}\phi_{,i}\hat{\phi}_{,j}\\&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\rho\dot{\hat{u}}_i\dot{\hat{u}}_i+c_{ijkl}\hat{u}_{k,l}\hat{u}_{i,j})+a_{ij}\hat{\theta}\dot{\hat{u}}_{i,j}\\&\quad+(h_1\dot{\hat{\theta}}-a_{i,j}\dot{\hat{u}}_{i,j})\hat{\theta}+a(k_{ij}\phi_{,i})_{,j}(k_{lm}\hat{\phi}_{,l})_{,m}+k_{ij}\phi_{,i}\phi_{,j}+d_1k_{ij}\phi_{,i}\dot{\phi}_{,j}\\&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\rho\dot{\hat{u}}_i\dot{\hat{u}}_i+c_{ijkl}\hat{u}_{k,l}\hat{u}_{i,j}+h_1(\hat{\theta}^2)+ad_1(K(\phi))^2+d_1k_{ij}\phi_{,i}\phi_{,j})\\&\quad+a(K(\phi))^2+k_{ij}\phi_{,i}\phi_{,j}.\end{aligned}$$

Phragmén–Lindelöf alternative

As far as this theory is concerned, the analysis starts by considering the function

$$F_\omega(z,t)=\int_0^t\int_{D(z)}\exp(-2\omega s)\left((c_{i1kl}\hat{u}_{k,l}+a_{i1}\hat{\theta})\dot{\hat{u}}_i+k_{i1}\phi_{,i}\hat{\phi}\right)\mathrm{d}a\,\mathrm{d}s. \tag{4.1}$$

We have

$$F_\omega(t,z+h)-F_\omega(t,z)=\frac{\exp(-2\omega t)}{2}\int_{R(z,z+h)}W_1^*\,\mathrm{d}x+\int_0^t\int_{R(z,z+h)}\exp(-2\omega s)W_2^*\,\mathrm{d}x\,\mathrm{d}s, \tag{4.2}$$

where $R(z,z+h)$ has already been defined,

$$W_1^*=\rho\dot{\hat{u}}_i\dot{\hat{u}}_i+c_{ijkl}\hat{u}_{i,j}\hat{u}_{k,l}+h_1(\hat{\theta})^2+d_1k_{ij}\phi_{,i}\phi_{,j}+d_1a(K(\phi))^2$$

and

$$W_2^*=\omega W_1^*+k_{ij}\phi_{,i}\phi_{,j}+a(K(\phi))^2.$$

We then obtain

$$\frac{\partial F_\omega(t,z)}{\partial z}=\frac{\exp(-2\omega t)}{2}\int_{D(z)}W_1^*\,\mathrm{d}a+\int_0^t\int_{D(z)}\exp(-2\omega s)W_2^*\,\mathrm{d}a\,\mathrm{d}s. \tag{4.3}$$

We need another function G. Here, we define the function G by

$$G(z,t)=a\omega h_0^*\int_0^t\int_{D(z)}\exp(-2\omega s)k_{i1}\hat{\phi}\hat{\phi}_{,i}\,\mathrm{d}a\,\mathrm{d}s.$$

We have

$$\frac{\partial G(z,t)}{\partial z}=a\omega h_0^*\int_0^t\int_{D(z)}\exp(-2\omega s)(k_{ij}\hat{\phi}_{,j}\hat{\phi}_{,i}+(k_{ij}\hat{\phi}_{,i})_{,j}\hat{\phi})\,\mathrm{d}a\,\mathrm{d}s, \tag{4.4}$$

As in Sect. 3, we define $H_\omega = F_\omega + G$. We also have

$$H_\omega(z+h,t) - H_\omega(z,t) = \frac{\exp(-2\omega t)}{2} \int\limits_{R(z,z+h)} W_1^* \,\mathrm{d}x + \int\limits_0^t \int\limits_{R(z,z+h)} \exp(-2\omega s) W_2 \,\mathrm{d}x \,\mathrm{d}s, \tag{4.5}$$

where

$$\begin{aligned} W_2 &= \omega(\rho \dot{\hat{u}}_i \dot{\hat{u}}_i + c_{ijkl}\hat{u}_{i,j}\hat{u}_{k,l} + d_1 k_{ij}\phi_{,i}\phi_{,j} + d_1 a(K(\phi))^2 \\ &\quad + \left(h_1 - \frac{h_0^*}{2}\right)(\hat{\theta})^2 + \frac{h_0^*}{2}(\hat{\phi}^2 + 2ak_{ij}\hat{\phi}_{,i}\hat{\phi}_{,j} + a^2(K(\hat{\phi}))^2)) + k_{ij}\phi_{,i}\phi_{,j} + a(K(\phi))^2. \end{aligned}$$

Similarly, we have

$$|F_\omega| \le |I_1| + |I_2| + |I_3|,$$

where

$$I_1 = \int\limits_0^t \int\limits_{D(z)} \exp(-2\omega s) c_{i1kl}\hat{u}_{k,l}\dot{\hat{u}}_i \,\mathrm{d}a \,\mathrm{d}s,$$

$$I_2 = \int\limits_0^t \int\limits_{D(z)} \exp(-2\omega s) a_{i1}\hat{\theta}\dot{\hat{u}}_i \,\mathrm{d}a \,\mathrm{d}s,$$

and

$$I_3 = \int\limits_0^t \int\limits_{D(z)} \exp(-2\omega s) k_{i1}\phi_{,i}\hat{\phi} \,\mathrm{d}a \,\mathrm{d}s.$$

We can estimate $I_1, \ldots, I_3$ in a similar way, and we have

$$\begin{aligned} |I_1| &\le \left(\int\limits_0^t \int\limits_{D(z)} \exp(-2\omega s) c_{ijkl}\hat{u}_{i,j}\hat{u}_{k,l} \,\mathrm{d}a \,\mathrm{d}s \int\limits_0^t \int\limits_{D(z)} \exp(-2\omega s) c_{ijkl}\dot{\hat{u}}_i n_j \dot{\hat{u}}_k n_l \,\mathrm{d}a \,\mathrm{d}s \right)^{1/2} \\ &\le \left(\frac{c_1}{4\rho_0\omega^2}\right)^{1/2} \left(\omega \int\limits_0^t \int\limits_{D(z)} \exp(-2\omega s) c_{ijkl}\hat{u}_{i,j}\hat{u}_{k,l} \,\mathrm{d}a \,\mathrm{d}s + \omega \int\limits_0^t \int\limits_{D(z)} \exp(-2\omega s) \rho \dot{\hat{u}}_i \dot{\hat{u}}_i \,\mathrm{d}a \,\mathrm{d}s \right), \\ |I_2| &\le \left(\int\limits_0^t \int\limits_{D(z)} \exp(-2\omega s) a_{ij}a_{ij}(\hat{\theta})^2 \,\mathrm{d}a \,\mathrm{d}s \int\limits_0^t \int\limits_{D(z)} \exp(-2\omega s) \dot{\hat{u}}_i \dot{\hat{u}}_i \,\mathrm{d}a \,\mathrm{d}s \right)^{1/2} \\ &\le \left(\frac{\beta}{2\rho_0\omega^2 h_0^*}\right)^{1/2} \left(\omega \int\limits_0^t \int\limits_{D(z)} \exp(-2\omega s) \left(h_1 - \frac{h_0^*}{2}\right)(\hat{\theta})^2 \,\mathrm{d}a \,\mathrm{d}s + \omega \int\limits_0^t \int\limits_{D(z)} \exp(-2\omega s) \rho \dot{\hat{u}}_i \dot{\hat{u}}_i \,\mathrm{d}a \,\mathrm{d}s \right), \\ |I_3| &\le \left(\int\limits_0^t \int\limits_{D(z)} \exp(-2\omega s) k_{ij}\phi_{,i}\phi_{,j} \,\mathrm{d}a \,\mathrm{d}s \int\limits_0^t \int\limits_{D(z)} \exp(-2\omega s) k_{11}\hat{\phi}^2 \,\mathrm{d}a \,\mathrm{d}s \right)^{1/2} \end{aligned}$$

$$+m^*\left(\int_0^t\int_{D(z)} \exp(-2\omega s)d_1 k_{ij}\phi_{,i}\phi_{,j}\,\mathrm{d}a\,\mathrm{d}s+\int_0^t\int_{D(z)}\frac{\omega h_0^*}{2}\exp(-2\omega s)(\hat{\phi})^2\,\mathrm{d}a\,\mathrm{d}s\right),$$

where

$$m^*=\left(\frac{\sup k_{11}}{2d_1 h_0^*\omega}\right)^{1/2}.$$

On the other hand

$$|G(z,t)|\le(2a)^{1/2}\left(\int_0^t\int_{D(z)}\exp(-2\omega s)\omega a h_0^* k_{ij}\hat{\phi}_{,i}\hat{\phi}_{,j}\,\mathrm{d}a\,\mathrm{d}s\int_0^t\int_{D(z)}\exp(-2\omega s)\omega\frac{h_0^*}{2}k_{11}\hat{\phi}^2\,\mathrm{d}a\,\mathrm{d}s\right)^{1/2}$$

$$\le m^{**}\left(\int_0^t\int_{D(z)}\exp(-2\omega s)\omega a h_0^* k_{ij}\hat{\phi}_{,i}\hat{\phi}_{,j}\,\mathrm{d}a\,\mathrm{d}s+\int_0^t\int_{D(z)}\frac{\omega h_0^*}{2}\exp(-2\omega s)(\hat{\phi})^2\,\mathrm{d}a\,\mathrm{d}s\right),$$

where

$$m^{**}=(a\sup k_{11}/2)^{1/2}.$$

From the previous inequalities, we see that

$$|H_\omega(z,t)|\le C_3\frac{\partial H_\omega(z,t)}{\partial z}, \tag{4.6}$$

for every t and $z\ge 0$, where

$$C_3=\max\left(\left(\frac{c_1}{4\rho_0\omega^2}\right)^{1/2}+\left(\frac{\beta}{2\rho_0\omega^2 h_0^*}\right)^{1/2},m^*+m^{**}\right).$$

From this equality on, the analysis is identical to the one proposed in the previous section. If we define the functions $E_\omega(z,t)$ and $\mathcal{E}(z,t)$ as in Sect. 3, but with the functions W_1^* and W_2 proposed in this section, we have the following result:

Theorem 4.1. *Let (u_i,ϕ) be a smooth solution of the problem defined by (2.9)–(2.11), the boundary conditions (2.13)–(2.14) and the initial conditions (2.15). Then, either this solution satisfies the growth estimate (3.11) or it satisfies the decay estimate*

$$\mathcal{E}(z,t)\le E_\omega(0,t)\exp(2\omega t-C_3^{-1}z),\ \ z\ge 0, \tag{4.7}$$

where the energy $\mathcal{E}$ is defined in (3.16) and E_ω is given by (3.15).

To have a complete description of this estimate, we need an upper bound on the amplitude term $E_\omega(0,t)$. This can be done by using an argument similar to the one proposed in Sect. 3. However, we do not do it here to avoid repetitions.

5. Conclusion

In this paper, we have analyzed the spatial behavior of the solutions for the thermoelasticity theory of Green–Lindsay with two temperatures and the thermoelasticity theory of Lord–Shulman with two temperatures. We have seen that the alternative for the solutions is of exponential type in both theories. In fact, the growth/decay are also quite similar, but of course depending on the constitutive functions and tensors. It is worth noting that from the first system we obtain the classical thermoelasticity when $a=\alpha=0$ and $h=0$. We then see that the growth/decay for the classical thermoelasticity could be faster in general. A similar comment can be done for the Lord–Shulman theory. Another relevant result is that both theories determine the infinite speed of propagation in view of the alternative result obtained in each case.

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