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Well-posedness and asymptotic behavior of Timoshenko beam system with dynamic boundary dissipative feedback of fractional derivative type

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Abstract. We consider the Timoshenko beam system with two dynamic control boundary conditions of fractional derivative type. We show that the system is not uniformly stable by a spectrum method and we establish the polynomial stability using the semigroup theory of linear operators and a result obtained by Borichev and Tomilov.

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1. Introduction

In this paper we investigate the existence and decay properties of solutions for the initial boundary value problem of the linear Timoshenko beam system of the type

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0 & \text{in} \quad (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) = 0 & \text{in} \quad (0,L)\times(0,+\infty), \end{cases} \tag{P}$$

where $(x,t)\in(0,L)\times(0,+\infty)$. This system is subject to the boundary conditions

$$\begin{array}{ll} \varphi(0,t)=0, \quad \psi(0,t)=0, & \text{in} \quad (0,+\infty), \\ m_1\varphi_{tt}(L,t)+K(\varphi_x+\psi)(L,t) = -\gamma_1\partial_t^{\alpha,\eta}\varphi(L,t) & \text{in} \quad (0,+\infty), \\ m_2\psi_{tt}(L,t)+b\psi_x(L,t) = -\gamma_2\partial_t^{\alpha,\eta}\psi(L,t) & \text{in} \quad (0,+\infty), \end{array}$$

where $\gamma_i>0, i=1,2$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order $\alpha, 0<\alpha<1$, with respect to the time variable. It is defined as follows

$$\partial_t^{\alpha,\eta}w(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\mathrm{e}^{-\eta(t-s)}\frac{\mathrm{d}w}{\mathrm{d}s}(s)\,\mathrm{d}s, \quad \eta\geq 0.$$

In other words, we investigate two dissipative effects at the boundary. The system is finally completed with initial conditions

$$\begin{cases} \varphi(x,0)=\varphi_0(x), \quad \varphi_t(x,0)=\varphi_1(x), \quad \psi(x,0)=\psi_0(x), \\ \psi_t(x,0)=\psi_1(x), \quad x\in(0,L), \end{cases}$$

where the initial data $(\varphi_0,\varphi_1,\psi_0,\psi_1)$ belong to a suitable function space.

A simple model describing the transverse vibration of a beam, which was developed in [23,24], is given by a system of coupled hyperbolic equations of the form

$$\begin{cases} \rho u_{tt}(x,t) = (K(u_x-\phi))_x & \text{in} \quad (0,L)\times(0,+\infty), \\ \tilde{\rho}\phi_{tt}(x,t) = (EI\psi_x)_x + K(u_x-\phi) & \text{in} \quad (0,L)\times(0,+\infty), \end{cases}$$

where t denotes the time variable, x is the space variable along the beam of length L, in its equilibrium configuration, u is the transverse displacement of the beam, and ϕ is the rotation angle of the filament of the beam. The coefficients $\rho, \tilde{\rho}, E, I$, and K are, respectively, the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus.

There are a number of publications concerning the stabilization of Timoshenko system with different kinds of damping (see [1,8–10,16,17,19,22]). Raposo et al. [22] proved the exponential decay of the solution for the following linear system of Timoshenko-type beam equations with linear frictional dissipative terms:

$$\begin{aligned} &\rho_1\varphi_{tt} - K(\varphi_x + \psi)_x + \mu_1\varphi_t = 0 \\ &\rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \widetilde{\mu_1}\psi_t = 0. \end{aligned}$$

Messaoudi and Mustafa [16] (see also [19]) considered the stabilization for the following Timoshenko system with nonlinear internal feedbacks:

$$\begin{aligned} &\rho_1\varphi_{tt} - K(\varphi_x + \psi)_x + g_1(\psi_t) = 0 \\ &\rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + g_2(\psi_t) = 0. \end{aligned}$$

Recently, Park and Kang [19] considered the stabilization of the Timoshenko system with weakly nonlinear internal feedbacks.

Kim and Renardy [9] considered (P) together with two boundary controls of the form

$$\begin{aligned} &K(\varphi_x + \psi)(L,t) = -\gamma_1\partial_t\varphi(L,t) && \text{in} \quad (0,+\infty) \\ &b\psi_x(L,t) = -\gamma_2\partial_t\psi(L,t) && \text{in} \quad (0,+\infty) \end{aligned}$$

and used the multiplier techniques to establish an exponential decay result for the natural energy of (P). In addition, a polynomial decay result was established by Yan [27] when considering two boundary frictional damping terms with polynomial growth near the origin. We also recall the result by Xu and Feng [26], where the authors proved a result similar to the one in [9] by adopting the spectral analysis approach.

Zietsman et al. [28] considered a one-dimensional hybrid structure consisting of a Timoshenko beam system (P) with a tip load attached to one free end. The beam is clamped at $x = 0$, while the tip load is fixed to the end $x = L$ in such a manner that the center of mass of the load is coincident with its point of attachment to the beam. We assume interaction between the beam and the load. Thus, the forces and moments within the vibrating beam are transmitted to the tip load which moves in accordance with Newton's law. Dissipation is introduced into the coupled model by applying feedback boundary moment and force controls on the shear and displacement velocities ψ_t and φ_t at $x = L$. Hence the system (P) is subject to the following boundary conditions

$$\begin{aligned} &m\varphi_{tt}(L,t) + K(\varphi_x + \psi)(L,t) = -\gamma_1\partial_t\varphi(L,t) && \text{in} \quad (0,+\infty), \\ &I_{\mathrm{m}}\psi_{tt}(L,t) + b\psi_x(L,t) = -\gamma_2\partial_t\psi(L,t) && \text{in} \quad (0,+\infty), \end{aligned}$$

where the coefficients m and I_{m} denote, respectively, the mass and the rotary inertia of the tip load. It is established an efficiency and accuracy of the finite element method for calculating the eigenvalues and eigenmodes.

In [18], Muñoz Rivera and Ávila studied the same problem as in [28]. They proved that the decay of the energy is not exponential, but polynomial. They used the Weyl's theorem for lack of exponential stability and Borichev–Tomilov theorem for establishing decay rate $E(t) \leq c/t, t \geq 0$.

Very recently in [15], Mercier and Régnier studied a more general problem than [18] (with constants k_1 and k_3 instead of K and b in boundary conditions). They proved that the decay of the energy is not exponential, but polynomial that is $E(t) \leq c/t, t \geq 0$. They used a semigroup theory with a frequency domain approach and Riesz basis property of the generalized eigenvector of the system.

The boundary feedback under the consideration here is of fractional type and is described by the fractional derivatives

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \mathrm{e}^{-\eta(t-s)} \frac{\mathrm{d}w}{\mathrm{d}s}(s)\,\mathrm{d}s$$

The order of our derivatives is between 0 and 1. Very little attention has been paid to this type of feedback. In addition to being nonlocal, fractional derivatives involve singular and nonintegrable kernels ($t^{-\alpha}, 0 < \alpha < 1$). Therefore, the employment of mathematical analysis tools, such as stability analysis, is very difficult.

It is well known (see [14]) that, as ∂_t, the fractional derivative ∂_t^α forces the system to become dissipative and the solution to converge the equilibrium state. Therefore, when applied on the boundary, we can consider them as controllers which help to suppress or attenuate the undesirable vibrations.

Nowadays, fractional calculus is a well-established theory with strong mathematical bases and its application has become a new interest in research areas such as electrical circuits, chemical processes, signal processing, bioengineering, viscoelasticity, and obviously control systems (see [20]).

Control of fractional order type is not only important from the theoretical point of view but also for applications. It is the generalization of the classical integer order control theory, which could lead to a more adequate modeling and more robust control performance. Indeed, it has been observed by experiments that many concepts cannot be described in Newtonian terms. For example, in viscoelasticity, due to the nature of the material microstructure, both elastic solid and viscous fluid like response qualities are involved. More precisely, the stress at each point and at each instant does not depend only on the present value of the strain but also on the entire temporal prehistory of the motion from 0 up to time t. Viscoelastic response occurs in a variety of materials, such as soils, concrete, rubber, cartilage, biological tissue, glasses, and polymers (see [3–5,12]).

Our purpose in this paper is to give a global solvability in Sobolev spaces and energy decay estimates of the strong solutions to the problem (P) for damping of fractional derivative type. To obtain global solutions to the problem (P), we use the argument combining the semigroup theory [7,11] with the energy estimate method. To prove decay estimates, we use a frequency domain approach and a theorem of Borichev and Tomilov.

2. Augmented model

This section is concerned with the reformulation of the model (P) into an augmented system. For that, we need the following claims.

Theorem 2.1. (see [13]) *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty, \quad 0 < \alpha < 1. \tag{1}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \phi(\xi,t) + \xi^2 \phi(\xi,t) + \eta\phi(\xi,t) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \quad \eta \geq 0, \quad t > 0, \tag{2}$$

$$\phi(\xi,0) = 0, \tag{3}$$

$$O(t) = (\pi)^{-1} \sin(\alpha\pi) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\,\mathrm{d}\xi \tag{4}$$

is given by

$$O = I^{1-\alpha,\eta} U. \tag{5}$$

where

$$[I^{\alpha,\eta}f](t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\mathrm{e}^{-\eta(t-\tau)}f(\tau)\,\mathrm{d}\tau$$

Lemma 2.1. *If* $\lambda\in D=\{\lambda\in\mathbb{C}:Re\lambda+\eta>0\}\cup\{\lambda\in\mathbb{C}:Im\lambda\neq 0\}$ *then*

$$F_\mu(\lambda)=\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\,\mathrm{d}\xi=\frac{\pi}{\sin\alpha\pi}(\lambda+\eta)^{\alpha-1}.$$

Proof. Let us set

$$f_\lambda(\xi)=\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}.$$

We have

$$\left|\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\right|\le\begin{cases}\dfrac{\mu^2(\xi)}{Re\lambda+\eta+\xi^2} & \text{or}\\ \dfrac{\mu^2(\xi)}{|Im\lambda|+\eta+\xi^2}\end{cases}$$

Then the function f_λ is integrable. Moreover,

$$\left|\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\right|\le\begin{cases}\dfrac{\mu^2(\xi)}{\eta_0+\eta+\xi^2} & \text{for all}\quad Re\lambda\ge\eta_0>-\eta\\ \dfrac{\mu^2(\xi)}{\tilde{\eta}_0+\xi^2} & \text{for all}\quad |Im\lambda|\ge\tilde{\eta}_0>0\end{cases}$$

From Theorem 1.16.1 in [25], the function

$$F_\mu: D\to\mathbb{C}\quad\text{is holomorphic.}$$

For a real number $\lambda>-\eta$, we have

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\,\mathrm{d}\xi&=\int_{-\infty}^{+\infty}\frac{|\xi|^{2\alpha-1}}{\lambda+\eta+\xi^2}\,\mathrm{d}\xi=\int_0^{+\infty}\frac{x^{\alpha-1}}{\lambda+\eta+x}\,\mathrm{d}x\ (\text{with}\,\xi^2=x)\\&=(\lambda+\eta)^{\alpha-1}\int_1^{+\infty}y^{-1}(y-1)^{\alpha-1}\,\mathrm{d}y\ (\text{with}\,y=x/(\lambda+\eta)+1)\\&=(\lambda+\eta)^{\alpha-1}\int_0^1 z^{-\alpha}(1-z)^{\alpha-1}\,\mathrm{d}z\ (\text{with}\,z=1/y)\\&=(\lambda+\eta)^{\alpha-1}B(1-\alpha,\alpha)=(\lambda+\eta)^{\alpha-1}\Gamma(1-\alpha)\Gamma(\alpha)=(\lambda+\eta)^{\alpha-1}\frac{\pi}{\sin\pi\alpha}.\end{aligned}$$

Both holomorphic functions F_μ and $\lambda\mapsto(\lambda+\eta)^{\alpha-1}\dfrac{\pi}{\sin\pi\alpha}$ coincide on the half line $]-\eta,\infty[$, hence on D following the principle of isolated zeroes. □

We are now in a position to reformulate system (P). Indeed, by using Theorem 2.1, system (P) may be recast into the augmented model:

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0 \\ \partial_t\phi_1(\xi,t) + (\xi^2+\eta)\phi_1(\xi,t) - \varphi_t(L,t)\mu(\xi) = 0 \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x+\psi) = 0 \\ \partial_t\phi_2(\xi,t) + (\xi^2+\eta)\phi_2(\xi,t) - \psi_t(L,t)\mu(\xi) = 0 \\ \varphi(0,t) = 0, \quad \psi(0,t) = 0, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \\ m_1\varphi_{tt}(L,t) + K(\varphi_x+\psi)(L,t) = -\zeta_1 \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_1(\xi,t)\,\mathrm{d}\xi, \quad \zeta_1 = \gamma_1(\pi)^{-1}\sin(\alpha\pi) \\ m_2\psi_{tt}(L,t) + b\psi_x(L,t) = -\zeta_2 \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi,t)\,\mathrm{d}\xi, \quad \zeta_2 = \gamma_2(\pi)^{-1}\sin(\alpha\pi). \end{cases} \tag{P$'$}$$

We define the energy associated to the solution of the problem (P') by the following formula:

$$E(t) = \frac{\rho_1}{2}\|\varphi_t\|_2^2 + \frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{b}{2}\|\psi_x\|_2^2 + \frac{K}{2}\|\varphi_x+\psi\|_2^2 + \frac{m_1}{2}|\varphi_t(L,t)|^2 + \frac{m_2}{2}|\psi_t(L,t)|^2 + (\pi)^{-1}\sin(\alpha\pi)\sum_{i=1}^{2}\frac{\gamma_i}{2}\int_{-\infty}^{+\infty}(\phi_i(\xi,t))^2\,\mathrm{d}\xi. \tag{6}$$

Lemma 2.2. *Let $(\varphi, \phi_1, \psi, \phi_2)$ be a regular solution of the problem (P'). Then, the energy functional defined by (6) satisfies*

$$E'(t) = -(\pi)^{-1}\sin(\alpha\pi)\sum_{i=1}^{2}\gamma_i\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi_i(\xi,t))^2\,\mathrm{d}\xi \le 0. \tag{7}$$

Remark 2.1. For an initial datum in $D(\mathcal{A})$ (see Theorem 3.1 below), we known that $(\varphi, \phi_1, \psi, \phi_2)$ is of class C^1 in time; thus, we can derive the energy $E(t)$.

Proof of Lemma 2.2. Multiplying the first equation in (P') by φ_t and the third equation by ψ_t, integrating over $(0, L)$ and using integration by parts, we get

$$\frac{1}{2}\rho_1\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_t\|_2^2 - K\int_0^L(\varphi_x+\psi)_x\varphi_t\mathrm{d}x = 0,$$

$$\frac{1}{2}\rho_2\frac{\mathrm{d}}{\mathrm{d}t}\|\psi_t\|_2^2 - b\int_0^L\psi_{xx}\psi_t\,\mathrm{d}x + K\int_0^L(\varphi_x+\psi)\psi_t\mathrm{d}x = 0.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho_1}{2}\|\varphi_t\|_2^2 + \frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{b}{2}\|\psi_x\|_2^2 + \frac{K}{2}\|\varphi_x+\psi\|_2^2 + \frac{m_1}{2}|\varphi_t(L,t)|^2 + \frac{m_2}{2}|\psi_t(L,t)|^2\right) + \zeta_1\varphi_t(L,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi,t)\,\mathrm{d}\xi + \zeta_2\psi_t(L,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi,t)\,\mathrm{d}\xi = 0. \tag{8}$$

Multiplying the second equation in (P') by $\gamma_1(\pi)^{-1}\sin(\alpha\pi)\phi_1$, the fourth equation in (P') by $\gamma_2(\pi)^{-1}\sin(\alpha\pi)\phi_2$ and integrating over $(-\infty, +\infty)$, to obtain:

$$\frac{\zeta_1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi_1\|_2^2+\zeta_1\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi_1(\xi,t))^2\,\mathrm{d}\xi-\zeta_1\varphi_t(L,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi,t)\,\mathrm{d}\xi=0,$$
$$\frac{\zeta_2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi_2\|_2^2+\zeta_2\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi_2(\xi,t))^2\,\mathrm{d}\xi-\zeta_2\psi_t(L,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi,t)\,\mathrm{d}\xi=0. \tag{9}$$

From (6), (8) and (9) we get

$$E'(t)=-\sum_{i=1}^{2}\zeta_i\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi_i(\xi,t))^2\,\mathrm{d}\xi.$$

This completes the proof of the lemma. □

3. Global existence

In this section we will give well-posedness results for problem (P') using semigroup theory. Let us introduce the semigroup representation of the Timoshenko system (P'). We consider the following condition of the right end contour of wave

$$\varphi_t(L,t)=\theta(t),\quad \psi_t(L,t)=\vartheta(t),\quad \text{for}\quad t>0 \tag{10}$$

where θ and ϑ solve the system

$$\begin{aligned}
&m_1\theta_t(t)+K(\varphi_x+\psi)(L,t)+\zeta_1\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi,t)\,\mathrm{d}\xi=0,\\
&m_2\vartheta_t(t)+b\psi_x(L,t)+\zeta_2\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi,t)\,\mathrm{d}\xi=0.
\end{aligned} \tag{11}$$

Let $U=(\varphi,\varphi_t,\phi_1,\theta,\psi,\psi_t,\phi_2,\vartheta)^{\mathrm{T}}$ and rewrite (P') as

$$\begin{cases} U'=\mathcal{A}U,\\ U(0)=(\varphi_0,\varphi_1,\phi_{01},\theta_0,\psi_0,\psi_1,\phi_{02},\vartheta_0), \end{cases} \tag{12}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix}\varphi\\ u\\ \phi_1\\ \theta\\ \psi\\ v\\ \phi_2\\ \vartheta\end{pmatrix}=\begin{pmatrix} u\\ \dfrac{K}{\rho_1}(\varphi_x+\psi)_x\\ -(\xi^2+\eta)\phi_1+u(L)\mu(\xi)\\ -\frac{K}{m_1}(\varphi_x+\psi)(L)-\frac{\zeta_1}{m_1}\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi)\,\mathrm{d}\xi\\ v\\ \dfrac{b}{\rho_2}\psi_{xx}-\dfrac{K}{\rho_2}(\varphi_x+\psi)\\ -(\xi^2+\eta)\phi_2+v(L)\mu(\xi)\\ -\frac{b}{m_2}\psi_x(L)-\frac{\zeta_2}{m_2}\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi)\,\mathrm{d}\xi \end{pmatrix} \tag{13}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (\varphi, u, \phi_1, \theta, \psi, v, \phi_2, \vartheta)^{\mathrm{T}} \quad \text{in} \quad \mathcal{H} : \varphi, \psi \in H^2(0,L) \cap H^1_*(0,L), u, v \in H^1_*(0,L), \\ \theta, \vartheta \in \mathbb{C}, -(\xi^2+\eta)\phi_1 + u(L)\mu(\xi), -(\xi^2+\eta)\phi_2 + v(L)\mu(\xi) \in L^2(-\infty,+\infty), \\ u(L) = \theta, v(L) = \vartheta, \\ |\xi|\phi_1, |\xi|\phi_2 \in L^2(-\infty,+\infty) \end{array} \right\}, \tag{14}$$

where the energy space $\mathcal{H}$ is defined as

$$\mathcal{H} = \left(H^1_*(0,L) \times L^2(0,L) \times L^2(-\infty,+\infty) \times \mathbb{C}\right)^2$$

where

$$H^1_*(0,L) = \{\varphi \in H^1(0,L) : \varphi(0) = 0\}.$$

For $U = (\varphi, u, \phi_1, \theta, \psi, v, \phi_2, \vartheta)^{\mathrm{T}}, \overline{U} = (\overline{\varphi}, \overline{u}, \overline{\phi}_1, \overline{\theta}, \overline{\psi}, \overline{v}, \overline{\phi}_2, \overline{\vartheta})^{\mathrm{T}}$, we define the following inner product in $\mathcal{H}$

$$\begin{aligned} \langle U, \overline{U} \rangle_{\mathcal{H}} &= \int_0^L \Big(\rho_1 u\overline{u} + \rho_2 v\overline{v} + b\psi_x\overline{\psi}_x + K(\varphi_x+\psi)(\overline{\varphi}_x + \overline{\psi}) \,\mathrm{d}x \\ &\quad + \sum_{i=1}^{2} \zeta_i \int_{-\infty}^{+\infty} \phi_i\overline{\phi}_i \,\mathrm{d}\xi + m_1\theta\overline{\theta} + m_2\vartheta\overline{\vartheta}. \end{aligned}$$

We show that the operator $\mathcal{A}$ generates a C_0- semigroup in $\mathcal{H}$. In this step, we prove that the operator $\mathcal{A}$ is dissipative. Let $U = (\varphi, u, \phi_1, \theta, \psi, v, \phi_2, \vartheta)^{\mathrm{T}}$. Using (12), (7), and the fact that

$$E(t) = \frac{1}{2}\|U\|^2_{\mathcal{H}}, \tag{15}$$

we get

$$\langle \mathcal{A}U, U \rangle_{\mathcal{H}} = -\sum_{i=1}^{2} \zeta_i \int_{-\infty}^{+\infty} (\xi^2+\eta)(\phi_i(\xi))^2 \,\mathrm{d}\xi. \tag{16}$$

Consequently, the operator $\mathcal{A}$ is dissipative. Now, we will prove that the operator $\lambda I - \mathcal{A}$ is surjective for $\lambda > 0$. For this purpose, let $(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^{\mathrm{T}} \in \mathcal{H}$, we seek $U = (\varphi, u, \phi_1, \theta, \psi, v, \phi_2, \vartheta)^{\mathrm{T}} \in D(\mathcal{A})$ solution of the following system of equations

$$\left\{ \begin{array}{l} \lambda\varphi - u = f_1, \\ \lambda u - \dfrac{K}{\rho_1}(\varphi_x+\psi)_x = f_2, \\ \lambda\phi_1 + (\xi^2+\eta)\phi_1 - u(L)\mu(\xi) = f_3, \\ \lambda\theta + \frac{K}{m_1}(\varphi_x+\psi)(L) + \frac{\zeta_1}{m_1}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_1(\xi)\,\mathrm{d}\xi = f_4. \\ \lambda\psi - v = f_5, \\ \lambda v - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x+\psi) = f_6, \\ \lambda\phi_2 + (\xi^2+\eta)\phi_2 - v(L)\mu(\xi) = f_7, \\ \lambda\vartheta + \frac{b}{m_2}\psi_x(L) + \frac{\zeta_2}{m_2}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi)\,\mathrm{d}\xi = f_8. \end{array} \right. \tag{17}$$

Suppose that we have found φ and ψ. Therefore, the first and the fifth equations in (17) give

$$\left\{ \begin{array}{l} u = \lambda\varphi - f_1, \\ v = \lambda\psi - f_5. \end{array} \right. \tag{18}$$

It is clear that $u \in H^1_*(0,L)$ and $v \in H^1_*(0,L)$. Furthermore, by (17) we can find ϕ_i $(i=1,2)$ as

$$\begin{cases} \phi_1 = \dfrac{f_3(\xi) + \mu(\xi)u(L)}{\xi^2+\eta+\lambda}, \\ \phi_2 = \dfrac{f_7(\xi) + \mu(\xi)v(L)}{\xi^2+\eta+\lambda}. \end{cases} \tag{19}$$

By using (17) and (18) the functions φ and ψ satisfying the following system

$$\begin{cases} \lambda^2\varphi - \dfrac{K}{\rho_1}(\varphi_x+\psi)_x = f_2 + \lambda f_1, \\ \lambda^2\psi - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x+\psi) = f_6 + \lambda f_5, \end{cases} \tag{20}$$

Solving system (20) is equivalent to finding $(\varphi,\psi) \in (H^2 \cap H^1_*(0,L))^2$ such that

$$\begin{cases} \displaystyle\int_0^L (\rho_1\lambda^2\varphi w - K(\varphi_x+\psi)_x w)\,\mathrm{d}x = \int_0^L \rho_1(f_2+\lambda f_1)w\,\mathrm{d}x, \\ \displaystyle\int_0^L (\rho_2\lambda^2\psi\chi - b\psi_{xx}\chi + K(\varphi_x+\psi)\chi)\,\mathrm{d}x = \int_0^L \rho_2(f_6+\lambda f_5)\chi\,\mathrm{d}x, \end{cases} \tag{21}$$

for all $(w,\chi) \in H^1_*(0,L) \times H^1_*(0,L)$. By using (21) and (19) the functions φ and ψ satisfying the following system

$$\begin{cases} \displaystyle\int_0^L (\rho_1\lambda^2\varphi w + K(\varphi_x+\psi)w_x)\,\mathrm{d}x + (\lambda m_1 + \tilde{\zeta}_1)u(L)w(L) \\ \qquad \displaystyle = \int_0^L \rho_1(f_2+\lambda f_1)w\,\mathrm{d}x - \zeta_1 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\,\mathrm{d}\xi\ w(L) + m_1 f_4 w(L), \\ \displaystyle\int_0^L (\rho_2\lambda^2\psi\chi + b\psi_x\chi_x + K(\varphi_x+\psi)\chi)\,\mathrm{d}x + (\lambda m_2 + \tilde{\zeta}_2)v(L)\chi(L) \\ \qquad \displaystyle = \int_0^L \rho_2(f_6+\lambda f_5)\chi\,\mathrm{d}x - \zeta_2 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_7(\xi)\,\mathrm{d}\xi\ \chi(L) + m_2 f_8 \chi(L) \end{cases} \tag{22}$$

where $\tilde{\zeta}_i = \zeta_i \displaystyle\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+\lambda}\,\mathrm{d}\xi$. Using again (18), we deduce that

$$\begin{cases} u(L) = \lambda\varphi(L) - f_1(L), \\ v(L) = \lambda\psi(L) - f_5(L). \end{cases} \tag{23}$$

Inserting (23) into (22), we get

$$\begin{cases} \int_0^L (\rho_1\lambda^2\varphi w + K(\varphi_x+\psi)w_x)\,\mathrm{d}x + \lambda(\lambda m_1+\tilde{\zeta}_1)\varphi(L)w(L) \\ \quad = \int_0^L \rho_1(f_2+\lambda f_1)w\,\mathrm{d}x - \zeta_1\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+\lambda}f_3(\xi)\,\mathrm{d}\xi\ w(L) + (\lambda m_1+\tilde{\zeta}_1)f_1(L)w(L) + m_1 f_4 w(L), \\ \int_0^L (\rho_2\lambda^2\psi\chi + b\psi_x\chi_x + K(\varphi_x+\psi)\chi)\,\mathrm{d}x + \lambda(\lambda m_2+\tilde{\zeta}_2)\psi(L)\chi(L) \\ \quad = \int_0^L \rho_2(f_6+\lambda f_5)\chi\,\mathrm{d}x - \zeta_2\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+\lambda}f_7(\xi)\,\mathrm{d}\xi\ \chi(L) + (\lambda m_2+\tilde{\zeta}_2)f_5(L)\chi(L) + m_2 f_8\chi(L). \end{cases} \tag{24}$$

Consequently, problem (24) is equivalent to the problem

$$a((\varphi,\psi),(w,\chi)) = L(w,\chi), \tag{25}$$

where the bilinear form $a : [H_*^1(0,L)\times H_*^1(0,L)]^2 \to \mathbb{R}$ and the linear form $L : H_*^1(0,L)\times H_*^1(0,L) \to \mathbb{R}$ are defined by

$$\begin{aligned} a((\varphi,\psi),(w,\chi)) = &\int_0^L (\rho_1\lambda^2\varphi w + K(\varphi_x+\psi)(w_x+\chi))\,\mathrm{d}x \\ &+ \int_0^L (\rho_2\lambda^2\psi\chi + b\psi_x\chi_x)\,\mathrm{d}x + \lambda(\lambda m_1+\tilde{\zeta}_1)\varphi(L)w(L) + \lambda(\lambda m_2+\tilde{\zeta}_2)\psi(L)\chi(L) \end{aligned}$$

and

$$\begin{aligned} L(w,\chi) = &\int_0^L \rho_1(f_2+\lambda f_1)w\,\mathrm{d}x - \zeta_1\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+\lambda}f_3(\xi)\,\mathrm{d}\xi\ w(L) + (\lambda m_1+\tilde{\zeta}_1)f_1(L)w(L) \\ &+ m_1 f_4 w(L) + \int_0^L \rho_2(f_6+\lambda f_5)\chi\,\mathrm{d}x - \zeta_2\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+\lambda}f_7(\xi)\,\mathrm{d}\xi\ \chi(L) \\ &+ (\lambda m_2+\tilde{\zeta}_2)f_5(L)\chi(L) + m_2 f_8\chi(L). \end{aligned}$$

It is easy to verify that a is continuous and coercive, and L is continuous. So applying the Lax–Milgram theorem, we deduce that for all $(w,\chi)\in H_*^1(0,L)\times H_*^1(0,L)$ problem (25) admits a unique solution $(\varphi,\psi)\in H_*^1(0,L)\times H_*^1(0,L)$. Applying the classical elliptic regularity, it follows from (24) that $(\varphi,\psi)\in H^2(0,L)\times H^2(0,L)$. Therefore, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$. Consequently, using Hille–Yosida theorem, we have the following results.

Theorem 3.1. (Existence and uniqueness)

(1) *If $U_0 \in D(\mathcal{A})$, then system (12) has a unique strong solution*

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) *If $U_0 \in \mathcal{H}$, then system (12) has a unique weak solution*

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

4. Lack of exponential stability

We first state three well-known theorems.

Theorem 4.1. [21] *Let $S(t) = \mathrm{e}^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space $\mathcal{H}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Theorem 4.2. [6] *Let $S(t) = \mathrm{e}^{\mathcal{A}t}$ be a C_0-semigroup on a Hilbert space $\mathcal{H}$. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \quad \text{and} \quad \sup_{|\beta|\geq 1} \frac{1}{\beta^l}\|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < M$$

for some l, then there exist c such that

$$\|\mathrm{e}^{\mathcal{A}t}U_0\|^2 \leq \frac{c}{t^{\frac{2}{l}}}\|U_0\|^2_{D(\mathcal{A})}.$$

Theorem 4.3. [2] *Let $\mathcal{A}$ be the generator of a uniformly bounded C_0 semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. If:*

(i) *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*
(ii) *The intersection of the spectrum $\sigma(\mathcal{A})$ with $i\mathbb{R}$ is at most a countable set.*

Then the semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically stable, i.e., $\|S(t)z\|_{\mathcal{H}} \to 0$ as $t \to \infty$ for any $z \in \mathcal{H}$.

Our main result is the following

Theorem 4.4. *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable.*

Proof. We will examine two cases.
Case 1 $\eta = 0$: We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $(\sin x, 0, 0, 0, 0, 0, 0, 0,)^{\mathrm{T}} \in \mathcal{H}$, and denoting by $(\varphi, u, \phi_1, \theta, \psi, v, \phi_2, \vartheta)^{\mathrm{T}}$ the image of $(\sin x, 0, 0, 0, 0, 0, 0, 0)^{\mathrm{T}}$ by $\mathcal{A}^{-1}$, we see that $\phi_1(\xi) = |\xi|^{\frac{2\alpha-5}{2}} \sin L$. But, then $\phi_1 \notin L^2(-\infty, +\infty)$, since $\alpha \in (0, 1)$ and so $(\varphi, u, \phi_1, \theta, \psi, v, \phi_2, \vartheta)^{\mathrm{T}} \notin D(\mathcal{A})$.
Case 2 $\eta \neq 0$: We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the Timoshenko system (P) from being exponentially stable. Indeed we first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$.

Let λ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (\varphi, u, \phi_1, \theta, \psi, v, \phi_2, \vartheta)^{\mathrm{T}}$. Then $\mathcal{A}U = \lambda U$ is equivalent to

$$\begin{cases} \lambda\varphi - u = 0, \\ \lambda u - \dfrac{K}{\rho_1}(\varphi_x + \psi)_x = 0, \\ \lambda\phi_1 + (\xi^2 + \eta)\phi_1 - u(L)\mu(\xi) = 0, \\ \lambda\theta + \frac{K}{m_1}(\varphi_x + \psi)(L) + \frac{\zeta_1}{m_1}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_1(\xi)\,\mathrm{d}\xi, \\ \lambda\psi - v = 0, \\ \lambda v - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x + \psi) = 0, \\ \lambda\phi_2 + (\xi^2 + \eta)\phi_2 - v(L)\mu(\xi) = 0, \\ \lambda\vartheta + \frac{b}{m_2}\psi_x(L) + \frac{\zeta_2}{m_2}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi)\,\mathrm{d}\xi. \end{cases} \tag{26}$$

From $(26)_1$–$(26)_2$ and $(26)_5$–$(26)_6$ for such λ, we find

$$\begin{cases} \lambda^2\varphi - \dfrac{K}{\rho_1}(\varphi_x + \psi)_x = 0, \\ \lambda^2\psi - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x + \psi) = 0. \end{cases} \tag{27}$$

Since $\theta = u(L)$ and $\vartheta = v(L)$, using $(26)_3$–$(26)_4$ and $(26)_7$–$(26)_8$, we get

$$\begin{cases} \dfrac{K}{m_1}(\varphi_x + \psi)(L) + \left(\lambda + \dfrac{\gamma_1}{m_1}(\lambda + \eta)^{\alpha-1}\right)\lambda\varphi(L) = 0, \\ \dfrac{b}{m_2}\psi_x(L) + \left(\lambda + \dfrac{\gamma_2}{m_2}(\lambda + \eta)^{\alpha-1}\right)\lambda\psi(L) = 0, \\ \varphi(0) = \psi(0) = 0. \end{cases} \tag{28}$$

We set

$$\tilde{\varphi} = (\varphi_x + \psi), \quad \tilde{\psi} = \psi_x.$$

(27) is equivalent to

$$\begin{cases} \lambda^2\varphi - \dfrac{K}{\rho_1}\tilde{\varphi}_x = 0, \\ \lambda^2\psi - \dfrac{b}{\rho_2}\tilde{\psi}_x + \dfrac{K}{\rho_2}\tilde{\varphi} = 0. \end{cases} \tag{29}$$

Then

$$\begin{cases} \left(\lambda^2 + \dfrac{K}{\rho_2}\right)\tilde{\varphi} - \dfrac{K}{\rho_1}\tilde{\varphi}_{xx} - \dfrac{b}{\rho_2}\tilde{\psi}_x = 0((29)_1 x + (29)_2), \\ \lambda^2\tilde{\psi} - \dfrac{b}{\rho_2}\tilde{\psi}_{xx} + \dfrac{K}{\rho_2}\tilde{\varphi}_x = 0. \end{cases} \tag{30}$$

From $(29)_2$ we have

$$\tilde{\varphi} = \frac{\rho_2}{K}\left(-\lambda^2\psi + \frac{b}{\rho_2}\tilde{\psi}_x\right)$$

$$\tilde{\varphi}_{xx} = \frac{\rho_2}{K}\left(-\lambda^2\psi_{xx} + \frac{b}{\rho_2}\tilde{\psi}_{xxx}\right).$$

Replacing this in $(30)_1$, we get

$$\psi'''' - \lambda^2\left(\frac{\rho_1}{K} + \frac{\rho_2}{b}\right)\psi'' + \frac{\rho_1}{K}\frac{\rho_2}{b}\lambda^2\left(\lambda^2 + \frac{K}{\rho_2}\right)\psi = 0. \tag{31}$$

The characteristic polynomial of (31) is

$$s^4 - \left(\frac{\rho_1}{K} + \frac{\rho_2}{b}\right)\lambda^2 s^2 + \frac{\rho_1}{K}\frac{\rho_2}{b}\lambda^2\left(\lambda^2 + \frac{K}{\rho_2}\right) = 0.$$

The solution ψ is given by

$$\psi(x) = \sum_{i=1}^{4} c_i \mathrm{e}^{t_i x} \tag{32}$$

where $c_i \in \mathbb{C}$ for all $1 \le i \le 4$ and

$$\begin{cases} t_1(\lambda) = \lambda\sqrt{\frac{\left(\frac{\rho_1}{K}+\frac{\rho_2}{b}\right)+\sqrt{\left(\frac{\rho_1}{K}-\frac{\rho_2}{b}\right)^2-\frac{4\rho_1}{b\lambda^2}}}{2}}, & t_2(\lambda) = -t_1(\lambda), \\ t_3(\lambda) = \lambda\sqrt{\frac{\left(\frac{\rho_1}{K}+\frac{\rho_2}{b}\right)-\sqrt{\left(\frac{\rho_1}{K}-\frac{\rho_2}{b}\right)^2-\frac{4\rho_1}{b\lambda^2}}}{2}}, & t_4(\lambda) = -t_3(\lambda). \end{cases}$$

From $(29)_1$ and $(30)_2$, we have

$$\varphi = \frac{K}{\rho_1}\frac{1}{\lambda^2}\tilde{\varphi}_x = \frac{\rho_2}{\rho_1}\frac{1}{\lambda^2}\left(-\lambda^2\psi_x + \frac{b}{\rho_2}\tilde{\psi}_{xx}\right).$$

Thus, the boundary conditions may be written as the following system:

$$\begin{aligned}
&\psi(0) = 0 \Longrightarrow \sum_{i=1}^{4} c_i = 0 \\
&\varphi(0) = 0 \Longrightarrow \sum_{i=1}^{4}\left(-\lambda^2 t_i + \frac{b}{\rho_2}\, t_i^3\right) c_i = 0 \\
&\frac{b}{m_2}\psi_x(L) + \left(\lambda + \frac{\gamma_2}{m_2}(\lambda+\eta)^{\alpha-1}\right)\lambda\psi(L) = 0 \Longrightarrow \sum_{i=1}^{4}\left(\frac{b}{m_2}t_i + \left(\lambda + \frac{\gamma_2}{m_2}(\lambda+\eta)^{\alpha-1}\right)\lambda\right)\mathrm{e}^{t_i L}c_i = 0 \\
&\frac{K}{m_1}(\varphi_x + \psi)(L) + \left(\lambda + \frac{\gamma_1}{m_1}(\lambda+\eta)^{\alpha-1}\right)\lambda\varphi(L) = 0 \Longrightarrow \\
&\sum_{i=1}^{4}\left(-\frac{1}{m_1}\lambda^2 - \frac{1}{\rho_1}\lambda\left(\lambda + \frac{\gamma_1}{m_1}(\lambda+\eta)^{\alpha-1}\right)t_i + \frac{b}{m_1\rho_2}t_i^2 + \frac{b}{\rho_1\rho_2}\frac{1}{\lambda}\left(\lambda + \frac{\gamma_1}{m_1}(\lambda+\eta)^{\alpha-1}\right)t_i^3\right)\mathrm{e}^{t_i L}c_i = 0 \\
&\mathcal{M}C(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ h_1(t_1) & h_1(t_2) & h_1(t_3) & h_1(t_4) \\ h_2(t_1)\mathrm{e}^{t_1 L} & h_2(t_2)\mathrm{e}^{t_2 L} & h_2(t_3)\mathrm{e}^{t_3 L} & h_2(t_4)\mathrm{e}^{t_4 L} \\ h_3(t_1)\mathrm{e}^{t_1 L} & h_3(t_2)\mathrm{e}^{t_2 L} & h_3(t_3)\mathrm{e}^{t_3 L} & h_3(t_4)\mathrm{e}^{t_4 L} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
h_1(r) &= -\lambda^2 r + \frac{b}{\rho_2}\, r^3, \\
h_2(r) &= \frac{b}{m_2}r + \left(\lambda + \frac{\gamma_2}{m_2}(\lambda+\eta)^{\alpha-1}\right)\lambda, \\
h_3(r) &= -\frac{1}{m_1}\lambda^2 - \frac{1}{\rho_1}\lambda\left(\lambda + \frac{\gamma_1}{m_1}(\lambda+\eta)^{\alpha-1}\right)r + \frac{b}{m_1\rho_2}r^2 + \frac{b}{\rho_1\rho_2}\frac{1}{\lambda}\left(\lambda + \frac{\gamma_1}{m_1}(\lambda+\eta)^{\alpha-1}\right)r^3.
\end{aligned}$$

Set $r_1^2 = \frac{\rho_2}{b}, r_2^2 = \frac{\rho_1}{K}$ and $l = K/b$. We will examine two cases.

Case 1 $r_1 = r_2$:
We start by the expansion of t_1 and t_3:

$$t_1(\lambda) = r_1\lambda + \left(\frac{i}{2}\sqrt{l}\right) + \frac{1}{8}\frac{l}{r_1}\frac{1}{\lambda} - \left(\frac{i}{16}\frac{\sqrt{l}l}{r_1^2}\right)\frac{1}{\lambda^2} - \left(\frac{5}{128}\frac{l^2}{r_1^3}\right)\frac{1}{\lambda^3} + \left(\frac{7i}{256}\frac{l^2\sqrt{l}}{r_1^4}\right)\frac{1}{\lambda^4} + O\left(\frac{1}{\lambda^5}\right) \tag{34}$$

$$t_3(\lambda) = r_1\lambda - \left(\frac{i}{2}\sqrt{l}\right) + \frac{1}{8}\frac{l}{r_1}\frac{1}{\lambda} + \left(\frac{i}{16}\frac{\sqrt{l}l}{r_1^2}\right)\frac{1}{\lambda^2} - \left(\frac{5}{128}\frac{l^2}{r_1^3}\right)\frac{1}{\lambda^3} - \left(\frac{7i}{256}\frac{l^2\sqrt{l}}{r_1^4}\right)\frac{1}{\lambda^4} + O\left(\frac{1}{\lambda^5}\right) \tag{35}$$

Using (34) and (35), we find the asymptotic development of:

$$\begin{aligned} h_3(t_1) = {} & i\frac{\sqrt{l}}{\rho_1}\lambda^2 + \left(-\frac{1}{2}\frac{l}{\rho_1 r_1} + i\frac{\sqrt{l}}{m_1 r_1}\right)\lambda + \frac{1}{8}i\frac{\left(\sqrt{l}\right)^3}{\rho_1 r_1^2} + \gamma_1\frac{i\sqrt{l}}{m_1\rho_1}\lambda^\alpha + \gamma_1\frac{-l-2i(1-\alpha)\sqrt{l}\eta r_1}{2m_1\rho_1 r_1}\lambda^{\alpha-1} \\ & + \frac{1}{16}\frac{l^2}{\rho_1 r_1^3}\frac{1}{\lambda} - \frac{5}{128}i\frac{l^{\frac{5}{2}}}{\rho_1 r_1^4}\frac{1}{\lambda^2} + \frac{1}{8}il^{\frac{3}{2}}\lambda^{\alpha-2}\frac{\gamma_1}{m_1\rho_1 r_1^2} + \frac{1}{2}i\lambda^{\alpha-2}\eta^2\sqrt{l}\gamma_1(\alpha-2)\frac{\alpha-1}{m_1\rho_1} \\ & - \frac{1}{2}\lambda^{\alpha-2}l\eta\gamma_1\frac{\alpha-1}{m_1\rho_1 r_1} + o\left(\frac{1}{\lambda^2}\right). \end{aligned} \tag{36}$$

$$\begin{aligned} h_3(t_2) = {} & -i\frac{\sqrt{l}}{\rho_1}\lambda^2 + \left(\frac{1}{2}\frac{l}{\rho_1 r_1} + i\frac{\sqrt{l}}{m_1 r_1}\right)\lambda - \frac{1}{8}i\frac{l^{\frac{3}{2}}}{\rho_1 r_1^2} + \gamma_1\frac{-i\sqrt{l}}{m_1\rho_1}\lambda^\alpha - \gamma_1\frac{-l-2i(1-\alpha)\sqrt{l}\eta r_1}{2m_1\rho_1 r_1}\lambda^{\alpha-1} \\ & - \frac{1}{16}\frac{l^2}{\rho_1 r_1^3}\frac{1}{\lambda} + \frac{5}{128}i\frac{l^{\frac{5}{2}}}{\rho_1 r_1^4}\frac{1}{\lambda^2} - \frac{1}{8}il^{\frac{3}{2}}\lambda^{\alpha-2}\frac{\gamma_1}{m_1\rho_1 r_1^2} - \frac{1}{2}i\lambda^{\alpha-2}\eta^2\sqrt{l}\gamma_1(\alpha-1)\frac{\alpha-2}{m_1\rho_1} \\ & + \frac{1}{2}\lambda^{\alpha-2}l\eta\gamma_1\frac{\alpha-1}{m_1\rho_1 r_1} + o\left(\frac{1}{\lambda^2}\right). \end{aligned} \tag{37}$$

$$\begin{aligned} h_3(t_3) = {} & -i\frac{\sqrt{l}}{\rho_1}\lambda^2 + \left(-\frac{1}{2}\frac{l}{\rho_1 r_1} - i\frac{\sqrt{l}}{m_1 r_1}\right)\lambda - \frac{1}{8}i\frac{l^{\frac{3}{2}}}{\rho_1 r_1^2} - \gamma_1\frac{i\sqrt{l}}{m_1\rho_1}\lambda^\alpha + \gamma_1\frac{-l+2i(1-\alpha)\sqrt{l}\eta r_1}{2m_1\rho_1 r_1}\lambda^{\alpha-1} \\ & + \frac{1}{16\rho_1 r_1^3}l^2\frac{1}{\lambda} + \frac{5}{128}i\frac{l^{\frac{5}{2}}}{\rho_1 r_1^4}\frac{1}{\lambda^2} - \frac{1}{8}il^{\frac{3}{2}}\lambda^{\alpha-2}\frac{\gamma_1}{m_1\rho_1 r_1^2} - \frac{1}{2}i\lambda^{\alpha-2}\eta^2\sqrt{l}\gamma_1(\alpha-2)\frac{\alpha-1}{m_1\rho_1} \\ & - \frac{1}{2}\lambda^{\alpha-2}l\eta\gamma_1\frac{\alpha-1}{m_1\rho_1 r_1} + o\left(\frac{1}{\lambda^2}\right). \end{aligned} \tag{38}$$

$$\begin{aligned} h_3(t_4) = {} & i\frac{\sqrt{l}}{\rho_1}\lambda^2 + \left(\frac{1}{2}\frac{l}{\rho_1 r_1} - i\frac{\sqrt{l}}{m_1 r_1}\right)\lambda + \frac{1}{8}i\frac{l^{\frac{3}{2}}}{\rho_1 r_1^2} + \gamma_1\frac{i\sqrt{l}}{m_1\rho_1}\lambda^\alpha - \gamma_1\frac{-l+2i(1-\alpha)\sqrt{l}\eta r_1}{2m_1\rho_1 r_1}\lambda^{\alpha-1} \\ & - \frac{1}{16\rho_1 r_1^3}l^2\frac{1}{\lambda} - \frac{5}{128}i\frac{l^{\frac{5}{2}}}{\rho_1 r_1^4}\frac{1}{\lambda^2} + \frac{1}{8}il^{\frac{3}{2}}\lambda^{\alpha-2}\frac{\gamma_1}{m_1\rho_1 r_1^2} + \frac{1}{2}i\lambda^{\alpha-2}\eta^2\sqrt{l}\gamma_1(\alpha-1)\frac{\alpha-2}{m_1\rho_1} \\ & + \frac{1}{2}\lambda^{\alpha-2}l\eta\gamma_1\frac{\alpha-1}{m_1\rho_1 r_1} + o\left(\frac{1}{\lambda^2}\right). \end{aligned} \tag{39}$$

$$\begin{aligned} h_2(t_1) = {} & \lambda^2 + b\frac{\lambda}{m_2}r_1 + \frac{1}{2}ib\frac{\sqrt{l}}{m_2} + \frac{\gamma_2}{m_2}\lambda^\alpha + \eta\gamma_2\frac{\alpha-1}{m_2}\lambda^{\alpha-1} + \frac{1}{8}b\frac{l}{m_2 r_1}\frac{1}{\lambda} + \frac{1}{2}\lambda^{\alpha-2}\eta^2\gamma_2(\alpha-1)\frac{\alpha-2}{m_2} \\ & - \frac{1}{16}ib\frac{l^{\frac{3}{2}}}{m_2 r_1^2}\frac{1}{\lambda^2} + o\left(\frac{1}{\lambda^2}\right). \end{aligned} \tag{40}$$

$$\begin{aligned} h_2(t_2) = {} & \lambda^2 - b\frac{\lambda}{m_2}r_1 - \frac{1}{2}ib\frac{\sqrt{l}}{m_2} + \frac{\gamma_2}{m_2}\lambda^\alpha + \eta\gamma_2\frac{\alpha-1}{m_2}\lambda^{\alpha-1} - \frac{1}{8}b\frac{l}{m_2 r_1}\frac{1}{\lambda} + \frac{1}{2}\lambda^{\alpha-2}\eta^2\gamma_2(\alpha-1)\frac{\alpha-2}{m_2} \\ & + \frac{1}{16}ib\frac{l^{\frac{3}{2}}}{m_2 r_1^2}\frac{1}{\lambda^2} + o\left(\frac{1}{\lambda^2}\right). \end{aligned} \tag{41}$$

$$h_2(t_3) = \lambda^2 + b\frac{\lambda}{m_2}r_1 - \frac{1}{2}ib\frac{\sqrt{l}}{m_2} + \frac{\gamma_2}{m_2}\lambda^\alpha + \eta\gamma_2\frac{\alpha-1}{m_2}\lambda^{\alpha-1} + \frac{1}{8}b\frac{l}{m_2 r_1}\frac{1}{\lambda} + \frac{1}{2}\lambda^{\alpha-2}\eta^2\gamma_2(\alpha-1)\frac{\alpha-2}{m_2} + \frac{1}{16}ib\frac{l^{\frac{3}{2}}}{m_2 r_1^2}\frac{1}{\lambda^2} + o\left(\frac{1}{\lambda^2}\right). \tag{42}$$

$$h_2(t_4) = \lambda^2 - b\frac{\lambda}{m_2}r_1 + \frac{1}{2}ib\frac{\sqrt{l}}{m_2} + \frac{\gamma_2}{m_2}\lambda^\alpha + \eta\gamma_2\frac{\alpha-1}{m_2}\lambda^{\alpha-1} - \frac{1}{8}b\frac{l}{m_2 r_1}\frac{1}{\lambda} + \frac{1}{2}\lambda^{\alpha-2}\eta^2\gamma_2(\alpha-1)\frac{\alpha-2}{m_2} - \frac{1}{16}ib\frac{l^{\frac{3}{2}}}{m_2 r_1^2}\frac{1}{\lambda^2} + o\left(\frac{1}{\lambda^2}\right). \tag{43}$$

$$h_1(t_1) = i\sqrt{l}\lambda^2 - \frac{1}{2}l\frac{\lambda}{r_1} + \frac{1}{8}i\frac{l^{\frac{3}{2}}}{r_1^2} + \frac{1}{16}\frac{l^2}{\lambda r_1^3} - \frac{5}{128}i\frac{l^{\frac{5}{2}}}{\lambda^2 r_1^4} + O\left(\frac{1}{\lambda^3}\right). \tag{44}$$

$$h_1(t_2) = -i\sqrt{l}\lambda^2 + \frac{1}{2}l\frac{\lambda}{r_1} - \frac{1}{8}i\frac{l^{\frac{3}{2}}}{r_1^2} - \frac{1}{16}\frac{l^2}{\lambda r_1^3} + \frac{5}{128}i\frac{l^{\frac{5}{2}}}{\lambda^2 r_1^4} + O\left(\frac{1}{\lambda^3}\right). \tag{45}$$

$$h_1(t_3) = -i\sqrt{l}\lambda^2 - \frac{1}{2}l\frac{\lambda}{r_1} - \frac{1}{8}i\frac{l^{\frac{3}{2}}}{r_1^2} + \frac{1}{16}\frac{l^2}{\lambda r_1^3} + \frac{5}{128}i\frac{l^{\frac{5}{2}}}{\lambda^2 r_1^4} + O\left(\frac{1}{\lambda^3}\right). \tag{46}$$

$$h_1(t_4) = i\sqrt{l}\lambda^2 + \frac{1}{2}l\frac{\lambda}{r_1} + \frac{1}{8}i\frac{l^{\frac{3}{2}}}{r_1^2} - \frac{1}{16}\frac{l^2}{\lambda r_1^3} - \frac{5}{128}i\frac{l^{\frac{5}{2}}}{\lambda^2 r_1^4} + O\left(\frac{1}{\lambda^3}\right). \tag{47}$$

Using the asymptotic development (36)–(47)

$$\begin{aligned}
f(\lambda) &= \mathrm{e}^{t_3+t_4}\Big(h_1(t_2) - (h_1(t_1)\Big)\Big(h_2(t_3)h_3(t_4) - h_3(t_3)h_2(t_4)\Big) \\
&\quad + \mathrm{e}^{t_1+t_3}\Big(h_1(t_2) - (h_1(t_4)\Big)\Big(h_2(t_1)h_3(t_3) - h_3(t_1)h_2(t_3)\Big) \\
&\quad + \mathrm{e}^{t_1+t_4}\Big(h_1(t_2) - (h_1(t_3)\Big)\Big(h_3(t_1)h_2(t_4) - h_2(t_1)h_3(t_4)\Big) \\
&\quad + \mathrm{e}^{t_2+t_3}\Big(h_1(t_4) - (h_1(t_1)\Big)\Big(h_2(t_2)h_3(t_3) - h_2(t_3)h_3(t_2)\Big) \\
&\quad + \mathrm{e}^{t_2+t_4}\Big(h_1(t_1) - (h_1(t_3)\Big)\Big(h_2(t_2)h_3(t_4) - h_3(t_2)h_2(t_4)\Big) \\
&\quad + \mathrm{e}^{t_1+t_2}\Big(h_1(t_4) - (h_1(t_3)\Big)\Big(h_2(t_1)h_3(t_2) - h_2(t_2)h_3(t_1)\Big) \\
&= -4l\frac{\mathrm{e}^{L(t_1+t_3)} + \mathrm{e}^{L(t_2+t_4)} - 2}{\rho_1}\lambda^6 - 4l\,(m_2\rho_1 + m_1\rho_2)\,\frac{\mathrm{e}^{L(t_1+t_3)} - \mathrm{e}^{L(t_2+t_4)}}{m_1 m_2 \rho_1 r_1}\lambda^5 \\
&\quad - 4l\,(\gamma_1 m_2 + \gamma_2 m_1)\,\frac{\mathrm{e}^{L(t_1+t_3)} + \mathrm{e}^{L(t_2+t_4)} - 2}{m_1 m_2 \rho_1}\lambda^{4+\alpha} \\
&\quad + \Bigg(-l^2\frac{\mathrm{e}^{L(t_1+t_3)} + \mathrm{e}^{L(t_1+t_4)} + \mathrm{e}^{L(t_2+t_3)} + \mathrm{e}^{L(t_2+t_4)} - 4}{\rho_1 r_1^2} - 4bl\frac{\mathrm{e}^{L(t_1+t_3)} + \mathrm{e}^{L(t_2+t_4)} + 2}{m_1 m_2} \\
&\quad + 2ibl^{\frac{3}{2}}\left(\frac{1}{m_1\rho_2} - \frac{1}{m_2\rho_1}\right)\left(\mathrm{e}^{L(t_1+t_4)} - \mathrm{e}^{L(t_2+t_3)}\right)\Bigg)\lambda^4 \\
&\quad - 4l\left(\eta\,(\gamma_1 m_2 + \gamma_2 m_1)\,(\alpha-1)\,\frac{\mathrm{e}^{L(t_1+t_3)} + \mathrm{e}^{L(t_2+t_4)} - 2}{m_1 m_2 \rho_1} + (\gamma_2\rho_1 + \gamma_1\rho_2)\,\frac{\mathrm{e}^{L(t_1+t_3)} - \mathrm{e}^{L(t_2+t_4)}}{m_1 m_2 \rho_1 r_1}\right)\lambda^{3+\alpha} \\
&\quad - \frac{1}{2}\frac{bl^2}{r_1}\left(\frac{1}{m_1\rho_2} + 5\frac{1}{m_2\rho_1}\right)\left(\mathrm{e}^{L(t_1+t_3)} - \mathrm{e}^{Lt_2+Lt_4}\right)\lambda^3 \\
&\quad - 4l\frac{\gamma_1\gamma_2}{m_1 m_2 \rho_1}(\mathrm{e}^{L(t_1+t_3)} + \mathrm{e}^{L(t_2+t_4)} - 2)\lambda^{2+2\alpha} + \frac{8l\eta(1-\alpha)\gamma_1\gamma_2}{m_1 m_2 \rho_1}(\mathrm{e}^{L(t_1+t_3)} + \mathrm{e}^{L(t_2+t_4)} - 2)\lambda^{1+2\alpha}
\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{1}{\rho_1}\left(\frac{\gamma_1}{m_1}+\frac{\gamma_2}{m_2}\right)\left(-4l\eta^2-\frac{l^2}{r_1^2}+6ld\eta^2-2ld^2\eta^2\right)(\mathrm{e}^{L(t_1+t_3)}+\mathrm{e}^{L(t_2+t_4)}-2)\right.\\
&+\frac{4lb(1-\alpha)\eta r_1}{m_1m_2}\left(\frac{\gamma_1}{\rho_1}+\frac{\gamma_2}{\rho_2}\right)(\mathrm{e}^{L(t_1+t_3)}-\mathrm{e}^{L(t_2+t_4)})-\frac{l^2}{r_1^2\rho_1}\left(\frac{\gamma_1}{m_1}+\frac{\gamma_2}{m_2}\right)(\mathrm{e}^{L(t_1+t_4)}+\mathrm{e}^{L(t_2+t_3)}-2)\\
&\left.-2ib\frac{l^{3/2}}{m_1m_2}\left(\frac{\gamma_1}{\rho_1}+\frac{\gamma_2}{\rho_2}\right)(\mathrm{e}^{L(t_1+t_4)}-\mathrm{e}^{L(t_2+t_3)})\right]\lambda^{2+\alpha}\\
&+\left[\frac{l^3}{\rho_1r_1^4}(\mathrm{e}^{L(t_1+t_3)}+\mathrm{e}^{L(t_2+t_4)}+\mathrm{e}^{L(t_1+t_4)}+\mathrm{e}^{L(t_2+t_3)}-4)\right.\\
&-\frac{bl^2}{\rho_1r_1^4}(\mathrm{e}^{L(t_1+t_3)}+\mathrm{e}^{L(t_2+t_4)}+\mathrm{e}^{L(t_1+t_4)}+\mathrm{e}^{L(t_2+t_3)}+4)\\
&\left.-\frac{1}{4}i\frac{l^{5/2}b}{r_1^2}\left(\frac{1}{m_1\rho_2}+\frac{3}{m_2\rho_1}\right)(\mathrm{e}^{L(t_1+t_4)}-\mathrm{e}^{L(t_2+t_3)})\right]\lambda^2+o(\lambda^{1+\alpha}) \qquad (48)
\end{aligned}
$$

$$
\begin{aligned}
=&-\frac{4l}{\rho_1}\lambda^6\left[(\mathrm{e}^{L(t_1+t_3)}+\mathrm{e}^{L(t_2+t_4)}-2)+\frac{(m_2\rho_1+m_1\rho_2)}{m_1m_2r_1}\frac{\mathrm{e}^{L(t_1+t_3)}-\mathrm{e}^{L(t_2+t_4)}}{\lambda}\right.\\
&+\frac{(\gamma_1m_2+\gamma_2m_1)}{m_1m_2}\frac{\mathrm{e}^{L(t_1+t_3)}+\mathrm{e}^{L(t_2+t_4)}-2}{\lambda^{2-\alpha}}\\
&+\left(l\frac{\mathrm{e}^{L(t_1+t_3)}+\mathrm{e}^{L(t_1+t_4)}+\mathrm{e}^{L(t_2+t_3)}+\mathrm{e}^{L(t_2+t_4)}-4}{4r_1^2}+b\rho_1\frac{\mathrm{e}^{L(t_1+t_3)}+\mathrm{e}^{L(t_2+t_4)}+2}{m_1m_2}\right.\\
&\left.-ib\frac{\rho_1}{2}l^{\frac{1}{2}}\left(\frac{1}{m_1\rho_2}-\frac{1}{m_2\rho_1}\right)\left(\mathrm{e}^{L(t_1+t_4)}-\mathrm{e}^{L(t_2+t_3)}\right)\right)\frac{1}{\lambda^2}\\
&+\left(\eta\left(\gamma_1m_2+\gamma_2m_1\right)(\alpha-1)\frac{\mathrm{e}^{L(t_1+t_3)}+\mathrm{e}^{L(t_2+t_4)}-2}{m_1m_2}-(\gamma_2\rho_1+\gamma_1\rho_2)\frac{\mathrm{e}^{L(t_1+t_3)}-\mathrm{e}^{L(t_2+t_4)}}{m_1m_2r_1}\right)\frac{1}{\lambda^{3-\alpha}}\\
&+\frac{1}{8}\frac{bl\rho_1}{r_1}\left(\frac{1}{m_1\rho_2}+5\frac{1}{m_2\rho_1}\right)\left(\mathrm{e}^{L(t_1+t_3)}-\mathrm{e}^{Lt_2+Lt_4}\right)\frac{1}{\lambda^3}\\
&+\frac{\gamma_1\gamma_2}{m_1m_2}(\mathrm{e}^{L(t_1+t_3)}+\mathrm{e}^{L(t_2+t_4)}-2)\frac{1}{\lambda^{4-2\alpha}}-2\eta(1-\alpha)\frac{\gamma_1\gamma_2}{m_1m_2}(\mathrm{e}^{L(t_1+t_3)}+\mathrm{e}^{L(t_2+t_4)}-2)\frac{1}{\lambda^{5-2\alpha}}\\
&-\left[\frac{1}{4}\left(\frac{\gamma_1}{m_1}+\frac{\gamma_2}{m_2}\right)\left(-4\eta^2-\frac{l}{r_1^2}+6\alpha\eta^2-2\alpha^2\eta^2\right)(\mathrm{e}^{L(t_1+t_3)}+\mathrm{e}^{L(t_2+t_4)}-2)\right.\\
&+b(1-\alpha)\eta\rho_1\frac{r_1}{m_1m_2}\left(\frac{\gamma_1}{\rho_1}+\frac{\gamma_2}{\rho_2}\right)(\mathrm{e}^{L(t_1+t_3)}-\mathrm{e}^{L(t_2+t_4)})-\frac{l}{4r_1^2}\left(\frac{\gamma_1}{m_1}+\frac{\gamma_2}{m_2}\right)(\mathrm{e}^{L(t_1+t_4)}+\mathrm{e}^{L(t_2+t_3)}-2)\\
&\left.-ib\rho_1\frac{l^{1/2}}{2m_1m_2}\left(\frac{\gamma_1}{\rho_1}+\frac{\gamma_2}{\rho_2}\right)(\mathrm{e}^{L(t_1+t_4)}-\mathrm{e}^{L(t_2+t_3)})\right]\frac{1}{\lambda^{4-\alpha}}\\
&-\left[\frac{l^2}{4r_1^4}(\mathrm{e}^{L(t_1+t_3)}+\mathrm{e}^{L(t_2+t_4)}+\mathrm{e}^{L(t_1+t_4)}+\mathrm{e}^{L(t_2+t_3)}-4)\right.\\
&-\frac{bl}{4r_1^4}(\mathrm{e}^{L(t_1+t_3)}+\mathrm{e}^{L(t_2+t_4)}+\mathrm{e}^{L(t_1+t_4)}+\mathrm{e}^{L(t_2+t_3)}+4)\\
&\left.\left.-\frac{1}{16}i\frac{l^{3/2}b\rho_1}{r_1^2}\left(\frac{1}{m_1\rho_2}+\frac{3}{m_2\rho_1}\right)(\mathrm{e}^{L(t_1+t_4)}-\mathrm{e}^{L(t_2+t_3)})\right]\frac{1}{\lambda^4}\right]+o\left(\frac{1}{\lambda^{5-\alpha}}\right).
\end{aligned}
$$

We set

$$\begin{aligned}\tilde{f}(\lambda) &= f_0(\lambda) + \frac{f_1(\lambda)}{\lambda} + \frac{f_2(\lambda)}{\lambda^{2-\alpha}} + \frac{f_3(\lambda)}{\lambda^2} + \frac{f_4(\lambda)}{\lambda^{3-\alpha}} + \frac{f_5(\lambda)}{\lambda^3} + \frac{f_6(\lambda)}{\lambda^{4-\alpha}} \\ &\quad + \frac{f_7(\lambda)}{\lambda^{4-2\alpha}} + \frac{f_8(\lambda)}{\lambda^{5-2\alpha}} + \frac{f_9(\lambda)}{\lambda^4} + o\left(\frac{1}{\lambda^{5-\alpha}}\right) \qquad (49)\end{aligned}$$

$$f_0(\lambda) = e^{L(t_1+t_3)} + e^{L(t_2+t_4)} - 2 = e^{-L(t_1+t_3)}(e^{L(t_1+t_3)} - 1)^2 \qquad (50)$$

$$\begin{aligned}f_1(\lambda) &= \frac{(m_2\rho_1 + m_1\rho_2)}{m_1m_2r_1}(e^{(t_1+t_3)L} - e^{-(t_1+t_3)L}) \\ &= \frac{(m_2\rho_1 + m_1\rho_2)}{m_1m_2r_1}e^{-(t_1+t_3)L}(e^{(t_1+t_3)L} - 1)(e^{(t_1+t_3)L} + 1) \qquad (51)\end{aligned}$$

$$\begin{aligned}f_2(\lambda) &= \frac{(\gamma_1m_2 + \gamma_2m_1)}{m_1m_2}(e^{L(t_1+t_3)} + e^{L(t_2+t_4)} - 2) \\ &= \frac{(\gamma_1m_2 + \gamma_2m_1)}{m_1m_2}e^{-L(t_1+t_3)}(e^{L(t_1+t_3)} - 1)^2 \qquad (52)\end{aligned}$$

$$\begin{aligned}f_3(\lambda) &= \Bigg(l\frac{e^{L(t_1+t_3)} + e^{L(t_1+t_4)} + e^{L(t_2+t_3)} + e^{L(t_2+t_4)} - 4}{4r_1^2} + b\rho_1\frac{e^{L(t_1+t_3)} + e^{L(t_2+t_4)} + 2}{m_1m_2} \\ &\quad - ib\frac{\rho_1}{2}l^{\frac{1}{2}}\left(\frac{1}{m_1\rho_2} - \frac{1}{m_2\rho_1}\right)\left(e^{L(t_1+t_4)} - e^{L(t_2+t_3)}\right)\Bigg) \\ &= e^{-L(t_1+t_3)}\Bigg(\frac{l}{4r_1^2}((e^{L(t_1+t_3)} - 1)^2 + (e^{Lt_1} - e^{Lt_3})^2) + \frac{b\rho_1}{m_1m_2}(e^{L(t_1+t_3)} + 1)^2 \\ &\quad - ib\frac{\rho_1}{2}l^{\frac{1}{2}}\left(\frac{1}{m_1\rho_2} - \frac{1}{m_2\rho_1}\right)\left(e^{2Lt_1} - e^{2Lt_3}\right)\Bigg) \qquad (53)\end{aligned}$$

$$f_4(\lambda) = e^{-L(t_1+t_3)}\left(\frac{\eta(\alpha-1)(\gamma_1m_2 + \gamma_2m_1)}{m_1m_2}(e^{L(t_1+t_3)} - 1)^2 + \frac{(\gamma_2\rho_1 + \gamma_1\rho_2)}{m_1m_2r_1}(e^{2L(t_1+t_3)} - 1)\right). \qquad (54)$$

$$f_5(\lambda) = -\frac{1}{2}\frac{bl^2}{r_1}\left(\frac{1}{m_1\rho_2} + 5\frac{1}{m_2\rho_1}\right)e^{-L(t_1+t_3)}\left(e^{2L(t_1+t_3)} - 1\right) \qquad (55)$$

$$\begin{aligned}f_6(\lambda) &= -\Bigg[\frac{1}{4}\left(\frac{\gamma_1}{m_1} + \frac{\gamma_2}{m_2}\right)\left(-4\eta^2 - \frac{l}{r_1^2} + 6\alpha\eta^2 - 2\alpha^2\eta^2\right)(e^{L(t_1+t_3)} - 1)^2 \\ &\quad + b(1-\alpha)\eta\rho_1\frac{r_1}{m_1m_2}\left(\frac{\gamma_1}{\rho_1} + \frac{\gamma_2}{\rho_2}\right)(e^{2L(t_1+t_3)} - 1) - \frac{l}{4r_1^2}\left(\frac{\gamma_1}{m_1} + \frac{\gamma_2}{m_2}\right)(e^{Lt_1} - e^{Lt_3})^2 \\ &\quad - ib\rho_1\frac{l^{1/2}}{2m_1m_2}\left(\frac{\gamma_1}{\rho_1} + \frac{\gamma_2}{\rho_2}\right)(e^{2Lt_1} - e^{2Lt_3})\Bigg] \qquad (56)\end{aligned}$$

$$f_7(\lambda) = \frac{\gamma_1\gamma_2}{m_1m_2}e^{-L(t_1+t_3)}(e^{L(t_1+t_3)} - 1)^2 \qquad (57)$$

$$f_8(\lambda) = -2\eta(1-\alpha)\frac{\gamma_1\gamma_2}{m_1m_2}e^{-L(t_1+t_3)}(e^{L(t_1+t_3)} - 1)^2 \qquad (58)$$

$$\begin{aligned}f_9(\lambda) &= -e^{-L(t_1+t_3)}\Bigg[\frac{l^2}{4r_1^4}((e^{L(t_1+t_3)} - 1)^2 + (e^{Lt_1} - e^{Lt_3})^2) \\ &\quad - \frac{bl}{4r_1^4}((e^{L(t_1+t_3)} + 1)^2 + (e^{Lt_1} + e^{Lt_3})^2) - \frac{1}{16}i\frac{l^{3/2}b\rho_1}{r_1^2}\left(\frac{1}{m_1\rho_2} + \frac{3}{m_2\rho_1}\right)(e^{2Lt_1} - e^{2Lt_3})\Bigg]. \qquad (59)\end{aligned}$$

□

Lemma 4.1. (Asymptotic behavior of the large eigenvalues of $\mathcal{A}$) *The large eigenvalues of $\mathcal{A}$ can be split into two families $(\lambda_k^j)_{k\in\mathbf{Z},|k|\geq k_0}, j=1,2,$ ($k_0\in\mathbb{N}$ chosen large enough). The following asymptotic expansions hold:*

$$\lambda_k^1=\frac{i}{Lr_1}k\pi+o(1),\quad \lambda_k^2=\frac{i}{Lr_1}k\pi+o(1). \tag{60}$$

Either $\lambda_k^1=\lambda_k^2$ and this root is of order 2, or $\lambda_k^1\neq\lambda_k^2$ and these two roots are simple.

Proof. The multiplicity of the roots of f_0 given by (60) is two, and λ is a root of f_0 if and only if

$$(t_1+t_3)L=2ik\pi.$$

Since $t_1+t_3=2r_1\lambda+\frac{1}{4}\frac{1}{r_1}l\frac{1}{\lambda}+o(\frac{1}{\lambda})$. we deduce that, for each $k\in\mathbf{Z}$, with $|k|$ large enough, corresponds a double root of f_0; denoted by λ_k^0 which satisfies

$$\lambda_k^0=\frac{i}{Lr_1}k\pi+O\left(\frac{1}{k}\right).$$

We will now use Rouché's theorem. Let $B_k=B(\frac{i}{Lr_1}k\pi,r_k)$ be the ball of centrum $ik\pi$ and radius $r_k=\frac{1}{k^{\frac{1}{4}}}$ and $\lambda\in\partial B_k$ (i.e., $\lambda=\frac{i}{Lr_1}k\pi+r_k\mathrm{e}^{i\theta},\theta\in[0,2\pi]$). Then we successively have:

$$\begin{aligned}L(t_1+t_3)(\lambda)&=2ik\pi+2Lr_1r_k\mathrm{e}^{i\theta}+O\left(\frac{1}{k}\right)\\ \mathrm{e}^{L(t_1+t_3)(\lambda)}&=\mathrm{e}^{2Lr_1r_k\mathrm{e}^{i\theta}+O\left(\frac{1}{k}\right)}\\ &=1+2Lr_1r_k\mathrm{e}^{i\theta}+O(r_k^2).\end{aligned}$$

and

$$\begin{aligned}f_0(\lambda)&=\left(1-2Lr_1r_k\mathrm{e}^{i\theta}+O(r_k^2)\right)\left(2Lr_1r_k\mathrm{e}^{i\theta}+O(r_k^2)\right)^2\\ &=\left(1-2Lr_1r_k\mathrm{e}^{i\theta}+O(r_k^2)\right)\left(4L^2r_1^2r_k^2\mathrm{e}^{2i\theta}+O(r_k^3)\right)\\ &=4r_1^2L^2r_k^2\mathrm{e}^{2i\theta}+O(r_k^3).\end{aligned}$$

It follows that there exists a positive constant c such that

$$\forall\lambda\in\partial B_k,\quad |f_0(\lambda)|\geq cr_k^2=\frac{c}{\sqrt{k}}.$$

Then we deduce from (49) that $|f(\lambda)-f_0(\lambda)|=O\left(\frac{1}{\lambda}\right)=O\left(\frac{1}{k}\right)$. It follows that, for $|k|$ large enough

$$\forall\lambda\in\partial B_k,\quad |f(\lambda)-f_0(\lambda)|<|f_0(\lambda)|.$$

Since the imaginary axis is an asymptote for the spectrum of $\mathcal{A}$ then system (12) is not uniformly stable. □

More information concerning the asymptotic behavior of the spectrum of $\mathcal{A}$ is given by:

Proposition 4.1. (Asymptotic expansions for the eigenvalues of $\mathcal{A}$) *Assume Condition*

$$\frac{\rho_1}{m_1}\neq\frac{\rho_2}{m_2}\quad or\quad L\sqrt{l}\neq 2k\pi,\quad k\in\mathbb{N}^*. \tag{H}$$

Then the large eigenvalues of the dissipative operator $\mathcal{A}$ are simple and can be split into two families $(\lambda_k^j)_{k\in\mathbf{Z},|k|\geq k_0}, j=1,2,(k_0\in\mathbb{N}$, chosen large enough). Moreover, we have the following asymptotic expansions for the eigenvalues of $\mathcal{A}$:

$$\lambda_k^1 = \frac{i}{Lr_1}k\pi + \frac{iq_1}{k} + \frac{\tilde{\alpha}_1}{k^{3-\alpha}} + \frac{\tilde{q}_1}{|k|^{3-\alpha}} + o\left(\frac{1}{k^{3-\alpha}}\right),\ q_1 \in \mathbb{R}, \tilde{\alpha}_1 \in i\mathbb{R}, \tilde{q}_1 \in \mathbb{R}, \tilde{q}_1 < 0, k \geq k_0$$
$$\lambda_k^1 = \overline{\lambda_{-k}^1}, k \leq -k_0.$$
$$\lambda_k^2 = \frac{i}{Lr_1}k\pi + \frac{iq_2}{k} + \frac{\tilde{\alpha}_2}{k^{3-\alpha}} + \frac{\tilde{q}_2}{|k|^{3-\alpha}} + o\left(\frac{1}{k^{3-\alpha}}\right),\ q_2 \in \mathbb{R}, \tilde{\alpha}_2 \in i\mathbb{R}, \tilde{q}_2 \in \mathbb{R}, \tilde{q}_2 < 0, k \geq k_0$$
$$\lambda_k^2 = \overline{\lambda_{-k}^2}, k \leq -k_0.$$

Proof. Let $\lambda_k = \lambda_k^j$ with $j = 1$ or $j = 2$. It follows

$$\lambda_k = \frac{i}{Lr_1}k\pi + \varepsilon_k. \tag{61}$$

Using (34)–(35), we get

$$\begin{aligned}
(t_1+t_3)L &= 2r_1\lambda_k + \frac{lL}{4r_1}\frac{1}{\lambda_k} - \frac{5}{64}\frac{L^2l^2}{r_1^3}\frac{1}{\lambda^3} + O\left(\frac{1}{\lambda^3}\right)\\
&= 2ik\pi + 2Lr_1\varepsilon_k - i\frac{lL^2}{4k\pi} + o\left(\frac{1}{k^2}\right) + o(\varepsilon_k)\\
2t_1L &= 2r_1\lambda_k + iL\sqrt{l} + \frac{lL}{4r_1}\frac{1}{\lambda_k} - \frac{i}{8}\frac{Ll\sqrt{l}}{r_1^2}\frac{1}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right)\\
&= 2ik\pi + 2Lr_1\varepsilon_k + iL\sqrt{l} - i\frac{lL^2}{4k\pi} + \frac{i}{8}\frac{L^3l\sqrt{l}}{\pi^2k^2} + o\left(\frac{1}{k^2}\right) + o(\varepsilon_k)\\
2t_3L &= 2r_1\lambda_k - iL\sqrt{l} + \frac{lL}{4r_1}\frac{1}{\lambda_k} + \frac{i}{8}\frac{Ll\sqrt{l}}{r_1^2}\frac{1}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right)\\
&= 2ik\pi + 2Lr_1\varepsilon_k - iL\sqrt{l} - i\frac{lL^2}{4k\pi} - \frac{i}{8}\frac{L^3l\sqrt{l}}{\pi^2k^2} + o\left(\frac{1}{k^2}\right) + o(\varepsilon_k).
\end{aligned} \tag{62}$$

It follows that

$$\begin{aligned}
e^{L(t_1+t_3)} &= 1 + 2Lr_1\varepsilon_k - i\frac{lL^2}{4k\pi} - \frac{l^2L^4}{32\pi^2k^2} - i\frac{lL^3r_1}{2\pi}\frac{\varepsilon_k}{k} + o(\varepsilon_k) + o\left(\frac{\varepsilon_k}{k}\right) + o\left(\frac{1}{k^2}\right)\\
e^{2Lt_1} &= e^{iL\sqrt{l}}\left(1 + 2Lr_1\varepsilon_k - i\frac{lL^2}{4k\pi} - \frac{l^2L^4}{32\pi^2k^2} - i\frac{lL^3r_1}{2\pi}\frac{\varepsilon_k}{k} + \frac{i}{8}\frac{L^3l\sqrt{l}}{\pi^2k^2} + o\left(\frac{\varepsilon_k}{k}\right) + o\left(\frac{1}{k^2}\right)\right)\\
e^{2Lt_3} &= e^{-iL\sqrt{l}}\left(1 + 2Lr_1\varepsilon_k - i\frac{lL^2}{4k\pi} - \frac{l^2L^4}{32\pi^2k^2} - i\frac{lL^3r_1}{2\pi}\frac{\varepsilon_k}{k} - \frac{i}{8}\frac{L^3l\sqrt{l}}{\pi^2k^2} + o\left(\frac{\varepsilon_k}{k}\right) + o\left(\frac{1}{k^2}\right)\right).
\end{aligned} \tag{63}$$

Using (49), inserting (63) into $f(\lambda_k)$ and keeping only the terms greater than or equal to $O(\frac{1}{k^2})$, we obtain after calculations

$$\begin{aligned}
f(\lambda_k) &= 4L^2r_1^2\varepsilon_k^2 - \left(4\tfrac{i}{\pi}\tfrac{L^2}{k}mr_1^2 + \tfrac{i}{\pi}\tfrac{L^3}{k}lr_1\right)\varepsilon_k - \left(\tfrac{1}{16\pi^2}\tfrac{L^4}{k^2}l^2 + \tfrac{1}{2\pi^2}\tfrac{L^3}{k^2}lmr_1\right.\\
&\quad \left.+ \frac{L^2r_1^2}{\pi^2k^2}\left(4\tilde{\tilde{m}} - 4A\sin^2\left(\frac{L\sqrt{l}}{2}\right) - B\sin(L\sqrt{l})\right)\right) + o(\varepsilon_k^2) + o\left(\frac{\varepsilon_k}{k}\right) + o\left(\frac{1}{k^2}\right) = 0,
\end{aligned} \tag{64}$$

where

$$m = \frac{m_2\rho_1 + m_1\rho_2}{m_1m_2r_1},\quad \tilde{\tilde{m}} = \frac{b\rho_1}{m_1m_2},\quad A = \frac{l}{4r_1^2},\quad B = b\rho_1\sqrt{l}\left(\frac{1}{m_1\rho_2} - \frac{1}{m_2\rho_1}\right).$$

Multiplying (64) by k^2 leads to:

$$4L^2r_1^2(k\varepsilon_k)^2 - i\left(4\frac{1}{\pi}L^2mr_1^2 + \frac{1}{\pi}L^3lr_1\right)(k\varepsilon_k)$$
$$-\left(\frac{L^4l^2}{16\pi^2} + \frac{L^3lmr_1}{2\pi^2} + \frac{L^2r_1^2}{\pi^2}\left(4\tilde{\tilde{m}} - 4A\sin^2\left(\frac{L\sqrt{l}}{2}\right) - B\sin(L\sqrt{l})\right)\right) + o(1) + o(k\varepsilon_k) + o(k^2\varepsilon_k^2) = 0.$$

Thus, $k\varepsilon_k$ is bounded and

$$4L^2r_1^2(k\varepsilon_k)^2 - i\left(4\frac{L^2mr_1^2}{\pi} + \frac{L^3lr_1}{\pi}\right)(k\varepsilon_k)$$
$$-\left(\frac{L^4l^2}{16\pi^2} + \frac{L^3lmr_1}{2\pi^2} + \frac{L^2r_1^2}{\pi^2}\left(4\tilde{\tilde{m}} - 4A\sin^2\left(\frac{L\sqrt{l}}{2}\right) - B\sin(L\sqrt{l})\right)\right) + o(1) = 0.$$

The previous equation has two solutions

$$k\varepsilon_k = \tfrac{1}{8\pi r_1}\left(4imr_1 - 4r_1\sqrt{4\tilde{\tilde{m}} - 2A + 2A\cos L\sqrt{l} - B\sin L\sqrt{l} - m^2} + iLl\right) \text{ or}$$
$$k\varepsilon_k = \tfrac{1}{8\pi r_1}\left(4imr_1 + 4r_1\sqrt{4\tilde{\tilde{m}} - 2A + 2A\cos L\sqrt{l} - B\sin L\sqrt{l} - m^2} + iLl\right).$$

It holds:

$$\varepsilon_k = \tfrac{1}{8\pi r_1 k}\left(4imr_1 - 4r_1\sqrt{4\tilde{\tilde{m}} - 2A + 2A\cos L\sqrt{l} - B\sin L\sqrt{l} - m^2} + iLl\right) \quad \text{or}$$
$$\varepsilon_k = \tfrac{1}{8\pi r_1 k}\left(4imr_1 + 4r_1\sqrt{4\tilde{\tilde{m}} - 2A + 2A\cos L\sqrt{l} - B\sin L\sqrt{l} - m^2} + iLl\right).$$

Set

$$\begin{aligned} P &= 4\tilde{\tilde{m}} - 2A + 2A\cos(L\sqrt{l}) - B\sin(L\sqrt{l}) - m^2 \\ &= 4\tilde{\tilde{m}} - m^2 - 2A + 2A\cos(L\sqrt{l}) - B\sin(L\sqrt{l}). \end{aligned}$$

As $r_1^2 = r_2^2 = \dfrac{\rho_2}{b}$, we deduce that

$$4\tilde{\tilde{m}} - m^2 = -\frac{1}{r_1^2}\left(\frac{\rho_1}{m_1} - \frac{\rho_2}{m_2}\right)^2.$$

Then

$$\begin{aligned} P &= -\frac{1}{r_1^2}\left(\frac{\rho_1}{m_1} - \frac{\rho_2}{m_2}\right)^2 - \frac{\sqrt{l}}{r_1^2}\left(\frac{\rho_1}{m_1} - \frac{\rho_2}{m_2}\right)\sin(L\sqrt{l}) - \frac{1}{2}\frac{l}{r_1^2} + \frac{1}{2}\frac{l}{r_1^2}\cos(L\sqrt{l}) \\ &= -\frac{1}{r_1^2}\left(\frac{\sqrt{l}}{2}\sin(L\sqrt{l}) + \left(\frac{\rho_1}{m_1} - \frac{\rho_2}{m_2}\right)\right)^2 - \frac{l}{4r_1^2}(\cos(L\sqrt{l}) - 1)^2. \end{aligned}$$

Hence

$$\varepsilon_k = \frac{i}{8\pi r_1 k}\left(4mr_1 - 2\sqrt{\left(\sqrt{l}\sin(L\sqrt{l}) + 2\left(\frac{\rho_1}{m_1} - \frac{\rho_2}{m_2}\right)\right)^2 + (\cos(L\sqrt{l}) - 1)^2} + Ll\right) + o\left(\frac{1}{k}\right) \quad \text{or}$$
$$\varepsilon_k = \frac{i}{8\pi r_1 k}\left(4mr_1 + 2\sqrt{\left(\sqrt{l}\sin(L\sqrt{l}) + 2\left(\frac{\rho_1}{m_1} - \frac{\rho_2}{m_2}\right)\right)^2 + (\cos(L\sqrt{l}) - 1)^2} + Ll\right) + o\left(\frac{1}{k}\right).$$

Step 2. From Step 1, we can write

$$\lambda_k^1 = \frac{i}{Lr_1}k\pi + i\frac{q_1}{k} + \frac{\varepsilon_k^1}{k},$$
$$\lambda_k^2 = \frac{i}{Lr_1}k\pi + i\frac{q_2}{k} + \frac{\varepsilon_k^2}{k},$$

where $\varepsilon_k^j = o(1)$.

$$\begin{aligned}
(t_1+t_3)L &= i\frac{2r_1Lq_j}{k} + \frac{2r_1L\varepsilon_k^j}{k} - i\frac{lL^2}{4k\pi} + i\frac{lL^3r_1}{4k^3\pi^2}q_j + \frac{lL^3r_1}{4k^3\pi^2}\varepsilon_k^j - i\frac{5}{64}\frac{l^2L^4}{k^3\pi^3}\\
Lt_1 &= k\pi i + i\frac{L\sqrt{l}}{2} + i\frac{L}{k}r_1q_j - \frac{1}{8}\frac{i}{\pi}\frac{L^2}{k}l - \frac{1}{2}\frac{L^2}{k^2}r_1^2q_j^2 + \frac{L}{k}\varepsilon_k^j r_1 + \frac{i}{8}\frac{L^3l\sqrt{l}}{\pi^2k^2}\\
Lt_3 &= k\pi i - i\frac{L\sqrt{l}}{2} + i\frac{L}{k}r_1q_j - \frac{1}{8}\frac{i}{\pi}\frac{L^2}{k}l - \frac{1}{2}\frac{L^2}{k^2}r_1^2q_j^2 + \frac{L}{k}\varepsilon_k^j r_1 - \frac{i}{8}\frac{L^3l\sqrt{l}}{\pi^2k^2}\\
\mathrm{e}^{(t_1+t_3)L} - 1 &= 2i\frac{L}{k}r_1q_j + 2\frac{L^2}{k^2}(\varepsilon_k^j)^2r_1^2 - \frac{1}{4}\frac{i}{\pi}\frac{L^2}{k}l - 2\frac{L^2}{k^2}r_1^2q_j^2 + 2\frac{L}{k}\varepsilon_k^j r_1 - \frac{1}{32\pi^2}\frac{L^4}{k^2}l^2 - \frac{5}{64}\frac{i}{\pi^3}\frac{L^4}{k^3}l^2\\
&\quad - \frac{5}{256\pi^4}\frac{L^6}{k^4}l^3 + 4i\frac{L^2}{k^2}\varepsilon_k^j r_1^2q_j + \frac{7}{32\pi^3}\frac{L^5}{k^4}l^2r_1q_j - \frac{1}{2}\frac{i}{\pi}\frac{L^3}{k^2}l\varepsilon_k^j r_1 + \frac{1}{4\pi^2}\frac{L^3}{k^3}l\varepsilon_k^j r_1 - \frac{1}{2\pi^2}\frac{L^4}{k^4}lr_1^2q_j^2\\
&\quad + \frac{1}{2\pi}\frac{L^3}{k^2}lr_1q_j + \frac{1}{4}\frac{i}{\pi^2}\frac{L^3}{k^3}lr_1q_j.\\
\mathrm{e}^{t_1L} &= (-1)^k\mathrm{e}^{\frac{iL\sqrt{l}}{2}}\left(1 + i\frac{L}{k}r_1q_j - \frac{1}{8}\frac{i}{\pi}\frac{L^2}{k}l - \frac{1}{2}\frac{L^2}{k^2}r_1^2q_j^2 + \frac{L}{k}\varepsilon_k^j r_1 - \frac{1}{128\pi^2}\frac{L^4}{k^2}l^2\right.\\
&\quad \left. + \frac{1}{16}\frac{i}{\pi^2}\frac{L^3}{k^2}l^{\frac{3}{2}} + i\frac{L^2}{k^2}\varepsilon_k^j r_1^2q_j - \frac{1}{8}\frac{i}{\pi}\frac{L^3}{k^2}l\varepsilon_k^j r_1 + \frac{1}{8\pi}\frac{L^3}{k^2}lr_1q_j\right)\\
\mathrm{e}^{t_3L} &= (-1)^k\mathrm{e}^{-\frac{iL\sqrt{l}}{2}}\left(1 + i\frac{L}{k}r_1q_j - \frac{1}{8}\frac{i}{\pi}\frac{L^2}{k}l - \frac{1}{2}\frac{L^2}{k^2}r_1^2q_j^2 + \frac{L}{k}\varepsilon_k^j r_1 - \frac{1}{128\pi^2}\frac{L^4}{k^2}l^2\right.\\
&\quad \left. - \frac{1}{16}\frac{i}{\pi^2}\frac{L^3}{k^2}l^{\frac{3}{2}} + i\frac{L^2}{k^2}\varepsilon_k^j r_1^2q_j - \frac{1}{8}\frac{i}{\pi}\frac{L^3}{k^2}l\varepsilon_k^j r_1 + \frac{1}{8\pi}\frac{L^3}{k^2}lr_1q_j\right).
\end{aligned}$$

Using (49), Taylor series and simplification in the term of order $1/k^2$ coming from Step 1, we get

$$\begin{aligned}
f(\lambda_k) &= \left(4\frac{L^2}{k^2}r_1^2\right)(\varepsilon_k^j)^2 + \left(8i\frac{L^2}{k^2}r_1^2q_j - 24\frac{L^3}{k^3}r_1^3q_j^2 - 4\frac{i}{\pi}\frac{L^2}{k^2}mr_1^2 - \frac{3}{8\pi^2}\frac{L^5}{k^3}l^2r_1 - \frac{i}{\pi}\frac{L^3}{k^2}lr_1 - \frac{2}{\pi^2}\frac{L^4}{k^3}lmr_1^2\right.\\
&\quad \left. + \frac{6}{\pi}\frac{L^4}{k^3}lr_1^2q_j + \frac{16}{\pi}\frac{L^3}{k^3}mr_1^3q_j\right)\varepsilon_k^j + \left(4\frac{L^4}{k^4}r_1^4q_j^4 - 8i\frac{L^3}{k^3}r_1^3q_j^3 + \frac{1}{64}\frac{i}{\pi^3}\frac{L^6}{k^3}l^3 - \frac{5}{128\pi^4}\frac{L^6}{k^4}l^3 + \frac{1}{1024\pi^4}\frac{L^8}{k^4}l^4\right.\\
&\quad + \frac{1}{8}\frac{i}{\pi^3}\frac{L^5}{k^3}l^2mr_1 - \frac{5}{32\pi^4}\frac{L^5}{k^4}l^2mr_1 + \frac{1}{64\pi^4}\frac{L^7}{k^4}l^3mr_1 + \frac{3}{8\pi^2}\frac{L^6}{k^4}l^2r_1^2q_j^2 - \frac{3}{8}\frac{i}{\pi^2}\frac{L^5}{k^3}l^2r_1q_j + \frac{7}{16\pi^3}\frac{L^5}{k^4}l^2r_1q_j\\
&\quad - \frac{1}{32\pi^3}\frac{L^7}{k^4}l^3r_1q_j + 3\frac{i}{\pi}\frac{L^4}{k^3}lr_1^2q_j^2 + 8\frac{i}{\pi}\frac{L^3}{k^3}mr_1^3q_j^2 - \frac{1}{\pi^2}\frac{L^4}{k^4}lr_1^2q_j^2 - \frac{2}{\pi}\frac{L^5}{k^4}lr_1^3q_j^3 - \frac{4}{\pi^2}\frac{L^3}{k^4}mr_1^3q_j^2\\
&\quad \left. - \frac{8}{\pi}\frac{L^4}{k^4}mr_1^4q_j^3 - 2\frac{i}{\pi^2}\frac{L^4}{k^3}lmr_1^2q_j + \frac{1}{\pi^3}\frac{L^4}{k^4}lmr_1^2q_j + \frac{3}{\pi^2}\frac{L^5}{k^4}lmr_1^3q_j^2 - \frac{3}{8\pi^3}\frac{L^6}{k^4}l^2mr_1^2q_j\right)\\
&\quad + s_{2-\alpha}\left(\frac{Lr_1}{i\pi}\right)^{2-\alpha}\left(-4\frac{L^2}{k^{4-\alpha}}r_1^2q_j^2 - \frac{1}{16\pi^2}\frac{L^4}{k^{4-\alpha}}l^2 + \frac{1}{\pi}\frac{L^3}{k^{4-\alpha}}lr_1q_j\right)\\
&\quad + s_{3-\alpha}\left(\frac{Lr_1}{i\pi}\right)^{3-\alpha}\left(4i\frac{L}{k^{4-\alpha}}r_1q_j - \frac{1}{2}\frac{i}{\pi}\frac{L^2}{k^{4-\alpha}}l\right) - s_{4-\alpha}\left(\frac{Lr_1}{ik\pi}\right)^{4-\alpha}\\
&\quad + s_{41}\left(\frac{Lr_1}{ik\pi}\right)^4 + s_{42}\left(\frac{Lr_1}{ik\pi}\right)^4 + s_{43}\left(\frac{Lr_1}{ik\pi}\right)^4 + +s_3\left(4i\frac{L}{k^4}r_1q_j - \frac{1}{2}\frac{i}{\pi}\frac{L^2}{k^4}l\right)\left(\frac{Lr_1}{i\pi}\right)^3
\end{aligned}$$

$$
\begin{aligned}
&+\frac{l}{4r_1^2}\left(\frac{Lr_1}{i\pi}\right)^2\left(2i\sin\left(\frac{L\sqrt{l}}{2}\right)\right)^2\left(2i\frac{L}{k^3}r_1q_j-\frac{1}{4}\frac{i}{\pi}\frac{L^2}{k^3}l\right)+\frac{b\rho_1}{m_1m_2}\left(8i\frac{L}{k^3}r_1q_j-\frac{i}{\pi}\frac{L^2}{k^3}l\right)\left(\frac{Lr_1}{i\pi}\right)^2\\
&+\frac{l}{4r_1^2}\left(\frac{Lr_1}{i\pi}\right)^2\left(-4\frac{L^2}{k^4}r_1^2q_j^2-\frac{1}{16\pi^2}\frac{L^4}{k^4}l^2+\frac{1}{\pi}\frac{L^3}{k^4}lr_1q_j-\sin(L\sqrt{l})\frac{1}{2}\frac{1}{\pi^2}\frac{L^3}{k^4}l^{\frac{3}{2}}-4\sin^2\left(\frac{L\sqrt{l}}{2}\right)\right.\\
&\times\left.\left(-3\frac{L^2}{k^4}r_1^2q_j^2-\frac{1}{32\pi^2}\frac{L^4}{k^4}l^2+\frac{1}{2\pi}\frac{L^3}{k^4}lr_1q_j\right)\right)+\frac{b\rho_1}{m_1m_2}\left(-12\frac{L^2}{k^4}r_1^2q_j^2-\frac{3}{16\pi^2}\frac{L^4}{k^4}l^2+\frac{3}{\pi}\frac{L^3}{k^4}lr_1q_j\right)\\
&\times\left(\frac{Lr_1}{i\pi}\right)^2+2is_2\left(\frac{Lr_1}{\pi}\right)^2\sin(L\sqrt{l})\left(2i\frac{L}{k^3}r_1q_j-\frac{1}{4}\frac{i}{\pi}\frac{L^2}{k^3}l\right)+2is_2(\frac{Lr_1}{\pi})^2\sin(L\sqrt{l})\\
&\times\left(-2\frac{L^2}{k^4}r_1^2q_j^2-\frac{1}{32\pi^2}\frac{L^4}{k^4}l^2+\frac{1}{2\pi}\frac{L^3}{k^4}lr_1q_j\right)+s_2\left(\frac{Lr_1}{i\pi}\right)^2\frac{i}{4}L^3\sqrt{l}l\frac{\cos L\sqrt{l}}{\pi^2k^4}+s_{2-\alpha}\left(\frac{Lr_1}{i\pi}\right)^{2-\alpha}\\
&\times\left(-\frac{i}{\pi}\frac{L^3}{k^{4-\alpha}}l\varepsilon_k^jr_1+8i\frac{L^2}{k^{4-\alpha}}\varepsilon_k^jr_1^2q_j\right)+s_{3-\alpha}\left(\frac{Lr_1}{i\pi}\right)^{3-\alpha}\frac{4L}{k^{4-\alpha}}\varepsilon_k^jr_1+2\frac{L^3l}{\pi^2k^3}r_1\sin^2\left(\frac{L\sqrt{l}}{2}\right)\varepsilon_k^j\\
&+\frac{b\rho_1}{m_1m_2}\left(8\frac{L}{k^3}\varepsilon_k^jr_1\right)\left(\frac{Lr_1}{i\pi}\right)^2+s_{2-\alpha}\left(\frac{Lr_1}{i\pi}\right)^{2-\alpha}\left(4\frac{L^2}{k^{4-\alpha}}(\varepsilon_k^j)^2r_1^2\right)+4is_2\frac{L^3r_1^3}{\pi^2k^3}\sin(L\sqrt{l})\varepsilon_k^j,
\end{aligned}
$$

where

$$
\begin{aligned}
&s_2=-ib\frac{\rho_1}{2}\sqrt{l}\left(\frac{1}{m_1\rho_2}-\frac{1}{m_2\rho_1}\right),\ s_{2-\alpha}=\frac{\gamma_1m_2+\gamma_2m_1}{m_1m_2},\ m=\frac{\rho_1m_2+\rho_2m_1}{m_1m_2r_1},s_{3-\alpha}=\frac{\gamma_1\rho_2+\gamma_2\rho_1}{m_1m_2r_1},\\
&s_3=\frac{1}{8}\frac{lb\rho_1}{r_1}\left(\frac{1}{m_1\rho_2}+\frac{5}{m_2\rho_1}\right),\ s_{4-\alpha}=\frac{b\sqrt{l}\rho_1}{m_1m_2}\left(\frac{\gamma_1}{\rho_1}+\frac{\gamma_2}{\rho_2}\right)\sin(L\sqrt{l})\\
&s_{41}=\frac{l^2}{r_1^4}\sin^2\left(\frac{L\sqrt{l}}{2}\right),\ s_{42}=\frac{bl^2}{r_1^4}\left(\cos^2\left(\frac{L\sqrt{l}}{2}\right)+1\right),\ s_{43}=-\frac{bl^2\rho_1}{8r_1^2}\sin(L\sqrt{l}).
\end{aligned}
$$

Considering only the dominant terms of $\frac{1}{k}$, the following is obtaining:

$$
\begin{aligned}
f(\lambda_k)=&\left(4\frac{L^2}{k^2}r_1^2\right)(\varepsilon_k^j)^2+\left(8i\frac{L^2}{k^2}r_1^2q_j-4\frac{i}{\pi}\frac{L^2}{k^2}mr_1^2-\frac{i}{\pi}\frac{L^3}{k^2}lr_1\right)\varepsilon_k^j-8i\frac{L^3}{k^3}r_1^3q_j^3+\frac{1}{64}\frac{i}{\pi^3}\frac{L^6}{k^3}l^3\\
&+\frac{1}{8}\frac{i}{\pi^3}\frac{L^5}{k^3}l^2mr_1-\frac{3}{8}\frac{i}{\pi^2}\frac{L^5}{k^3}l^2r_1q_j+3\frac{i}{\pi}\frac{L^4}{k^3}lr_1^2q_j^2+8\frac{i}{\pi}\frac{L^3}{k^3}mr_1^3q_j^2-2\frac{i}{\pi^2}\frac{L^4}{k^3}lmr_1^2q_j\\
&+s_{2-\alpha}\left(\frac{Lr_1}{i\pi}\right)^{2-\alpha}\left(-4\frac{L^2}{k^{4-\alpha}}r_1^2q_j^2-\frac{1}{16\pi^2}\frac{L^4}{k^{4-\alpha}}l^2+\frac{1}{\pi}\frac{L^3}{k^{4-\alpha}}lr_1q_j\right)+s_{3-\alpha}\\
&\times\left(\frac{Lr_1}{i\pi}\right)^{3-\alpha}\left(4i\frac{L}{k^{4-\alpha}}r_1q_j-\frac{1}{2}\frac{i}{\pi}\frac{L^2}{k^{4-\alpha}}l\right)\\
&-s_{4-\alpha}\left(\frac{Lr_1}{ik\pi}\right)^{4-\alpha}+\frac{l}{4r_1^2}\left(\frac{Lr_1}{i\pi}\right)^2\left(2i\sin\left(\frac{L\sqrt{l}}{2}\right)\right)^2\left(2i\frac{L}{k^3}r_1q_j-\frac{1}{4}\frac{i}{\pi}\frac{L^2}{k^3}l\right)\\
&+\frac{b\rho_1}{m_1m_2}\left(8i\frac{L}{k^3}r_1q_j-\frac{i}{\pi}\frac{L^2}{k^3}l\right)\left(\frac{Lr_1}{i\pi}\right)^2+2is_2\left(\frac{Lr_1}{\pi}\right)^2\sin(L\sqrt{l})\left(2i\frac{L}{k^3}r_1q_j-\frac{1}{4}\frac{i}{\pi}\frac{L^2}{k^3}l\right).
\end{aligned}
$$

We remark that

$$
\begin{aligned}
&-8i\frac{L^3}{k^3}r_1^3q_j^3+\frac{1}{64}\frac{i}{\pi^3}\frac{L^6}{k^3}l^3+\frac{1}{8}\frac{i}{\pi^3}\frac{L^5}{k^3}l^2mr_1-\frac{3}{8}\frac{i}{\pi^2}\frac{L^5}{k^3}l^2r_1q_j+3\frac{i}{\pi}\frac{L^4}{k^3}lr_1^2q_j^2+8\frac{i}{\pi}\frac{L^3}{k^3}mr_1^3q_j^2\\
&-2\frac{i}{\pi^2}\frac{L^4}{k^3}lmr_1^2q_j+\frac{l}{4r_1^2}\left(\frac{Lr_1}{i\pi}\right)^2\left(2i\sin\left(\frac{L\sqrt{l}}{2}\right)\right)^2\left(2i\frac{L}{k^3}r_1q_j-\frac{1}{4}\frac{i}{\pi}\frac{L^2}{k^3}l\right)\\
&+\frac{b\rho_1}{m_1m_2}\left(8i\frac{L}{k^3}r_1q_j-\frac{i}{\pi}\frac{L^2}{k^3}l\right)\left(\frac{Lr_1}{i\pi}\right)^2+2is_2\left(\frac{Lr_1}{\pi}\right)^2\sin(L\sqrt{l})\left(2i\frac{L}{k^3}r_1q_j-\frac{1}{4}\frac{i}{\pi}\frac{L^2}{k^3}l\right)=0.
\end{aligned}
$$

Then ε_k^j satisfy

$$f(\lambda_k) = \left(4\frac{L^2}{k^2}r_1^2\right)(\varepsilon_k^j)^2 + \left(8i\frac{L^2}{k^2}r_1^2 q_j - 4\frac{i}{\pi}\frac{L^2}{k^2}mr_1^2 - \frac{i}{\pi}\frac{L^3}{k^2}lr_1\right)\varepsilon_k^j$$
$$+ s_{2-\alpha}\left(\frac{Lr_1}{i\pi}\right)^{2-\alpha}\left(-4\frac{L^2}{k^{4-\alpha}}r_1^2 q_j^2 - \frac{1}{16\pi^2}\frac{L^4}{k^{4-\alpha}}l^2 + \frac{1}{\pi}\frac{L^3}{k^{4-\alpha}}lr_1 q_j\right)$$
$$+ s_{3-\alpha}\left(\frac{Lr_1}{i\pi}\right)^{3-\alpha}\left(4i\frac{L}{k^{4-\alpha}}r_1 q_j - \frac{1}{2}\frac{i}{\pi}\frac{L^2}{k^{4-\alpha}}l\right) - s_{4-\alpha}\left(\frac{Lr_1}{ik\pi}\right)^{4-\alpha} \tag{65}$$

Multiplying (65) by k^4 leads to:

$$f(\lambda_k) = \left(4L^2r_1^2\right)(k\varepsilon_k^j)^2 + k\left(8iL^2r_1^2q_j - 4\tfrac{i}{\pi}L^2mr_1^2 - \tfrac{i}{\pi}L^3lr_1\right)(k\varepsilon_k^j) + s_{2-\alpha}\left(\tfrac{Lr_1}{i\pi}\right)^{2-\alpha}k^\alpha\Big(-4L^2r_1^2q_j^2$$
$$- \tfrac{1}{16\pi^2}L^4l^2 + \tfrac{1}{\pi}L^3lr_1q_j\Big) + s_{3-\alpha}\left(\tfrac{Lr_1}{i\pi}\right)^{3-\alpha}k^\alpha\left(4iLr_1q_j - \tfrac{1}{2}\tfrac{i}{\pi}L^2l\right) - s_{4-\alpha}k^\alpha\left(\tfrac{Lr_1}{i\pi}\right)^{4-\alpha} + o(1) = 0.$$

Hence ε_k^1 and ε_k^2 satisfy

$$\left(4L^2r_1^2\right)(k\varepsilon_k^1)^2 - 4k\frac{i}{\pi}L^2\sqrt{\theta}r_1^2(k\varepsilon_k^1) + I_1k^\alpha\left(\frac{Lr_1}{i\pi}\right)^{2-\alpha}\frac{L^2r_1^2}{\pi^2} + o(1) = 0$$
$$\left(4L^2r_1^2\right)(k\varepsilon_k^2)^2 + 4k\frac{i}{\pi}L^2\sqrt{\theta}r_1^2(k\varepsilon_k^2) + I_2k^\alpha\left(\frac{Lr_1}{i\pi}\right)^{2-\alpha}\frac{L^2r_1^2}{\pi^2} + o(1) = 0,$$

where

$$I_1 = 2s_{3-\alpha}(m-\sqrt{\theta}) - s_{2-\alpha}(m-\sqrt{\theta})^2 + s_{4-\alpha}$$
$$I_2 = 2s_{3-\alpha}(m+\sqrt{\theta}) - s_{2-\alpha}(m+\sqrt{\theta})^2 + s_{4-\alpha}$$
$$\theta = \frac{1}{r_1^2}\left(\frac{\sqrt{l}}{2}\sin(L\sqrt{l}) + \left(\frac{\rho_1}{m_1} - \frac{\rho_2}{m_2}\right)\right)^2 + \frac{l}{4r_1^2}(\cos(L\sqrt{l}) - 1)^2$$
$$k\varepsilon_k^1 = -\frac{I_1}{4\pi\sqrt{\theta}}\left(\frac{Lr_1}{\pi}\right)^{2-\alpha}\left(\cos(1-\alpha)\frac{\pi}{2} - i\cos(1-\alpha)\frac{\pi}{2}\right)\frac{1}{k^{1-\alpha}} + o(1)$$
$$k\varepsilon_k^2 = \frac{I_2}{4\pi\sqrt{\theta}}\left(\frac{Lr_1}{\pi}\right)^{2-\alpha}\left(\cos(1-\alpha)\frac{\pi}{2} - i\cos(1-\alpha)\frac{\pi}{2}\right)\frac{1}{k^{1-\alpha}} + o(1).$$

Then

$$\varepsilon_k^1 = -\frac{I_1}{4\pi\sqrt{\theta}}\left(\frac{Lr_1}{\pi}\right)^{2-\alpha}\left(\cos(1-\alpha)\frac{\pi}{2} - i\cos(1-\alpha)\frac{\pi}{2}\right)\frac{1}{k^{2-\alpha}} + o\left(\frac{1}{k^{2-\alpha}}\right)$$
$$\varepsilon_k^2 = \frac{I_2}{4\pi\sqrt{\theta}}\left(\frac{Lr_1}{\pi}\right)^{2-\alpha}\left(\cos(1-\alpha)\frac{\pi}{2} - i\cos(1-\alpha)\frac{\pi}{2}\right)\frac{1}{k^{2-\alpha}} + o\left(\frac{1}{k^{2-\alpha}}\right).$$

Since all the eigenvalues of $\mathcal{A}$ are on the left of the imaginary axis, necessarily $I_1 > 0$ and $I_2 < 0$. Note that, if $\gamma_1 = \gamma_2 = 0$ then $I_1 = I_2 = 0$.

Remark 4.1. If condition (H) does not hold, we can study the asymptotic behavior of the spectrum of $\mathcal{A}$, but the calculation is long.

Case 2 $r_1 \neq r_2$:
We start by the expansion of t_1 and t_3:

$$t_1 = r_1\lambda - \frac{l}{2}\frac{r_2^2}{r_1(r_1^2 - r_2^2)}\frac{1}{\lambda} - \frac{l^2}{8}\frac{r_2^4(5r_1^2 - r_2^2)}{r_1^3(r_1^2 - r_2^2)^3}\frac{1}{\lambda^3} + O\left(\frac{1}{\lambda^5}\right). \tag{66}$$

$$t_3 = r_2\lambda + \frac{l}{2}\frac{r_2}{(r_1^2 - r_2^2)}\frac{1}{\lambda} + \frac{l^2}{8}\frac{r_2(5r_2^2 - r_1^2)}{(r_1^2 - r_2^2)^3}\frac{1}{\lambda^3} + O\left(\frac{1}{\lambda^5}\right). \tag{67}$$

$$h_1(t_1) = -l\lambda\frac{r_2^2}{r_1(r_1^2-r_2^2)} - \frac{1}{2}l^2r_2^4\frac{r_1^2+r_2^2}{r_1^3\left(r_1^2-r_2^2\right)^3}\frac{1}{\lambda} + O\left(\frac{1}{\lambda^3}\right). \tag{68}$$

$$h_1(t_2) = l\lambda\frac{r_2^2}{r_1(r_1^2-r_2^2)} + \frac{1}{2}l^2r_2^4\frac{r_1^2+r_2^2}{r_1^3(r_1^2-r_2^2)^3}\frac{1}{\lambda} + O\left(\frac{1}{\lambda^3}\right). \tag{69}$$

$$h_1(t_3) = -\lambda^3(r_1^2-r_2^2)\frac{r_2}{r_1^2} - \frac{1}{2}lr_2\frac{r_1^2-3r_2^2}{r_1^2\left(r_1^2-r_2^2\right)}\lambda + \frac{1}{8}l^2r_2\frac{r_1^4+9r_2^4-2r_1^2r_2^2}{r_1^2\left(r_1^2-r_2^2\right)^3}\frac{1}{\lambda} + O\left(\frac{1}{\lambda^3}\right). \tag{70}$$

$$h_1(t_4) = \lambda^3\left(r_1^2-r_2^2\right)\frac{r_2}{r_1^2} + \frac{1}{2}lr_2\frac{r_1^2-3r_2^2}{r_1^2\left(r_1^2-r_2^2\right)}\lambda - \frac{1}{8}l^2r_2\frac{r_1^4+9r_2^4-2r_1^2r_2^2}{r_1^2\left(r_1^2-r_2^2\right)^3}\frac{1}{\lambda} + O\left(\frac{1}{\lambda^3}\right). \tag{71}$$

$$h_2(t_1) = \lambda^2 + \frac{br_1}{m_2}\lambda + \frac{\gamma_2}{m_2}\lambda^\alpha + \eta\lambda^{\alpha-1}\gamma_2\frac{\alpha-1}{m_2} - \frac{1}{2}\frac{blr_2^2}{m_2r_1\left(r_1^2-r_2^2\right)}\frac{1}{\lambda} + O\left(\frac{1}{\lambda^{2-\alpha}}\right). \tag{72}$$

$$h_2(t_2) = \lambda^2 - \frac{br_1}{m_2}\lambda + \frac{\gamma_2}{m_2}\lambda^\alpha + \eta\lambda^{\alpha-1}\gamma_2\frac{\alpha-1}{m_2} + \frac{1}{2}\frac{blr_2^2}{m_2r_1\left(r_1^2-r_2^2\right)}\frac{1}{\lambda} + O\left(\frac{1}{\lambda^{2-\alpha}}\right). \tag{73}$$

$$h_2(t_3) = \lambda^2 + \frac{br_2}{m_2}\lambda + \frac{\gamma_2}{m_2}\lambda^\alpha + \eta\lambda^{\alpha-1}\gamma_2\frac{\alpha-1}{m_2} + \frac{1}{2}bl\frac{r_2}{m_2\left(r_1^2-r_2^2\right)}\frac{1}{\lambda} + O\left(\frac{1}{\lambda^{2-\alpha}}\right). \tag{74}$$

$$h_2(t_4) = \lambda^2 - \frac{br_2}{m_2}\lambda + \frac{\gamma_2}{m_2}\lambda^\alpha + \eta\lambda^{\alpha-1}\gamma_2\frac{\alpha-1}{m_2} - \frac{1}{2}bl\frac{r_2}{m_2\left(r_1^2-r_2^2\right)}\frac{1}{\lambda} + O\left(\frac{1}{\lambda^{2-\alpha}}\right). \tag{75}$$

$$\begin{aligned} h_3(t_1) = &-\frac{lr_2^2}{\rho_1r_1\left(r_1^2-r_2^2\right)}\lambda - \frac{lr_2^2}{m_1r_1^2\left(r_1^2-r_2^2\right)} - l\lambda^{\alpha-1}\gamma_1\frac{r_2^2}{m_1\rho_1r_1\left(r_1^2-r_2^2\right)} \\ &-\frac{1}{2}l^2r_2^4\frac{r_1^2+r_2^2}{\rho_1r_1^3\left(r_1^2-r_2^2\right)^3}\frac{1}{\lambda} + o\left(\frac{1}{\lambda}\right). \end{aligned} \tag{76}$$

$$\begin{aligned} h_3(t_2) = &\frac{lr_2^2}{\rho_1r_1\left(r_1^2-r_2^2\right)}\lambda - l\frac{r_2^2}{m_1r_1^2\left(r_1^2-r_2^2\right)} + l\lambda^{\alpha-1}\gamma_1\frac{r_2^2}{m_1\rho_1r_1\left(r_1^2-r_2^2\right)} \\ &+\frac{1}{2}l^2r_2^4\frac{r_1^2+r_2^2}{\rho_1r_1^3\left(r_1^2-r_2^2\right)^3}\frac{1}{\lambda} + o\left(\frac{1}{\lambda}\right). \end{aligned} \tag{77}$$

$$\begin{aligned} h_3(t_3) = &-\lambda^3\left(r_1^2-r_2^2\right)\frac{r_2}{\rho_1r_1^2} - \lambda^2\left(r_1^2-r_2^2\right)\frac{1}{m_1r_1^2} - \lambda^{1+\alpha}\left(r_1^2-r_2^2\right)\frac{\gamma_1r_2}{m_1\rho_1r_1^2} - \frac{1}{2}l\lambda r_2\frac{r_1^2-3r_2^2}{\rho_1r_1^2\left(r_1^2-r_2^2\right)} \\ &-\eta\lambda^\alpha\gamma_1r_2\left(r_1^2-r_2^2\right)\frac{\alpha-1}{m_1\rho_1r_1^2} + l\frac{r_2^2}{r_1^2m_1\left(r_1^2-r_2^2\right)} - \frac{1}{2}l\lambda^{\alpha-1}\gamma_1r_2\frac{r_1^2-3r_2^2}{m_1\rho_1r_1^2\left(r_1^2-r_2^2\right)} \\ &-\frac{1}{2}\eta^2\lambda^{\alpha-1}\gamma_1r_2\left(r_1^2-r_2^2\right)(\alpha-1)\frac{\alpha-2}{m_1\rho_1r_1^2} + o\left(\frac{1}{\lambda^{1-\alpha}}\right). \end{aligned} \tag{78}$$

$$\begin{aligned} h_3(t_4) = &\lambda^3\left(r_1^2-r_2^2\right)\frac{r_2}{\rho_1r_1^2} - \lambda^2\left(r_1^2-r_2^2\right)\frac{1}{m_1r_1^2} + \lambda^{1+\alpha}\gamma_1r_2\left(r_1-r_2\right)\frac{r_1+r_2}{m_1\rho_1r_1^2} + \frac{1}{2}l\lambda r_2\frac{r_1^2-3r_2^2}{\rho_1r_1^2\left(r_1^2-r_2^2\right)} \\ &+\eta\lambda^\alpha\gamma_1r_2\left(r_1^2-r_2^2\right)\frac{\alpha-1}{m_1\rho_1r_1^2} + l\frac{r_2^2}{r_1^2m_1\left(r_1^2-r_2^2\right)} + \frac{1}{2}l\lambda^{\alpha-1}\gamma_1r_2\frac{r_1^2-3r_2^2}{m_1\rho_1r_1^2\left(r_1^2-r_2^2\right)} \\ &+\frac{1}{2}\eta^2\lambda^{\alpha-1}\gamma_1r_2\left(r_1^2-r_2^2\right)(\alpha-1)\frac{\alpha-2}{m_1\rho_1r_1^2} + o\left(\frac{1}{\lambda^{1-\alpha}}\right). \end{aligned} \tag{79}$$

Using the asymptotic development (68)–(79)

$$\begin{aligned} f(\lambda) = &\lambda^8\frac{r_2^2}{\rho_1r_1^4}(r_1-r_2)^2(r_1+r_2)^2[\mathrm{e}^{L(t_1+t_3)} - \mathrm{e}^{L(t_1+t_4)} - \mathrm{e}^{L(t_2+t_3)} + \mathrm{e}^{L(t_2+t_4)}] \\ &+\lambda^7\frac{r_2}{m_1m_2\rho_1r_1^4}(r_1-r_2)^2(r_1+r_2)^2[(m_2\rho_1+bm_1r_1r_2)(\mathrm{e}^{L(t_1+t_3)} - \mathrm{e}^{L(t_2+t_4)}) \end{aligned}$$

$$
\begin{aligned}
&+ (m_2\rho_1 - bm_1r_1r_2)(\mathrm{e}^{L(t_1+t_4)} - \mathrm{e}^{L(t_2+t_3)})] \\
&+ \lambda^{6+\alpha}\frac{r_2^2(\gamma_1 m_2 + \gamma_2 m_1)}{m_1 m_2 \rho_1 r_1^4}(r_1^2 - r_2^2)^2[\mathrm{e}^{L(t_1+t_3)} - \mathrm{e}^{L(t_1+t_4)} - \mathrm{e}^{L(t_2+t_3)} + \mathrm{e}^{L(t_2+t_4)}] + O(\lambda^2) \\
= {} & \lambda^8 \frac{r_2^2}{\rho_1 r_1^4}(r_1 - r_2)^2(r_1 + r_2)^2 \Bigg[(\mathrm{e}^{L(t_1+t_3)} - \mathrm{e}^{L(t_1+t_4)} - \mathrm{e}^{L(t_2+t_3)} + \mathrm{e}^{L(t_2+t_4)}) \\
&+ ((m_2\rho_1 + bm_1r_1r_2)(\mathrm{e}^{L(t_1+t_3)} - \mathrm{e}^{L(t_2+t_4)}) + (m_2\rho_1 - bm_1r_1r_2)(\mathrm{e}^{L(t_1+t_4)} - \mathrm{e}^{L(t_2+t_3)}))\frac{1}{m_1 m_2 r_2 \lambda} \\
&+ \frac{(\gamma_1 m_2 + \gamma_2 m_1)}{m_1 m_2}(\mathrm{e}^{L(t_1+t_3)} - \mathrm{e}^{L(t_1+t_4)} - \mathrm{e}^{L(t_2+t_3)} + \mathrm{e}^{L(t_2+t_4)})\frac{1}{\lambda^{2-\alpha}} + O\left(\frac{1}{\lambda^2}\right)\Bigg]. \qquad (80)
\end{aligned}
$$

We set

$$
\tilde{\tilde{f}}(\lambda) = f_0(\lambda) + \frac{f_1(\lambda)}{\lambda} + \frac{f_2(\lambda)}{\lambda^{2-\alpha}} + +O\left(\frac{1}{\lambda^3}\right), \qquad (81)
$$

where

$$
f_0(\lambda) = \mathrm{e}^{L(t_1+t_3)} - \mathrm{e}^{L(t_1+t_4)} - \mathrm{e}^{L(t_2+t_3)} + \mathrm{e}^{L(t_2+t_4)} = \mathrm{e}^{-L(t_1+t_3)}(\mathrm{e}^{2Lt_1} - 1)(\mathrm{e}^{2Lt_3} - 1) \qquad (82)
$$

$$
\begin{aligned}
f_1(\lambda) &= \frac{(m_2\rho_1 + bm_1r_1r_2)}{m_1 m_2 r_2}(\mathrm{e}^{L(t_1+t_3)} - \mathrm{e}^{L(t_2+t_4)}) + \frac{(m_2\rho_1 - bm_1r_1r_2)}{m_1 m_2 r_2}(\mathrm{e}^{L(t_1+t_4)} - \mathrm{e}^{L(t_2+t_3)}) \\
&= \mathrm{e}^{-L(t_1+t_3)}\left[\frac{(m_2\rho_1 + bm_1r_1r_2)}{m_1 m_2 r_2}(\mathrm{e}^{2L(t_1+t_3)} - 1) + \frac{(m_2\rho_1 - bm_1r_1r_2)}{m_1 m_2 r_2}(\mathrm{e}^{2Lt_1} - \mathrm{e}^{2Lt_3})\right] \qquad (83)
\end{aligned}
$$

$$
f_2(\lambda) = \frac{(\gamma_1 m_2 + \gamma_2 m_1)}{m_1 m_2}\mathrm{e}^{-L(t_1+t_3)}(\mathrm{e}^{2Lt_1} - 1)(\mathrm{e}^{2Lt_3} - 1) \qquad (84)
$$

□

Lemma 4.2. (Asymptotic behavior of the large eigenvalues of $\mathcal{A}$) *The large eigenvalues of $\mathcal{A}$ can be split into two families $(\lambda_k^j)_{k\in\mathbf{Z},|k|\geq k_0}, j = 1, 2$, ($k_0 \in \mathbb{N}$ chosen large enough). The following asymptotic expansions hold:*

$$
\lambda_k^1 = \frac{i}{Lr_1}k\pi + o(1), \quad \lambda_k^2 = \frac{i}{Lr_2}k\pi + o(1), \qquad (85)
$$

and these two roots are simple.

Proof. From (85), f_0 has two families of roots that we denote λ_k^0 and μ_k^0. Now, we prove that

$$
f_0(\lambda) = 0 \quad \text{if and only if} \quad 2t_1L = 2ik\pi \quad \text{and} \quad 2t_3L = 2ik'\pi,\ k, k' \in \mathbf{Z}.
$$

Indeed, Suppose that

$$
t_1L = ik\pi \quad \text{and} \quad t_3L \neq ik'\pi,\ k, k' \in \mathbf{Z}.
$$

Then

$$
\mathcal{M} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ h_1(t_1) & -h_1(t_1) & h_1(t_3) & -h_1(t_3) \\ h_2(t_1)(-1)^k & h_2(t_2)(-1)^k & h_2(t_3)\mathrm{e}^{t_3L} & h_2(t_4)\mathrm{e}^{-t_3L} \\ h_3(t_1)(-1)^k & h_3(t_2)(-1)^k & h_3(t_3)\mathrm{e}^{t_3L} & h_3(t_4)\mathrm{e}^{-t_3L} \end{pmatrix}
$$

We can check that $h_1(t_1) \neq 0$ and $h_1(t_3) \neq 0$ for λ large enough. Since $t_3L \neq ik'\pi$ for all $k' \in \mathbf{Z}$, then using Gaussian elimination for $\mathcal{M}$, we get

$$
c_1 = c_2 = c_3 = c_4 = 0.
$$

which is a contradiction with $\|U\|_{\mathcal{H}} = 1$. Similarly if

$$
t_1L \neq ik\pi \quad \text{and} \quad t_3L = ik'\pi,\ k, k' \in \mathbf{Z}.
$$

we get $U \equiv 0$. We conclude that

$$f_0(\lambda) = 0 \quad \text{if and only if} \quad t_1 L = ik\pi \quad \text{and} \quad t_3 L = ik'\pi,\ k, k' \in \mathbf{Z}.$$

Then from (66) and (67), the large roots of f_0 satisfy the following asymptotic equations

$$\lambda_k^0 = \frac{i}{Lr_1} k\pi + O\left(\frac{1}{k}\right) \forall |k| \geq k_0$$
$$\lambda_k^1 = \frac{i}{Lr_2} k'\pi + O\left(\frac{1}{k'}\right) |k'| \geq k_0'.$$

We will now use Rouché's theorem. Let $B_k = B(\frac{i}{Lr_1} k\pi, r_k)$ be the ball of centrum $\frac{ik\pi}{Lr_1}$ and radius $r_k = \frac{1}{k^{\frac{1}{4}}}$ and $\lambda \in \partial B_k$ (i.e., $\lambda = \frac{i}{Lr_1} k\pi + r_k \mathrm{e}^{i\theta}, \theta \in [0, 2\pi]$). Then we successively have:

$$\begin{aligned} 2Lt_1(\lambda) &= 2ik\pi + 2Lr_1 r_k \mathrm{e}^{i\theta} + O\left(\frac{1}{k}\right) \\ \mathrm{e}^{2Lt_1(\lambda)} &= \mathrm{e}^{2Lr_1 r_k \mathrm{e}^{i\theta} + O\left(\frac{1}{k}\right)} \\ &= 1 + 2Lr_1 r_k \mathrm{e}^{i\theta} + O(r_k^2). \end{aligned}$$

and

$$\begin{aligned} f_0(\lambda) &= (2Lr_1 r_k \mathrm{e}^{i\theta} + O(r_k^2)) \left(\left(2Lr_2 r_k \mathrm{e}^{i\theta} + O\left(\frac{1}{k}\right)\right)\right) \\ &= 4L^2 r_1 r_2 r_k^2 \mathrm{e}^{2i\theta} + O(r_k^3). \end{aligned}$$

It follows that there exists a positive constant c such that

$$\forall \lambda \in \partial B_k,\ |f_0(\lambda)| \geq c r_k^2 = \frac{c}{\sqrt{k}}.$$

Then we deduce from (81) that $|f(\lambda) - f_0(\lambda)| = O\left(\frac{1}{\lambda}\right) = O\left(\frac{1}{k}\right)$. It follows that, for $|k|$ large enough

$$\forall \lambda \in \partial B_k,\ |f(\lambda) - f_0(\lambda)| < |f_0(\lambda)|,$$

Since the imaginary axis is an asymptote for the spectrum of $\mathcal{A}$ then system (12) is not uniformly stable. □

5. Asymptotic stability

5.1. Strong stability of the system

In this part, we use a general criteria of Theorem 4.3 to show the strong stability of the C_0-semigroup $\mathrm{e}^{t\mathcal{A}}$ associated to the wave system (P') in the absence of the compactness of the resolvent of $\mathcal{A}$. Our main result is the following theorem:

Theorem 5.1. *The C_0-semigroup $\mathrm{e}^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$; i.e., for all $U_0 \in \mathcal{H}$, the solution of (12) satisfies*

$$\lim_{t \to \infty} \|\mathrm{e}^{t\mathcal{A}} U_0\|_{\mathcal{H}} = 0.$$

Lemma 5.1. *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

Proof. We will argue by contraction. Let us suppose that there $\lambda \in \mathbb{R}, \lambda \neq 0$ and $U \neq 0$, such that $\mathcal{A}U = i\lambda U$. Then, we get

$$\begin{cases} i\lambda\varphi - u = 0, \\ i\lambda u - \dfrac{K}{\rho_1}(\varphi_x + \psi)_x = 0, \\ i\lambda\phi_1 + (\xi^2 + \eta)\phi_1 - u(L)\mu(\xi) = 0, \\ i\lambda\theta + \frac{K}{m_1}(\varphi_x + \psi)(L) + \frac{\zeta_1}{m_1}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_1(\xi)\,\mathrm{d}\xi = 0. \\ i\lambda\psi - v = 0, \\ i\lambda v - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x + \psi) = 0, \\ i\lambda\phi_2 + (\xi^2 + \eta)\phi_2 - v(L)\mu(\xi) = 0, \\ i\lambda\vartheta + \frac{b}{m_2}\psi_x(L) + \frac{\zeta_2}{m_2}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi)\,\mathrm{d}\xi = 0. \end{cases} \tag{86}$$

Then, from (16) we have

$$\phi_i \equiv 0, \ \ i = 1, 2. \tag{87}$$

From $(86)_3$ and $(86)_7$, we have

$$u(L) = v(L) = 0. \tag{88}$$

Hence, from $(86)_1, (86)_5, (86)_4$ and $(86)_8$ we obtain

$$\varphi(L) = \psi(L) = 0 \quad \text{and} \quad \varphi_x(L) = \psi_x(L) = 0. \tag{89}$$

From (86), we have

$$\begin{cases} -\lambda^2\rho_1\varphi - K(\varphi_x + \psi)_x = 0, \\ -\lambda^2\rho_2\psi - b\psi_{xx} + K(\varphi_x + \psi) = 0, \end{cases} \tag{90}$$

Consider $X = (\varphi, \psi, \varphi_x, \psi_x)$. Then we can rewrite (89) and (90) as the initial value problem

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}X = \mathcal{A}X \\ &X(L) = 0 \end{aligned} \tag{91}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-\lambda^2\rho_1}{K} & 0 & 0 & -1 \\ 0 & \frac{-\rho_2\lambda^2 + K}{b} & \frac{K}{b} & 0 \end{pmatrix}$$

By the Picard theorem for ordinary differential equations the system (91) has a unique solution $X = 0$. Therefore $\varphi = 0, \psi = 0$. It follows from (86), that $u = 0, v = 0, \theta = 0, \vartheta = 0$, i.e., $U = 0$.

The condition (ii) of Theorem 4.3 will be satisfied if we show that $\sigma(\mathcal{A}) \cap \{i\mathbb{R}\}$ is at most a countable set. We will prove that the operator $i\lambda I - \mathcal{A}$ is surjective for $\lambda \neq 0$. For this purpose, let $(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^{\mathrm{T}} \in \mathcal{H}$, we seek $U = (\varphi, u, \phi_1, \theta, \psi, v, \phi_2, \vartheta)^{\mathrm{T}} \in D(\mathcal{A})$ solution of solution of the following equation

$$(i\lambda - \mathcal{A})U = F.$$

Equivalently, we have the following system

$$\begin{cases} i\lambda\varphi - u = f_1, \\ i\lambda u - \dfrac{K}{\rho_1}(\varphi_x+\psi)_x = f_2, \\ i\lambda\phi_1 + (\xi^2+\eta)\phi_1 - u(L)\mu(\xi) = f_3, \\ i\lambda\theta + \frac{K}{m_1}(\varphi_x+\psi)(L) + \frac{\zeta_1}{m_1}\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi)\,\mathrm{d}\xi = f_4. \\ i\lambda\psi - v = f_5, \\ i\lambda v - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x+\psi) = f_6, \\ i\lambda\phi_2 + (\xi^2+\eta)\phi_2 - v(L)\mu(\xi) = f_7, \\ i\lambda\vartheta + \frac{b}{m_2}\psi_x(L) + \frac{\zeta_2}{m_2}\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi)\,\mathrm{d}\xi = f_8. \end{cases} \tag{92}$$

We get

$$\begin{cases} -\lambda^2\varphi - \dfrac{K}{\rho_1}(\varphi_x+\psi)_x = f_2 + i\lambda f_1, \\ -\lambda^2\psi - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x+\psi) = f_6 + i\lambda f_5. \end{cases} \tag{93}$$

Solving system (93) is equivalent to finding $(\varphi,\psi)\in(H^2\cap H^1_*(0,L))^2$ such that

$$\begin{cases} \displaystyle\int_0^L(-\rho_1\lambda^2\varphi w - K(\varphi_x+\psi)_x w)\,\mathrm{d}x = \int_0^L \rho_1(f_2+i\lambda f_1)w\,\mathrm{d}x, \\ \displaystyle\int_0^L(-\rho_2\lambda^2\psi\chi - b\psi_{xx}\chi + K(\varphi_x+\psi)\chi)\,\mathrm{d}x = \int_0^L \rho_2(f_6+i\lambda f_5)\chi\,\mathrm{d}x, \end{cases} \tag{94}$$

for all $(w,\chi)\in H^1_*(0,L)\times H^1_*(0,L)$. By using (21) and (19) the functions φ and ψ satisfying the following system

$$\begin{cases} \displaystyle\int_0^L(-\rho_1\lambda^2\varphi w + K(\varphi_x+\psi)w_x)\,\mathrm{d}x + (i\lambda m_1+\tilde{\zeta}_1)u(L)w(L) \\ \displaystyle\quad = \int_0^L \rho_1(f_2+i\lambda f_1)w\,\mathrm{d}x - \zeta_1\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_3(\xi)\,\mathrm{d}\xi\ w(L) + m_1 f_4 w(L), \\ \displaystyle\int_0^L(-\rho_2\lambda^2\psi\chi + b\psi_x\chi_x + K(\varphi_x+\psi)\chi)\,\mathrm{d}x + (i\lambda m_2+\tilde{\zeta}_2)v(L)\chi(L) \\ \displaystyle\quad = \int_0^L \rho_2(f_6+i\lambda f_5)\chi\,\mathrm{d}x - \zeta_2\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_7(\xi)\,\mathrm{d}\xi\ \chi(L) + m_2 f_8\chi(L), \end{cases} \tag{95}$$

where $\tilde{\zeta}_i = \zeta_i\displaystyle\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta+i\lambda}\,\mathrm{d}\xi$. Using again (18), we deduce that

$$\begin{cases} u(L) = i\lambda\varphi(L) - f_1(L), \\ v(L) = i\lambda\psi(L) - f_5(L). \end{cases} \tag{96}$$

Inserting (96) into (95), we get

$$\begin{cases} \displaystyle\int_0^L (-\rho_1\lambda^2\varphi w + K(\varphi_x+\psi)w_x)\,\mathrm{d}x + i\lambda(i\lambda m_1+\tilde{\zeta}_1)\varphi(L)w(L) \\ \quad \displaystyle = \int_0^L \rho_1(f_2+i\lambda f_1)w\,\mathrm{d}x - \zeta_1\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_3(\xi)\,\mathrm{d}\xi\ w(L) + (i\lambda m_1+\tilde{\zeta}_1)f_1(L)w(L) + m_1 f_4 w(L), \\ \displaystyle\int_0^L (-\rho_2\lambda^2\psi\chi + b\psi_x\chi_x + K(\varphi_x+\psi)\chi)\,\mathrm{d}x + i\lambda(i\lambda m_2+\tilde{\zeta}_2)\psi(L)\chi(L) \\ \quad \displaystyle = \int_0^L \rho_2(f_6+i\lambda f_5)\chi\,\mathrm{d}x - \zeta_2\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+i\lambda}f_7(\xi)\,\mathrm{d}\xi\ \chi(L) + (i\lambda m_2+\tilde{\zeta}_2)f_5(L)\chi(L) + m_2 f_8\chi(L). \end{cases} \tag{97}$$

We can rewrite (97) as

$$-(L_\lambda U, V)_{H_R^1} + (U,V)_{H_R^1} = l(V) \tag{98}$$

where

$$H_R^1(0,L) = H_*^1(0,L)\times H_*^1(0,L),$$

with the inner product defined by

$$(U,V)_{H_R^1} = \int_0^L K(\varphi_x+\psi)(w_x+\chi) + b\psi_x\chi_x\,\mathrm{d}x$$

$$(L_\lambda U, V)_{H_R^1} = \lambda^2\int_0^L (\rho_1\varphi w + \rho_2\psi\chi + \ \mathrm{d}x - i\lambda((i\lambda m_1+\tilde{\zeta}_1)\varphi(L)w(L) + (i\lambda m_2+\tilde{\zeta}_2)\psi(L)\chi(L)).$$

Using the compactness embedding from $L^2(0,L)$ into $H^{-1}(0,L)$ and from $H_*^1(0,L)$ into $L^2(0,L)$ we deduce that the operator L_λ is compact from $L^2(0,L)$ into $L^2(0,L)$. Consequently, by Fredholm alternative, proving the existence of U solution of (98) reduces to proving that 1 is not an eigenvalue of L_λ. Indeed if 1 is an eigenvalue, then there exists $U\neq 0$, such that

$$(L_\lambda U, V)_{H_R^1} = (U,V)_{H_R^1} \quad \forall V\in H_R^1. \tag{99}$$

In particular for $V=U$, it follows that

$$\begin{aligned} &\lambda^2\left[\rho_1\|\varphi\|^2_{L^2(0,L)} + \rho_2\|\psi\|^2_{L^2(0,L)}\right] - i\lambda((i\lambda m_1+\tilde{\zeta}_1)|\varphi(L)|^2 + (i\lambda m_2+\tilde{\zeta}_2)|\psi(L)|^2) \\ &\quad = K\|\varphi_x+\psi\|^2_{L^2(0,L)} + b\|\psi_x\|^2_{L^2(0,L)}. \end{aligned}$$

Hence, we have

$$\varphi(L) = \psi(L) = 0. \tag{100}$$

From (99), we obtain

$$\varphi_x(L) = \psi_x(L) = 0 \tag{101}$$

and

$$\begin{cases} -\lambda^2\varphi - \dfrac{K}{\rho_1}(\varphi_x+\psi)_x = 0, \\ -\lambda^2\psi - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x+\psi) = 0. \end{cases} \tag{102}$$

Consider $X = (\varphi, \psi, \varphi_x, \psi_x)$. Then we can rewrite (102), (100) and (101) as the initial value problem

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x} X = \mathcal{B} X \\ &X(L) = 0 \end{aligned} \tag{103}$$

where

$$\mathcal{B} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-\lambda^2 \rho_1}{K} & 0 & 0 & -1 \\ 0 & \frac{-\rho_2 \lambda^2 + K}{b} & \frac{K}{b} & 0 \end{pmatrix}$$

By the Picard theorem for ordinary differential equations the system (103) has a unique solution $X = 0$. Therefore $\varphi = 0, \psi = 0$. It follows from (86), that $u = 0, v = 0, \theta = 0, \vartheta = 0$, i.e., $U = 0$. □

Lemma 5.2. *If $\eta \neq 0$, we have*

$$0 \in \rho(\mathcal{A}).$$

Proof. From (92)

$$\begin{cases} -u = f_1, \\ -\dfrac{K}{\rho_1}(\varphi_x + \psi)_x = f_2, \\ (\xi^2 + \eta)\phi_1 - u(L)\mu(\xi) = f_3, \\ \frac{K}{m_1}(\varphi_x + \psi)(L) + \frac{\zeta_1}{m_1} \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_1(\xi)\,\mathrm{d}\xi = f_4. \\ -v = f_5, \\ -\dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x + \psi) = f_6, \\ (\xi^2 + \eta)\phi_2 - v(L)\mu(\xi) = f_7, \\ \frac{b}{m_2}\psi_x(L) + \frac{\zeta_2}{m_2} \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi)\,\mathrm{d}\xi = f_8. \end{cases} \tag{104}$$

$$\begin{cases} \displaystyle\int_0^L K(\varphi_x + \psi) w_x \,\mathrm{d}x \\ \quad = \displaystyle\int_0^L \rho_1 f_2 w \,\mathrm{d}x - \zeta_1 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta} f_3(\xi)\,\mathrm{d}\xi \; w(L) + \tilde{\zeta}_1 f_1(L) w(L) + m_1 f_4 w(L), \\ \displaystyle\int_0^L (b\psi_x \chi_x + K(\varphi_x + \psi)\chi)\,\mathrm{d}x \\ \quad = \displaystyle\int_0^L \rho_2 f_6 \chi \,\mathrm{d}x - \zeta_2 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta} f_7(\xi)\,\mathrm{d}\xi \; \chi(L) + \tilde{\zeta}_2 f_5(L)\chi(L) + m_2 f_8 \chi(L). \end{cases} \tag{105}$$

Consequently, problem (105) is equivalent to the problem

$$a_\eta((\varphi, \psi), (w, \chi)) = L_\eta(w, \chi) \tag{106}$$

where the bilinear form $a_\eta : [H^1_*(0,L)\times H^1_*(0,L)]^2 \to \mathbb{R}$ and the linear form $L_\eta : H^1_*(0,L)\times H^1_*(0,L) \to \mathbb{R}$ are defined by

$$a_\eta((\varphi,\psi),(w,\chi)) = \int_0^L K(\varphi_x+\psi)(w_x+\chi)\,\mathrm{d}x + \int_0^L b\psi_x\chi_x\,\mathrm{d}x.$$

and

$$\begin{aligned} L_\eta(w,\chi) = & \int_0^L \rho_1 f_2 w\,\mathrm{d}x - \zeta_1 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta} f_3(\xi)\,\mathrm{d}\xi\ w(L) + \tilde{\zeta}_1 f_1(L) w(L) \\ & + \int_0^L \rho_2 f_6 \chi\,\mathrm{d}x - \zeta_2 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta} f_7(\xi)\,\mathrm{d}\xi\ \chi(L) + \tilde{\zeta}_2 f_5(L)\chi(L) \\ & + m_1 f_4 w(L) + m_2 f_8 \chi(L). \end{aligned}$$

It is easy to verify that a_η is continuous and coercive, and L_η is continuous. So applying the Lax–Milgram theorem, we deduce that for all $(w,\chi,\zeta) \in H^1_*(0,L)\times H^1_*(0,L)$ problem (25) admits a unique solution $(\varphi,\psi) \in H^1_*(0,L)\times H^1_*(0,L)$. Applying the classical elliptic regularity, it follows from (24) that $(\varphi,\psi)\in H^2(0,L)\times H^2(0,L)$. Therefore, the operator $\mathcal{A}$ is surjective. □

5.2. Residual spectrum of $\mathcal{A}$

Lemma 5.3. *Let $\mathcal{A}$ be defined by (13). Then*

$$\mathcal{A}^* \begin{pmatrix} \varphi \\ u \\ \phi_1 \\ \theta \\ \psi \\ v \\ \phi_2 \\ \vartheta \end{pmatrix} = \begin{pmatrix} -u \\ -\frac{K}{\rho_1}(\varphi_x+\psi)_x \\ -(\xi^2+\eta)\phi_1 - u(L)\mu(\xi) \\ \frac{K}{m_1}(\varphi_x+\psi)(L) + \frac{\zeta_1}{m_1}\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi)\,\mathrm{d}\xi \\ -v \\ -\frac{b}{\rho_2}\psi_{xx} + \frac{K}{\rho_2}(\varphi_x+\psi) \\ -(\xi^2+\eta)\phi_2 - v(L)\mu(\xi) \\ \frac{b}{m_2}\psi_x(L) + \frac{\zeta_2}{m_2}\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi)\,\mathrm{d}\xi \end{pmatrix} \tag{107}$$

with domain

$$D(\mathcal{A}^*) = \left\{ \begin{array}{l} (\varphi,u,\phi_1,\theta,\psi,v,\phi_2) \quad in \quad \mathcal{H} : \varphi,\psi \in H^2(0,L)\cap H^1_*(0,L), u,v\in H^1_*(0,L), \\ \theta,\vartheta\in\mathbb{C}, u(L)=\theta, v(L)=\vartheta, \\ -(\xi^2+\eta)\phi_1 - u(L)\mu(\xi), -(\xi^2+\eta)\phi_2 - v(L)\mu(\xi) \in L^2(-\infty,+\infty), \\ |\xi|\phi_1, |\xi|\phi_2 \in L^2(-\infty,+\infty) \end{array} \right\} \tag{108}$$

Proof. Let $U=(\varphi,u,\phi_1,\theta,\psi,v,\phi_2,\vartheta)^{\mathrm{T}}$ and $V=(\tilde{\varphi},\tilde{u},\tilde{\phi_1},\tilde{\theta},\tilde{\psi},\tilde{v},\tilde{\phi_2},\tilde{\vartheta})^{\mathrm{T}}$. We have $\langle\mathcal{A}U,V\rangle_{\mathcal{H}}=\langle U,\mathcal{A}^*V\rangle_{\mathcal{H}}$.

$$\begin{aligned}
\langle\mathcal{A}U,V\rangle_{\mathcal{H}} &= K\int_0^L \tilde{u}(\varphi_x+\psi)_x\,\mathrm{d}x + b\int_0^L \tilde{v}\psi_{xx}\,\mathrm{d}x - K\int_0^L \tilde{v}(\varphi_x+\psi)\,\mathrm{d}x + K\int_0^L(\tilde{\varphi}_x+\tilde{\psi})(u_x+v)\,\mathrm{d}x\\
&\quad + b\int_0^L \tilde{\psi}_x v_x\,\mathrm{d}x + \zeta_1\int_{-\infty}^{+\infty}[-(\xi^2+\eta)\phi_1+u(L)\mu(\xi)]\tilde{\phi}_1\,\mathrm{d}\xi + \zeta_2\int_{-\infty}^{+\infty}[-(\xi^2+\eta)\phi_2\\
&\quad + v(L)\mu(\xi)]\tilde{\phi}_2\,\mathrm{d}\xi + m_1\left(-\frac{K}{m_1}(\varphi_x+\psi)(L)-\frac{\zeta_1}{m_1}\int_{-\infty}^{+\infty}\mu(\xi)\phi_1(\xi)\,\mathrm{d}\xi\right)\tilde{\theta}\\
&\quad + m_2\left(-\frac{b}{m_2}\psi_x(L)-\frac{\zeta_2}{m_2}\int_{-\infty}^{+\infty}\mu(\xi)\phi_2(\xi)\,\mathrm{d}\xi\right)\tilde{\vartheta}\\
&= -K\int_0^L(\tilde{u}_x+\tilde{v})(\varphi_x+\psi)\,\mathrm{d}x + K(\varphi_x+\psi)(L)\tilde{u}(L) - b\int_0^L \tilde{v}_x\psi_x\,\mathrm{d}x\\
&\quad + b\psi_x(L)\tilde{v}(L) + K(\tilde{\varphi}_x+\tilde{\psi})(L)u(L) - K\int_0^L(\tilde{\varphi}_x+\tilde{\psi})_x u\,\mathrm{d}x + K\int_0^L(\tilde{\varphi}_x+\tilde{\psi})_x v\,\mathrm{d}x\\
&\quad - b\int_0^L\tilde{\psi}_{xx}v\,\mathrm{d}x + b\tilde{\psi}_x(L)v(L) + \zeta_1 u(L)\int_{-\infty}^{+\infty}\mu(\xi)\tilde{\phi}_1\,\mathrm{d}\xi + \zeta_2 u(L)\int_{-\infty}^{+\infty}\mu(\xi)\tilde{\phi}_2\,\mathrm{d}\xi\\
&\quad - \zeta_1\int_{-\infty}^{+\infty}\phi_1[(\xi^2+\eta)\tilde{\phi}_1+\tilde{\theta}\mu(\xi)]\,\mathrm{d}\xi - \zeta_2\int_{-\infty}^{+\infty}\phi_2[(\xi^2+\eta)\tilde{\phi}_2+\tilde{\vartheta}\mu(\xi)]\,\mathrm{d}\xi\\
&\quad - K(\varphi_x+\psi)(L)\tilde{\theta} - b\psi_x(L)\tilde{\vartheta}
\end{aligned}$$

As $\theta=u(L),\vartheta=v(L)$ and if we set $\tilde{\theta}=\tilde{u}(L),\tilde{\vartheta}=\tilde{v}(L)$, we find

$$\begin{aligned}
\langle\mathcal{A}U,V\rangle_{\mathcal{H}} &= -K\int_0^L(\tilde{u}_x+\tilde{v})(\varphi_x+\psi)\,\mathrm{d}x - b\int_0^L\tilde{v}_x\psi_x\,\mathrm{d}x - K\int_0^L(\tilde{\varphi}_x+\tilde{\psi})_x u\,\mathrm{d}x\\
&\quad + \int_0^L(-b\tilde{\psi}_{xx}+K(\tilde{\varphi}_x+\tilde{\psi})_x)v\,\mathrm{d}x\\
&\quad + u(L)\left(K(\tilde{\varphi}_x+\tilde{\psi})(L)+\zeta_1\int_{-\infty}^{+\infty}\mu(\xi)\tilde{\phi}_1\,\mathrm{d}\xi\right) + v(L)\left(b\tilde{\psi}_x(L)+\zeta_2\int_{-\infty}^{+\infty}\mu(\xi)\tilde{\phi}_2\,\mathrm{d}\xi\right)\\
&\quad - \zeta_1\int_{-\infty}^{+\infty}\phi_1[(\xi^2+\eta)\tilde{\phi}_1+\tilde{u}(L)\mu(\xi)]\,\mathrm{d}\xi - \zeta_2\int_{-\infty}^{+\infty}\phi_2[(\xi^2+\eta)\tilde{\phi}_2+\tilde{v}(L)\mu(\xi)]\,\mathrm{d}\xi.
\end{aligned}$$

□

Theorem 5.2. *$\sigma_r(\mathcal{A}) = \emptyset$, where $\sigma_r(\mathcal{A})$ denotes the set of residual spectrum of $\mathcal{A}$.*

Since $\lambda \in \sigma_r(\mathcal{A}), \overline{\lambda} \in \sigma_p(\mathcal{A}^*)$ the proof will be accomplished if we can show that $\sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}^*)$. This is because obviously the eigenvalues of $\mathcal{A}$ are symmetric on the real axis. From (107), the eigenvalue problem $\mathcal{A}^*Z = \lambda Z$ for $\lambda \in \mathbb{C}$ and $0 \neq Z = (\varphi, u, \phi_1, \theta, \psi, v, \phi_2, \vartheta) \in D(\mathcal{A}^*)$ we have

$$\begin{cases} \lambda\varphi + u = 0, \\ \lambda u + \dfrac{K}{\rho_1}(\varphi_x + \psi)_x = 0, \\ \lambda\phi_1 + (\xi^2 + \eta)\phi_1 + u(L)\mu(\xi) = 0, \\ \lambda\theta - \frac{K}{m_1}(\varphi_x + \psi)(L) - \frac{\zeta_1}{m_1}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_1(\xi)\,d\xi = 0. \\ \lambda\psi + v = 0, \\ \lambda v + \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{K}{\rho_2}(\varphi_x + \psi) = 0, \\ \lambda\phi_2 + (\xi^2 + \eta)\phi_2 + v(L)\mu(\xi) = 0, \\ \lambda\vartheta - \frac{b}{m_2}\psi_x(L) - \frac{\zeta_2}{m_2}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi)\,d\xi = 0. \end{cases} \tag{109}$$

From $(109)_1$ and $(109)_2$, $(109)_5$ and $(109)_6$, we get

$$\begin{cases} -\lambda^2 u + \dfrac{K}{\rho_1}(\varphi_x + \psi)_x = 0, \\ -\lambda^2 v + \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{K}{\rho_2}(\varphi_x + \psi) = 0, \end{cases} \tag{110}$$

As $\theta = u(L) = -\lambda\varphi(L)$ and $\vartheta = v(L) = -\lambda\psi(L)$, we deduce from $(109)_3$ and $(109)_4$, $(109)_7$ and $(109)_8$ that

$$\begin{aligned} &\left(\lambda + \frac{\gamma_1}{m_1}(\lambda + \eta)^{\alpha-1}\right)\lambda\varphi(L) + \frac{K}{m_1}(\varphi_x + \psi)(L) = 0 \\ &\left(\lambda + \frac{\gamma_2}{m_2}(\lambda + \eta)^{\alpha-1}\right)\lambda\psi(L) + \frac{b}{m_2}\psi_x(L) = 0 \end{aligned} \tag{111}$$

System (110)–(111) is exactly the eigenvalue problem of $\mathcal{A}$. Hence $\mathcal{A}^*$ has the same eigenvalues as $\mathcal{A}$. The proof is complete. □

5.3. Polynomial stability (for $\eta \neq 0$)

Theorem 5.3. *The semigroup $S_{\mathcal{A}}(t)_{t\geq 0}$ is polynomially stable and*

$$\|S_{\mathcal{A}}(t)U_0\|_{\mathcal{H}} \leq \frac{1}{t^{1/(4-2\alpha)}}\|U_0\|_{D(\mathcal{A})}$$

Proof. We will need to study the resolvent equation $(i\lambda - \mathcal{A})U = F$, for $\lambda \in \mathbb{R}$, namely

$$\begin{cases} i\lambda\varphi - u = f_1, \\ i\lambda u - \dfrac{K}{\rho_1}(\varphi_x + \psi)_x = f_2, \\ i\lambda\phi_1 + (\xi^2 + \eta)\phi_1 - u(L)\mu(\xi) = f_3, \\ i\lambda\theta + \frac{K}{m_1}(\varphi_x + \psi)(L) + \frac{\zeta_1}{m_1}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_1(\xi)\,\mathrm{d}\xi = f_4. \\ i\lambda\psi - v = f_5, \\ i\lambda v - \dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}(\varphi_x + \psi) = f_6, \\ i\lambda\phi_2 + (\xi^2 + \eta)\phi_2 - v(L)\mu(\xi) = f_7, \\ i\lambda\vartheta + \frac{b}{m_2}\psi_x(L) + \frac{\zeta_2}{m_2}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi)\,\mathrm{d}\xi = f_8. \end{cases} \tag{112}$$

where $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^{\mathrm{T}}$. Taking inner product in $\mathcal{H}$ with U and using (16) we get

$$|Re\,\langle \mathcal{A}U, U\rangle\,| \le \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{113}$$

This implies that

$$\sum_{i=1}^{2} \zeta_i \int_{-\infty}^{+\infty} (\xi^2 + \eta)(\phi_i(\xi, t))^2\,\mathrm{d}\xi \le \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{114}$$

and, applying $(112)_{1,4,7}$, we obtain

$$||\lambda||\varphi(L)| - |f_1(L)||^2 \le |u(L)|^2.$$

We deduce that

$$|\lambda|^2|\varphi(L)|^2 \le c|f_1(L)|^2 + c|u(L)|^2.$$

Moreover, from $(112)_4$, we have

$$K(\varphi_x + \psi)(L) = -im_1\lambda u(L) - \zeta_1 \int_{-\infty}^{+\infty} \mu(\xi)\phi_1(\xi)\,\mathrm{d}\xi + m_1 f_4.$$

Then

$$\begin{aligned} K^2|(\varphi_x + \psi)(L)|^2 &\le 2m_1^2|\lambda|^2|u(L)|^2 + 2m_1^2 f_4^2 + 2\zeta_1^2 \left|\int_{-\infty}^{+\infty} \mu(\xi)\phi_1(\xi)\,\mathrm{d}\xi\right|^2 \\ &\le 2m_1^2|\lambda|^2|u(L)|^2 + 2m_1^2 f_4^2 + 2\zeta_1^2 \left(\int_{-\infty}^{+\infty} (\xi^2 + \eta)^{-1}|\mu(\xi)|^2\,\mathrm{d}\xi\right) \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi_1(\xi)|^2\,\mathrm{d}\xi \\ &\le 2m_1^2|\lambda|^2|u(L)|^2 + c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c'\|F\|_{\mathcal{H}}^2. \end{aligned} \tag{115}$$

From $(112)_3$, we obtain

$$u(L)\mu(\xi) = (i\lambda + \xi^2 + \eta)\phi_1 - f_3(\xi). \tag{116}$$

By multiplying $(116)_1$ by $(i\lambda + \xi^2 + \eta)^{-1}\mu(\xi)$, we get

$$(i\lambda + \xi^2 + \eta)^{-1}u(L)\mu^2(\xi) = \mu(\xi)\phi_1 - (i\lambda + \xi^2 + \eta)^{-1}\mu(\xi)f_3(\xi). \tag{117}$$

Hence, by taking absolute values of both sides of (117), integrating over the interval $]-\infty,+\infty[$ with respect to the variable ξ and applying Cauchy-Schwartz inequality, we obtain

$$S|u(L)| \leq U\left(\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_1|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}} + V\left(\int_{-\infty}^{+\infty}|f_3(\xi)|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}} \tag{118}$$

where

$$S = \int_{-\infty}^{+\infty}(|\lambda|+\xi^2+\eta)^{-1}|\mu(\xi)|^2\,\mathrm{d}\xi$$

$$U = \left(\int_{-\infty}^{+\infty}(\xi^2+\eta)^{-1}|\mu(\xi)|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}}$$

$$V = \left(\int_{-\infty}^{+\infty}(|\lambda|+\xi^2+\eta)^{-2}|\mu(\xi)|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}}.$$

Thus, by using again the inequality $2PQ \leq P^2+Q^2, P \geq 0, Q \geq 0$, we get

$$S^2|u(L)|^2 \leq 2U^2\left(\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi_1|^2\,\mathrm{d}\xi\right) + 2V^2\left(\int_{-\infty}^{+\infty}|f_3(\xi)|^2\,\mathrm{d}\xi\right). \tag{119}$$

We deduce that

$$|u(L)|^2 \leq c|\lambda|^{2-2\alpha}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c\|F\|^2_{\mathcal{H}}. \tag{120}$$

Similarly, we have

$$b^2|\psi_x(L)|^2 \leq 2m_2^2|\lambda|^2|v(L)|^2 + c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c'\|F\|^2_{\mathcal{H}}. \tag{121}$$

$$|v(L)|^2 \leq c|\lambda|^{2-2\alpha}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c\|F\|^2_{\mathcal{H}}. \tag{122}$$

Let us introduce the following notation

$$\mathcal{I}_\varphi(\alpha) = \rho_1|u(\alpha)|^2 + K|\varphi_x(\alpha)|^2$$
$$\mathcal{I}_\psi(\alpha) = \rho_2|v(\alpha)|^2 + b|\psi_x(\alpha)|^2$$
$$\mathcal{E}_\varphi(L) = \int_0^L q(x)\mathcal{I}_\varphi(s)\,\mathrm{d}s, \quad \mathcal{E}_\psi(L) = \int_0^L \mathcal{I}_\psi(s)\,\mathrm{d}s.$$

□

Lemma 5.4. *Let $q \in H^1(0,L)$. We have that*

$$\mathcal{E}_\varphi(L) = [q\mathcal{I}_\varphi]_0^L + 2K\ Re\int_0^L q\psi_x\overline{\varphi}_x\,\mathrm{d}x + R_1. \tag{123}$$

and

$$\mathcal{E}_\psi(L) = [q\mathcal{I}_\psi]_0^L - K[q|\psi|^2]_0^L - 2K\ Re\int_0^L q\varphi_x\overline{\psi}_x\,\mathrm{d}x + K\int_0^L q'|\psi|^2\,\mathrm{d}x + R_2. \tag{124}$$

where R_i satisfies

$$|R_1| \leq C\mathcal{E}_\varphi(L) + \|q^{1/2}F\|^2_{\mathcal{H}}$$
$$|R_2| \leq C\mathcal{E}_\psi(L) + \|q^{1/2}F\|^2_{\mathcal{H}}.$$

for a positive constant C.

Proof. To get (123), let us multiply the equation $(112)_2$ by $q\overline{\varphi}_x$ Integrating on $(0, L)$ we obtain

$$i\lambda\rho_1 \int_0^L uq\overline{\varphi}_x \,\mathrm{d}x - K \int_0^L (\varphi_x + \psi)_x q\overline{\varphi}_x \,\mathrm{d}x = \rho_1 \int_0^L f_2 q\overline{\varphi}_x \,\mathrm{d}x$$

or

$$-\rho_1 \int_0^L uq(\overline{i\lambda\varphi_x}) \,\mathrm{d}x - K \int_0^L q\varphi_{xx}\overline{\varphi}_x \,\mathrm{d}x - K \int_0^L q\psi_x\overline{\varphi}_x \,\mathrm{d}x = \rho_1 \int_0^L f_2 q\overline{\varphi}_x \,\mathrm{d}x.$$

Since $i\lambda\varphi_x = u_x + f_{1x}$ taking the real part in the above equality results in

$$-\frac{\rho_1}{2} \int_0^L q\frac{\mathrm{d}}{\mathrm{d}x}|u|^2 \,\mathrm{d}x - \frac{K}{2} \int_0^L q\frac{\mathrm{d}}{\mathrm{d}x}|\varphi_x|^2 \,\mathrm{d}x = \rho_1 Re \int_0^L f_2 q\overline{\varphi}_x \,\mathrm{d}x + \rho_1 Re \int_0^L uq\overline{f}_{1x} \,\mathrm{d}x + K\ Re \int_0^L q\psi_x\overline{\varphi}_x \,\mathrm{d}x.$$

Performing an integration by parts we get

$$\int_0^L q'(s)[\rho_1|u(s)|^2 + K|\varphi_x(s)|^2] \,\mathrm{d}s = [q\mathcal{I}_\varphi]_0^L + 2K\ Re \int_0^L q\psi_x\overline{\varphi}_x \,\mathrm{d}x + R_1$$

where

$$R_1 = 2\rho_1 Re \int_0^L f_2 q\overline{\varphi}_x \,\mathrm{d}x + 2\rho_1 Re \int_0^L uq\overline{f}_{1x} \,\mathrm{d}x.$$

Similarly, multiplying equation $(112)_5$ by $q\overline{\varphi}_x$, integrating on $(0, L)$ and taking the real part we obtain

$$i\lambda\rho_2 \int_0^L vq\overline{\psi}_x \,\mathrm{d}x - b \int_0^L \psi_{xx} q\overline{\psi}_x \,\mathrm{d}x + K \int_0^L (\varphi_x + \psi) q\overline{\psi}_x \,\mathrm{d}x = \rho_2 \int_0^L f_6 q\overline{\psi}_x \,\mathrm{d}x$$

or

$$-\rho_2 \int_0^L vq(\overline{i\lambda\psi_x}) \,\mathrm{d}x - b \int_0^L q\psi_{xx}\overline{\psi}_x \,\mathrm{d}x + K \int_0^L q\varphi_x\overline{\psi}_x \,\mathrm{d}x + K \int_0^L q\psi\overline{\psi}_x \,\mathrm{d}x = \rho_2 \int_0^L f_6 q\overline{\psi}_x \,\mathrm{d}x.$$

Since $i\lambda\psi_x = v_x + f_{5x}$ taking the real part in the above equality results in

$$-\frac{\rho_2}{2} \int_0^L q\frac{\mathrm{d}}{\mathrm{d}x}|v|^2 \,\mathrm{d}x - \frac{b}{2} \int_0^L q\frac{\mathrm{d}}{\mathrm{d}x}|\psi_x|^2 \,\mathrm{d}x = \rho_2 Re \int_0^L f_6 q\overline{\psi}_x \,\mathrm{d}x$$
$$+ \rho_2\ Re \int_0^L qv\overline{f}_{5x} \,\mathrm{d}x - K\ Re \int_0^L q\varphi_x\overline{\psi}_x \,\mathrm{d}x - \frac{K}{2} \int_0^L q\frac{\mathrm{d}}{\mathrm{d}x}|\psi|^2 \,\mathrm{d}x.$$

Performing an integration by parts we get

$$\int_0^L q'(s)[\rho_2|v(s)|^2 + b|\psi_x(s)|^2] \,\mathrm{d}s = [q\mathcal{I}_\psi]_0^L - K[q|\psi|^2]_0^L - 2K\ Re \int_0^L q\varphi_x\overline{\psi}_x \,\mathrm{d}x + K \int_0^L q'|\psi|^2 \,\mathrm{d}x + R_2$$

where

$$R_2 = 2\rho_2 Re \int_0^L f_6 q \overline{\psi}_x \,\mathrm{d}x + 2\rho_2 \ Re \int_0^L q v \overline{f}_{5x} \,\mathrm{d}x.$$

If we take $q(x) = \int_0^x \mathrm{e}^{ns}\,\mathrm{d}s = \frac{\mathrm{e}^{nx}-1}{n}$ (Here n will be chosen large enough) in Lemma 5.4 we arrive at

$$\begin{aligned}\mathcal{E}_\varphi(L) + \mathcal{E}_\psi(L) &= q(L)\mathcal{I}_\varphi(L) + 2K \ Re \int_0^L q\psi_x\overline{\varphi}_x \,\mathrm{d}x q(L)\mathcal{I}_\psi(L) - Kq(L)|\psi(L)|^2 \\ &\quad + K\int_0^L q'(x)|\psi|^2\,\mathrm{d}x - 2K \ Re \int_0^L q\varphi_x\overline{\psi}_x\,\mathrm{d}x + R_1 + R_2 \\ &= q(L)\mathcal{I}_\varphi(L) + q(L)\mathcal{I}_\psi(L) - Kq(L)|\psi(L)|^2 + \ K\int_0^L q'(x)|\psi|^2\,\mathrm{d}x + R_1 + R_2\end{aligned}$$

Also, we have

$$\begin{aligned}|R_1| &\le 2\rho_1 \int_0^L q(x)(|u(s)|^2 + |\varphi_x(s)|^2)\,\mathrm{d}s + 2\rho_1 \int_0^L q(x)(|f_2(s)|^2 + |f_{1x}(s)|^2)\,\mathrm{d}s \\ &\le C\frac{\mathrm{e}^{Ln}}{n}\|F\|^2_{\mathcal{H}} + \frac{c'}{n}\mathcal{E}_\varphi(L)\end{aligned} \tag{125}$$

and

$$\begin{aligned}|R_2| &\le 2\rho_2 \int_0^L q(x)(|v(s)|^2 + |\psi_x(s)|^2)\,\mathrm{d}s + 2\rho_1 \int_0^L q(x)(|f_6(s)|^2 + |f_{5x}(s)|^2)\,\mathrm{d}s \\ &\le C\frac{\mathrm{e}^{Ln}}{n}\|F\|^2_{\mathcal{H}} + \frac{c'}{n}\mathcal{E}_\psi(L)\end{aligned} \tag{126}$$

Using Lemma 5.4 and the Young inequality we get

$$\mathcal{E}_\varphi(L) + \mathcal{E}_\psi(L) \le q(L)\mathcal{I}_\varphi(L) + q(L)\mathcal{I}_\psi(L) + \ K\int_0^L q'(x)|\psi|^2\,\mathrm{d}x + c\|F\|^2_{\mathcal{H}}$$

for a positive constant C. It results by (115), (120), (121), and (122) that we can find a positive constant C such that

$$\mathcal{E}_\varphi(L) + \mathcal{E}_\psi(L) \le K\int_0^L |\psi|^2\,\mathrm{d}x + c(|\lambda|^{4-2\alpha} + |\lambda|^{2-2\alpha} + 1)\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c(|\lambda|^2 + 1)\|F\|^2_{\mathcal{H}}$$

for $\lambda \neq 0$. Since that $\varphi = \frac{u+f_1}{i\lambda}$ and $\psi = \frac{v+f_4}{i\lambda}$ we obtain

$$\mathcal{E}_\varphi(L) + \mathcal{E}_\psi(L) \le c(|\lambda|^{4-2\alpha} + |\lambda|^{2-2\alpha} + 1)\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c(|\lambda|^2 + 1)\|F\|^2_{\mathcal{H}} + \frac{c}{|\lambda|^2}\|U\|^2_{\mathcal{H}} + \frac{c}{|\lambda|^2}\|F\|^2_{\mathcal{H}}.$$

Since that

$$\int_{-\infty}^{+\infty} (\phi_i(\xi))^2\,\mathrm{d}\xi \le C\int_{-\infty}^{+\infty} (\xi^2 + \eta)(\phi_i(\xi))^2\,\mathrm{d}\xi$$

for $\lambda \neq 0$. If $|\lambda| > 1$ we get

$$\|U\|_{\mathcal{H}}^2 \leq |\lambda|^{8-4\alpha}\|F\|_{\mathcal{H}}^2.$$

It follows that

$$\frac{1}{|\lambda|^{(4-2\alpha)}}\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C, \quad \forall \lambda \in \mathbb{R},$$

for a positive constant C. The conclusion then follows by applying Theorem 4.2. □

Remark 5.1. 1. By Proposition 4.1, the spectrum of $\mathcal{A}$ is at the left of the imaginary axis, but approaches this axis. Hence, the decay of the energy depends on the asymptotic behavior of the real part of these eigenvalues, since Proposition 4.1 shows a behavior like $k^{-(3-\alpha)}$, we can expect a decay rate (optimal) of the energy of order $t^{-2/(3-\alpha)}$. We unfortunately were not able to prove this optimal decay rate by Borichev–Tomilov theorem. In Theorem 5.3, we obtain decay rate of order $t^{-1/(2-\alpha)}$ which is less better. But, it is interesting to remark that both energy decay in Theorem 5.3 and Proposition 4.1 approach t^{-1} (as $\alpha \to 1$) which is the energy decay given in [15,18].

2. Estimation of decay rate in the case $\eta = 0$ is open. As $\lambda = 0$ is a spectral value, both techniques used in [15,18] do not work. In the future, we try other methods, in particular some tools from observability theory. Another technic is the use of Laplace transform and representation of solutions by Mittag–Leffler Functions.
3. It seems to be interesting to study a global decaying solutions of hyperbolic systems (strong and weakly) under control of fractional derivative type. We think that the interaction of the hyperbolicity (order of multiplicity) and the number of dissipative terms have an effect on the result.

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