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Analysis on a diffusive SIS epidemic model with logistic source

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Abstract. In this paper, we are concerned with an SIS epidemic reaction–diffusion model with logistic source in spatially heterogeneous environment. We first discuss some basic properties of the parabolic system, including the uniform upper bound of solutions and global stability of the endemic equilibrium when spatial environment is homogeneous. Our primary focus is to determine the asymptotic profile of endemic equilibria (when exist) if the diffusion (migration) rate of the susceptible or infected population is small or large. Combined with the results of Li et al. (J Differ Equ 262:885–913, 2017) where the case of linear source is studied, our analysis suggests that varying total population enhances persistence of infectious disease.

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1. Introduction

We consider the following susceptible-infected-susceptible epidemic system:

$$\begin{cases} \dfrac{\partial S}{\partial t} - d_S \Delta S = a(x)S - b(x)S^2 - \beta(x)\dfrac{SI}{S+I} + \gamma(x)I, & x \in \Omega,\ t > 0, \\ \dfrac{\partial I}{\partial t} - d_I \Delta I = \beta(x)\dfrac{SI}{S+I} - \gamma(x)I, & x \in \Omega,\ t > 0, \\ \dfrac{\partial S}{\partial \nu} = \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ S(x,0) = S_0(x),\ I(x,0) = I_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here, S and I, respectively, stand for the density of susceptible and infected individuals; d_S and d_I are positive constants measuring the motility of susceptible and infected populations, respectively; β and γ are positive Hölder continuous functions on $\overline{\Omega}$ accounting for the rates of disease transmission and recovery, respectively. The nonlinear term $a(x)S - b(x)S^2$ for positive Hölder functions a and b represents that the susceptible population is subject to logistic growth. The habitat $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, and the homogeneous Neumann boundary conditions mean that no population flux crosses the boundary $\partial\Omega$.

It is straightforward to verify that $SI/(S + I)$ is a Lipschitz continuous function of S and I in the open first quadrant. Therefore, we can extend it to the entire first quadrant by defining it to be zero whenever $S = 0$ or $I = 0$. Throughout the paper, it is assumed that initially, S_0 and I_0 are nonnegative continuous functions on $\overline{\Omega}$, and there is a positive number of infected individuals, i.e., $\int_\Omega I_0(x)dx > 0$.

To capture the effect of spatial heterogeneity, Allen et al. [3] proposed the following SIS epidemic reaction–diffusion model.

$$\begin{cases} \dfrac{\partial S}{\partial t} - d_S\Delta S = -\beta(x)\dfrac{SI}{S+I} + \gamma(x)I, & x \in \Omega,\ t > 0, \\ \dfrac{\partial I}{\partial t} - d_I\Delta I = \beta(x)\dfrac{SI}{S+I} - \gamma(x)I, & x \in \Omega,\ t > 0, \\ \dfrac{\partial S}{\partial \nu} = \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ S(x,0) = S_0(x),\ I(x,0) = I_0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

A basic reproduction number $\mathcal{R}_0$ via variational characterization was defined in [3], and it was proved that the disease will become extinct whenever $\mathcal{R}_0 < 1$; that is, the unique disease-free equilibrium (DFE) $(N/|\Omega|, 0)$ is globally asymptotically stable if $\mathcal{R}_0 < 1$. On the other hand, [22] confirmed the persistence of disease provided $\mathcal{R}_0 > 1$, when the environment is temporally periodic. Practically, we are more concerned with the question whether restricting the motility of susceptible or infected population will help to eliminate the infectious disease or not. To mathematically study the effect of diffusion rates, the authors in [3] first investigated the existence and uniqueness of endemic equilibrium (EE) and then demonstrated that restricting the migration rate of susceptible individuals shall eradicate the disease entirely, provided the environment can be modified to include low-risk sites. Additionally, it was conjectured that the unique EE should be globally stable and Peng and Liu [19] confirmed it in some special scenarios. Further results on asymptotic profiles of the EE were obtained in [18,21]. We also refer interested readers to [2,5–9,13,14,23,25] and references therein for more works related to model (1.2) and for similar SIS epidemic models.

We notice that one of the main features of model (1.2) is that the total number of population is conserved for all time $t > 0$ in the sense that

$$\int_\Omega (S(x,t) + I(x,t))dx = \int_\Omega (S_0(x) + I_0(x))dx, \quad \forall t > 0,$$

provided that the initial total population

$$\int_\Omega (S_0(x) + I_0(x))dx$$

is fixed; see [3]. However, model (1.1), due to the introduction of the logistic source of susceptible population, no longer possesses this property. Furthermore, the logistic source also enables $(0, 0)$ to become a trivial steady state of (1.1) and thus brings new difficulty for theoretical analysis. As a result, our mathematical treatment, especially the existence of endemic equilibrium, is significantly different from that in [3] or [14].

As in most mathematical models in population biology, compared to linear source, logistic source seems to be a more reasonable choice describing birth/death rate since it models the overcrowding effect in this context. Hence, we are motivated to study SIS epidemic model (1.1) subject to logistic growth for susceptible individuals in spatially heterogeneous environment. Using the same basic reproduction number $\mathcal{R}_0$ as that in [3], our mathematical results imply that the infectious disease will be eliminated if $\mathcal{R}_0 < 1$ and the unique DFE is globally asymptotically stable, at least when a and b are positive constants; see Theorem 2.2 and Remark 2.1. On the other hand, it is difficult to obtain even local stability of the trivial equilibrium $(0, 0)$ since system (1.1) cannot be linearized there. For the time being, we are unable to rule out the possibility that $(0, 0)$ may be globally asymptotically stable. And consequently, we cannot prove the persistence of disease in the case of $\mathcal{R}_0 > 1$ either, as in [14,22]. Nevertheless, we can indeed show the global attractivity of the unique positive constant steady state when $\mathcal{R}_0 > 1$ in homogeneous environment; see Theorem 2.3.

It is obvious that only nonnegative steady states of (1.1) are of physical interest. The stability of the trivial steady state $(0, 0)$ is left for future investigation and is not discussed in this paper. A *disease-free equilibrium* (DFE) is a steady state in which the I-component is identically zero over Ω, whereas an *endemic equilibrium* (EE) is a steady state in which the I-component is positive for some $x \in \Omega$. To study the effects of population motility on disease elimination/persistence, we consider the asymptotic profiles of endemic equilibria when the diffusion (or migration) rate of the susceptible or infected population is small or large. Assume that the habitat includes some *high-risk* sites, i.e., locations where $\beta(x) > \gamma(x)$. Then, our mathematical results indicate that slow motility (or migration) of the infected individuals shall drive the infectious disease to become extinct only in *low-risk* (where $\beta(x) < \gamma(x)$) and *moderate-risk* (where $\beta(x) = \gamma(x)$) sites but not in *high-risk* sites, whereas the disease persists throughout the entire habitat if the movement of susceptible population is controlled to be small. This result is consistent with that in [14] where an SIS epidemic model with linear source is considered. However, it is in sharp contrast with model (1.2), for which the optimal strategy of eliminating the disease is to restrict the movement of susceptible individuals, i.e., restricting the movement of the susceptible can eradicate the disease completely throughout the habitat, while restricting the movement of the infected can eliminate the disease solely in *low-risk* and *moderate-risk* sites; see [3,18,21]. On the other hand, if the habitat Ω is of *high-risk* type, meaning $\int_\Omega \beta > \int_\Omega \gamma$, then for both (1.2) and (1.1), the results of [3,14,18,21] and the current paper show that fast movement of the susceptible or the infected tends to homogenize the corresponding population density to a positive constant and hence large motility of the susceptible or infected individuals does not help to eliminate the disease at all.

Let us also compare our results here with those in [14]. First of all, for global boundedness of solutions to the corresponding parabolic systems, the linear nature of the model in [14] enables us to see that L^1-bound is sufficient to ensure the L^∞-bound (see [1, Theorem 3.1] and [22, Lemma 3.1]), while for system (1.1) with quadratic nonlinearity, we need L^p-bound for larger $p > 1$, and this is achieved through the method of mathematical induction; see Theorem 2.1. Secondly, since we are not able to show that the trivial steady state $(0, 0)$ may be globally asymptotically stable, to prove the existence of EE, we need an extra condition beyond $\mathcal{R}_0 > 1$; see Theorem 3.1. Moreover, the existence of EE is proved here by using degree theory, while that in [14] is done by dynamical system theory. Lastly, for the asymptotic profiles of EE, our analysis resembles that in [14] and biological interpretations of the main mathematical results remain the same, although new technical difficulties appear because of the logistic term.

In summary, the above discussion, together with the mathematical results in [14], suggests that, in stark contrast to epidemic model (1.2) with constant total population number, an extra birth/death term for the susceptibles, either a linear one or a logistic one, tends to enhance the persistence of disease and makes the infectious disease more difficult to control; varying total population invalidates the strategy of restricting the motility of susceptible or infected individuals for disease elimination. Thus, in this scenario, for decision makers, more effective control measures should be taken into account in order to eradicate the infectious disease.

The paper is organized as follows. Section 2 is dedicated to the study of parabolic system (1.1), where we first establish uniform boundedness of global solutions. Then we turn to the global dynamics of (1.1): The stability of DFE and EE is discussed in Sects. 2.2 and 2.3, respectively. In Sect. 3, we prove the existence of EE by using a topological degree-type argument. We also establish a priori estimates of positive equilibrium of (1.1), which will be used repeatedly in the forthcoming section. In Sect. 4, we investigate the asymptotic profiles of EE as the motility of susceptible or infected individuals is small or large. Finally, in "Appendix," we discuss two strongly related elliptic problems, whose existence and uniqueness of positive solutions, together with the asymptotic behavior of solutions, are obtained, as the diffusion rate shrinks.

In the rest of the paper, for notational convenience, we denote

$$g^* = \max_{x \in \overline{\Omega}} g(x) \quad \text{and} \quad g_* = \min_{x \in \overline{\Omega}} g(x),$$

for $g = a,\ b,\ \beta,\ \gamma$.

2. Properties of solutions to (1.1)

We first notice that the standard theory for parabolic equations, combined with our assumption on the initial data, guarantees that (1.1) admits a unique classical solution $(S,I)\in C^{2,1}(\overline{\Omega}\times(0,\infty))$. Moreover, it follows from the strong maximum principle and the Hopf boundary lemma for parabolic equations that both $S(x,t)$ and $I(x,t)$ are positive for $x\in\overline{\Omega}$ and $t\in(0,\infty)$.

2.1. Uniform boundedness

In this subsection, we will establish the uniform bound of solutions of (1.1). The following well-known Young's inequality will be used:

$$ab\le \epsilon a^{\tau}+\epsilon^{-\frac{\tau'}{\tau}}b^{\tau'}, \tag{2.1}$$

where $a,b,\epsilon>0$, $\tau,\tau'>1$ and $\frac{1}{\tau}+\frac{1}{\tau'}=1$.

Theorem 2.1. *There exists a positive constant C, independent of initial data, such that*

$$\|S(\cdot,t)\|_{L^{\infty}(\Omega)}+\|I(\cdot,t)\|_{L^{\infty}(\Omega)}\le C,\ \forall t\ge T, \tag{2.2}$$

for some large time $T>0$.

Proof. First of all, we will prove the following assertion: for any positive integer k, there exists a positive constant $C=C(k)$ independent of initial data such that

$$\|S(\cdot,t)\|_{L^{k}(\Omega)}+\|I(\cdot,t)\|_{L^{k}(\Omega)}\le C,\ \forall t\ge T, \tag{2.3}$$

for some large time $T>0$.

We shall employ the method of mathematical induction to derive (2.3). We first show that (2.3) holds for $k=1$. To this aim, it follows from (1.1) and (2.1) that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}(S+2I) &= \int_{\Omega}\frac{\beta SI}{S+I}-\int_{\Omega}\gamma I+\int_{\Omega}(aS-bS^2)\\
&\le \int_{\Omega}\beta^* S-\gamma_*\int_{\Omega}I+\int_{\Omega}\left(a^*S-b_*S^2\right)\\
&\le (\beta^*+a^*)\int_{\Omega}S-\gamma_*\int_{\Omega}I-b_*\int_{\Omega}\left(\epsilon S-\frac{\epsilon^2}{4}\right)\\
&\le \mu-\nu\int_{\Omega}(S+2I),
\end{aligned}$$

where

$$\mu=\frac{b_*\epsilon^2|\Omega|}{4},\quad \nu=\frac{\gamma_*}{2}$$

by taking $\epsilon>0$ satisfying

$$\beta^*+a^*-\epsilon b_*=-\frac{\gamma_*}{2}.$$

The above differential inequality yields

$$\int_{\Omega}(S(x,t)+2I(x,t))\le e^{-\nu t}\int_{\Omega}(S_0(x)+2I_0(x))+\frac{\mu}{\nu}\left(1-e^{-\nu t}\right),\quad \forall t\ge 0.$$

This obviously implies that (2.3) holds.

We now assume that (2.3) is true for $k-1$. Then it remains to show that (2.3) also holds for k. Multiplying the first equation in (1.1) by S^{k-1} and integrating by parts, we find

$$\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega S^k + d_S(k-1)\int_\Omega S^{k-2}|\nabla S|^2 = \int_\Omega \left[-\beta\frac{S^k I}{S+I} + \gamma I S^{k-1} + (aS - bS^2)S^{k-1}\right].$$

In the same fashion,

$$\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega I^k + d_I(k-1)\int_\Omega I^{k-2}|\nabla I|^2 = \int_\Omega \left(\beta\frac{I^k S}{S+I} - \gamma I^k\right).$$

In view of the above two equalities, we can get from (2.1) and (2.3) with k replaced by $k-1$ that

$$\begin{aligned}
&\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(S^k + I^k\right) + (k-1)\int_\Omega \left(d_S S^{k-2}|\nabla S|^2 + d_I I^{k-2}|\nabla I|^2\right)\\
&\le \int_\Omega \left[\beta\frac{I}{S+I}I^{k-1}S + \gamma I S^{k-1} - \gamma I^k + S^{k-1}\left(aS - bS^2\right)\right]\\
&\le \int_\Omega \left[\beta^* I^{k-1}S + \gamma^* I S^{k-1} - \gamma_* I^k + a^* S^k - b_* S^{k-1}\left(\varepsilon S - \frac{\varepsilon^2}{4}\right)\right]\\
&\le \int_\Omega \left[\beta^*\left(\varepsilon' I^k + C(\varepsilon',k)S^k\right) + \gamma^*\left(\varepsilon' I^k + C(\varepsilon',k)S^k\right) - \gamma_* I^k + a^* S^k - b_*\varepsilon S^k\right] + \frac{\varepsilon^2 b_*}{4}\int_\Omega S^{k-1}\\
&\le \left[(\beta^* + \gamma^*)\,\varepsilon' - \gamma_*\right]\int_\Omega I^k + \left[(\beta^* + \gamma^*)\,C(\varepsilon',k) + a^* - b_*\varepsilon\right]\int_\Omega S^k + \frac{\varepsilon^2 b_*}{4}\int_\Omega S^{k-1},
\end{aligned} \tag{2.4}$$

for any $\varepsilon, \varepsilon' > 0$ and some positive constant $C(\varepsilon', k)$.

In (2.4), we first take $\varepsilon' > 0$ such that $(\beta^* + \gamma^*)\,\varepsilon' - \gamma_* = -\gamma_*/2$. For this $\varepsilon' > 0$, we then choose $\varepsilon > 0$ fulfilling $(\beta^* + \gamma^*)\,C(\varepsilon', k) + a^* - b_*\varepsilon = -\gamma_*/2$. Hence, it follows from (2.4) and the induction hypothesis that

$$\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(S^k + I^k\right) \le -\frac{\gamma_*}{2}\int_\Omega \left(S^k + I^k\right) + \overline{C}, \quad t \ge 0,$$

where the positive constant $\overline{C}$ is independent of the initial data for large t. Thus, this differential inequality yields (2.3).

In view of (2.3), one can use [22, Lemma 3.1] to conclude uniform bound (2.2) of any solutions to (1.1). The proof is complete. □

2.2. Global stability of DFE

As in [24], we define the basic reproduction number $\mathcal{R}_0$ for system (1.1) which is the spectral radius of the associated operator L of [24] by taking $m = 1$, $F(x) = \beta(x)$, $V(x) = \gamma(x)$, $d_I(x) = d_I$ there. Furthermore, by [24, Theorem 3.2] and the variational characterization of $\mathcal{R}_0$, one immediately sees that $\mathcal{R}_0$ is given by

$$\mathcal{R}_0 = \sup_{0\ne\varphi\in H^1(\Omega)} \frac{\int_\Omega \beta\varphi^2}{\int_\Omega \left(d_I|\nabla\varphi|^2 + \gamma\varphi^2\right)}.$$

Observe that $\mathcal{R}_0$ is independent of the diffusion rate d_S.

It is well known that the logistic-type elliptic problem

$$\begin{cases} -d_S\Delta S = a(x)S - b(x)S^2, & x \in \Omega, \\ \dfrac{\partial S}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{2.5}$$

admits a unique positive solution $\tilde{S}$, which is globally asymptotically stable for the corresponding parabolic equation [19, Lemma 2.1]. Therefore, $(\tilde{S}, 0)$ is a semitrivial steady state of (1.1), which we call the DFE. Moreover, after the extension of the function $SI/(S+I)$ so that it is zero once $S = 0$ or $I = 0$, it is obvious that $(0, 0)$ is a trivial steady state of (1.1).

On the other hand, let (λ^*, ψ^*) be the principal eigenpair of the eigenvalue problem

$$\begin{cases} d_I\Delta u + (\beta - \gamma)u + \lambda u = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{2.6}$$

Then, we have the following properties of $\mathcal{R}_0$, which were established in [3].

Proposition 2.1. *The following assertions hold.*

(a) *$\mathcal{R}_0$ is a monotone decreasing function of d_I with $\mathcal{R}_0 \to \max\{\beta(x)/\gamma(x) : x \in \overline{\Omega}\}$ as $d_I \to 0$ and $\mathcal{R}_0 \to \int_\Omega \beta / \int_\Omega \gamma$ as $d_I \to \infty$;*

(b) *If $\int_\Omega \beta(x)dx < \int_\Omega \gamma(x)dx$ and $\beta - \gamma$ changes sign, then there exists a threshold value $d_I^* \in (0, \infty)$ such that $\mathcal{R}_0 > 1$ for $d_I < d_I^*$ and $\mathcal{R}_0 < 1$ for $d_I > d_I^*$;*

(c) *If $\int_\Omega \beta(x)dx \geq \int_\Omega \gamma(x)dx$, then $\mathcal{R}_0 > 1$ for all $d_I > 0$;*

(d) *$\mathcal{R}_0 > 1$ when $\lambda^* < 0$, $\mathcal{R}_0 = 1$ when $\lambda^* = 0$, and $\mathcal{R}_0 < 1$ when $\lambda^* > 0$.*

It turns out that the stability of the DFE $(\tilde{S}, 0)$ is completely determined by the size of $\mathcal{R}_0$.

Proposition 2.2. *The DFE $(\tilde{S}, 0)$ is stable if $\mathcal{R}_0 < 1$, and it is unstable if $\mathcal{R}_0 > 1$.*

Proof. The analysis is rather standard; especially the stability of the DFE for $\mathcal{R}_0 < 1$ follows directly from [24, Theorem 3.1(ii)]. Thus, the details are omitted here. □

The following result suggests that the disease will become extinct if $\mathcal{R}_0 < 1$.

Theorem 2.2. *Let (S, I) be the unique solution of (1.1). If $\mathcal{R}_0 < 1$, then*

$$\lim_{t\to\infty} I(x,t) = 0$$

uniformly for $x \in \overline{\Omega}$. If additionally $\gamma(x) \geq, \not\equiv \beta(x)$ on $\overline{\Omega}$, then

$$\lim_{t\to\infty} (S(x,t) - \tilde{S}(x)) = 0$$

uniformly for $x \in \overline{\Omega}$, where $\tilde{S}$ is the unique positive solution of (2.5).

Proof. When $\mathcal{R}_0 < 1$, the assertion $I(x,t) \to 0$ uniformly for $x \in \overline{\Omega}$ as $t \to \infty$ follows from the same argument of [3, Lemma 2.5]. It remains to determine the limit of S. Given any small $\epsilon > 0$, we can find a large time $T_0 > 0$ such that $I(x,t) \leq \epsilon$ for all $x \in \overline{\Omega}$ as $t \geq T_0$. It is clear that S is a lower solution to the following problem

$$\begin{cases} \dfrac{\partial w}{\partial t} - d_S\Delta w = a(x)w - b(x)w^2 + \epsilon\gamma(x), & x \in \Omega,\ t > T_0, \\ \dfrac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > T_0, \\ w(x, T_0) = S(x, T_0) > 0, & x \in \Omega. \end{cases} \tag{2.7}$$

Denote by $\overline{w}$ the unique solution to (2.7). Thus, the well-known parabolic comparison principle gives $S(x,t) \le \overline{w}(x,t),\ \forall (x,t) \in \overline{\Omega} \times [T_0,\infty)$. By Lemma 5.1, we know that the steady-state problem corresponding to (2.7):

$$\begin{cases} -d_S \Delta w = a(x)w - b(x)w^2 + \epsilon\gamma(x), & x \in \Omega, \\ \dfrac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega \end{cases}$$

admits a unique positive solution, denoted by $w_{1,\epsilon}$. In addition, some standard analysis enables us to conclude that $\overline{w}(x,t) \to w_{1,\epsilon}(x)$ uniformly on $\overline{\Omega}$ as $t \to \infty$ and $w_{1,\epsilon}(x) \to \tilde{S}(x)$ uniformly on $\overline{\Omega}$ as $\epsilon \to 0$. Hence,

$$\limsup_{t\to\infty} S(x,t) \le \tilde{S}(x) \ \text{ uniformly for } x \in \overline{\Omega}. \tag{2.8}$$

If $\gamma(x) \ge, \not\equiv \beta(x)$ on $\overline{\Omega}$, (which can ensure $\mathcal{R}_0 < 1$,) it is observed that S is an upper solution to

$$\begin{cases} \dfrac{\partial w}{\partial t} - d_S \Delta w = a(x)w - b(x)w^2, & x \in \Omega,\ t > 1, \\ \dfrac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 1, \\ w(x,1) = S(x,1) > 0, & x \in \Omega. \end{cases} \tag{2.9}$$

Denote by $\underline{w}$ the unique solution to (2.9). So we have $S(x,t) \ge \underline{w}(x,t),\ \forall (x,t) \in \overline{\Omega} \times [1,\infty)$. It is also known that $\underline{w}(x,t) \to \tilde{S}(x)$ uniformly on $\overline{\Omega}$ as $t \to \infty$. As a result, we obtain

$$\liminf_{t\to\infty} S(x,t) \ge \tilde{S}(x) \ \text{ uniformly for } x \in \overline{\Omega}. \tag{2.10}$$

A combination of (2.8) and (2.10) yields the limit of S. □

Remark 2.1. We remark here that in the special case when both a and b are positive constants, the unique DFE $(a/b, 0)$ is globally asymptotically stable, provided $\mathcal{R}_0 < 1$. In fact, as above, for any small $\epsilon > 0$, we can take a large time $T_0 > 0$ fulfilling $I(x,t) \le \epsilon$ for all $x \in \overline{\Omega}$ and $t \ge T_0$. Let $\delta = \min_{x\in\overline{\Omega}} S(x,T_0) > 0$ and by diminishing $\epsilon > 0$ if necessary, we may assume

$$\frac{a - \sqrt{a - 4b\beta^*\epsilon}}{2b} < \delta. \tag{2.11}$$

For $x \in \Omega$ and $t > T_0$, it holds

$$\frac{\partial S}{\partial t} - d_S \Delta S = aS - bS^2 - \beta \frac{SI}{S+I} + \gamma I \ge aS - bS^2 - \beta^*\epsilon.$$

Consider the following ordinary differential equation.

$$\begin{cases} \bar{S}' = a\bar{S} - b\bar{S}^2 - \beta^*\epsilon, & t > T_0, \\ \bar{S}(T_0) = \delta. \end{cases} \tag{2.12}$$

Simple analysis shows that the unique solution $\bar{S}(t)$ of (2.12) satisfies

$$\lim_{t\to\infty} \bar{S}(t) = \frac{a + \sqrt{a^2 - 4b\beta^*\epsilon}}{2b},$$

thanks to (2.11). An application of the comparison principle then yields

$$\liminf_{t\to\infty} S(x,t) \ge \lim_{t\to\infty} \bar{S}(t) = \frac{a + \sqrt{a^2 - 4b\beta^*\epsilon}}{2b} \ \text{ uniformly for } x \in \overline{\Omega}.$$

This combined with the arbitrariness of small ϵ and (2.8) indicates that $S(x,t) \to \tilde{S}(x) \equiv a/b$ uniformly for $x \in \overline{\Omega}$ as $t \to \infty$.

Remark 2.2. We suspect that the conclusion of Theorem 2.2 remains valid as long as $\mathcal{R}_0 < 1$. We also remark that when $\mathcal{R}_0 > 1$, we are unable to show the persistence property as in [14]; the mathematical difficulty lies in that we cannot rule out the possibility that the solution (S, I) of (1.1) may converge to $(0, 0)$. Even the local stability of this trivial equilibrium seems to be a rather delicate issue since one cannot linearize the system there.

2.3. Global stability of EE

In this subsection, we consider the global stability of the endemic equilibrium of (1.1), in the case that all of β, γ, a and b are positive constants. In view of Theorem 2.2, it seems that (1.1) admits no EE if $\mathcal{R}_0 < 1$. Therefore, throughout this section, we always assume that $\mathcal{R}_0 > 1$, and that all the coefficients β, γ, a, b are positive constants. It is easy to see that $(\hat{S}, \hat{I})$ is the unique constant EE if and only if $\beta > \gamma$ (i.e., $\mathcal{R}_0 = \frac{\beta}{\gamma} > 1$), where

$$(\hat{S}, \hat{I}) = \left(\frac{a}{b}, \quad \frac{a(\beta - \gamma)}{b\gamma}\right).$$

In fact, we have

Theorem 2.3. *For any d_S, $d_I > 0$, assume that the positive constants $\beta > \gamma$. Then the EE $(\hat{S}, \hat{I})$ is globally attractive.*

Proof. For any solution (S, I) of (1.1), we now choose the following Volterra-type Lyapunov functional

$$\mathcal{L}(t) = \int_\Omega L(S(x,t), I(x,t))$$

with $L(S, I) = S - \hat{S}\ln S + I - \hat{I}\ln I$ (the same Lyapunov functional has been used in [14]). Then, for all $t > 0$,

$$\begin{aligned}
\dot{\mathcal{L}}(t) &= \int_\Omega [L_S(S,I)S_t + L_I(S,I)I_t] \\
&= \int_\Omega \left[\left(1 - \frac{\hat{S}}{S}\right)(d_S \Delta S) + \left(1 - \frac{\hat{I}}{I}\right)(d_I \Delta I)\right] \\
&\quad + \int_\Omega \left[\left(1 - \frac{\hat{S}}{S}\right)\left(-\beta\frac{SI}{S+I} + \gamma I + aS - bS^2\right) + \left(1 - \frac{\hat{I}}{I}\right)\left(\beta\frac{SI}{S+I} - \gamma I\right)\right] \\
&= -\int_\Omega \left(d_S \frac{\hat{S}}{S^2}|\nabla S|^2 + d_I \frac{\hat{I}}{I^2}|\nabla I|^2\right) \\
&\quad + \int_\Omega \left[\left(S - \hat{S}\right)(a - bS) - I\left(\beta\frac{S}{S+I} - \gamma\right)\left(\frac{\hat{I}}{I} - \frac{\hat{S}}{S}\right)\right] \\
&= -\int_\Omega \left(d_S \frac{\hat{S}}{S^2}|\nabla S|^2 + d_I \frac{\hat{I}}{I^2}|\nabla I|^2\right) - \int_\Omega \left[b\left(S - \hat{S}\right)^2 + \beta\frac{(S\hat{I} - \hat{S}I)^2}{S\,(S+I)\,(\hat{S}+\hat{I})}\right] \\
&\le 0,
\end{aligned}$$

where we have used the equalities that

$$a = b\hat{S} \quad \text{and} \quad \gamma = \beta\frac{\hat{S}}{\hat{S}+\hat{I}}.$$

As a result, $\mathcal{L}(t)$ is a Lyapunov functional for (1.1), namely, $\dot{\mathcal{L}}(t) < 0$, $\forall t > 0$ along all trajectories except at $(\hat{S}, \hat{I})$ where $\dot{\mathcal{L}}(t) = 0$, $\forall t > 0$. By some standard arguments, we easily see that

$$(S(x,t), I(x,t)) \to (\hat{S}, \hat{I}) \ \text{ in } \left[L^2(\Omega)\right]^2, \quad \text{as } t \to \infty.$$

Recall that both $\|S(\cdot,t)\|_{L^\infty(\Omega)}$ and $\|I(\cdot,t)\|_{L^\infty(\Omega)}$ are bounded due to Proposition 2.1. Hence, by [4, Theorem A2], we have

$$\|S(\cdot,t)\|_{C^2(\overline{\Omega})} + \|I(\cdot,t)\|_{C^2(\overline{\Omega})} \le C_0, \quad \forall t \ge 1,$$

for some positive constant C_0. Thus, the Sobolev embedding theorem allows one to assert

$$(S(x,t), I(x,t)) \to (\hat{S}, \hat{I}) \ \text{ in } \left(L^\infty(\Omega)\right)^2, \quad \text{as } t \to \infty,$$

that is, $(\hat{S}, \hat{I})$ attracts all solutions of (1.1). □

3. Existence of EE

From now on, we are concerned with the steady state (namely, EE) of (1.1), which is a positive solution to the following elliptic system:

$$\begin{cases} -d_S\Delta S = aS - bS^2 - \dfrac{\beta(x)SI}{S+I} + \gamma(x)I, & x \in \Omega, \\ -d_I\Delta I = \dfrac{\beta(x)SI}{S+I} - \gamma(x)I, & x \in \Omega, \\ \dfrac{\partial S}{\partial \nu} = \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

In this section, we will establish a sufficient condition which ensures the existence of EE. Let us first present some a priori bounds of S and I, which do not depend on d_S, $d_I > 0$. Clearly, there exist positive constants μ and ν, independent of d_S, $d_I > 0$, such that

$$aS - bS^2 \le \mu - \nu S^2, \quad \forall S > 0.$$

Integrating the two PDEs of (3.1), respectively, yields

$$\int_\Omega \beta\frac{SI}{S+I} = \int_\Omega \gamma I + \int_\Omega (aS - bS^2) \le \int_\Omega \gamma I + \int_\Omega \left(\mu - \nu S^2\right)$$

and

$$\int_\Omega \beta\frac{SI}{S+I} = \int_\Omega \gamma I.$$

Consequently, we have

$$0 \le \mu|\Omega| - \nu\int_\Omega S^2,$$

and therefore,

$$\int_\Omega S^2 \le \frac{\mu}{\nu}|\Omega|. \tag{3.2}$$

As we shall see in the sequel, this actually can allow one to derive better estimates for I. In fact, multiplying the second equation in (3.1) by I^k for $k>0$ and integrating by parts, we have

$$0 \le kd_I \int_\Omega I^{k-1}|\nabla I|^2 = \int_\Omega \frac{\beta S I^{k+1}}{S+I} - \int_\Omega \gamma I^{k+1}.$$

As a result,

$$\begin{aligned}\gamma_* \int_\Omega I^{k+1} &\le \int_\Omega \gamma I^{k+1} \le \int_\Omega \beta \frac{I}{S+I} S I^k \le \beta^* \int_\Omega S I^k \\ &\le \beta^* \left(\int_\Omega I^{k\cdot\frac{k+1}{k}}\right)^{\frac{k}{k+1}} \left(\int_\Omega S^{k+1}\right)^{\frac{1}{k+1}} \\ &\le \frac{\gamma_*}{2}\int_\Omega I^{k+1} + C\int_\Omega S^{k+1}, \end{aligned} \tag{3.3}$$

where we have used Hölder inequality and Young's inequality (2.1). Taking $k=1$ and recalling (3.2), we obtain

$$\int_\Omega I^2 \le C. \tag{3.4}$$

Hereafter, the positive constant C is independent of d_S, d_I and may vary from line to line.

Now multiplying the first equation of (3.1) by S^m for $m>0$ and integrating by parts, we have

$$0 \le md_S \int_\Omega S^{m-1}|\nabla S|^2 = \int_\Omega aS^{m+1} - \int_\Omega bS^{m+2} - \int_\Omega \frac{\beta S^{m+1} I}{S+I} + \int_\Omega \gamma I S^m,$$

from which it follows that

$$b_* \int_\Omega S^{m+2} \le a^* \int_\Omega S^{m+1} + \gamma^* \int_\Omega I S^m \le a^* \int_\Omega S^{m+1} + \gamma^* \left(\int_\Omega S^{m\cdot\frac{m+1}{m}}\right)^{\frac{m}{m+1}} \left(\int_\Omega I^{m+1}\right)^{\frac{1}{m+1}}. \tag{3.5}$$

Taking $m=1$ and in view of (3.2) and (3.4), we find

$$\int_\Omega S^3 \le C. \tag{3.6}$$

Now with (3.6) at hand, we then take $k=2$ in (3.3) to conclude that

$$\int_\Omega I^3 \le C.$$

In the same fashion, by setting $m=2$ in (3.5) and then $k=3$ in (3.3), we see

$$\int_\Omega S^4 \le C \quad \text{and} \quad \int_\Omega I^4 \le C.$$

Thus, the above iteration procedure eventually yields the following preliminary result.

Lemma 3.1. *Let (S,I) be any EE of (3.1). Then, for any $p>0$, there exists a positive constant $C=C(p)$ independent of $d_S, d_I>0$ satisfying*

$$\int_\Omega S^p \le C \quad \textit{and} \quad \int_\Omega I^p \le C.$$

To obtain the existence of positive solutions to (3.1), we shall employ the topological degree argument. For our purpose, for the parameter $\delta \in [0,1]$, we consider the following auxiliary problem:

$$\begin{cases} -d_S\Delta S = [(1-\delta)a_0 + \delta a]\, S - [(1-\delta)b_0 + \delta b]\, S^2 \\ \qquad\qquad -\dfrac{[(1-\delta)\beta_0 + \delta\beta(x)]\, SI}{S+I} + [(1-\delta)\gamma_0 + \delta\gamma(x)]\, I, & x \in \Omega, \\ -d_I\Delta I = \dfrac{[(1-\delta)\beta_0 + \delta\beta(x)]\, SI}{S+I} - [(1-\delta)\gamma_0 + \delta\gamma(x)]\, I, & x \in \Omega, \\ \dfrac{\partial S}{\partial \nu} = \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{3.7}$$

where a_0, b_0, β_0, γ_0 are given positive constants satisfying $a_0 > \gamma_0$, $\beta_0 > \gamma_0$ and $\beta_0 - \gamma_0 > \beta - \gamma$ on $\overline{\Omega}$.

Before stating our result, we need to give a preliminary result as follows.

Lemma 3.2. *Assume that $\mathcal{R}_0 > 1$. Then, for any constant $\eta \in (0,1)$, the principal eigenvalue $\hat{\lambda}$ of the eigenvalue problem*

$$\begin{cases} d_I\Delta\psi + [(1-\eta)(\beta_0 - \gamma_0) + \eta((\beta-\gamma)]\psi + \lambda\psi = 0, & x \in \Omega, \\ \dfrac{\partial \psi}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{3.8}$$

satisfies $\hat{\lambda} < 0$.

Proof. We can choose φ^* to be an eigenfunction corresponding to λ^* with $\varphi^* > 0$ on $\overline{\Omega}$; see (2.6). Since $\mathcal{R}_0 > 1$, $\lambda^* < 0$ due to Proposition 2.1 (d). Given $\eta \in (0,1)$, by direct calculations, we see that $\underline{u} = \eta\varphi^* + (1-\eta) > 0$ satisfies

$$\begin{aligned} -d_I\Delta\underline{u} &= \eta[\lambda^*\varphi^* - (\gamma-\beta)\varphi^*] \\ &< -[\eta(\gamma-\beta) + (1-\eta)(\gamma_0 - \beta_0)][\eta\varphi^* + (1-\eta)] \\ &= -[\eta(\gamma-\beta) + (1-\eta)(\gamma_0 - \beta_0)]\underline{u}, \quad \forall x \in \Omega \end{aligned} \tag{3.9}$$

Here, we used the facts of $\beta_0 > \gamma_0$, $\beta_0 - \gamma_0 > \beta - \gamma$ on $\overline{\Omega}$ and $\lambda^* < 0$.

Let $\hat{\varphi} > 0$ be the principal eigenfunction of (3.8). We now multiply (3.9) by $\hat{\varphi}$ and then integrate to conclude that

$$d_I \int_\Omega \nabla\underline{u} \cdot \nabla\hat{\varphi} < \int_\Omega [\eta(\beta-\gamma) + (1-\eta)(\beta_0 - \gamma_0)]\, \underline{u}\hat{\varphi}. \tag{3.10}$$

On the other hand, if we multiply the PDE satisfied by $(\hat{\lambda}, \hat{\varphi})$ by $\underline{u}$ and integrate, we are led to

$$d_I \int_\Omega \nabla\underline{u} \cdot \nabla\hat{\varphi} = \int_\Omega [\eta(\beta-\gamma) + (1-\eta)(\beta_0 - \gamma_0)]\, \underline{u}\hat{\varphi} + \hat{\lambda} \int_\Omega \hat{\varphi}\underline{u}. \tag{3.11}$$

A combination of (3.10) and (3.11) gives $\hat{\lambda} \int_\Omega \varphi^* \underline{u} < 0$, which clearly infers $\hat{\lambda} < 0$. □

Our existence result can be stated as follows.

Theorem 3.1. *Assume that $\mathcal{R}_0 > 1$ and*

$$\min_{x \in \overline{\Omega}} \left(a(x) + \frac{\max\{\min_{\overline{\Omega}}(\beta - \gamma),\ 0\}}{\gamma^*}\gamma(x) - \beta(x) \right) > 0. \tag{3.12}$$

Then, (3.1) admits at least one positive solution.

Proof. In the following, we denote by C a positive constant which is independent of δ and may vary from line to line. Throughout this argument, we use $(S, I) = (S_\delta, I_\delta)$ to represent a positive solution of (3.7).

First of all, similar to Lemma 3.1, the standard regularity theory for elliptic equations concludes that

$$\|S\|_{L^\infty(\Omega)} \le C, \quad \|I\|_{L^\infty}(\Omega) \le C. \tag{3.13}$$

We now show

$$S(x),\, I(x) > C, \quad \forall x \in \overline{\Omega}. \tag{3.14}$$

Let $S(x_0) = \min_{x\in\overline{\Omega}} S(x)$. Then it follows from the maximum principle [16, Proposition 2.2] and the first equation of (3.7) that

$$\overline{a}S(x_0) - \overline{b}S^2(x_0) - \frac{\overline{\beta}S(x_0)I(x_0)}{S(x_0)+I(x_0)} + \overline{\gamma}I(x_0) \le 0.$$

where $\overline{a} := (1-\delta)a_0 + \delta a(x_0)$, $\overline{b} := (1-\delta)b_0 + \delta b(x_0)$, $\overline{\beta} := (1-\delta)\beta_0 + \delta\beta(x_0)$, $\overline{\gamma} := (1-\delta)\gamma_0 + \delta\gamma(x_0)$.

Consequently,

$$\overline{a}S(x_0) + \overline{\gamma}I(x_0) \le \overline{b}S^2(x_0) + \frac{\overline{\beta}S(x_0)I(x_0)}{S(x_0)+I(x_0)}. \tag{3.15}$$

Similarly, by setting $I(y_0) = \min_{x\in\overline{\Omega}} I(x)$, we get from the second equation of (3.7) that

$$\frac{[(1-\delta)\beta_0 + \delta\beta(y_0)]S(y_0)}{S(y_0)+I(y_0)} \le (1-\delta)\gamma_0 + \delta\gamma(y_0).$$

This implies that

$$[(1-\delta)(\beta_0-\gamma_0) + \delta(\beta(y_0)-\gamma(y_0))]S(y_0) \le [(1-\delta)\gamma_0 + \delta\gamma(y_0)]I(y_0),$$

hence,

$$I(x) \ge I(y_0) \ge \frac{[(1-\delta)(\beta_0-\gamma_0) + \delta(\beta(y_0)-\gamma(y_0))]}{(1-\delta)\gamma_0 + \delta\gamma(y_0)} S(x_0). \tag{3.16}$$

Due to (3.12) and $\beta_0 > \gamma_0$, using (3.15) and (3.16), simple analysis shows that

$$S(x) \ge S(x_0) \ge C > 0.$$

We then verify that there exists a positive C such that $I(x) \ge C$, $\forall x \in \Omega$. We argue by contradiction and suppose that there exists a sequence $\{\delta_n\}$ with $\delta_n \to \widetilde{\delta}$ such that the corresponding solution sequence $(S_n, I_n) := (S_{\delta_n} I_{\delta_n})$ of (3.7) with $\delta = \delta_n$ satisfies

$$\min_{\overline{\Omega}} I_n \to 0, \quad \text{as } n \to \infty, \tag{3.17}$$

We observe that I_n satisfies following problem:

$$\begin{cases} -d_I\Delta I_n = \dfrac{[(1-\delta_n)\beta_0 + \delta_n\beta(x)]S_nI_n}{S_n+I_n} - [(1-\delta_n)\gamma_0 + \delta_n\gamma(x)]I_n, & x \in \Omega, \\ \dfrac{\partial I_n}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.18}$$

According to the Harnack-type inequality (see, e.g., [15] or [20, Lemma 2.2]) as applied to (3.18), it is easily seen that

$$I_n \to 0 \quad \text{uniformly on } \overline{\Omega}, \quad \text{as } n \to \infty, \tag{3.19}$$

Define

$$\widetilde{I}_n := \frac{I_n}{\|I_n\|_{L^\infty(\Omega)}},$$

then $\|\widetilde{I}_n\|_{L^\infty(\Omega)} = 1$ for all $n \geq 1$, and $\widetilde{I}_n$ satisfies

$$\begin{cases} -d_I \Delta \widetilde{I}_n = \left\{ \dfrac{[(1-\delta_n)\beta_0 + \delta_n \beta(x)]\, S_n}{S_n + I_n} - [(1-\delta_n)\gamma_0 + \delta_n \gamma(x)] \right\} \widetilde{I}_n, & x \in \Omega, \\ \dfrac{\partial \widetilde{I}_n}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.20}$$

By a standard compactness argument for elliptic equations, together with the fact $S_n(x) \geq C, \forall x \in \Omega,\ n \geq 1$, after passing to a further subsequence if necessary, we may assume that

$$\widetilde{I}_n \to I^{**} \quad \text{uniformly on } \overline{\Omega}, \quad \text{as } n \to \infty,$$

where $I^{**} \in C^1(\overline{\Omega})$ with $I^{**} \geq 0$ on $\overline{\Omega}$ and $\|I^{**}\|_{L^\infty(\Omega)} = 1$. From (3.17), (3.19) and (3.20), it follows that I^{**} solves

$$\begin{cases} -d_I \Delta I^{**} = \left\{ \Big[(1-\widetilde{\delta})\beta_0 + \widetilde{\delta}\beta(x)\Big] - \Big[(1-\widetilde{\delta})\gamma_0 + \widetilde{\delta}\gamma(x)\Big] \right\} I^{**}, & x \in \Omega, \\ \dfrac{\partial I^{**}}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.21}$$

Using the Harnack-type inequality again, one has $I^{**} > 0$ on $\overline{\Omega}$. In the following, we consider separately the cases of $\widetilde{\delta} = 0$, $\widetilde{\delta} \in (0,1)$ and $\widetilde{\delta} = 1$.

If $\widetilde{\delta} = 0$, (3.21) becomes

$$\begin{cases} -d_I \Delta I^{**} = (\beta_0 - \gamma_0) I^{**}, & x \in \Omega, \\ \dfrac{\partial I^{**}}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.22}$$

Integrating the first equation of (3.22) yields

$$\int_\Omega (\beta_0 - \gamma_0) I^{**} = 0,$$

which is a contradiction with $I^{**} > 0$ and $\beta_0 > \gamma_0$.

If $\widetilde{\delta} \in (0,1)$, Lemma 3.2 tells us that $\widetilde{\lambda} < 0$, where $\widetilde{\lambda}$ is the principal eigenvalue of the problem

$$\begin{cases} -d_I \Delta \psi + \Big[(1-\widetilde{\delta})(\gamma_0 - \beta_0) + \widetilde{\delta}(\gamma - \beta)\Big]\psi = \lambda \psi, & x \in \Omega, \\ \dfrac{\partial \psi}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

However, (3.21) implies that $\widetilde{\lambda} = 0$, an obvious contradiction.

If $\widetilde{\delta} = 1$, then (3.21) becomes

$$\begin{cases} -d_I \Delta I^{**} = (\beta(x) - \gamma(x)) I^{**}, & x \in \Omega, \\ \dfrac{\partial I^{**}}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

Thus, it follows that the principal eigenvalue λ^* of (2.6) satisfies $\lambda^* = 0$. Again, this is a contradiction with our assumption that $\lambda^* < 0$ due to $\mathcal{R}_0 > 1$. Hence, (3.14) is proved.

Finally, we prove the existence of the positive solution. Let us denote

$$\Theta = \{(S, I) \in C(\overline{\Omega}) \times C(\overline{\Omega}) \ : \ C_1 < S,\ I < C_2\},$$

where C_1 and C_2 can be found to be independent of $\delta \in [0,1]$ by (3.13) and (3.14). Thus, for such C_1 and C_2, (3.7) has no positive solution $(S, I) \in \partial\Theta$. For $\delta \in [0,1]$, we also define the operator

$$\mathbf{B}(\delta, (S, I)) = (-\Delta + \mathbf{I})^{-1}\Big(h_1(\delta, (S, I)),\ h_2(\delta, (S, I))\Big),$$

where $(-\Delta + \mathbf{I})^{-1}$ stands for the inverse operator of $-\Delta + \mathbf{I}$ subject to Neumann boundary condition over $\partial\Omega$ and

$$\begin{aligned} &h_1(\delta, (S, I)) \\ &= S + d_S^{-1}\left\{[(1-\delta)a_0 + \delta a]\,S - [(1-\delta)b_0 + \delta b]\,S^2 - \frac{[(1-\delta)\beta_0 + \delta\beta]\,SI}{S+I} + [(1-\delta)\gamma_0 + \delta\gamma]\,I\right\}, \end{aligned}$$

and

$$h_2(\delta, (S, I)) = I + d_I^{-1}\left\{\frac{[(1-\delta)\beta_0 + \delta\beta]\,SI}{S+I} - [(1-\delta)\gamma_0 + \delta\gamma]\,I\right\}.$$

It is well known that $\mathbf{B}$ is a compact operator from $[0, 1] \times \Theta$ to $C(\overline{\Omega}) \times C(\overline{\Omega})$. In addition,

$$(S, I) \neq \mathbf{B}(\delta, (S, I)), \quad \forall \delta \in [0, 1] \text{ and } (S, I) \notin \partial\Theta.$$

As a result, the topological degree $\deg(\mathbf{I}-\mathbf{B}(\delta, \cdot), \Theta)$ is well defined, which is also independent of $\delta \in [0, 1]$.

In light of Theorem 2.3, we notice that

$$(\hat{S}_0, \hat{I}_0) = \left(\frac{a_0}{b_0}, \ \frac{a_0(\beta_0 - \gamma_0)}{b_0\gamma_0}\right)$$

is the unique fixed point of $\mathbf{B}(0, \cdot)$ in Θ, and thus

$$\deg(\mathbf{I} - \mathbf{B}(0, \cdot), \Theta) = \operatorname{index}(\mathbf{I} - \mathbf{B}(0, \cdot), (\hat{S}_0, \hat{I}_0)).$$

Furthermore, a straightforward calculation shows that $(\hat{S}_0, \hat{I}_0)$ is linearly stable as the unique positive constant steady state of (1.1) with (a, b, β, γ) replaced by $(a_0, b_0, \beta_0, \gamma_0)$. Hence, by the well-known Leray-Schauder degree formula (see, e.g., Theorem 2.8.1 in [17]), we get

$$\deg(\mathbf{I} - \mathbf{B}(0, \cdot), \Theta) = \operatorname{index}(\mathbf{I} - \mathbf{B}(0, \cdot), (\hat{S}_0, \hat{I}_0)) = 1.$$

Therefore, it follows that

$$\deg(\mathbf{I} - \mathbf{B}(1, \cdot), \Theta) = \deg(\mathbf{I} - \mathbf{B}(0, \cdot), \Theta) = 1,$$

which in turn implies that $\mathbf{B}(1, \cdot)$ has at least one fixed point in Θ (equivalently, (3.1) admits at least one positive solution). This finishes the proof. □

4. Asymptotic profiles of EE

In this section, we are concerned with the asymptotic behavior of the EE of (1.1) when d_S or d_I is large or small.

4.1. The case of $d_S \longrightarrow 0$

Our result of this subsection reads as follows.

Theorem 4.1. *Assume that $\mathcal{R}_0 > 1$ and*

$$\min_{x\in\overline{\Omega}}(a(x) - \beta(x)) > 0. \tag{4.1}$$

Fix $d_I > 0$, and let $d_S \to 0$, then every positive solution (S, I) of (3.1) satisfies (up to a subsequence of $d_S \to 0$)

$$(S, I) \to (S^*, I^*) \quad \text{uniformly on } \overline{\Omega},$$

where $S^*(x) = G(x, I^*(x))$ is the unique positive root of $h(\tau) = 0$ with

$$h(\tau) = -b(x)\tau^3 + [a(x) - b(x)I^*(x)]\,\tau^2 + I^*(x)\,[a(x) - \beta(x) + \gamma(x)]\,\tau + \gamma(x)\,(I^*(x))^2,$$

and I^* is a positive solution to

$$\begin{cases} -d_I\Delta I^* = \dfrac{\beta(x)G(x,I^*)I^*}{G(x,I^*)+I^*} - \gamma(x)I^*, & x\in\Omega, \\ \dfrac{\partial I^*}{\partial\nu} = 0, & x\in\partial\Omega. \end{cases} \tag{4.2}$$

Proof. We first notice that (4.1) surely implies (3.12) and hence according to Theorem 3.1, there exists at least one positive solution of (3.1). In the following, we divide our argument into three steps for the sake of clarity.

Step 1: Estimates for lower bound of S and upper bound of I. Let $S(x_0)=\min_{x\in\overline{\Omega}}S(x)$. Then it follows from the maximum principle [16, Proposition 2.2] and the first equation of (3.1) that

$$a(x_0)S(x_0)-b(x_0)S^2(x_0)-\frac{\beta(x_0)S(x_0)I(x_0)}{S(x_0)+I(x_0)}+\gamma(x_0)I(x_0)\le 0.$$

Consequently,

$$a(x_0)S(x_0)<a(x_0)S(x_0)+\gamma(x_0)I(x_0)\le b(x_0)S^2(x_0)+\beta(x_0)S(x_0),$$

which gives

$$S(x)\ge S(x_0)\ge \frac{\min_{\overline{\Omega}}(a-\beta)}{b^*}=:m>0, \tag{4.3}$$

thanks to (4.1)

As for I, observe that it solves

$$\begin{cases} \Delta I+\left[\dfrac{\beta(x)S}{d_I(S+I)}-\dfrac{\gamma(x)}{d_I}\right]I=0, & x\in\Omega, \\ \dfrac{\partial I}{\partial\nu}=0, & x\in\partial\Omega. \end{cases}$$

According to the Harnack-type inequality (see, e.g., [15] or [20, Lemma 2.2]), we get

$$\max_{\overline{\Omega}} I\le C\min_{\overline{\Omega}} I. \tag{4.4}$$

Hereafter, C represents a positive constant independent of $d_S>0$ which may vary from line to line. Thus, we obtain from (4.4) and Lemma 3.1 that

$$\max_{\overline{\Omega}} I\le C\min_{\overline{\Omega}} I\le \frac{C}{|\Omega|}\int_\Omega I\le C. \tag{4.5}$$

Step 2: Convergence of I. Recall that I satisfies

$$\begin{cases} -d_I\Delta I+\gamma(x)I=\dfrac{\beta S}{S+I}I, & x\in\Omega, \\ \dfrac{\partial I}{\partial\nu}=0, & x\in\partial\Omega, \end{cases}$$

By (4.5), we have

$$\left\|\frac{\beta S}{S+I}I\right\|_{L^p(\Omega)}\le C,\quad \forall\, p>1.$$

From the standard L^p-estimate for elliptic equations (see, e.g., [11]), it then follows that

$$\|I\|_{W^{2,p}(\Omega)}\le C\quad\text{for any given } p>1.$$

Taking p to be sufficiently large, we see from the Sobolev embedding that

$$\|I\|_{C^{1+\alpha}(\overline{\Omega})}\le C\ \text{ for some } 0<\alpha<1.$$

As a result, there exists a subsequence of $d_S \to 0$, say $d_n := d_{S,n}$, satisfying $d_n \to 0$ as $n \to \infty$, and a corresponding positive solution (S_n, I_n) of (3.1) with $d_S = d_n$, such that

$$I_n \to I^* \text{ uniformly on } \overline{\Omega}, \quad \text{as } n \to \infty,$$

where $0 \le I^* \in C^1(\overline{\Omega})$. Due to (4.4),

$$\text{either } I^* \equiv 0 \ \text{ on } \overline{\Omega} \ \text{ or } \ I^* > 0 \ \text{ on } \overline{\Omega}. \tag{4.6}$$

Suppose the former holds in (4.6); that is,

$$I_n \to 0 \ \text{ uniformly on } \overline{\Omega}, \quad \text{as } n \to \infty. \tag{4.7}$$

Then for arbitrarily small $\epsilon > 0$, we have

$$0 \le I_n(\cdot) \le \epsilon \quad \text{for all large } n.$$

This fact, together with the first equation of (1.1), implies that for all large n, S_n satisfies

$$-d_n \Delta S_n \le a(x)S_n - b(x)S_n^2 + \gamma^* \epsilon, \ x \in \Omega; \quad \frac{\partial S_n}{\partial \nu} = 0, \ x \in \partial\Omega \tag{4.8}$$

and

$$-d_n \Delta S_n \ge a(x)S_n - b(x)S_n^2 - \beta^* \epsilon, \ x \in \Omega; \quad \frac{\partial S_n}{\partial \nu} = 0, \ x \in \partial\Omega.$$

It follows from (4.8) and Lemma 5.1 that $S_n(x) \le u_n(x)$, where u_n is the unique positive solution of (5.1) with d replaced by d_n and κ replaced by $\gamma^*\epsilon$. Moreover,

$$\limsup_{n\to\infty} S_n(x) \le \lim_{n\to\infty} u_n(x) = g^\epsilon(x), \tag{4.9}$$

where $g^\epsilon(x)$ is the unique positive root of $h^\epsilon(\tau) := a(x)\tau - b(x)\tau^2 + \gamma^*\epsilon$. Elementary analysis shows that, for all small $\epsilon > 0$,

$$\frac{a(x)}{b(x)} < g^\epsilon(x) < \frac{a(x)}{b(x)} + \left(\frac{\gamma^*\epsilon}{b_*}\right)^{\frac{1}{2}}. \tag{4.10}$$

On the other hand, consider the auxiliary problem

$$-d_n \Delta v = a(x)v - b(x)v^2 - \beta^* \epsilon, \ x \in \Omega; \quad \frac{\partial v}{\partial \nu} = 0, \ x \in \partial\Omega. \tag{4.11}$$

For m defined via (4.3), Lemma 5.2 tells us that (4.11) possesses a unique positive solution larger than m, denoted by v_n. Straightforward calculations show that m is a lower solution of (4.11) for all small $\epsilon > 0$, while $S_n \ge m$ is an upper solution, thanks to (4.3). Thus, for all large n, $m \le v_n \le S_n$ on $\overline{\Omega}$. As a result,

$$\liminf_{n\to\infty} S_n(x) \ge \lim_{n\to\infty} v_n(x) = g_\epsilon(x) \ \text{ uniformly on } \overline{\Omega}, \tag{4.12}$$

with $g_\epsilon(x)$ being the larger positive root of $h_\epsilon(\tau) := a(x)\tau - b(x)\tau^2 - \beta^*\epsilon$. Furthermore, it can be easily checked that, for all small $\epsilon > 0$,

$$\frac{a(x)}{b(x)} - \left(\frac{\beta^*\epsilon}{b_*}\right)^{\frac{1}{2}} < g_\epsilon(x) < \frac{a(x)}{b(x)}. \tag{4.13}$$

In light of (4.9) and (4.12), together with (4.10) and (4.13), we obtain

$$S_n(x) \to \frac{a(x)}{b(x)} \quad \text{uniformly on } \overline{\Omega}, \quad \text{as } n \to \infty. \tag{4.14}$$

Notice that I_n verifies

$$-d_I \Delta I_n = \frac{\beta(x) S_n I_n}{S_n + I_n} - \gamma(x) I_n, \ x \in \Omega; \quad \frac{\partial I_n}{\partial \nu} = 0, \ x \in \partial\Omega. \tag{4.15}$$

Define

$$\tilde{I}_n := \frac{I_n}{\|I_n\|_{L^\infty(\Omega)}}.$$

Then $\|\tilde{I}_n\|_{L^\infty(\Omega)} = 1$ for all $n \geq 1$, and $\tilde{I}_n$ solves

$$-d_I \Delta \tilde{I}_n = \left[\frac{\beta(x) S_n}{S_n + I_n} - \gamma(x)\right] \tilde{I}_n, \ x \in \Omega; \quad \frac{\partial \tilde{I}_n}{\partial \nu} = 0, \ x \in \partial\Omega. \tag{4.16}$$

As before, by a standard compactness argument for elliptic equations, after passing to a further subsequence if necessary, we may assume that

$$\tilde{I}_n \to \tilde{I} \ \text{ in } C^1(\overline{\Omega}), \ \text{ as } n \to \infty,$$

where $0 \leq \tilde{I} \in C^1(\overline{\Omega})$ with $\|\tilde{I}\|_{L^\infty(\Omega)} = 1$. From (4.7), (4.14) and (4.16), it follows that $\tilde{I}$ satisfies

$$-d_I \Delta \tilde{I} = [\beta(x) - \gamma(x)]\, \tilde{I}, \ x \in \Omega; \quad \frac{\partial \tilde{I}}{\partial \nu} = 0, \ x \in \partial\Omega.$$

An application of the Harnack-type inequality (see, [15] or [20, Lemma 2.2]) yields $\tilde{I} > 0$ on $\overline{\Omega}$. As a result, this implies $\lambda^* = 0$, where λ^* is the principal eigenvalue of (2.6), contradicting our assumption $\mathcal{R}_0 > 1$ (note that $\mathcal{R}_0$ does not depend on d_S) and Proposition 2.1 (d). Thus, (4.7) cannot occur, and so $I^* > 0$ on $\overline{\Omega}$. That is,

$$I_n \to I^* > 0 \ \text{ uniformly on } \overline{\Omega}, \ \text{ as } n \to \infty. \tag{4.17}$$

Step 3: Convergence of S. Consider the equation satisfied by S_n:

$$\begin{cases} -d_n \Delta S_n = a(x) S_n - b(x) S_n^2 - \dfrac{\beta(x) S_n I_n}{S_n + I_n} + \gamma(x) I_n, & x \in \Omega, \\ \dfrac{\partial S_n}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{4.18}$$

It follows from (4.17) that for any small $\epsilon > 0$, we have

$$0 < I^* - \epsilon \leq I_n \leq I^* + \epsilon, \tag{4.19}$$

on $\overline{\Omega}$ for all large n. Thus,

$$\begin{aligned} & a(x) S_n - b(x) S_n^2 - \frac{\beta(x) S_n I_n}{S_n + I_n} + \gamma(x) I_n \\ & \leq a(x) S_n - b(x) S_n^2 - \frac{\beta(x) S_n (I^* - \epsilon)}{S_n + (I^* - \epsilon)} + \gamma(x)(I^* + \epsilon) \\ & = \frac{h_\epsilon(x, S_n(x))}{S_n + (I^* - \epsilon)}, \end{aligned}$$

with

$$h_\epsilon(x, \tau) := -b\tau^3 - b\,(I^* - \epsilon)\,\tau^2 + a\tau^2 + [(a - \beta)(I^* - \epsilon) + \gamma(I^* + \epsilon)]\,\tau + \gamma(I^* - \epsilon)(I^* + \epsilon).$$

To emphasize the dependence on τ, we write $h_\epsilon(\tau)$ instead of $h_\epsilon(x, \tau)$ for each fixed $x \in \Omega$. Direct calculations show that

$$\begin{aligned} & h'_\epsilon(\tau) = -3b\tau^2 - 2b(I^* - \epsilon)\tau + 2a\tau + [(a - \beta)(I^* - \epsilon) + \gamma(I^* + \epsilon)], \\ & h''_\epsilon(\tau) = -6b\tau - 2b(I^* - \epsilon) + 2a, \\ & h'''_\epsilon(\tau) = -6b. \end{aligned}$$

As $h'''_\epsilon(\tau) < 0$ for all $\tau > 0$, we have $h''_\epsilon(\tau)$ is strictly decreasing with respect to $\tau > 0$. Consequently, $h''_\epsilon(\tau)$ changes sign at most once on $(0, \infty)$. We consider two different cases.

Case 1: $h''_\epsilon(\tau)$ changes sign exactly once. This means $h''_\epsilon(0) > 0$ since $h''_\epsilon(\infty) = -\infty$. Notice $h'_\epsilon(0) > 0$ due to (4.1) and $h'_\epsilon(\infty) = -\infty$. It follows that $h'_\epsilon(\tau)$ is strictly increasing and then strictly decreasing with a positive maximum value for $\tau \in (0,\infty)$. Also observe that $h_\epsilon(0) > 0$ and $h_\epsilon(\infty) = -\infty$. We then conclude that $h_\epsilon(\tau)$ is also strictly increasing and then strictly decreasing with a positive maximum value for $\tau \in (0,\infty)$. As a result, $h_\epsilon(\tau) = 0$ has a unique positive root. Case 2: $h''_\epsilon(\tau)$ does not change sign. This means $h''_\epsilon(\tau) < 0$ for all $\tau \in (0,\infty)$. Hence $h_\epsilon(\tau)$ is strictly decreasing with respect to $\tau \in (0,\infty)$. Since $h'_\epsilon(0) > 0$ and $h'_\epsilon(\infty) = -\infty$, then $h'_\epsilon(\tau)$ changes sign exactly once (from positive to negative). Again using the fact that $h_\epsilon(0) > 0$ and $h_\epsilon(\infty) = -\infty$, we see $h_\epsilon(\tau)$ is strictly increasing and then decreasing with a positive maximum value. In either case, we can conclude that $h_\epsilon(\tau) = 0$ possesses a unique positive root, denoted by $g_\epsilon(x, I^*(x))$. Moreover, we can write $h_\epsilon(\tau) = (g_\epsilon(x, I^*(x)) - \tau)\,\tilde{h}_\epsilon(\tau)$ with $\tilde{h}_\epsilon(\tau) > 0$ for all $x \in \overline{\Omega}$ and $\tau > 0$.

Let $S_n(x_n) = \max_{\overline{\Omega}} S_n$. Then the maximum principle [16, Proposition 2.2] applied to (4.18) yields

$$b(x_n)S_n^2(x_n) - a(x_n)S_n(x_n) - \gamma(x_n)I_n(x_n) \le 0,$$

which indicates that

$$S_n(x) \le S_n(x_n) \le \frac{a(x_n) + \sqrt{a^2(x_n) + 4b(x_n)\gamma(x_n)I_n(x_n)}}{2b(x_n)} \le \frac{a^* + \sqrt{(a^*)^2 + 4b^*\gamma^*(I^*+1)}}{2b_*}, \tag{4.20}$$

for large n.

For large n, we consider the following auxiliary problem

$$-d_n\Delta w = \frac{(g_\epsilon(x, I^*(x)) - w)\tilde{h}_\epsilon(w)}{w + (I^* - \epsilon)},\ x \in \Omega; \quad \frac{\partial w}{\partial \nu} = 0,\ x \in \partial\Omega. \tag{4.21}$$

Observe that S_n is a subsolution to (4.21) and any sufficiently large positive constant C is a supersolution. Moreover, we can take C large enough so that $S_n \le C$ on $\overline{\Omega}$ thanks to (4.20). Hence, (4.21) has at least one positive solution, denoted by w_n, satisfying $S_n \le w_n \le C$ on $\overline{\Omega}$. Upon an application of the maximum principle [16, Proposition 2.2] again, one can see that any positive solution $\overline{w}_n$ of (4.21) fulfills

$$\min_{x\in\overline{\Omega}} g_\epsilon(x, I^*(x)) \le \min_{x\in\overline{\Omega}} \overline{w}_n(x) \le \overline{w}_n(x) \le \max_{x\in\overline{\Omega}} \overline{w}_n(x) \le \max_{x\in\overline{\Omega}} g_\epsilon(x, I^*(x)), \quad \forall x \in \overline{\Omega}.$$

By similar arguments to those in the proof of Lemma 5.1 (see also [10, Lemma 2.4]), together with the facts that $g_\epsilon > 0$ and $\tilde{h}_\epsilon > 0$ on $\overline{\Omega}$, we know that any positive solution $\overline{w}_n$ of (4.21) satisfies

$$\overline{w}_n \to g_\epsilon(x, I^*(x)) \ \text{ uniformly on } \overline{\Omega}, \text{ as } n \to \infty,$$

which, combined with $S_n \le w_n \le C$ on $\overline{\Omega}$, yields

$$\limsup_{n\to\infty} S_n(x) \le g_\epsilon(x, I^*(x)) \ \text{ uniformly on } \overline{\Omega}. \tag{4.22}$$

On the other hand, by (4.19), for all large n we have

$$\begin{aligned} &a(x)S_n - b(x)S_n^2 - \frac{\beta S_n I_n}{S_n + I_n} + \gamma I_n \\ &\ge a(x)S_n - b(x)S_n^2 - \frac{\beta S_n(I^*+\epsilon)}{S_n + (I^*+\epsilon)} + \gamma(I^* - \epsilon) \\ &= \frac{h^\epsilon(x, S_n(x))}{S_n + (I^*+\epsilon)}, \end{aligned}$$

with

$$h^\epsilon(x,\tau) := -b\tau^3 - b\,(I^*+\epsilon)\,\tau^2 + a\tau^2 + [(a-\beta)(I^*+\epsilon) + \gamma(I^*-\epsilon)]\,\tau + \gamma(I^*-\epsilon)(I^*+\epsilon).$$

Arguing similarly as before, we are led to

$$\liminf_{n\to\infty} S_n(x) \ge g^\epsilon(x, I^*(x)) \text{ uniformly on } \overline{\Omega}, \tag{4.23}$$

where $g^\epsilon(x, I^*(x))$ is the unique positive root of $h^\epsilon(x,\tau)=0$. As

$$\lim_{\epsilon\to 0} g_\epsilon(x, I^*(x)) = \lim_{\epsilon\to 0} g^\epsilon(x, I^*(x)) = G(x, I^*(x)),$$

it follows readily from (4.22) and (4.23) that

$$S_n(x) \to G(x, I^*(x)) \text{ uniformly on } \overline{\Omega}, \text{ as } n\to\infty.$$

Furthermore, because of (4.15), it can be easily seen that I^* satisfies (4.2). The proof is finally complete. □

4.2. The case of $d_I \to 0$

This subsection is devoted to the investigation of the asymptotic behavior of positive solutions of (3.1) with $d_S > 0$ being fixed and $d_I \to 0$. In light of Proposition 2.1(a) and (3.12), we assume that $\{\beta(x) > \gamma(x):\ x\in\overline{\Omega}\}$ is nonempty so that $\mathcal{R}_0 > 1$ for all small d_I. Our main result reads as follows.

Theorem 4.2. *Assume that the set $\{\beta(x) > \gamma(x):\ x\in\overline{\Omega}\}$ is nonempty and that (3.12) holds. Fix $d_S>0$ and let $d_I\to 0$, then every positive solution (S,I) of (3.1) satisfies*

$$(S,I)\to(S_*,I_*) \quad \textit{uniformly on } \overline{\Omega},$$

where

$$I_*(x) := \frac{(\beta(x)-\gamma(x))_+}{\gamma(x)} S_*(x),$$

and S_* is the unique positive solution of

$$\begin{cases} -d_S\Delta S_* = a(x)S_* - b(x)S_*^2 - \dfrac{\beta(x)S_*I_*}{S_*+I_*} + \gamma(x)I_*, & x\in\Omega,\\ \dfrac{\partial S_*}{\partial\nu} = 0, & x\in\partial\Omega. \end{cases}$$

Proof. According to the standard elliptic L^p theory, it follows from the S-equation and Lemma 3.1 that

$$\|S\|_{W^{2,p}(\Omega)} \le C,$$

for any $p>1$. Hereafter, C represents a positive constant independent of small $d_I>0$. Then for sufficiently large p, the Sobolev embedding theory guarantees that for some $\alpha\in(0,1)$, it holds

$$\|S\|_{C^{1+\alpha}(\overline{\Omega})} \le C. \tag{4.24}$$

Moreover, up to a subsequence of $d_I\to 0$, say $d_n := d_{I,n}\to 0$ with $d_n\to 0$ as $n\to\infty$, the corresponding positive solution sequence (S_n, I_n) of (3.1) with $d_I = d_n$ satisfies

$$S_n \to S_* \ \text{ in } C^1(\overline{\Omega}), \text{ as } n\to\infty, \tag{4.25}$$

where $0\le S_*\in C^1(\overline{\Omega})$.

Let $I(x_0) = \max_{x\in\overline{\Omega}} I(x)$. Then for all $d_I>0$, in light of the maximum principle [16, Proposition 2.2] and the I-equation, we obtain

$$\frac{\beta(x_0)S(x_0)I(x_0)}{S(x_0)+I(x_0)} - \gamma(x_0)I(x_0) \ge 0,$$

from which it follows

$$I(x)\le I(x_0) \le \frac{\beta(x_0)-\gamma(x_0)}{\gamma(x_0)}S(x_0) \le \left(\max_{\overline{\Omega}}\frac{\beta-\gamma}{\gamma}\right)\left(\max_{\overline{\Omega}} S\right) \le C, \quad \forall x\in\overline{\Omega}, \tag{4.26}$$

where we have used (4.24).

By (3.1), it can be easily seen that

$$\begin{cases} -\Delta(d_S S + d_I I) = a(x)S - b(x)S^2, & x \in \Omega, \\ \dfrac{\partial(d_S S + d_I I)}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

Denote $(d_S S + d_I I)(x_1) = \min_{x\in\overline{\Omega}}(d_S S + d_I I)(x)$. Then invoking the maximum principle [16, Proposition 2.2] again, we are led to

$$a(x_1)S(x_1) - b(x_1)S^2(x_1) \le 0 \ \text{ and so } S(x_1) \ge \frac{a_*}{b^*}.$$

Therefore,

$$(d_S S + d_I I)(x) \ge d_S S(x_1) \ge \frac{d_S a_*}{b^*}, \quad \forall x \in \overline{\Omega}. \tag{4.27}$$

By sending $n \to \infty$ in (4.27) and using (4.25) and (4.26), we conclude that $S_* > 0$ in $\overline{\Omega}$.

The rest of the argument is the same as Step 4 in the proof of [14, Theorem 5.2] and hence is omitted. □

4.3. The case of $d_S \to \infty$

When $d_S \to \infty$, we have

Theorem 4.3. *Assume that $\mathcal{R}_0 > 1$ and (3.12) holds. Fix $d_I > 0$ and let $d_S \to \infty$, then every positive solution (S, I) of (3.1), up to a subsequence of d_S, satisfies*

$$(S, I) \to (S^\infty, I^\infty) \quad \text{uniformly on } \overline{\Omega},$$

where S^∞ is a positive constant and $I^\infty > 0$ on $\overline{\Omega}$, and (S^∞, I^∞) solves

$$\begin{cases} \displaystyle\int_\Omega \left[a(x)S^\infty - b(x)\left(S^\infty\right)^2 - \frac{\beta(x)S^\infty I^\infty}{S^\infty + I^\infty} + \gamma(x)I^\infty\right] = 0, \\ -d_I \Delta I^\infty = \dfrac{\beta(x)S^\infty I^\infty}{S^\infty + I^\infty} - \gamma(x)I^\infty, & x \in \Omega, \\ \dfrac{\partial I^\infty}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{4.28}$$

Proof. We rewrite the first equation of (3.1) as

$$\begin{cases} -\Delta S = \dfrac{1}{d_S}\left[a(x)S - b(x)S^2 - \dfrac{\beta(x)SI}{S+I} + \gamma(x)I\right], & x \in \Omega, \\ \dfrac{\partial S}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{4.29}$$

In view of Lemma 3.1, by a standard compactness argument, we see that (4.24) and (4.26) remain valid for some positive constant C independent of $d_S > 1$ which may vary in different places below. Furthermore, there exists a subsequence of d_S, labeled by d_n with $d_n \to \infty$ as $n \to \infty$ such that the corresponding positive solution (S_n, I_n) of (3.1) for $d_S = d_n$ satisfies $S_n \to S^\infty \ge 0$ in $C^1(\overline{\Omega})$ as $n \to \infty$. Moreover, from (4.29), it can be easily verified that S_∞ solves

$$-\Delta S^\infty = 0, \ x \in \Omega; \quad \frac{\partial S^\infty}{\partial \nu} = 0, \ x \in \partial\Omega.$$

Clearly, $S^\infty \ge 0$ on $\overline{\Omega}$ must be a constant. Moreover, as in the proof of Theorem 3.1, (3.12) is sufficient to guarantee that S_n is bounded from below by a positive constant. Consequently, we must have $S^\infty > 0$.

Since $S_n \to S^\infty > 0$ in $C^1(\overline{\Omega})$, as before, it follows from the second equation of (3.1) that, by passing to a further subsequence if necessary,

$$I_n \to I^\infty \text{ in } C^1(\overline{\Omega}), \quad \text{as } n \to \infty,$$

with $0 \le I^\infty \in C^1(\overline{\Omega})$ fulfilling

$$-d_I \Delta I^\infty = \frac{\beta(x) S^\infty I^\infty}{S^\infty + I^\infty} - \gamma(x) I^\infty, \ x \in \Omega; \quad \frac{\partial I^\infty}{\partial \nu} = 0, \ x \in \partial\Omega.$$

Moreover, as in deriving (4.17), we can claim that $I^\infty > 0$ on $\overline{\Omega}$. The first equation of (4.28) is valid since

$$\int_\Omega \left[a(x) S_n - b(x) S_n^2 - \frac{\beta(x) S_n I_n}{S_n + I_n} + \gamma(x) I_n \right] = 0, \quad \forall n \ge 1.$$

□

4.4. The case of $d_I \to \infty$

Finally, we discuss the limiting behavior of positive solutions of (3.1) when $d_I \to \infty$. To ensure the existence of positive solutions of (3.1) for all large d_I, we need to assume that $\int_\Omega \beta > \int_\Omega \gamma$ due to Proposition 2.1.

Theorem 4.4. *Assume that* $\int_\Omega \beta > \int_\Omega \gamma$ *and* (3.12) *holds. Fix* $d_S > 0$ *and let* $d_I \to \infty$, *then every positive solution* (S, I) *of* (3.1), *up to a subsequence of* d_I, *satisfies*

$$(S, I) \to (S_\infty, I_\infty) \quad \textit{uniformly on } \overline{\Omega},$$

where I_∞ *is a positive constant and* $S_\infty > 0$ *on* $\overline{\Omega}$, *and* (S_∞, I_∞) *solves*

$$\begin{cases} -d_S \Delta S_\infty = a(x) S_\infty - b(x) S_\infty^2 - \dfrac{\beta(x) S_\infty I_\infty}{S_\infty + I_\infty} + \gamma(x) I_\infty, & x \in \Omega, \\ \dfrac{\partial S_\infty}{\partial \nu} = 0, & x \in \partial\Omega, \\ \displaystyle\int_\Omega \left[\frac{\beta(x) S_\infty}{S_\infty + I_\infty} - \gamma(x) \right] = 0. \end{cases} \tag{4.30}$$

Proof. In view of Lemma 3.1, a compactness argument as in the proof of Theorem 4.3 yields that there exists a sequence of $d_I \to \infty$, say d_n satisfying $d_n \to \infty$ as $n \to \infty$, such that the corresponding positive solution sequence (S_n, I_n) of (3.1) fulfills

$$(S_n, I_n) \to (S_\infty, I_\infty) \quad \text{in } C^1\left(\overline{\Omega}\right), \text{ as } d_I \to \infty,$$

where I_∞ is a nonnegative constant and $0 < S_\infty \in C^1\left(\overline{\Omega}\right)$, thanks to (3.12) ensuring the positive lower bound of S_n.

Now, we shall prove that I_∞ must be a positive constant and (S_∞, I_∞) solves (4.30). Suppose on the contrary that $I_\infty = 0$. Let $\tilde{I}_n = \frac{I_n}{\|I_n\|_{L^\infty(\Omega)}}$. Then $\|\tilde{I}_n\|_{L^\infty(\Omega)} = 1$ for all $n \ge 1$, and $\tilde{I}_n$ satisfies

$$-d_n \Delta \tilde{I}_n = \left[\frac{\beta(x) S_n}{S_n + I_n} - \gamma(x) \right] \tilde{I}_n, \ x \in \Omega; \quad \frac{\partial \tilde{I}_n}{\partial \nu} = 0, \ x \in \partial\Omega. \tag{4.31}$$

As before, a compactness argument yields that, after passing to a further subsequence if necessary,

$$\tilde{I}_n \to 1 \quad \text{in } C^1\left(\overline{\Omega}\right), \text{ as } n \to \infty. \tag{4.32}$$

On the other hand, from (4.31) it follows that

$$\int_\Omega \left[\frac{\beta(x)S_n}{S_n+I_n} - \gamma(x)\right]\tilde{I}_n = 0, \quad \forall n \geq 1.$$

By sending $d_n \to \infty$, this, together with (4.32) and $I_n \to 0$ while $S_n \to S_\infty > 0$ on $\overline{\Omega}$, leads to

$$\int_\Omega (\beta(x) - \gamma(x)) = 0,$$

which is a contradiction with our assumption. Therefore, I_∞ must be a positive constant. Furthermore, from the first equation of (3.1) and the fact that

$$\int_\Omega \left[\frac{\beta(x)S_nI_n}{S_n+I_n} - \gamma(x)I_n\right] = 0,$$

it is easily seen that (S_∞, I_∞) satisfies (4.30). □

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5. Appendix

Lemma 5.1. *Let $d, \kappa > 0$ be constants. Then the following problem*

$$\begin{cases} -d\Delta u = a(x)u - b(x)u^2 + \kappa, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega \end{cases} \tag{5.1}$$

admits a unique positive solution, denoted by u_d. Furthermore, $u_d(x) \to g(x)$ uniformly on $\overline{\Omega}$ as $d \to 0$, where $g(x)$ is the unique positive root of $h(\tau) := a(x)\tau - b(x)\tau^2 + \kappa = 0$.

Proof. It is easily checked that a small positive constant is a lower solution of (5.1), while a large positive constant is an upper solution. Thus, a positive solution exists according to the lower–upper solution method. Moreover, a minimal solution u_1 and a maximal solution u_2 exist. To show uniqueness, it suffices to prove $u_1 \equiv u_2$. Suppose that $u_1 \leq u_2, u_1 \not\equiv u_2$. Then multiplying the u_1-equation by u_2 and integrating by parts, we are led to

$$d\int_\Omega \nabla u_1 \cdot \nabla u_2 = \int_\Omega \left[a(x)u_1u_2 - b(x)u_1^2u_2 + \kappa u_2\right]. \tag{5.2}$$

Similarly, multiplying the u_2-equation by u_1 and integrating by parts, we have

$$d\int_\Omega \nabla u_1 \cdot \nabla u_2 = \int_\Omega \left[a(x)u_1u_2 - b(x)u_2^2u_1 + \kappa u_1\right]. \tag{5.3}$$

As a result of (5.2) and (5.3), there holds

$$\int_{\Omega} \left[b(x)u_1u_2+\kappa\right](u_1-u_2)=0.$$

This contradiction implies the uniqueness of positive solution to (5.1).

We now prove the asymptotic behavior of u_d under the extra assumption that $a, b \in C^2\left(\overline{\Omega}\right)$ (so that $g \in C^2\left(\overline{\Omega}\right)$). Denote $d(x)=\text{dist}(x,\partial\Omega)$ and choose $c>0$ such that $|\nabla g| \le c$ on $\overline{\Omega}$. Notice that since $\partial\Omega$ is smooth, then $d(x)$ is C^2 near $\partial\Omega$ and $\frac{\partial d}{\partial\nu}(x)<0$ for $x\in\partial\Omega$. We construct a pair of upper and lower solutions of (5.1) as follows.

For any given small $\epsilon>0$, define

$$\underline{u}=(1-\epsilon)g(x)-f(x)>0 \quad \text{and} \quad \overline{u}=(1+\epsilon)g(x)+f(x)\ge\underline{u},$$

with $f(x)$ fulfilling the following conditions:

(i) $f\in C^2\left(\overline{\Omega}\right)$ and $0\le f\le\epsilon$ on $\overline{\Omega}$;

(ii) $f(x)=\frac{\epsilon}{2}-Md(x)$ when x is close to $\partial\Omega$, where $M>0$ is chosen to satisfy

$$-(1\pm\epsilon)c-M\frac{\partial d}{\partial\nu}(x)\ge 0, \quad \forall x\in\partial\Omega.$$

Note that on $\partial\Omega$, it holds

$$\frac{\partial\underline{u}}{\partial\nu}=(1-\epsilon)\frac{\partial g}{\partial\nu}+M\frac{\partial d}{\partial\nu}\le(1-\epsilon)c+M\frac{\partial d}{\partial\nu}\le 0. \tag{5.4}$$

On the other hand, it is easily checked that

$$g(x)>\frac{a(x)}{2b(x)}, \quad \forall x\in\overline{\Omega}.$$

Hence, we can pick $c_1>0$ such that

$$g(x)>\frac{a(x)}{2b(x)}+c_1, \quad \forall x\in\overline{\Omega}. \tag{5.5}$$

For $x\in\Omega$, with some $\xi(x)\in[(1-\epsilon)g(x)-f(x),g(x)]$, in accordance with the definition of $g(x)$ and (5.5), we have

$$\begin{aligned}
&-d\Delta\underline{u}-a(x)\underline{u}+b(x)\underline{u}^2-\kappa\\
&=-d(1-\epsilon)\Delta g+d\Delta f-a(x)(1-\epsilon)g(x)+a(x)f(x)+b(x)\left[(1-\epsilon)g(x)-f(x)\right]^2-\kappa\\
&=-d(1-\epsilon)\Delta g+d\Delta f+a(x)\left[\epsilon g(x)+f(x)\right]+b(x)\left[(1-\epsilon)g(x)-f(x)\right]^2-b(x)g^2(x)\\
&=-d(1-\epsilon)\Delta g+d\Delta f+a(x)\left[\epsilon g(x)+f(x)\right]-2b(x)\xi(x)[\epsilon g(x)+f(x)]\\
&\le -d(1-\epsilon)\Delta g+d\Delta f+2b(x)[\epsilon g(x)+f(x)]\left\{g(x)-c_1-[(1-\epsilon)g(x)-f(x)]\right\}\\
&\le 0,
\end{aligned} \tag{5.6}$$

by first fixing ϵ small, and then $d>0$ small. Thus, (5.4), combined with (5.6), indicates that $\underline{u}$ is indeed a lower solution of (5.1) for small $d>0$.

In a similar fashion, one can show that $\overline{u}$ is an upper solution for all sufficiently small $d>0$. So it is necessary that $\underline{u}\le u_d\le\overline{u}$ on $\overline{\Omega}$. That is, for any small $\epsilon>0$, if $d>0$ is small enough, it holds

$$(1-\epsilon)g(x)-\epsilon\le u_d\le(1+\epsilon)g(x)+\epsilon \quad \text{on } \overline{\Omega}.$$

This clearly implies that $u_d(x)\to g(x)$ uniformly on $\overline{\Omega}$ as $d\to 0$.

Now for the general case $a,b\in C^{\alpha}(\overline{\Omega})$. Since $C^2(\overline{\Omega})$ is dense in $C^{\alpha}(\overline{\Omega})$, for any given small $\epsilon>0$, we can find $a_i,b_i\in C^2(\overline{\Omega})$, such that $0<a_1(x)<a(x)<a_2(x)$, $0<b_2(x)<b(x)<b_1(x)$, and $|a(x)-a_i(x)|\le\epsilon$, $|b(x)-b_i(x)|\le\epsilon$, $i=1,2$. Let u_i and g_i, respectively, be the unique positive solution of (5.1) and $h_i(\tau)=0$ with a,b replaced by a_i,b_i. Using what was just proved, it follows that $u_i\to g_i$ uniformly on

$\overline{\Omega}$ as $d \to 0$. Clearly, u_1 and u_2 form a lower–upper solution pair of (5.1). Moreover, $u_1 < u_2$ for small $d > 0$ since $g_1 < g_2$. Thus, thanks to the uniqueness of positive solution to (5.1), $u_1(x) \le u(x) \le u_2(x)$ on $\overline{\Omega}$. Consequently,

$$\limsup_{d\to 0} u(x) \le \lim_{d\to 0} u_2(x) = g_2(x),$$
$$\liminf_{d\to 0} u(x) \ge \lim_{d\to 0} u_1(x) = g_1(x)$$

uniformly on $\overline{\Omega}$. The conclusion follows readily from the arbitrariness of small $\epsilon > 0$. □

Lemma 5.2. *Let $d > 0$. Then for any constant $0 < m < \frac{a_*}{b^*}$, there exists $\kappa_0 = \kappa_0(m) > 0$ such that for any $0 < \kappa \le \kappa_0$, the problem*

$$\begin{cases} -d\Delta u = a(x)u - b(x)u^2 - \kappa, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega \end{cases} \tag{5.7}$$

admits a unique positive solution u_d larger than m. Furthermore, $u_d(x) \to g^+(x)$ uniformly on $\overline{\Omega}$ as $d \to 0$, where $g^+(x)$ is the larger positive root of $h(\tau) := a(x)\tau - b(x)\tau^2 - \kappa = 0$.

Proof. Straightforward calculations show that there exists a constant κ_1 satisfying $0 < \kappa_1 < b_* m^2$, such that m is a lower solution of (5.7) for all $0 < \kappa < \kappa_1$. Obviously, a sufficiently large positive constant is an upper solution. Thus, the lower–upper solution argument ensures that (5.7) has a minimal solution u_1 and a maximal solution u_2, with $u_1,\ u_2 \ge m$ on $\overline{\Omega}$.

To prove the uniqueness, it suffices to show $u_1 \equiv u_2$. Proceeding similarly as in the proof of Lemma 5.1, we find

$$\int_\Omega [b(x)u_1u_2 - \kappa]\,(u_1 - u_2) = 0.$$

Due to $0 < \kappa < b_* m^2$ and $u_1,\ u_2 \ge m$, this is impossible. Hence the uniqueness is established.

It is easily checked that for all $0 < \kappa < \frac{a_*^2}{4b^*}$, $h(\tau) = a(x)\tau - b(x)\tau^2 - \kappa = 0$ admits exactly two positive roots, with one smaller than $\frac{a(x)}{2b(x)}$ and one larger than $\frac{a(x)}{2b(x)}$. Let $\kappa_0 = \min\{\kappa_1,\ \frac{a_*^2}{4b^*}\} > 0$. The asymptotic behavior of u_d can be obtained in the same fashion as in the argument of Lemma 5.1, and hence we omit the details. □

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