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Note on limit cycles for m-piecewise discontinuous polynomial Liénard differential equations

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Abstract. In this paper, we study the limit cycles for m-piecewise discontinuous polynomial Liénard differential systems of degree n with $m/2$ straight lines passing through the origin whose slopes are $\tan(\alpha + 2j\pi/m)$ for $j = 0, 1, \ldots, m/2 - 1$, and prove that for any positive even number m, if $\sin(m\alpha/2) \neq 0$, then there always exists such a system possessing at least $\left[\frac{1}{2}(n - \frac{m-2}{2})\right]$ limit cycles. This result verifies a conjecture proposed by Llibre and Teixerira (Z Angew Math Phys 66:51–66, 2015).

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1. Introduction and statement of the main results

In recent years, the non-smooth differential systems have been studied extensively. They appear and play an intrinsic role in a wide range of science areas, not only in Mathematics, but also in Physics and Engineering, for instance, in control systems, mechanical systems, nonlinear oscillations, and particular electrical circuits. For more details on them, one can see [1,2,7] and the references therein. No matter what in the theories or in the applications of non-smooth differential systems, the detection of limit cycles is of fundamental importance.

A kind of typical non-smooth differential systems is the following so-called m-piecewise discontinuous Liénard polynomial differential systems of degree n,

$$\begin{cases} \dot{x} = y + \operatorname{sgn}(g_m(x,y))F(x), \\ \dot{y} = -x, \end{cases} \tag{1}$$

where $F(x)$ is a polynomial of degree n and the zero set of the function $\operatorname{sgn}(g_m(x,y))$ with positive even number m is the union of $m/2$ different straight lines passing through the origin of coordinates dividing the plane into sectors of angle $2\pi/m$. Here, $\operatorname{sgn}(z)$ denotes the sign function, i.e.,

$$\operatorname{sgn}(z) = \begin{cases} -1 & \text{if } z < 0, \\ 0 & \text{if } z = 0, \\ 1 & \text{if } z > 0. \end{cases}$$

The above systems in some sense generalize the class of Liénard differential systems to the non-smooth differential systems; this is just the reason for the name.

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In a recent paper [5], the authors studied the limit cycles of systems (1). For $m = 0, 2, 4$, they proved that there always exists a system (1) of degree n such that it has $\left[\frac{n-1}{2}\right]$, $\left[\frac{n}{2}\right]$, $\left[\frac{n-1}{2}\right]$ limit cycles, respectively, where $[\cdot]$ denotes the integer part function, and for general integers m, they presented a conjecture which can be rewrite as follows:

Conjecture. *For any even number $m \geq 6$, there always exists a system (1) of degree n such that it can have at least $\left[\frac{1}{2}(n - \frac{m-2}{2})\right]$ limit cycles.*

In this paper, we shall prove that this conjecture is correct indeed. Moreover, if we assume that, without loss of generality, the slopes of the $m/2$ straight lines of $g_m(x, y) = 0$ are $\tan(\alpha + 2j\pi/m)$ for $j = 0, 1, \ldots, m/2 - 1$, we will show that how the number of limit cycles for systems (1) of degree n depends on α, i.e., we have the following theorem:

Theorem 1.1. *For any positive even number m, if $\sin(m\alpha/2) \neq 0$, then there always exists a system (1) of degree n such that it can have at least $\left[\frac{1}{2}(n - \frac{m-2}{2})\right]$ limit cycles.*

In [5], the authors also presented some analytical results and numerical computations for small $m = 6, 8, 10$ and $\alpha = \pi/2, \pi/8, \pi/2$, respectively, but lack of rigorous proof. Their results become the particular examples of Theorem 1.1 with its proof.

2. Proof of the main result

2.1. Proof of Theorem 1.1

Proof of Theorem 1.1. For proving Theorem 1.1, it suffices to construct such a system. For this, similarly to [5], we also consider systems (1) with a small parameter ϵ,

$$\begin{cases} \dot{x} = y + \epsilon\,\mathrm{sgn}(g_m(x, y))F(x), \\ \dot{y} = -x, \end{cases} \tag{2}$$

where $F(x) = \sum_{i=0}^{n} a_i x^i,\ a_n \neq 0$. In polar coordinates above systems become

$$\begin{cases} \dot{r} = \epsilon\,\mathrm{sgn}(g_m(r\cos\theta, r\sin\theta))\cos\theta F(r\cos\theta), \\ \dot{\theta} = -1 - \epsilon\,\mathrm{sgn}(g_m(r\cos\theta, r\sin\theta))\frac{1}{r}\sin\theta F(r\cos\theta), \end{cases}$$

Taking θ as the new independent variable system, they can be written as

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}\theta} &= -\epsilon\,\mathrm{sgn}(g_m(r\cos\theta, r\sin\theta))\cos\theta F(r\cos\theta) + O\left(\epsilon^2\right), \\ &= \epsilon f(\theta, r) + \epsilon^2 f_1(\theta, r, \epsilon), \end{aligned} \tag{3}$$

where $f(\theta, r) = -\mathrm{sgn}(g_m(r\cos\theta, r\sin\theta))\cos\theta F(r\cos\theta)$.

In references [3,4], the averaging theory for studying limit cycles of discontinuous piecewise differential systems has been developed. We can apply the first-order averaging theorem(Theorem A of [3]) for discontinuous systems to systems (3). That is, if the first-order averaged function

$$f_0(r) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta, r)\mathrm{d}\theta$$

is not equal to zero identically, then the number of limit cycles of system (2) is equal to the number of the simple positive zeros of $f_0(r)$. Without loss of generality, we assume that $g_m(x, y) = 0$ consists of $m/2$ lines $\theta = \alpha + 2j\pi/m$ for $j = 0, 1, \ldots, m/2 - 1$ and $\operatorname{sgn}(g_m(r\cos\theta, r\sin\theta)) = (-1)^j$ if $\theta \in (\alpha + 2j\pi/m, \alpha + 2(j+1)\pi/m)$ for $j = 0, 1, \ldots, m-1$. Then $f_0(r)$ can be expressed as

$$\begin{aligned} f_0(r) &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta, r)\mathrm{d}\theta = \frac{1}{2\pi}\int_\alpha^{\alpha+2\pi} f(\theta, r)\,\mathrm{d}\theta \\ &= -\frac{1}{2\pi}\sum_{i=0}^{n} a_i r^i \int_\alpha^{\alpha+2\pi} \operatorname{sgn}(g_m(r\cos\theta, r\sin\theta))\cos^{i+1}\theta\,\mathrm{d}\theta \\ &= -\frac{1}{2\pi}\sum_{i=0}^{n} a_i r^i \sum_{j=0}^{m-1} \int_{\alpha+\frac{2j\pi}{m}}^{\alpha+\frac{2(j+1)\pi}{m}} (-1)^j\cos^{i+1}\theta\,\mathrm{d}\theta. \end{aligned}$$

Denoting by

$$d_i = \frac{1}{2\pi}\sum_{j=0}^{m-1} \int_{\alpha+\frac{2j\pi}{m}}^{\alpha+\frac{2(j+1)\pi}{m}} (-1)^{j+1}\cos^{i+1}\theta\,\mathrm{d}\theta, \tag{4}$$

we have $f_0(r) = \sum_{i=0}^{n} a_i d_i r^i$.

By the Descartes theorem below (more details can be seen in [5]), in order to study the simple zeros of the polynomial $f_0(r)$, we must know if the constants d_i which depend on m are zero or not. According to Propositions 5 and 6 of [5], if $m = 4k$, where k is a positive integer, then $d_i = 0$ for even number i, and the polynomial $f_0(r)$ is odd; if $m = 4k + 2$ is not a multiple of 4, then $d_i = 0$ for odd number i, and the polynomial $f_0(r)$ is even.

Descartes Theorem. *Consider the real polynomial $p(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \cdots + a_{i_r}x^{i_r}$ with $r > 1$, $0i_1 < i_2 < \cdots < i_r$ and the numbers a_{i_j} are not simultaneously zeros for $j \in 1, 2, \ldots, r$. When $a_{i_j}a_{i_{j+1}} < 0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of signs is m, then $p(x)$ has at most m positive real roots. Moreover, it is always possible to choose the coefficients of $p(x)$ in such a way that $p(x)$ has exactly $r - 1$ positive real roots.*

Finally, the problem of proving Theorem 1.1 is reduced to decide how many d_i are not zero for a given m. This can be finished easily using the following two theorems.

Theorem 2.1. *For $m = 4k + 2$, $l \in \mathbb{Z}$, then the following statements hold*:

1. *if $\sin m\alpha/2 \neq 0$, then $d_{2l} = 0$ for $0 \leq l < k$, and $d_{2l} \neq 0$ for all $l \geq k$;*
2. *if $\sin m\alpha/2 = 0$, then $d_{2l} = 0$ for all l.*

Theorem 2.2. *For $m = 4k$, $l \in \mathbb{Z}$, then the following statements hold*:

1. *if $\sin m\alpha/2 \neq 0$, then $d_{2l+1} = 0$ for $0 \leq l < k - 1$, and $d_{2l+1} \neq 0$ for all $l \geq k - 1$;*
2. *if $\sin m\alpha/2 = 0$, then $d_{2l+1} = 0$ for all l.*

□

2.2. Proof of Theorems 2.1 and 2.2

We first introduce two simple but useful lemmas. Denoting by

$$\nu = \mathrm{e}^{-\frac{\pi}{k}i} = \cos\left(\frac{\pi}{k}\right) - i\sin\left(\frac{\pi}{k}\right), \quad i = \sqrt{-1},$$

we have

Lemma 2.1. *If $j=(2p+1)k$, p is an integer, then*

$$1+\left(-\nu^j\right)+\left(-\nu^j\right)^2+\cdots+\left(-\nu^j\right)^{2k-1}=2k;$$

and if j is other integer, then

$$1+\left(-\nu^j\right)+\left(-\nu^j\right)^2+\cdots+\left(-\nu^j\right)^{2k-1}=0.$$

Proof. Note that $\nu^j=-1$ if and only if j/k is odd, i.e. $j=(2p+1)k$ with $p\in\mathbb{Z}$. Therefore

$$\sum_{m=0}^{2k-1}\nu^j=\begin{cases}2k, & j=(2p+1)k\\ \frac{1-\left(-\nu^j\right)^{2k}}{1+\nu^j}=0, & j\neq(2p+1)k\end{cases}.$$

□

Lemma 2.2. *If $\sin\omega>0$ (resp. $\sin\omega<0$), then for any natural number p, we have $\sum_{j=0}^{p}\sin((2j+1)\omega)\geq 0$ (resp. $\sum_{j=0}^{p}\sin((2j+1)\omega)\leq 0$).*

Proof. The conclusion comes from

$$\sum_{j=0}^{p}\sin((2j+1)\omega)=\frac{1}{\sin\omega}\sum_{j=0}^{p}\frac{\cos(2j\omega)-\cos((2j+2)\omega)}{2}=\frac{1-\cos((2p+2)\omega)}{2\sin\omega}.$$

□

Now we begin to prove Theorems 2.1 and 2.2.

Proof of Theorem 2.1. If $m=4k+2$, one can rewrite d_{2l} as the following

$$d_{2l}=\frac{1}{2\pi}\int\limits_{\alpha}^{\alpha+\frac{\pi}{2k+1}}\sum_{j=0}^{4k+1}(-1)^{j+1}\cos^{2l+1}\left(\theta-\frac{j\pi}{2k+1}\right)\mathrm{d}\theta.$$

Denote by

$$w=\mathrm{e}^{-\frac{\pi}{2k+1}i},\quad z=\mathrm{e}^{\theta i},\quad C_{n,k}=\binom{n}{k}=\frac{n!}{k!(n-k)!}.$$

Since

$$\cos\left(\theta-\frac{j\pi}{2k+1}\right)=\frac{zw^j+\bar{z}\bar{w}^j}{2},$$

and

$$\begin{aligned}\cos^{2l+1}\left(\theta-\frac{j\pi}{2k+1}\right)&=\left(\frac{zw^j+\bar{z}\bar{w}^j}{2}\right)^{2l+1}\\&=\frac{1}{2^{2l+1}}\sum_{i=0}^{2l+1}C_{2l+1,i}\left(zw^j\right)^i\left(\bar{z}\bar{w}^j\right)^{2l+1-i}\\&=\frac{1}{2^{2l+1}}\sum_{i=0}^{2l+1}C_{2l+1,i}z^{2i-2l-1}w^{j(2i-2l-1)},\end{aligned}$$

then by Lemma 2.1, the following equalities hold:

$$\begin{aligned}\sum_{j=0}^{4k+1}(-1)^{j+1}\cos^{2l+1}\left(\theta-\frac{j\pi}{2k+1}\right)&=-\frac{1}{2^{2l+1}}\sum_{i=0}^{2l+1}C_{2l+1,i}z^{2i-2l-1}\sum_{j=0}^{4k+1}\left(-w^{2i-2l-1}\right)^j\\&=-\frac{1}{2^{2l+1}}\sum_{0\leq i\leq 2l+1,2k+1|2i-2l-1}C_{2l+1,i}z^{2i-2l-1}(4k+2).\end{aligned}$$

Letting $2i-2l-1=(2s+1)(2k+1)$ and $h_s=(2l+1+(2s+1)(2k+1))/2\in\mathbb{Z}$, $s\in\mathbb{Z}$, we have

$$\begin{aligned}
&\sum_{j=0}^{4k+1}(-1)^{j+1}\cos^{2l+1}\left(\theta-\frac{j\pi}{2k+1}\right)\\
&\quad=-\frac{2k+1}{2^{2l}}\sum_{-(2l+1)\le(2s+1)(2k+1)\le 2l+1}C_{2l+1,h_s}z^{(2s+1)(2k+1)}\\
&\quad=-\frac{2k+1}{2^{2l}}\sum_{0\le(2s+1)(2k+1)\le 2l+1}C_{2l+1,h_s}\left(z^{(2s+1)(2k+1)}+z^{-(2s+1)(2k+1)}\right)\\
&\quad=-\frac{2(2k+1)}{2^{2l}}\sum_{0\le s\le\frac{l-k}{2k+1}}C_{2l+1,h_s}\cos\left((2s+1)(2k+1)\theta\right).
\end{aligned}$$

If $0\le l\le k-1$, then there does not exist $s\in\mathbb{Z}$ so that $-(2l+1)\le(2s+1)(2k+1)\le 2l+1$; thus,

$$\sum_{j=0}^{4k+1}(-1)^{j+1}\cos^{2l+1}\left(\theta-\frac{j\pi}{2k+1}\right)=0,$$

and this implies $d_{2l}=0$.

If $l\ge k$, setting $\omega=(2k+1)\alpha$, then

$$\begin{aligned}
d_{2l}&=\frac{1}{2\pi}\int_{\alpha}^{\alpha+\frac{\pi}{2k+1}}\sum_{j=0}^{4k+1}(-1)^{j+1}\cos^{2l+1}\left(\theta-\frac{j\pi}{2k+1}\right)\,\mathrm{d}\theta\\
&=-\frac{2k+1}{2^{2l}\pi}\sum_{0\le s\le\frac{l-k}{2k+1}}C_{2l+1,h_s}\int_{\alpha}^{\alpha+\frac{\pi}{2k+1}}\cos\left((2s+1)(2k+1)\theta\right)\,\mathrm{d}\theta\\
&=-\frac{1}{2^{2l}\pi}\sum_{0\le s\le\frac{l-k}{2k+1}}C_{2l+1,h_s}\frac{\sin\left((2s+1)(2k+1)\left(\alpha+\frac{\pi}{2k+1}\right)\right)-\sin\left((2s+1)(2k+1)\alpha\right)}{2s+1}\\
&=\frac{1}{2^{2l-1}\pi}\sum_{0\le s\le\frac{l-k}{2k+1}}\frac{C_{2l+1,h_s}}{2s+1}\sin\left((2s+1)\omega\right).
\end{aligned}\tag{5}$$

If $\sin\omega\neq 0$, without loss of generality, we can assume that $\sin\omega>0$. Noticing that $(2l+1)/2\le h_s\le 2l+1$, $C_{2l+1,h_s}/(2s+1)$ decreases along with the increase in s. Then for any natural number p such that $0\le(2p+1)(2k+1)\le 2l+1$, by Lemma 2.2, we have

$$\begin{aligned}
&\sum_{s=0}^{p}\frac{C_{2l+1,h_s}}{2s+1}\sin\left((2s+1)\omega\right)\\
&\quad=\frac{C_{2l+1,h_p}}{2p+1}\sum_{s=0}^{p}\sin\left((2s+1)\omega\right)+\left(\frac{C_{2l+1,h_{p-1}}}{2p-1}-\frac{C_{2l+1,h_p}}{2p+1}\right)\sum_{s=0}^{p-1}\sin\left((2s+1)\omega\right)\\
&\qquad+\cdots+\left(C_{2l+1,h_0}-\frac{C_{2l+1,h_1}}{3}\right)\sin\omega\\
&\quad\ge\left(C_{2l+1,h_0}-\frac{C_{2l+1,h_1}}{3}\right)\sin\omega>0.
\end{aligned}$$

Thus, $d_{2l}>0$, and the first statement holds.

If $\sin\omega=0$, then $\sin\left((2s+1)\omega\right)=0$, $d_{2l}=0$, i.e. the second statement holds. $\square$

Proof of Theorem 2.2. When $m = 4k$, we can rewrite d_{2l+1} as the following form

$$d_{2l+1} = \frac{1}{2\pi} \int_{\alpha}^{\alpha+\frac{\pi}{2k}} \sum_{j=0}^{4k-1} (-1)^{j+1} \cos^{2l+2}\left(\theta - \frac{j\pi}{2k}\right) \mathrm{d}\theta.$$

Now we take

$$w = \mathrm{e}^{-\frac{\pi}{2k}i}, \qquad z = \mathrm{e}^{\theta i}.$$

Since

$$\cos\left(\theta - \frac{j\pi}{2k}\right) = \frac{zw^j + \bar{z}\bar{w}^j}{2},$$

and

$$\begin{aligned}\cos^{2l+2}\left(\theta - \frac{j\pi}{2k}\right) &= \left(\frac{zw^j + \bar{z}\bar{w}^j}{2}\right)^{2l+2} \\ &= \frac{1}{2^{2l+2}} \sum_{i=0}^{2l+2} C_{2l+2,i} \left(zw^j\right)^i \left(\bar{z}\bar{w}^j\right)^{2l+2-i} \\ &= \frac{1}{2^{2l+2}} \sum_{i=0}^{2l+2} C_{2l+2,i} z^{2i-2l-2} w^{j(2i-2l-2)},\end{aligned}$$

then by Lemma 2.1, we obtain

$$\begin{aligned}\sum_{j=0}^{4k-1} (-1)^{j+1} \cos^{2l+2}\left(\theta - \frac{j\pi}{2k}\right) &= -\frac{1}{2^{2l+2}} \sum_{i=0}^{2l+2} C_{2l+2,i} z^{2i-2l-2} \sum_{j=0}^{4k-1} \left(-w^{2i-2l-2}\right)^j \\ &= -\frac{k}{2^{2l}} \sum_{0\le i\le 2l+2,\ \frac{i-l-1}{k} \text{is odd}} C_{2l+2,i} z^{2i-2l-2} \\ &= -\frac{k}{2^{2l}} \sum_{-(l+1)\le(2s+1)k\le l+1} C_{2l+2,h_s} \cos((2s+1)2k\theta) \\ &= -\frac{k}{2^{2l-1}} \sum_{0\le(2s+1)k\le l+1} C_{2l+2,h_s} \cos((2s+1)2k\theta),\end{aligned}$$

where $(i-l-1)/k = 2s+1$ and $h_s = (2s+1)k + l + 1$, $s \in \mathbb{Z}$.

If $0 \le l < k-1$, then there does not exist $s \in \mathbb{Z}$ so that $0 \le (2s+1)k \le l+1$; thus,

$$\sum_{j=0}^{4k-1} (-1)^{j+1} \cos^{2l+2}\left(\theta - \frac{j\pi}{2k}\right) = 0,$$

and this implies $d_{2l+1} = 0$.

If $l \geq k-1$, setting $\omega = 2k\alpha$, then we have

$$
\begin{aligned}
d_{2l+1} &= \frac{1}{2\pi} \int\limits_{\alpha}^{\alpha+\frac{\pi}{2k}} \sum_{j=0}^{4k-1} (-1)^{j+1} \cos^{2l+2}\left(\theta - \frac{j\pi}{2k}\right) \mathrm{d}\theta \\
&= -\frac{k}{2^{2l}\pi} \int\limits_{\alpha}^{\alpha+\frac{\pi}{2k}} \left(\sum_{0 \leq s \leq \frac{l-k+1}{2k}} C_{2l+2,h_s} \cos\left((2s+1)2k\theta\right) \right) \mathrm{d}\theta \\
&= \frac{1}{2^{2l}\pi} \sum_{0 \leq s \leq \frac{l-k+1}{2k}} C_{2l+2,h_s} \frac{\sin\left((2s+1)2k\alpha\right)}{2s+1} \\
&= \frac{1}{2^{2l}\pi} \sum_{0 \leq s \leq \frac{l-k+1}{2k}} \frac{C_{2l+2,h_s}}{2s+1} \sin\left((2s+1)\omega\right). \qquad (6)
\end{aligned}
$$

If $\sin\omega \neq 0$, similarly to the case $m = 4k+2$, we assume $\sin\omega > 0$. Here $C_{2l+2,h_s}/(2s+1)$ still decreases along with the increase in s. Then for any natural number p such that $0 \leq (2p+1)k \leq l+1$, by Lemma 2.2, we have

$$
\begin{aligned}
&\sum_{s=0}^{p} \frac{C_{2l+2,h_s}}{2s+1} \sin\left((2s+1)\omega\right) \\
&\quad = \frac{C_{2l+2,h_p}}{2p+1} \sum_{s=0}^{p} \sin\left((2s+1)\omega\right) + \left(\frac{C_{2l+2,h_{p-1}}}{2p-1} - \frac{C_{2l+2,h_p}}{2p+1} \right) \sum_{s=0}^{p-1} \sin\left((2s+1)\omega\right) \\
&\qquad + \cdots + \left(C_{2l+2,h_0} - \frac{C_{2l+2,h_1}}{3} \right) \sin\omega \\
&\quad \geq \left(C_{2l+2,h_0} - \frac{C_{2l+2,h_1}}{3} \right) \sin\omega > 0.
\end{aligned}
$$

Thus, $d_{2l+1} > 0$.

If $\sin\omega = 0$, then $\sin\left((2s+1)\omega\right) = 0$, $d_{2l+1} = 0$. The proof is finished. □

Remark 2.1. Noticing that the conditions imposed on $g_m(x,y)$, the plane is divided into m congruent sectors of angle $2\pi/m$ by the discontinuity set $g_m(x,y) = 0$. This symmetry of the regions is the main reason leading to the lower bound $\left[\frac{1}{2}(n - \frac{m-2}{2})\right]$ decreases with the number m when n is fixed. Indeed, in [6], the author showed that, without this symmetry, that is, if the discontinuity set $g_m(x,y) = 0$ consists of m rays starting at the origin whose slopes can be taken freely, then there exists an α-piecewise discontinuous polynomial Liénard differential system of degree n which has n limit cycles.

Remark 2.2. From the construction of system (2), it is not difficult to see that, for sufficiently small ϵ, each of the limit cycles considered above intersects the discontinuity set of system (2) only at crossing points, while for the original system (1), there may also be another type of limit cycles, which are not considered here, intersecting the sliding region of the discontinuity set. One can find the definitions of the crossing region and the sliding region in [3].

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