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Investigation on the effects of temperature dependency of material parameters on a thermoelastic loading problem

Anil Kumar and Santwana Mukhopadhyay

Abstract. The present work is concerned with the investigation of thermoelastic interactions inside a spherical shell with temperature-dependent material parameters. We employ the heat conduction model with a single delay term. The problem is studied by considering three different kinds of time-dependent temperature and stress distributions applied at the inner and outer surfaces of the shell. The problem is formulated by considering that the thermal properties vary as linear function of temperature that yield nonlinear governing equations. The problem is solved by applying Kirchhoff transformation along with integral transform technique. The numerical results of the field variables are shown in the different graphs to study the influence of temperature-dependent thermal parameters in various cases. It has been shown that the temperature-dependent effect is more prominent in case of stress distribution as compared to other fields and also the effect is significant in case of thermal shock applied at the two boundary surfaces of the spherical shell.

Mathematics Subject Classification. 35A22, 65Mxx, 74B10, 74F05.

Keywords. Three-phase-lag heat conduction, Heat conduction with a delay, Temperature dependency of material parameters, Kirchhoff transformation.

1. Introduction

Earlier in thermoelasticity theory it was assumed that the all thermal parameters are free from temperature. But at high temperature, considering many practical and theoretical results of the materials Noda [1] in 1991 reported a detail review on temperature-dependent material properties and showed that thermal conductivity of the materials decreases exponentially with temperature. In recent years, different researchers consider the generalized thermoelasticity theories by taking into account that thermal parameters vary with temperature. It must be mentioned here that earlier in 1918, considering only shear modulus depending on temperature, a thermoelastic model was solved by Suhara [2] and the effects of temperature dependency of shear modulus were investigated. Youssef et al. [3] discussed the dependency of modulus of elasticity and thermal conductivity of the material on temperature in generalized thermoelasticity theory for an unbounded medium with a spherical cavity. Othman [4–7] investigated the thermoelastic interactions in two-dimensional thermoelastic problems with temperature-dependent elastic moduli. Zenkour and Abbas et al. [8] analyzed a problem with density and thermoelastic properties depending on temperature and discussed some characteristic features of temperature-dependent properties of the materials.

The classical theory of thermoelasticity was given by Biot [9] and earlier it was widely being employed to study problems in coupled thermomechanics. This theory removes the drawbacks of uncoupled theory that mechanical disturbance does not affect the temperature field of the medium. But this thermoelasticity theory is based on parabolic-type heat conduction equation and indicates an infinite speed of thermal wave which is really not true in practical sense. Several experimental works are also carried out to verify that the classical heat conduction law is inadequate in some typical situations, specially those which involve short

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time and/or high heat flux. During last few years, various efforts are made to understand this physical drawback in classical thermoelasticity theory and subsequently some alternative theories are proposed to deal with this issue. The very first generalized thermoelastic theory was proposed by Lord and Shulman [10] in which Fourier's law of heat conduction is replaced by the Maxwell–Cattaneo law that includes one time relaxation parameter. After this generalized theory, Green and Lindsay [11] introduced an alternative thermoelasticity theory with two relaxation parameters of time that were introduced in the constitutive equation of stress and the entropy. However, Fourier's law of heat conduction remained unchanged in case of centrosymmetric properties. Experimental as well as theoretical studies verify that the effects of relaxation parameters are prominent in certain cases, although it is very small and negligible in many engineering applications. Both the above theories predict the finite speeds of propagation of heat waves. Later on, Green and Naghdi [12–14] proposed three different generalized thermoelasticity theories termed as GN-I, GN-II and GN-III theories by introducing a new term named thermal displacement gradient in the constitutive relations. Further the linear form of GN-I is equivalent to the classical Fourier's law of heat conduction.

Tzou [15] and Chandrasekharaiah [16] developed new thermoelasticity theory with two phase-lag parameters and termed it as thermoelasticity with dual-phase-lag effects. Tzou [15] replaced classical Fourier's law $\vec{q} = -K\vec{\nabla}\theta$ with $\vec{q}\,(P, t + \tau_q) = -K\vec{\nabla}\theta\,(P, t + \tau_\theta)$ where τ_q and τ_θ are two delay times. Further, Roychoudhuri [17] modified the GN-III model with the introduction of three different phase-lags. The constitutive relation of heat flux and temperature gradient is considered in the form

$$\vec{q}\,(P, t + \tau_q) = -\left[K\vec{\nabla}\theta\,(P, t + \tau_\theta) + K^*\vec{\nabla}\nu\,(P, t + \tau_\nu)\right]$$

where τ_ν is another delay time, $\vec{\nabla}\nu$ is the thermal displacement gradient with $\dot{\nu} = \theta$ and K^* is a material constant.

The thermoelasticity theories with phase-lags have drawn the attention of several researchers in order to obtain the wellposedness of solutions under these cases. Some qualitative results are reported in this direction. Quintanilla and Racke [18] discussed the stability, and Hetnarski and Ignaczak [19] show the theoretical significance of dual-phase-lag generalized thermoelasticity theories. Recently in 2011, Quintanilla [20] has reformulated the three-phase-lag model in an alternative way by defining $\tau_\nu > \tau_q = \tau_\theta$ and $\tau = \tau_q - \tau_\nu$ in the above equation. By combining this with energy equation $-\nabla\vec{q} = c\,\dot{\theta}$, he obtained the heat conduction equation with a single delay term in the form

$$c\ddot{\nu}(t) = K\triangle\theta(t) + K^*\triangle\nu(t + \tau)$$

which is termed as exact heat conduction with a delay.

He further discussed Taylor's series approximation of the heat conduction equation in a form

$$c\,\ddot{\nu} = K\triangle\theta + K^*\left(\triangle\nu + \tau\triangle\theta + \cdots + \frac{\tau^l}{l!}\triangle\theta^{(l-1)}\right)$$

From the above equation with $l = 0$ and $l = 1$, we get the known thermoelasticity theories, while for $l = 2$, a different thermoelastic theory is obtained as follows

$$\ddot{\nu} - \frac{K^*\tau^2}{2}\triangle\,\ddot{\nu} = (K + K^*\tau)\,\triangle\,\theta + K^*\,\triangle\,\nu$$

Subsequently, Leseduarte and Quintanilla [21] investigated the stability and spatial behavior of the solutions of this newly proposed model with single delay term. A Phragmen–Lindelof-type alternative is obtained, and it has been shown that the solutions either decay in an exponential way or blow up at infinity in an exponential way. The obtained results are extended to a thermoelasticity theory by considering the Taylor series approximation of the equation of heat conduction to the delay term and Phragmen–Lindelof-type alternative is obtained for the forward and backward in time equations. Kumari and Mukhopadhyay [22] made an attempt to establish some important theorems in this context. A uniqueness theorem has been established for an anisotropic body, and a variational principle as well as a reciprocity principle is established too. Recently, an uniqueness theorem and instability of solutions for this model under the relaxed assumption that the elasticity tensor can be negative have been proved by Quintanilla [23]. For the half-space, a detailed analysis of analytical and numerical results under the current theory is provided by Kumar and Mukhopadhyay [24].

In the present work, we aimed at the investigation of thermoelastic interactions in a temperature-dependent spherical shell under the recently proposed modified thermoelasticity theory by Quintanilla [20]: an exact heat conduction model with a single delay term. The thermal properties of the medium under the present thermoelasticity theory is taken as linear function of temperature. We consider the problem to be studied under three different kinds of boundary conditions. Due to the consideration of varying material properties, the governing equations reduce to nonlinear differential equations. We apply Kirchhoff transformation along with Laplace integral technique to solve the problems. Inversion of Laplace transform carried out by a numerical approach gives the final solution for different field variables inside the medium. The numerical results of the field variables are shown in different graphs to study the influence of temperature-dependent thermal parameters in the context of new model.

2. Problem formulation

We consider an isotropic elastic medium with temperature-dependent material properties and employ the thermoelasticity theory based on the heat conduction model with a delay term as recently given by Leseduarte and Quintanilla [21] to consider the thermoelastic interactions in the absence of any body forces or heat sources. The basic governing equations in usual indicial notation therefore can be written as follows:

The equation of motion:

$$\mu u_{i,jj} + (\lambda+\mu)\, u_{j,ji} - \gamma\theta_{,i} = \rho \ddot{u}_i \tag{1}$$

The equation of heat conduction:

$$\frac{\partial}{\partial t}\left(K\theta_{,i}\right)_{,i} + \left(1 + \tau\frac{\partial}{\partial t} + \frac{\tau^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(K^*\theta_{,i}\right)_{,i} = \frac{\partial}{\partial t}\left[\rho c_E\frac{\partial\theta}{\partial t} + T_0\gamma\frac{\partial e}{\partial t}\right] \tag{2}$$

The equation of stress–strain–temperature relation:

$$\sigma_{ij} = 2\mu e_{ij} + \left(\lambda e_{kk} - \gamma\theta\right)\delta_{ij} \tag{3}$$

where u_i are the components of displacement vector, t is the time, e_{ij} are the components of elastic strain tensor, $e = e_{ii}$ is the dilatation, σ_{ij} are the components of stress tensor,$\theta = T - T_0$, i.e., θ is the temperature variation above the uniform reference temperature, T_0. λ and μ are the Lamé's constants, ρ is the mass density, $\gamma = (3\lambda + 2\mu)\alpha_{\mathrm{t}}$, where α_{t} is the coefficient of linear thermal expansion, K, K^* are the thermal conductivity and conductivity rate, respectively. η is the thermal diffusivity, where $\eta = \frac{\rho c_E}{K}$ and c_E is the specific heat at constant strain. τ is the delay parameter [11].

We consider a spherical shell of inner radius a and outer radius b, initially at uniform temperature T_0. Considering the center of the shell at the origin, introducing spherical polar coordinates (r, ϑ, φ), and assuming spherical symmetry, Eqs. (1)–(3) reduce to

$$(\lambda + 2\mu)\frac{\partial e}{\partial r} - \gamma\frac{\partial\theta}{\partial r} = \rho\frac{\partial^2 u}{\partial t^2} \tag{4}$$

$$\frac{\partial}{\partial t}\left[K\nabla^2\theta + \frac{\partial K}{\partial\theta}\left(\frac{\partial\theta}{\partial r}\right)^2\right] + \left(1 + \tau\frac{\partial}{\partial t} + \frac{\tau^2}{2}\frac{\partial^2}{\partial t^2}\right)\left[K^*\nabla^2\theta + \frac{\partial K^*}{\partial\theta}\left(\frac{\partial\theta}{\partial r}\right)^2\right]$$
$$= \rho c_E\frac{\partial^2\theta}{\partial t^2} + \rho\frac{\partial c_E}{\partial\theta}\left(\frac{\partial\theta}{\partial t}\right)^2 + \gamma T_0\frac{\partial^2 e}{\partial t^2} \tag{5}$$

$$\sigma_{rr} = 2\mu\frac{\partial u}{\partial r} + \lambda e - \gamma\theta \tag{6}$$

$$\sigma_{\phi\phi} = \sigma_{\nu\nu} = 2\mu\frac{u}{r} + \lambda e - \gamma\theta \tag{7}$$

where u is the single nonzero component of displacement vector for the present problem.

We assume that material properties like, the thermal conductivity, K, and conductivity rate, K^*, vary with the temperature and assume that they are varying linearly in the form as

$$K(\theta) = K_0(1 + K_1\theta) \tag{8}$$

$$K^*(\theta) = K_0^*(1 + K_1\theta) \tag{9}$$

where K_1 is a constant and it is zero at reference temperature. K_0 and K_0^* are the thermal conductivity and thermal conductivity rate at reference temperature, T_0, respectively. For the simplicity of the problem, specific heat, c_E, and other material parameters are assumed to be independent of temperature.

In view of Eqs. (8, 9), we find that Eq. (5) is nonlinear, and therefore to tackle the nonlinearity we consider a new function Φ that is expressed in terms of temperature, θ with Kirchhoff transformation as

$$\Phi = \frac{1}{K_0}\int_0^\theta K(p)\mathrm{d}p = \frac{1}{K_0^*}\int_0^\theta K^*(p)\mathrm{d}p = \theta + \frac{1}{2}K_1\theta^2 \tag{10}$$

Hence, by using Eqs. (8)–(10) and the fact that $\mid \theta/T_0 \mid << 1$, Eqs. (4)–(7), respectively, reduce to

$$(\lambda + 2\mu)\frac{\partial e}{\partial r} - \gamma\frac{\partial \Phi}{\partial r} = \rho\frac{\partial^2 u}{\partial t^2} \tag{11}$$

$$K_0\nabla^2\dot{\Phi} + K_0^*\left(1 + \tau\frac{\partial}{\partial t} + \frac{\tau^2}{2}\frac{\partial^2}{\partial t^2}\right)\nabla^2\Phi = \eta K_0\frac{\partial^2\Phi}{\partial t^2} + \gamma T_0\frac{\partial^2 e}{\partial t^2} \tag{12}$$

$$\sigma_{rr} = 2\mu\frac{\partial u}{\partial r} + \lambda e - \gamma\Phi \tag{13}$$

$$\sigma_{\phi\phi} = \sigma_{\nu\nu} = 2\mu\frac{u}{r} + \lambda e - \gamma\Phi \tag{14}$$

Now, we use the following symbols and notations to make Eqs. (11)–(14) dimensionless:

$r' = c_0\eta r$, $u' = c_0\eta u$, $t' = c_0^2\eta t$, $\tau' = c_0^2\eta\tau$, $\Phi' = \frac{\Phi}{T_0}$, $e' = e$, $\sigma'_{ij} = \frac{\sigma_{ij}}{(\lambda_0+2\mu_0)}$, $c_0^2 = \frac{(\lambda+2\mu)}{\rho}$, $K^* = \frac{c_E(\lambda+2\mu)}{4}$, $a_0 = \frac{K_0^*}{K_0c_0^2\eta}$, $a_1 = \frac{\gamma T_0}{(\lambda+2\mu)}$, $a_2 = \frac{\gamma}{K_0\eta}$, $\beta^2 = \frac{\lambda}{(\lambda+2\mu)}$.

Therefore, after dropping the primes for clarity, Eqs. (11)–(14) change to their dimensionless forms as follows:

$$\frac{\partial e}{\partial r} - a_1\frac{\partial\Phi}{\partial r} = \frac{\partial^2 u}{\partial t^2} \tag{15}$$

$$\nabla^2\dot{\Phi} + a_0\left(1 + \tau\frac{\partial}{\partial t} + \frac{\tau^2}{2}\frac{\partial^2}{\partial t^2}\right)\nabla^2\Phi = \frac{\partial^2\Phi}{\partial t^2} + a_2\frac{\partial^2 e}{\partial t^2} \tag{16}$$

$$\sigma_{rr} = \left(1 - \beta^2\right)\frac{\partial u}{\partial r} + \beta^2 e - a_1\Phi \tag{17}$$

$$\sigma_{\phi\phi} = \sigma_{\nu\nu} = \left(1 - \beta^2\right)\frac{u}{r} + \beta^2 e - a_1\Phi \tag{18}$$

3. Solution of the problem

Applying Laplace transform to Eqs. (15)–(18) with homogeneous initial conditions, we obtain

$$\frac{\partial \bar{e}}{\partial r} - a_1 \frac{\partial \bar{\Phi}}{\partial r} = s^2 \bar{u} \tag{19}$$

$$\nabla^2 \bar{\Phi} = \frac{2s^2}{b_0(s)} \bar{\Phi} + \frac{2a_2 s^2}{b_0(s)} \bar{e} \tag{20}$$

$$\bar{\sigma}_{rr} = \left(1 - \beta^2\right) \frac{\partial \bar{u}}{\partial r} + \beta^2 \bar{e} - a_1 \bar{\Phi} \tag{21}$$

$$\bar{\sigma}_{\phi\phi} = \bar{\sigma}_{\nu\nu} = \left(1 - \beta^2\right) \frac{\bar{u}}{r} + \beta^2 \bar{e} - a_1 \bar{\Phi} \tag{22}$$

where $b_0(s) = a_0 \tau^2 s^2 + 2\left(1 + a_0 \tau\right) s + 2a_0$.
Now, taking divergence of Eq. (19), we get

$$\nabla^2 \bar{e} - a_1 \nabla^2 \bar{\Phi} = s^2 \bar{e} \tag{23}$$

We employ Eqs. (20) and (23) to get

$$\nabla^2 \bar{e} = \frac{2a_1 s^2}{b_0(s)} \bar{\Phi} + \left(\frac{2\epsilon s^2}{b_0(s)} + s^2\right) \bar{e} \tag{24}$$

where $\epsilon = a_1 a_2$. Applying ∇^2 operator on Eq. (24) , we get

$$\nabla^4 \bar{e} = \frac{2a_1 s^2}{b_0(s)} \nabla^2 \bar{\Phi} + \left(\frac{2\epsilon s^2}{b_0(s)} + s^2\right) \nabla^2 \bar{e} \tag{25}$$

With the help of Eqs. (23) and (25), we find

$$\left[\nabla^4 - b_1(s) \nabla^2 + b_2(s)\right] \bar{e} = 0 \tag{26}$$

where $b_1(s) = \frac{2s^2}{b_0(s)} + \frac{2\epsilon s^2}{b_0(s)} + s^2$ and $b_2(s) = \frac{2s^4}{b_0(s)}$.
Applying $\left(\nabla^2 - s^2\right)$ operator on Eq. (20) and using Eq. (23), we get

$$\left[\nabla^4 - b_1(s) \nabla^2 + b_2(s)\right] \bar{\Phi} = 0 \tag{27}$$

Now, Eqs. (26) and (27) can be rewritten as

$$\left(\nabla^2 - n_1^2\right)\left(\nabla^2 - n_2^2\right)\left(\bar{\Phi}, \bar{e}\right) = 0 \tag{28}$$

where n_1^2 and n_2^2 satisfy the equation

$$n^4 - b_1(s) n^2 + b_2(s) = 0 \tag{29}$$

Clearly, (28) represents the modified spherical Bessel equations. Hence, the general solution of Eq. (28) can be obtained as

$$\bar{\Phi} = \frac{1}{\sqrt{r}} \sum_{i=1}^{2} \left[A_i I_{1/2}\left(n_i r\right) + B_i K_{1/2}\left(n_i r\right)\right] \tag{30}$$

$$\bar{e} = \frac{1}{\sqrt{r}} \sum_{i=1}^{2} \left[C_i I_{1/2}\left(n_i r\right) + D_i K_{1/2}\left(n_i r\right)\right] \tag{31}$$

where A_i, B_i, C_i and D_i are arbitrary constants and $I_\alpha(r)$, $K_\alpha(r)$ are the representations of modified Bessel functions of order α of the first and second kinds, respectively.

Using Eqs. (23), (30) and (31) , we get

$$A_i = f_i C_i \text{ and } B_i = f_i D_i. \tag{32}$$

where $f_i = \frac{n_i^2 - s^2}{a_1 n_i^2}$, $i = 1, 2$.

With the help of Eqs. (19), (21), (22), (30) and (31), we get the solution of other physical variables in Laplace transform domain as follows:

$$\bar{u} = \frac{1}{\sqrt{r}} \sum_{i=1}^{2} \left[\frac{C_i}{n_i} I_{3/2}(n_i r) - \frac{D_i}{n_i} K_{3/2}(n_i r) \right] \tag{33}$$

$$\bar{\sigma}_{rr} = \frac{1}{\sqrt{r}} \sum_{i=1}^{2} \left[C_i \left\{ \frac{s^2}{n_i^2} I_{1/2}(n_i r) - \frac{2(1-\beta^2)}{(n_i r)} I_{3/2}(n_i r) \right\} + D_i \left\{ \frac{s^2}{n_i^2} K_{1/2}(n_i r) - \frac{2(1-\beta^2)}{(n_i r)} K_{3/2}(n_i r) \right\} \right] \tag{34}$$

$$\bar{\sigma}_{\phi\phi} = \bar{\sigma}_{\nu\nu} = \frac{1}{\sqrt{r}} \sum_{i=1}^{2} \left[C_i \left\{ \left(\beta^2 - \frac{n_i^2 - s^2}{n_i^2} \right) I_{1/2}(n_i r) + \frac{(1-\beta^2)}{(n_i r)} I_{3/2}(n_i r) \right\} \right] + \frac{1}{\sqrt{r}} \sum_{i=1}^{2} \left[D_i \left\{ \left(\beta^2 - \frac{n_i^2 - s^2}{n_i^2} \right) K_{1/2}(n_i r) - \frac{(1-\beta^2)}{(n_i r)} K_{3/2}(n_i r) \right\} \right] \tag{35}$$

4. Applications of the problem

Case-I: Unit step increase in temperature and zero stress on the boundary of an elastic spherical shell

We consider the thermoelastic spherical shell with initial conditions as homogeneous, and it is assumed that the inner and outer boundaries $r = a$ and $r = b$ of the spherical shell are traction free and are subjected to a unit step increase in temperature. Therefore, the boundary conditions in the dimensionless forms can be written as:

$$\theta(r,t) = \theta_1^* H(t) \text{ and } \sigma(r,t) = 0 \text{ at } r = a \tag{36}$$

$$\theta(r,t) = \theta_2^* H(t) \text{ and } \sigma(r,t) = 0 \text{ at } r = b \tag{37}$$

where θ_1^* and θ_2^* are two constant temperatures and $H(t)$ is the Heaviside unit step function.

Therefore, using Eq. (10) and applying Laplace transform to the boundary conditions given by (36) and (37), we find that

$$\bar{\Phi}(a,s) = \frac{\theta_1^*}{s} \left(1 + \frac{1}{2} K_1 \theta_1^* \right), \quad \bar{\Phi}(b,s) = \frac{\theta_2^*}{s} \left(1 + \frac{1}{2} K_1 \theta_2^* \right), \quad \bar{\sigma}(a,s) = 0 = \bar{\sigma}(b,s). \tag{38}$$

Case-II: Exponential variation in temperature and zero stress of the boundary of an elastic spherical shell

It is assumed that both the inner boundary $r = a$ and outer boundary $r = b$ of the spherical shell are traction free and the inner boundary is subjected to an exponential variation in temperature, whereas the outer boundary is maintained as insulated as follows:

$$\theta(r,t)=\frac{t^2}{t_s}\mathrm{e}^{-t/t_s},\ t>0 \text{ and } \sigma(r,t)=0 \text{ at } r=a \tag{39}$$

$$\frac{\partial\theta(r,t)}{\partial r}=0 \text{ and } \sigma(r,t)=0 \text{ at } r=b \tag{40}$$

where t_s is a constant parameter to control the nature of temperature prescribed on the inner boundary. Hence, using Eqs. (10), (39) and (40), we get

$$\bar{\Phi}(a,s)=\frac{2t_s}{(1+st_s)^3}+\frac{3K_1t_s}{4\,(2+st_s)^5},\,\bar{\Phi}(b,s)=0,\bar{\sigma}(a,s)=0=\bar{\sigma}(b,s). \tag{41}$$

Case-III: Sinusoidal varying temperature and zero displacement at the boundary of an elastic spherical shell

Now, we assume that the inner boundary $r=a$ of the spherical shell is rigidly fixed and is subjected to a sinusoidal variation in temperature as follows:

$$\theta(r,t)=\begin{cases}\theta_0\sin\left(\frac{\pi t}{t_0}\right), & 0<t<t_0\\ 0, & \text{otherwise}\end{cases},\quad t>0 \text{ and } u(r,t)=0,\text{at } r=a \tag{42}$$

The outer boundary is also kept rigidly fixed and insulated, i.e.,

$$\frac{\partial\theta(r,t)}{\partial r}=0 \text{ and } u(r,t)=0 \text{ at } r=b \tag{43}$$

where t_0 is a constant that controls the range of temperature to be positive on the inner boundary. Therefore, we get in this case

$$\bar{\Phi}(a,s)=\frac{\theta_0\pi t_0}{(\pi^2+s^2t_0^2)}\left(1+\mathrm{e}^{-st_0}\right)+\frac{K_1\theta_0^2\pi^2}{s\,(4\pi^2+s^2t_0^2)}\left(1-\mathrm{e}^{-st_0}\right),\bar{\Phi}(b,s)=0, \tag{44}$$

$$\bar{u}(a,s)=0=\bar{u}(b,s). \tag{45}$$

Now, for the Case-I, from Eqs. (30), (34) and (38) we obtain a linear system of four equations in four unknowns as given by

$$\sum_{i=1}^{2}\left[f_iI_{1/2}(n_ia)\,C_i+f_iK_{1/2}(n_ia)\,D_i\right]=\frac{\sqrt{a}\theta_1^*}{s}\left(1+\frac{1}{2}K_1\theta_1^*\right) \tag{46}$$

$$\sum_{i=1}^{2}\left[f_iI_{1/2}(n_ib)\,C_i+f_iK_{1/2}(n_ib)\,D_i\right]=\frac{\sqrt{b}\theta_2^*}{s}\left(1+\frac{1}{2}K_1\theta_2^*\right) \tag{47}$$

$$\sum_{i=1}^{2}\left\{\left[\frac{s^2}{n_i^2}aI_{1/2}(n_ia)-\frac{2\left(1-\beta^2\right)}{n_i}I_{3/2}(n_ia)\right]C_i+\left[\frac{s^2}{n_i^2}aK_{1/2}(n_ia)+\frac{2\left(1-\beta^2\right)}{n_i}K_{3/2}(n_ia)\right]D_i\right\}=0 \tag{48}$$

$$\sum_{i=1}^{2}\left\{\left[\frac{s^2}{n_i^2}bI_{1/2}(n_ib)-\frac{2\left(1-\beta^2\right)}{n_i}I_{3/2}(n_ib)\right]C_i+\left[\frac{s^2}{n_i^2}bK_{1/2}(n_ib)+\frac{2\left(1-\beta^2\right)}{n_i}K_{3/2}(n_ib)\right]D_i\right\}=0 \tag{49}$$

After solving Eqs. (46)–(49), we can find the unknowns C_i and D_i, hence the A_i and $B_i,\ i=1,2$ from Eq. (32) and this completes the solution of the present problem in Case-I in Laplace transform domain. We can obtain the solution for temperature θ by using Eqs. (10) and (30) in the Laplace transform domain. The solutions in Laplace transform domain for the cases II and III can also be obtained in the similar way.

5. Numerical results and discussion

The solution in the physical domain can be derived by inverting the solutions obtained in the Laplace transform domain as found out in the previous section. However, it is difficult to find the solutions in space–time, (r, t) domain analytically since the solutions of the physical field variables $\bar{\theta}$, $\bar{u}$, $\bar{\sigma}_{rr}$ and $\bar{\sigma}_{\phi\phi}$ have the complicated expressions in Laplace transform parameter s. Therefore, we find the Laplace inversion for the physical variables temperature, displacement, radial stress and shear stress with the help of MATLAB software and by employing a suitable numerical method of Laplace inversion. We employ here the method proposed by Graver–Stehfest et al. [25,26] in which if $\overline{f(s)}$ is the Laplace transform of the function $f(t)$, then

$$f(t) = \frac{\ln(2)}{2} \sum_{k=1}^{N} V_k \overline{f}\left(k \frac{\ln(2)}{t}\right) \tag{50}$$

where N is the suitable positive integer and V_k is given by

$$V_k = (-1)^{(k+N/2)} \sum_{j=[(k+1)/2]}^{\min(k,N/2)} \frac{j^{\frac{N}{2}}\,(2j)!}{\left(\frac{N}{2}-j\right)!\; j!\; (j-1)!\,(k-j)!\,(2j-k)!} \tag{51}$$

We assume that the spherical shell is made of copper material and the physical data points for which are taken as below [27].

$$\lambda = 7.76 \times 10^{10}\,\mathrm{N\,m^{-2}},\ \mu = 3.86 \times 10^{10}\,\mathrm{N\,m^{-2}},\ \alpha_{\mathrm{t}} = 1.78 \times 10^{-5}\,\mathrm{K^{-1}},\ \eta = 8886.73\,\mathrm{s\,m^{-2}},$$
$$c_E = 383.1\,\mathrm{J\,Kg^{-1}\,K^{-1}},\ \rho = 8954\,\mathrm{Kg\,m^{-3}},\ T_0 = 293\,\mathrm{K}.$$

We assume the following dimensionless values of the constants:

$$\tau = 0.01, \quad t_s = 0.2, \quad t_0 = 1.0, \quad \theta_0 = 1, \quad \theta_1^* = 1, \quad \theta_2^* = 1.$$

To analyze the numerically computed solutions for nondimensional temperature, displacement, radial stress and tangential stress in space-time domain inside the spherical shell, the results under three different cases are displayed in Figs. 1a–d to 3a–d. In each figure, we plotted the graphs for the fields at three different times, $t = 0.30$, $t = 0.35$, $t = 0.40$ and for three different values of the coefficient of temperature-dependent effect, $K_1(0.0, -0.3, -0.5)$. Specially, we aim to understand the effect of temperature dependency on the solutions at various times of interaction. The specific features related to the effect of temperature dependency under various cases of prescribed boundary conditions arising out from our investigation are highlighted as follows:

Case-I:

In this case, we find the effect of temperature-dependent material properties on the distributions of various fields inside the spherical shell when the inner and outer surfaces of the shell are subjected to thermal shock. The variations of displacement, temperature, radial stress and circumferential stress are shown in Fig. 1a–d, respectively. Figure 1a shows that the variation in displacement is prominently affected only near the boundaries and through the middle region of the shell, the effect of time- and temperature-dependent property on displacement is negligible. The amplitude of displacement u increases with time t. It is further evident that at higher time, the absolute value of displacement decreases with larger numerical value of K_1 and for smaller time, the dependency of thermal parameters is negligible for displacement u.

Figure 1b shows the variation of temperature, θ in spherical shell for Case-I. It is clear from Fig. 1b that for a fixed time, the temperature starts increasing from inner boundary of the shell and after getting maximum value starts decreasing till the outer boundary of the shell. In this case, the temperature field is more sensitive to the temperature-dependent material properties. The temperature increases with the

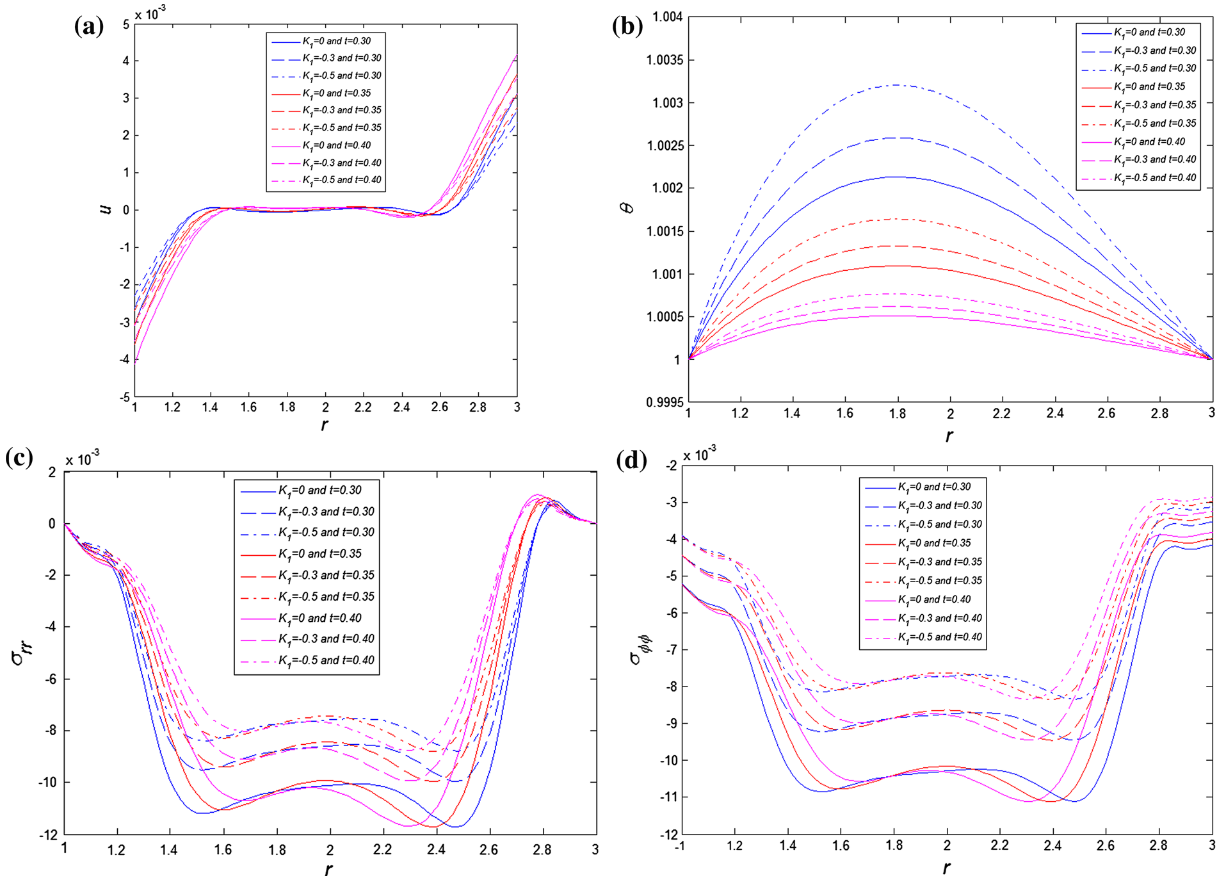

Fig. 1. **a** Variation of displacement, u versus r for the Case-I. **b** Variation of temperature, θ versus r for the Case-I. **c** Variation of radial stress, σ_{rr} versus r for the Case-I. **d** Variation of tangential stress, $\sigma_{\phi\phi}$ versus r for the Case-I

increase of the temperature-dependent coefficient K_1. It can also be seen from Fig. 1b that temperature distribution is in agreement with the boundary conditions given for the problem.

Radial stress σ_{rr} and shear stress $\sigma_{\phi\phi}$ are shown in Fig. 1c, d, respectively, and it is observed that both the stress components show significant variations near the boundaries of the shell and in the middle region of the shell. The stresses increase with higher negative value of K_1 for all time. One important fact is observed that the effect of temperature dependency of stresses is independent of time, implying that at any time, a similar effect of temperature dependency is observed for the stress components in Case-I.

Case-II:

Figure 2a–d displays the variation of field variables for the exponentially varying temperature on the inner boundary. It is observed that there is no adequate effect of temperature-dependent material properties on any physical quantities excluding the temperature. At any time, the same effect of temperature dependency is noted for all the field variables. Figure 2a shows the distribution of displacement u. It can be seen that the effective value of displacement u increases with increase in time. Displacement tends to zero through the radial distance r for all times. Figure 2b shows the variation of temperature with radial distance, and it can be seen that values of temperature increase with respect to the time and K_1 both.

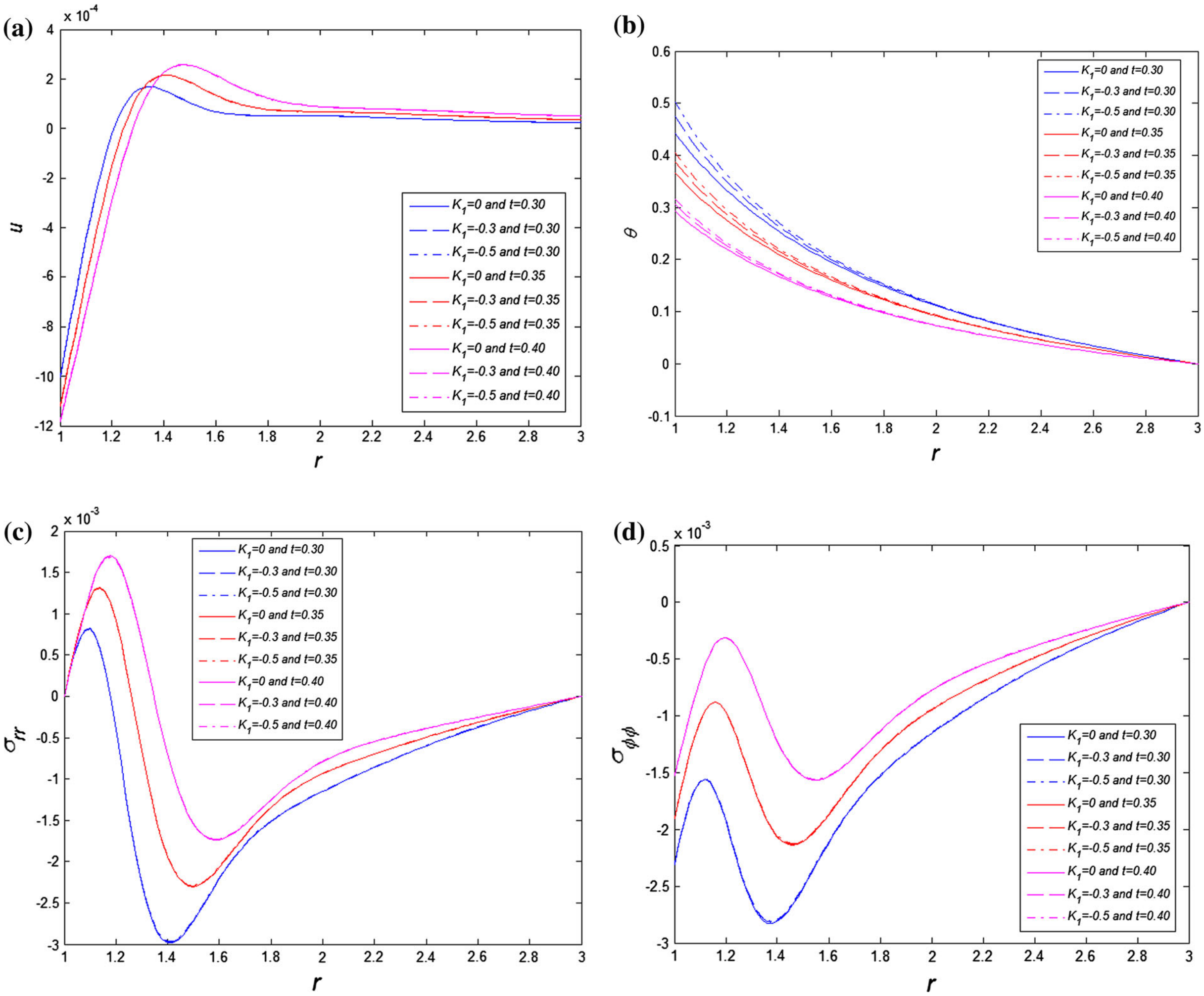

FIG. 2. **a** Variation of displacement, u versus r for the Case-II. **b** Variation of temperature, θ versus r for the Case-II. **c** Variation of radial stress, σ_{rr} versus r for the Case-II. **d** Variation of tangential stress, $\sigma_{\phi\phi}$ versus r for the Case-II

Figure 2c,d measures the variation of stress components, and it can be seen that radial stress σ_{rr} satisfies the boundary condition. It could be observed that the effective regions of stress components increase with increase in time t.

Case-III:

Distributions of displacement, temperature and stress components for the current generalized thermoelasticity in Case-III are displayed in Fig. 3a–d, respectively. It is observed that like Case-I, the influence of temperature-dependent properties is prominent in this case. From Fig. 3a, it is clear that the displacement is in agreement with the boundary conditions. The displacement, u, is positive throughout the distance r and gets a local maximum value within middle region of the shell. The effect of temperature-dependent property is more prominent at higher times. The displacement increases with time implying that the

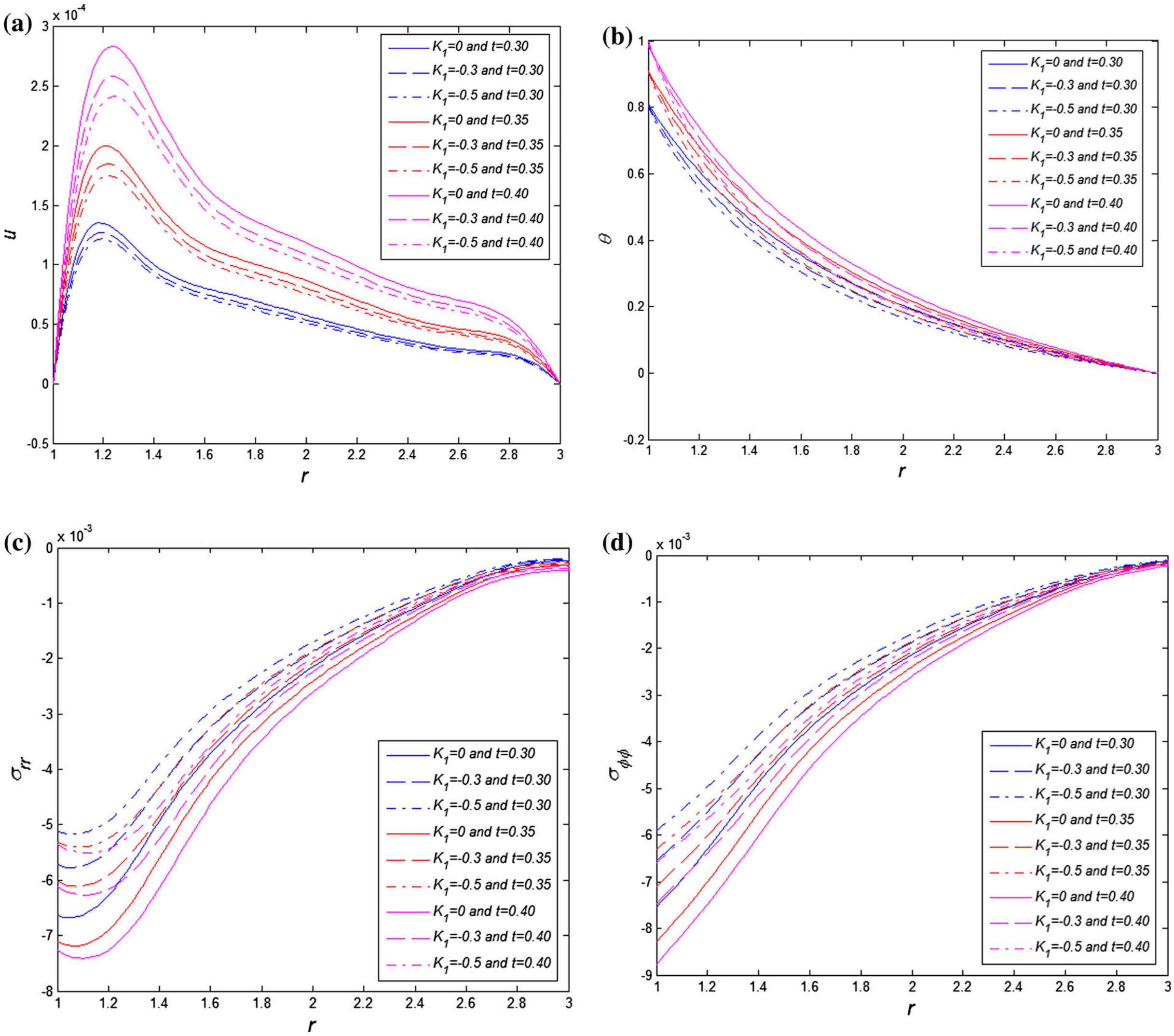

Fig. 3. **a** Variation of displacement, u versus r for the Case-III. **b** Variation of temperature, θ versus r for the Case-III. **c** Variation of radial stress, σ_{rr} versus r for the Case-III. **d** Variation of tangential stress, $\sigma_{\phi\phi}$ versus r for the Case-III

region of influence for u increases with the time. However, it decreases with larger numerical value of K_1.

Figure 3b shows the variation for temperature, θ. It is depicted in the graph that the temperature is influenced significantly by the temperature-dependent property of the material. The temperature, θ, increases with time and also with the increase of the coefficient K_1. The variation of stress components in Case-III is shown in Fig. 3c,d, respectively. The stresses are compressive in nature throughout region of the medium, and the influence of temperature-dependent properties on stress distributions is much prominent in this case. This effect on stresses is higher near the inner boundary of the shell, and it approaches to zero as $r \to b$, where b is the outer radius of the spherical shell.

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Anil Kumar and Santwana Mukhopadhyay
Department of Mathematical Sciences
Indian Institute of Technology (BHU)
Varanasi 221005
India
e-mail: anilkumar12891@gmail.com

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