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Global existence and finite time blow-up for a class of thin-film equation

Zhihua Dong and Jun Zhou

Abstract. This paper deals with a class of thin-film equation, which was considered in Li et al. (Nonlinear Anal Theory Methods Appl 147:96–109, 2016), where the case of lower initial energy ($J(u_0) \leq d$ and d is a positive constant) was discussed, and the conditions on global existence or blow-up are given. We extend the results of this paper on two aspects: Firstly, we consider the upper and lower bounds of blow-up time and asymptotic behavior when $J(u_0) < d$; secondly, we study the conditions on global existence or blow-up when $J(u_0) > d$.

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1. Introduction and Main Results

In this paper, we consider the following thin-film equation:

$$\begin{cases} u_t + u_{xxxx} - \left(|u_x|^{p-2}u_x\right)_x = |u|^{q-1}u - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega |u|^{q-1}u\mathrm{d}x, & x \in \Omega,\ t > 0, \\ u_x = u_{xxx} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}$ is an open interval, $p > 1, q > \max\{1, p-1\}$, $u_0 \in H$ and

$$H \triangleq \left\{ \phi \in H^2(\Omega) \,\middle|\, \phi_x|_{\partial\Omega} = 0, \int_\Omega \phi\mathrm{d}x = 0 \right\}.$$

Throughout this paper, we denote by $\|\cdot\|_s$ the $L^s(\Omega)$ norm for $1 \leq s \leq \infty$, and it is easy to see H with the norm $\|u_{xx}\|_2$ is a Banach space. Since $H \hookrightarrow L^{q+1}$ continuously, we denote by A the optimal embedding constant, i.e.,

$$\frac{1}{A} = \inf_{u \in H\setminus\{0\}} \frac{\|u_{xx}\|_2}{\|u\|_{q+1}}. \tag{1.2}$$

In order to review the previous results precisely, we define some notations, functionals and sets as follows:

$$J(u) \triangleq \frac{1}{2}\|u_{xx}\|_2^2 + \frac{1}{p}\|u_x\|_p^p - \frac{1}{q+1}\|u\|_{q+1}^{q+1},$$

$$I(u) \triangleq \|u_{xx}\|_2^2 + \|u_x\|_p^p - \|u\|_{q+1}^{q+1},$$

$$\mathcal{N} \triangleq \{u \in H | I(u) = 0, \|u_{xx}\|_2 \neq 0\},$$

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$$\begin{aligned} \mathcal{N}_+ &\triangleq \{u \in H \mid I(u) > 0\}, \\ \mathcal{N}_- &\triangleq \{u \in H \mid I(u) < 0\}, \\ J^\alpha &\triangleq \{u \in H \mid J(u) < \alpha\}, \end{aligned} \tag{1.3}$$

where α is a constant. Then, the mountain pass level d is (see [15])

$$d \triangleq \min_{u\in\mathcal{N}} J(u) = \min_{u\in H\setminus\{0\}} \max_{s\geq 0} J(\text{su}). \tag{1.4}$$

Finally, we let

$$E_1 \triangleq \frac{q-1}{2(q+1)} A^{-\frac{2(q+1)}{q-1}} > 0, \tag{1.5}$$

$$\alpha_1 \triangleq A^{-\frac{q+1}{q-1}}, \tag{1.6}$$

where A is given in (1.2), and

$$\begin{aligned} J_0(u) &\triangleq J(u) - \frac{1}{p}\|u_x\|_p^p = \frac{1}{2}\|u_{xx}\|_2^2 - \frac{1}{q+1}\|u\|_{q+1}^{q+1}, \\ I_0(u) &\triangleq \|u_{xx}\|_2^2 - \|u\|_{q+1}^{q+1}. \end{aligned} \tag{1.7}$$

Problem (1.1) describes a series of physical phenomena (see [14,15,32]). One characteristic of problem (1.1) is the nonlocal source $|u|^{q-1}u - \frac{1}{|\Omega|}\int_\Omega |u|^{q-1}u\mathrm{d}x$, and there are a lot of papers dealing with this kind of evolution equations (see [6,28,29] for the heat equations, see [10,12,13,20,24,25] for the p-Laplace equations, see [1,2,30] for the porous medium equations).

Another problem related to (1.1) is the following problem:

$$\begin{cases} u_t + u_{xxxx} = |u|^{q-1}u - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega |u|^{q-1}u\mathrm{d}x, & x \in \Omega,\ t > 0, \\ u_x = u_{xxx} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.8}$$

which was studied in (see [26,34]). The authors got the following conclusions:

- (i) The weak solution of problem (1.8) blows up in finite time if $J_0(u_0) \leq 0$ or $0 < J_0(u_0) \leq E_1$ and $I_0(u_0) < 0$;
- (ii) The weak solution of problem (1.8) exists globally if $0 < J_0(u_0) < E_1$ and $I_0(u_0) > 0$ or $J_0(u_0) = E_1$ and $I_0(u_0) \geq 0$.
- (iii) The blow-up time T satisfies

$$T \leq \frac{(q+1)|\Omega|^{\frac{q-1}{2}}\|u_0\|_2^{-(q-1)}}{(q-1)^2\left[1 - \left((q+1)\left(\frac{1}{2} - \frac{J_0(u_0)}{\alpha_1^2}\right)\right)^{-\frac{q+1}{q-1}}\right]}$$

when $0 < J_0(u_0) < E_1$ and $\|u_{0xx}\|_2 > \alpha_1$.

Problem (1.1) was firstly studied by Li et al. [15]. Next, we will introduce the main results of this paper. Firstly, we give the definition of the weak solutions to (1.1).

Definition 1. [15] A function $u \in L^\infty(0,T;H)$ with $u_t \in L^2\left(0,T;\widetilde{L^2(\Omega)}\right)$ is called a weak solution of problem (1.1), if $u(x,0) = u_0 \in H$ and $u(x,t)$ satisfies problem (1.1) in the following sense, i.e.,

$$\int_0^t \int_\Omega \left[u_t\phi + u_{xx}\phi_{xx} + |u_x|^{p-2}u_x\phi_x - \left(|u|^{q-1}u - \frac{1}{|\Omega|}\int_\Omega |u|^{q-1}u\mathrm{d}x\right)\phi\right]\mathrm{d}x\mathrm{d}\tau = 0, \quad \forall t \in (0,T) \tag{1.9}$$

for any $\phi \in L^2(0,T;H^2(\Omega))$ with $\phi_x|_{\partial\Omega} = 0$, where

$$\widetilde{L^2(\Omega)} \triangleq \left\{ u \in L^2(\Omega) \,\middle|\, \int\limits_{\Omega} u\mathrm{d}x = 0 \right\}.$$

The main results of [15] are the following four theorems.

Theorem 1. [15] *If $J(u_0) < d$ and $I(u_0) > 0$, then the weak solution of problem (1.1) exists globally. Moreover, there exist a constant $C > 0$ such that $\|u\|_2^2 \le \|u_0\|_2^2 \mathrm{e}^{-Ct}$, and u does not vanish in finite time.*

Theorem 2. [15] *If $J(u_0) < d$ and $I(u_0) < 0$, then the weak solution of problem (1.1) blows up at a finite time T, that is*

$$\lim_{t\to T} \int\limits_0^t \|u(\tau)\|_2^2 \mathrm{d}\tau = +\infty. \tag{1.10}$$

Theorem 3. [15] *If $J(u_0) = d$ and $I(u_0) \ge 0$, then the weak solution of problem (1.1) exists globally and $I(u(t)) \ge 0$ for all $t \ge 0$. Moreover, if $I(u) > 0$, then the solution does not vanish and there exist constants C_1 and C_2 such that $\|u\|_2^2 \le C_1 \mathrm{e}^{-C_2 t}$. If not, the solution vanishes in finite time.*

Theorem 4. [15] *If $J(u_0) = d$ and $I(u_0) < 0$, then the weak solution of problem (1.1) blows up at a finite time T, that is*

$$\lim_{t\to T} \int\limits_0^t \|u(\tau)\|_2^2 = +\infty. \tag{1.11}$$

In summary, in [15], the authors studied the conditions on global existence or blow-up when $J(u_0) \le d$. But there are two problems unsolved. Firstly, there is no estimates of blow-up time or asymptotic behavior for the blow-up solutions, which are important to study blow-up problems (see [4,5,7,9,11,16–19,21–23,27,31,33]); Secondly, when $J(u_0) > d$, whether the solution exists globally or blows up in finite time is unconsidered. The main task of this paper is to study these two problems.

In order to introduce the main results of this paper, we need some preparations. Firstly, we compare the values of d and E_1. It follows from (1.2) and (1.4) that

$$\begin{aligned} d &\ge \min_{u\in H\setminus\{0\}} \max_{s\ge 0} J_0(\mathrm{su}) \\ &= \min_{u\in H\setminus\{0\}} J_0(\mathrm{su}) \Bigg|_{s=\sqrt[q-1]{\frac{\|u_{xx}\|_2^2}{\|u\|_{q+1}^{q+1}}}} \\ &= \frac{q-1}{2(q+1)} \min_{u\in H\setminus\{0\}} \left(\frac{\|u_{xx}\|_2}{\|u\|_{q+1}} \right)^{\frac{2(q+1)}{q-1}} \\ &= \frac{q-1}{2(q+1)} A^{-\frac{2(q+1)}{q-1}} = E_1. \end{aligned}$$

By definition of $J(u)$, $\mathcal{N}$, J^α and d, we can get

$$\mathcal{N}_\alpha \triangleq \mathcal{N} \cap J^\alpha \equiv \left\{ u \in \mathcal{N} \,\middle|\, \left(\frac{1}{2} - \frac{1}{q+1}\right) \|u_{xx}\|_2^2 + \left(\frac{1}{p} - \frac{1}{q+1}\right) \|u_x\|_p^p < \alpha \right\} \ne \emptyset \quad \text{for all } \alpha > d. \tag{1.12}$$

Since $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, we denote by $\mu > 0$ the optimal embedding constant. Let $u \in H$, then $u_x \in H_0^1(\Omega)$, so we get

$$\|u_x\|_p \le \mu \|u_{xx}\|_2.$$

By (1.4), we can obtain

$$
\begin{aligned}
d &= \min_{u\in\mathcal{N}}\left\{\frac{1}{2}\|u_{xx}\|_2^2+\frac{1}{p}\|u_x\|_p^p-\frac{1}{q+1}\|u\|_{q+1}^{q+1}\right\}\\
&= \min_{u\in\mathcal{N}}\left\{\left(\frac{1}{2}-\frac{1}{q+1}\right)\|u_{xx}\|_2^2+\left(\frac{1}{p}-\frac{1}{q+1}\right)\|u_x\|_p^p\right\}\\
&\le \min_{u\in\mathcal{N}}\left\{\left(\frac{1}{2}-\frac{1}{q+1}\right)\|u_{xx}\|_2^2+\left(\frac{1}{p}-\frac{1}{q+1}\right)\mu^p\|u_{xx}\|_2^p\right\}\\
&= \left(\frac{1}{2}-\frac{1}{q+1}\right)\left(\min_{u\in\mathcal{N}}\|u_{xx}\|_2\right)^2+\left(\frac{1}{p}-\frac{1}{q+1}\right)\mu^p\left(\min_{u\in\mathcal{N}}\|u_{xx}\|_2\right)^p,
\end{aligned}
$$

then there exists a unique positive constant σ depending on p,q,μ,d such that $\min_{u\in\mathcal{N}}\|u_{xx}\|_2\ge\sigma$. Therefore,

$$
\operatorname{dist}(0,\mathcal{N})=\min_{u\in\mathcal{N}}\|u_{xx}\|_2:=\kappa\ge\sigma>0. \tag{1.13}
$$

For any $u\in\mathcal{N}_-$, i.e., $I(u)<0$, we have $\|u_{xx}\|_2\neq 0$ and $\|u_x\|_p\neq 0$. Combining the definition of $\mathcal{N}$ and (1.2) we can obtain

$$
\begin{aligned}
&\|u_{xx}\|_2^2<\|u\|_{q+1}^{q+1}\le A^{q+1}\|u_{xx}\|_2^{q+1}=A^{q+1}\|u_{xx}\|_2^{q-1}\|u_{xx}\|_2^2,\\
&\|u_{xx}\|_2>\alpha_1
\end{aligned} \tag{1.14}
$$

where α_1 is defined in (1.6). The above inequality yields

$$
\operatorname{dist}(0,\mathcal{N}_-)=\min_{u\in\mathcal{N}_-}\|u_{xx}\|_2\ge\alpha_1>0. \tag{1.15}
$$

We now define

$$
\lambda_\alpha\triangleq\inf\{\|u\|_2\,|u\in\mathcal{N}_\alpha\},\qquad \Lambda_\alpha\triangleq\sup\{\|u\|_2\mid u\in\mathcal{N}_\alpha\}\quad\text{for all }\alpha>d. \tag{1.16}
$$

Clearly, we have the following monotonicity properties

$$
\alpha\mapsto\lambda_\alpha \text{ is nonincreasing},\qquad \alpha\mapsto\Lambda_\alpha \text{ is nondecreasing}.
$$

For $\delta>0$, we define some modified functionals and sets as follows:

$$
\begin{aligned}
I_\delta(u)&\triangleq\delta\|u_{xx}\|_2^2+\delta\|u_x\|_p^p-\|u\|_{q+1}^{q+1},\\
N_\delta&\triangleq\{u\in H|I_\delta(u)=0,\|u_{xx}\|_2\neq 0\}.
\end{aligned}
$$

The modified potential wells and their corresponding sets are defined, respectively, by

$$
\begin{aligned}
d(\delta)&\triangleq\inf_{u\in\mathcal{N}_\delta}J(u),\\
W_\delta(u)&\triangleq\{u\in H\,|I_\delta(u)>0,J(u)<d(\delta)\}\cup\{0\},\\
V_\delta(u)&\triangleq\{u\in H\,|I_\delta(u)<0,J(u)<d(\delta)\}.
\end{aligned}
$$

Finally, we introduce some sets as following:

$$
\begin{aligned}
\mathcal{B}&\triangleq\{u_0\in H|\text{ the solution } u=u(t) \text{ of (1.1) blows up in finite time }\},\\
\mathcal{G}&\triangleq\{u_0\in H|\text{ the solution } u=u(t) \text{ of (1.1) exists for all } t>0\},\\
\mathcal{G}_0&\triangleq\{u_0\in\mathcal{G}|u(t)\longmapsto 0 \text{ in } H \text{ as } t\to\infty\}.
\end{aligned} \tag{1.17}
$$

With the above preparations, we can introduce the main results of this paper. The first result is about the estimate of the lower and upper bounds of the blow-up time, asymptotic behavior when $J(u_0)<d$.

Theorem 5. *Let $J(u_0)$, $I(u_0)$, $\mathcal{B}$ be defined in (1.3) and (1.17). If $J(u_0) < d$, $I(u_0) < 0$ and $q < 9$, then $u_0 \in \mathcal{B}$. Moreover, it holds*

$$T \geq \frac{\|u_0\|_2^{2-2\gamma}}{2(\gamma-1)C_0} \tag{1.18}$$

and

$$\|u\|_2 \geq [2C_0(\gamma-1)]^{-\frac{1}{2(\gamma-1)}}(T-t)^{-\frac{1}{2(\gamma-1)}}, \tag{1.19}$$

where T is the blow-up time,

$$\gamma = \frac{3q+5}{9-q} > 1, \quad C_0 = \hat{C}^{\frac{8(q+1)}{9-q}}, \tag{1.20}$$

$\hat{C}$ is the optimal constant of the Gagliardo–Nirenberg's inequality [3, 8]:

$$\|\phi\|_{q+1} \leq \hat{C}\|\phi_{xx}\|_2^{(1-\theta)}\|\phi\|_2^{\theta}, \quad \forall \phi \in H, \tag{1.21}$$

in which,

$$\theta = \frac{3q+5}{4(q+1)} \in (0,1). \tag{1.22}$$

For the upper bounds of blow-up time and asymptotic behavior, we cannot calculate them but for $J(u_0) < E_1$, which is smaller than d and is given in (1.5).

Theorem 6. *Let $J(u_0)$, $I(u_0)$, $\mathcal{B}$ be defined in (1.3) and (1.17). If $J(u_0) < E_1$ and $I(u_0) < 0$, then $u_0 \in \mathcal{B}$. Moreover, it holds*

$$T \leq \begin{cases} -\dfrac{\|u_0\|_2^2}{(q^2-1)J(u_0)}, & \text{if } J(u_0) < 0; \\ \dfrac{(q+1)|\Omega|^{\frac{q-1}{2}}\|u_0\|_2^{-(q-1)}}{(q-1)^2\left[1-\left((q+1)\left(\frac{1}{2}-\frac{J_0(u_0)}{\alpha_1^2}\right)\right)^{-\frac{q+1}{q-1}}\right]}, & \text{if } 0 \leq J(u_0) < E_1, \end{cases} \tag{1.23}$$

and

$$\|u\|_2 \leq \begin{cases} \left[(q^2-1)\|u_0\|_2^{-(q+1)}(-J(u_0))\right]^{-\frac{1}{q-1}}(T-t)^{-\frac{1}{q-1}}, & \text{if } J(u_0) < 0; \\ \dfrac{|\Omega|(q+1)^{\frac{1}{q-1}}}{(q-1)^{\frac{1}{q-1}}}\left[1-\left(\dfrac{1}{(q+1)\left(\frac{1}{2}-\frac{J_0(u_0)}{\alpha_1^2}\right)}\right)^{\frac{q+1}{q-1}}\right]^{-\frac{1}{q-1}}(T-t)^{-\frac{1}{q-1}}, & \text{if } 0 \leq J(u_0) < E_1. \end{cases} \tag{1.24}$$

where T is the blow-up time, $J_0(u_0)$ is defined in (1.7), E_1 and α_1 are given in (1.5) and (1.14), respectively.

Remark 1. By [34, Remark 1.2], we know that $(q+1)\left(\frac{1}{2}-\frac{J_0(u_0)}{\alpha_1^2}\right) > 1$. So (1.23) and (1.24) make sense for $0 \leq J(u_0) < E_1$.

At last, we give the conditions to ensure the solution exists globally or blows up in finite time with $J(u_0) > d$.

Theorem 7. *Let $\lambda_{J(u_0)}, \Lambda_{J(u_0)}, \mathcal{N}_+, \mathcal{N}_-, \mathcal{G}_0, \mathcal{B}$ be the constants or sets defined in (1.3), (1.16) and (1.17). Assume $J(u_0) > d$, then the following conclusions hold*

(i) *If $u_0 \in \mathcal{N}_+$ and $\|u_0\|_2 \leq \lambda_{J(u_0)}$, then $u_0 \in \mathcal{G}_0$;*
(ii) *If $u_0 \in \mathcal{N}_-$ and $\|u_0\|_2 \geq \Lambda_{J(u_0)}$, then $u_0 \in \mathcal{B}$.*

Remark 2. By (2.1), we know that there indeed exists u_0 satisfying the conditions of the above theorem.

By using (ii) of Theorem 7, we can get the following two corollaries.

Corollary 1. *Assume $J(u_0) > d$ and let $\mathcal{N}_-, \mathcal{B}$ be the sets defined in (1.3) and (1.17), respectively.*

(i) *If*

$$|\Omega|^{\frac{q-1}{2}} \frac{2(q+1)}{q-1} J(u_0) \begin{cases} \leq \|u_0\|_2^{q+1}, & \text{if } 1 < p < 2; \\ < \|u_0\|_2^{q+1}, & \text{if } p = 2, \end{cases} \tag{1.25}$$

then $u_0 \in \mathcal{N}_- \cap \mathcal{B}$.

(ii) *If $p > 2$ and*

$$|\Omega|^{\frac{q-1}{2}} \frac{p(q+1)}{q+1-p} J(u_0) \leq \|u_0\|_2^{q+1}, \tag{1.26}$$

then $u_0 \in \mathcal{N}_- \cap \mathcal{B}$.

The second corollary indicates that there exists blow-up solutions to (1.1) for any high initial energy.

Corollary 2. *For any $M > d$, there exists $u_M \in \mathcal{N}_-$ such that $J(u_M) = M$ and $u_M \in \mathcal{B}$.*

The organizations of the remaining of this paper are as follows: In Sect. 2, we give some preliminaries, which are important for our proofs. In Sect. 3, we give the proofs of the theorems and corollaries.

2. Preliminaries

In this section, we will give some useful lemmas and propositions for our later proofs. Throughout this section, we will use the notations, sets, functionals and constants defined in Sect. 1.

Lemma 1. [15] *Assume $u \in H^2(\Omega), 0 < J(u) < d$, and $0 < \delta_1 < 1 < \delta_2$ satisfy the equation $d(\delta) = J(u)$, then the sign of $I_\delta(u)$ does not change for $\delta_1 < 1 < \delta_2$.*

Remark 3. By [15, Lemma 2.3], we know that there indeed exist δ_1 and δ_2 satisfying $0 < \delta_1 < 1 < \delta_2$ and $d(\delta) = J(u)$ when $0 < J(u) < d$.

Lemma 2. [15] *Assume that u is a weak solution of problem (1.1) with $0 < J(u_0) < d$. Let $0 < \delta_1 < 1 < \delta_2$ be the two roots of the equation $d(\delta) = J(u_0)$ and T is the maximal existence time.*

(i) *If $I(u_0) > 0$, then $u \in W(\delta)$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$;*
(ii) *If $I(u_0) < 0$, then $u \in V(\delta)$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$.*

Lemma 3. [15] *Let $u \in H$ and*

$$r(\delta) = \left(\frac{\delta}{A^{q+1}}\right)^{\frac{1}{q-1}},$$

where A is defined in (1.2). Then

(i) *if $0 < \|u_{xx}\|_2 < r(\delta)$, then $I_\delta(u) > 0$;*
(ii) *if $I_\delta(u) < 0$, then $\|u_{xx}\|_2 > r(\delta)$;*
(iii) *if $I_\delta(u) = 0$, then $\|u_{xx}\|_2 = 0$ or $\|u_{xx}\|_2 \geq r(\delta)$.*

Proposition 1. *The two constants λ_α and Λ_α defined in (1.16) satisfy the following relationship:*

$$0 < \lambda_\alpha < \Lambda_\alpha < +\infty. \tag{2.1}$$

Proof. If $u \in \mathcal{N}$, then it follows the definition of $\mathcal{N}$ and (1.21) that

$$\|u_{xx}\|_2^2 \le \|u\|_{q+1}^{q+1} \le C\|u_{xx}\|_2^{(1-\theta)(q+1)}\|u\|_2^{\theta(q+1)},$$

where $C = \hat{C}^{q+1}$, i.e.,

$$\|u_{xx}\|_2^{\beta} \le C\|u\|_2^{\rho} \text{ for all } u \in \mathcal{N}, \tag{2.2}$$

where $\beta = \frac{9-q}{4}$ and $\rho = \frac{3q+5}{4}$.

Combining with (1.13) and (2.2), we have

$$\begin{aligned} \lambda_\alpha &= \inf_{u\in\mathcal{N}_\alpha} \|u\|_2 \ge \inf_{u\in\mathcal{N}} \|u\|_2 \\ &\ge C^{-\frac{1}{\rho}} \left(\inf_{u\in\mathcal{N}} \|u_{xx}\|_2\right)^{\frac{\beta}{\rho}} \\ &= C^{-\frac{1}{\rho}}\kappa^{\frac{\beta}{\rho}} > 0. \end{aligned}$$

Furthermore, since $p > 1$, by (1.2), (1.12) and Hölder's inequality, we can obtain

$$\begin{aligned} \Lambda_\alpha &= \sup_{u\in\mathcal{N}_\alpha} \|u\|_2 \\ &\le |\Omega|^{\frac{q-1}{2(q+1)}} \sup_{u\in\mathcal{N}_\alpha} \|u\|_{q+1} \\ &= |\Omega|^{\frac{q-1}{2(q+1)}} \left[\sup_{u\in\mathcal{N}_\alpha} \left(\|u_{xx}\|_2^2 + \|u_x\|_p^p\right)\right]^{\frac{1}{q+1}} \\ &\le \begin{cases} |\Omega|^{\frac{q-1}{2(q+1)}} \left[\dfrac{2\alpha(q+1)}{q-1}\right]^{\frac{1}{q+1}}, & \text{if } 1 < p \le 2; \\ |\Omega|^{\frac{q-1}{2(q+1)}} \left[\dfrac{p\alpha(q+1)}{q+1-p}\right]^{\frac{1}{q+1}}, & \text{if } p > 2, \end{cases} \\ &< +\infty. \end{aligned}$$

Then the result follows. □

Lemma 4. *Let u be the weak solution of (1.1), then it holds*

$$\|u(t)\|_2^2 = \|u_0\|_2^2 - 2\int_0^t I(u(\tau))\mathrm{d}\tau \tag{2.3}$$

and

$$\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + J(u(t)) = J(u_0). \tag{2.4}$$

Proof. Let $\phi = u$ in (1.9). Noting that $\int_\Omega u\mathrm{d}x = 0$, we get

$$\int_0^t \int_\Omega \left(u_\tau u + |u_{xx}|^2 + |u_x|^p - |u|^{q+1}\right) \mathrm{d}x\mathrm{d}\tau = 0,$$

which leads (2.3).

Now, we consider (2.4). Firstly we assume u is smooth enough such that $u_t \in L^2(0, T; H)$. Let $\phi = u_t$ in (1.9), note that $\int_\Omega u_t\mathrm{d}x = 0$, we have

$$\int_0^t \int_\Omega \left[|u_\tau|^2 + u_{xx}u_{\tau xx} + |u_x|^{p-2}u_x u_{\tau x} - |u|^{q-1}uu_\tau\right] \mathrm{d}x\mathrm{d}\tau = 0,$$

hence, we have

$$\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + \frac{1}{2}\|u_{xx}\|_2^2 + \frac{1}{p}\|u_x\|_p^p - \frac{1}{q+1}\|u\|_{q+1}^{q+1} = \frac{1}{2}\|u_{0xx}\|_2^2 + \frac{1}{p}\|u_{0x}\|_p^p - \frac{1}{q+1}\|u_0\|_{q+1}^{q+1},$$

i.e., (2.4) holds. Since $L^2\left(0,T;\widetilde{L^2(\Omega)}\right)$ is dense in $L^2(0,T;H)$, by density argument and Definition 1, we know that (2.4) holds for weak solutions of problem (1.1). □

Lemma 5. *The following results hold true.*

(i) $J(u) > 0$ *for any* $u \in \mathcal{N}_+$*;*
(ii) *For all* $u \in \mathcal{N}$*, we have* $J(u) = \max_{s\geq 0} J(su)$*;*
(iii) *For any* $\alpha > 0$*, it holds*

$$\|u_{xx}\|_2^2 < \frac{2(q+1)}{q-1}\alpha, \quad \forall u \in J^\alpha \cap \mathcal{N}_+. \tag{2.5}$$

Proof. Case (i). Since $u \in \mathcal{N}_+$ and $q > \max\{1, p-1\}$, we can obtain

$$\begin{aligned} J(u) &= \frac{1}{2}\|u_{xx}\|_2^2 + \frac{1}{p}\|u_x\|_p^p - \frac{1}{q+1}\|u\|_{q+1}^{q+1} \\ &> \frac{1}{q+1}\left(\|u_{xx}\|_2^2 + \|u_x\|_p^p - \|u\|_{q+1}^{q+1}\right) \\ &= \frac{1}{q+1} I(u) \\ &> 0. \end{aligned}$$

Case (ii). Since $u \in \mathcal{N}$, it follows from the definitions of $J(u)$ and $I(u)$ in (1.3) that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} J(\mathrm{su}) &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s^2}{2}\|u_{xx}\|_2^2 + \frac{s^p}{p}\|u_x\|_p^p - \frac{s^{q+1}}{q+1}\|u\|_{q+1}^{q+1}\right) \\ &= s^q\left(s^{1-q}\|u_{xx}\|_2^2 + s^{p-1-q}\|u_x\|_p^p - \|u\|_{q+1}^{q+1}\right) \\ &= s^q\left[\|u_{xx}\|_2^2 + \|u_x\|_p^p - \|u\|_{q+1}^{q+1} + \left(s^{1-q}-1\right)\|u_{xx}\|_2^2 + \left(s^{p-1-q}-1\right)\|u_x\|_p^p\right] \\ &= s^q\left[I(u) + \left(s^{1-q}-1\right)\|u_{xx}\|_2^2 + \left(s^{p-1-q}-1\right)\|u_x\|_p^p\right] \\ &= s\left(1-s^{q-1}\right)\|u_{xx}\|_2^2 + s^{p-1}\left(1-s^{q-p+1}\right)\|u_x\|_p^p. \end{aligned}$$

Since $p > 1, q > \max\{1, p-1\}$, we obtained

$$\frac{\mathrm{d}}{\mathrm{d}s} J(\mathrm{su}) \begin{cases} > 0, & \text{if } 0 < s < 1; \\ = 0, & \text{if } s = 1; \\ < 0, & \text{if } s > 1. \end{cases}$$

Hence, we get $J(u) = \max_{s\geq 0} J(\mathrm{su})$.

Case (iii). For any $u \in J^\alpha \cap \mathcal{N}_+$, we have $J(u) < \alpha$ and $I(u) > 0$, then by $p > 1, q > \max\{1, p-1\}$, we get

$$\begin{aligned} \alpha > J(u) &= \frac{1}{2}\|u_{xx}\|_2^2 + \frac{1}{p}\|u_x\|_p^p - \frac{1}{q+1}\|u\|_{q+1}^{q+1} \\ &= \frac{1}{q+1}\left(\|u_{xx}\|_2^2 + \|u_x\|_p^p - \|u\|_{q+1}^{q+1}\right) + \frac{q-1}{2(q+1)}\|u_{xx}\|_2^2 + \frac{q+1-p}{p(q+1)}\|u_x\|_p^p \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{q+1} I(u) + \frac{q-1}{2(q+1)}\|u_{xx}\|_2^2 + \frac{q+1-p}{p(q+1)}\|u_x\|_p^p \\
&> \frac{q-1}{2(q+1)}\|u_{xx}\|_2^2,
\end{aligned}
$$

which implies (2.5). □

Lemma 6. *If $0 \leq J(u_0) < E_1$ and $I(u_0) < 0$, then $\|u_{0xx}\| > \alpha_1$, where E_1 and α_1 are positive constants given in (1.5) and (1.6), respectively. Let u be the weak solution of problem (1.1) with initial value u_0, then there is a positive constant $\alpha_2 > \alpha_1$ such that*

$$\|u_{xx}(\cdot, t)\|_2 \geq \alpha_2, \quad t \geq 0, \tag{2.6}$$

and

$$\|u\|_{q+1} \geq A\alpha_2, \quad t \geq 0, \tag{2.7}$$

where A is a positive constant given in (1.2). Moreover,

$$\frac{\alpha_2}{\alpha_1} \geq \left[(q+1)\left(\frac{1}{2} - \frac{J_0(u_0)}{\alpha_1^2}\right)\right]^{1/(q-1)} > 1, \tag{2.8}$$

where $J_0(u_0)$ is defined in (1.7).

Proof. By (1.14), we get $\|u_{0xx}\| > \alpha_1$. The remaining proof is similar to [34, Lemma 2.2]. Although [34, Lemma 2.2] only considered the case $0 < J_0(u_0) < E_1$, one can check the lemma also hold for all $0 \leq J_0(u_0) < E_1$, and we omit it. □

The following lemma is similar to [34, Lemma 2.3], and we omit the proof.

Lemma 7. *Let $M(u) = E_1 - J(u)$. Assume the assumptions in Lemma 6 hold. Then, $M(u)$ satisfies the following estimates:*

$$0 < M(u_0) \leq M(u) \leq \frac{1}{q+1}\|u\|_{q+1}^{q+1}. \tag{2.9}$$

3. Proof of the Theorems

Proof of Theorem 5. By Theorem 2, the solution u of problem (1.1) blows up at some finite time T. From (2.4), we get

$$J(u(t)) \leq J(u_0) \tag{3.1}$$

holds for all $t \in (0, T)$.

Firstly, we consider the lower bound of T. Let

$$H(t) = \frac{1}{2}\|u\|_2^2. \tag{3.2}$$

By (2.3), we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} H(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|u\|_2^2\right) = -I(u) = -\|u_{xx}\|_2^2 - \|u_x\|_p^p + \|u\|_{q+1}^{q+1}. \tag{3.3}$$

Now we will prove $I(u) < 0$ for all $t \in [0, T)$. Otherwise, there must be a $t_0 \in (0, T)$ such that $I(u(t_0)) = 0$ and $I(u) < 0$ for $t \in [0, t_0)$. From Lemma 3(ii), $\|u_{xx}\| > r(1)$ for $t \in [0, t_0)$ and $\|u(t_0)_{xx}\| \geq r(1)$. The above two facts about $u(t_0)$ imply $u(t_0) \in \mathcal{N}$. Hence, by the definition of d, we have $J(u(t_0)) \geq d$. However, it follows from (3.1) that $J(u(t_0)) \leq J(u_0) < d$, a contradiction. So we have $I(u) < 0$ for all $t \in [0, T)$, i.e.,

$$\|u_{xx}\|_2^2 \leq \|u_{xx}\|_2^2 + \|u_x\|_p^p < \|u\|_{q+1}^{q+1}.$$

Combining the above inequality with (1.21) we obtain

$$\begin{aligned}\|u\|_{q+1}^{q+1} &\le \hat{C}^{q+1}\left(\|u_{xx}\|_2^2\right)^{\frac{(1-\theta)(q+1)}{2}}\left(\|u\|_2^2\right)^{\frac{\theta(q+1)}{2}}\\ &< \hat{C}^{q+1}\left(\|u\|_{q+1}^{q+1}\right)^{\frac{(1-\theta)(q+1)}{2}}\left(\|u\|_2^2\right)^{\frac{\theta(q+1)}{2}},\end{aligned} \tag{3.4}$$

which implies

$$\left(\|u\|_{q+1}^{q+1}\right)^{1-\frac{(1-\theta)(q+1)}{2}} \le \hat{C}^{q+1}\left(\|u\|_2^2\right)^{\frac{\theta(q+1)}{2}}. \tag{3.5}$$

By the value of θ in (1.22) and $q<9$, we get

$$1-\frac{(1-\theta)(q+1)}{2}=\frac{9-q}{8}>0,$$
$$\frac{\theta(q+1)}{2}=\frac{3q+5}{8}.$$

Then (3.5) becomes

$$\|u\|_{q+1}^{q+1} \le C_0\left(\|u\|_2^2\right)^{\gamma}, \tag{3.6}$$

where

$$C_0=\hat{C}^{\frac{8(q+1)}{9-q}}, \qquad \gamma=\frac{3q+5}{9-q}>1.$$

By combining (3.3) and (3.6) we get

$$\begin{aligned}H'(t) &= -\|u_{xx}\|_2^2-\|u_x\|_p^p+\|u\|_{q+1}^{q+1}\\ &\le \|u\|_{q+1}^{q+1}\\ &\le C_0\left(\|u\|_2^2\right)^{\gamma}\\ &= 2^{\gamma}C_0[H(t)]^{\gamma}.\end{aligned} \tag{3.7}$$

We can prove that $H(t)>0$ for any $t\in[0,T)$, if not, then there exists a t_0 such that $\|u(t_0)\|_2^2=0$, which contradicts (3.4). Therefore, by (3.7) we have

$$[H(t)]^{-\gamma}H'(t)\le 2^{\gamma}C_0. \tag{3.8}$$

Integrating above inequality from 0 to t, we get

$$\frac{1}{1-\gamma}\left[H(t)^{1-\gamma}-H(0)^{1-\gamma}\right]\le 2^{\gamma}C_0t.$$

Since $\gamma>1$, it follows

$$H(0)^{1-\gamma}-H(t)^{1-\gamma}\le 2^{\gamma}(\gamma-1)C_0t.$$

By (1.10), we obtain $\lim_{t\to T}H(t)=+\infty$. Since $\gamma>1$, letting $t\to T$ in the above inequality, we obtain

$$H(0)^{1-\gamma}=2^{\gamma-1}\|u_0\|_2^{2-2\gamma}\le 2^{\gamma}(\gamma-1)C_0T.$$

Then, (1.18) follows. Similarly, integrating (3.8) from t to T, we can get (1.19). □

Proof of Theorem 6. By Theorem 2, the solution u of problem (1.1) blows up at some finite time T. We divide the remaining proof into two cases.

Case 1: $J(u_0)<0$. Let $G(t)=-(q+1)J(u)$, then by (2.4), we get

$$\begin{cases} G'(t)=(q+1)\|u_t\|_2^2>0, & 0<t<T,\\ G(0)=-(q+1)J(u_0)>0, \end{cases}$$

which implies $G(t)>0$ for all $t\in(0,T)$.

Since $p>1, q>\max\{1,p-1\}$, it follows from $I(u_0)<0$ and (3.3) that

$$\begin{cases} H'(t)=-I(u)\geq G(t)>0, & 0<t<T,\\ H(0)=\frac{1}{2}\|u_0\|_2^2>0, \end{cases}$$

which implies $H(t)>0$ for all $t\in(0,T)$.

By Schwartz's inequality, we obtain

$$\begin{aligned} H(t)G'(t) &= \frac{q+1}{2}\|u\|_2^2\|u_t\|_2^2\\ &\geq \frac{q+1}{2}\left(\int_\Omega uu_t\mathrm{d}x\right)^2\\ &= \frac{q+1}{2}[H'(t)]^2\\ &\geq \frac{q+1}{2}H'(t)G(t). \end{aligned}$$

The above inequality can be rewritten as

$$\frac{G'(t)}{G(t)}\geq \frac{q+1}{2}\frac{H'(t)}{H(t)}.$$

Integrating the above inequality from 0 to t, we get

$$\frac{G(t)}{[H(t)]^{\frac{q+1}{2}}}\geq \frac{G(0)}{[H(0)]^{\frac{q+1}{2}}}.$$

By using $H'(t)\geq G(t)$ again, we get

$$\frac{H'(t)}{[H(t)]^{\frac{q+1}{2}}}\geq \frac{G(0)}{[H(0)]^{\frac{q+1}{2}}}. \tag{3.9}$$

Integrating (3.9) from 0 to T, note that $\lim_{t\to T}H(t)=+\infty$, we get

$$T\leq \frac{2H(0)}{(q-1)G(0)}=-\frac{\|u_0\|_2^2}{(q^2-1)J(u_0)}.$$

Similarly, integrating (3.9) from t to T, we can obtain

$$\|u(t)\|_2\leq \left[(q^2-1)\|u_0\|_2^{-(q+1)}(-J(u_0))\right]^{-\frac{1}{q-1}}(T-t)^{-\frac{1}{q-1}}.$$

Case 2: $0\leq J(u_0)<E_1$. The proof is similar to the proof of [34, Theorem 1.1], we give the details for reader's convenient. By (1.3) and Lemma 7, the functional $H(t)$ satisfies

$$H'(t)=-2E_1+2M(u)+\frac{1}{p}\|u_x\|_p^p+\frac{q-1}{q+1}\|u\|_{q+1}^{q+1}. \tag{3.10}$$

By (1.5), (1.14) and (2.7), we get

$$\begin{aligned} 2E_1 &= \frac{q-1}{q+1}A^{-\frac{2(q+1)}{q-1}}=\frac{q-1}{q+1}\left(AA^{-\frac{q+1}{q-1}}\right)^{q+1}\\ &= \frac{q-1}{q+1}(A\alpha_1)^{q+1}=\frac{q-1}{q+1}\left(\frac{\alpha_1}{\alpha_2}\right)^{q+1}(A\alpha_2)^{q+1}\\ &\leq \frac{q-1}{q+1}\left(\frac{\alpha_1}{\alpha_2}\right)^{q+1}\|u\|_{q+1}^{q+1}. \end{aligned} \tag{3.11}$$

Then, it follows from (3.10) and (3.11) that

$$H'(t) \geq \tilde{C}\|u\|_{q+1}^{q+1} + 2M(u) + \frac{1}{p}\|u_x\|_p^p, \tag{3.12}$$

where

$$\tilde{C} = \frac{q-1}{q+1}\left[1 - \left(\frac{\alpha_1}{\alpha_2}\right)^{q+1}\right].$$

By Hölder's inequality, we have

$$H^{\frac{q+1}{2}}(t) \leq \overline{C}\|u\|_{q+1}^{q+1} \tag{3.13}$$

with

$$\overline{C} = 2^{-\frac{q+1}{2}} \times |\Omega|^{\frac{q-1}{2}}. \tag{3.14}$$

So by (2.9), (3.12) and (3.13), we obtain

$$H'(t) \geq CH^{\frac{q+1}{2}}(t) \tag{3.15}$$

with $C = \tilde{C}/\overline{C}$, which means

$$\begin{aligned} H(t) &\geq \left(H^{-\frac{q-1}{2}}(0) - \frac{q-1}{2}Ct\right)^{-\frac{2}{q-1}} \\ &= \left(2^{\frac{q-1}{2}}\|u_0\|_2^{-(q-1)} - \frac{q-1}{2}Ct\right)^{-\frac{2}{q-1}}. \end{aligned}$$

Let

$$T^* = \frac{2^{\frac{q+1}{2}}}{C(q-1)}\|u_0\|_2^{-(q-1)} \in (0, \infty), \tag{3.16}$$

then $H(t)$ blows up at some finite time $T \leq T^*$. Next we estimate T. By (2.8), (3.16) and the values of $\tilde{C}, \overline{C}, C$, we have

$$\begin{aligned} T &\leq \frac{\overline{C}2^{\frac{q+1}{2}}}{\tilde{C}(q-1)}\|u_0\|_2^{-(q-1)} = \frac{(q+1)|\Omega|^{\frac{q-1}{2}}\|u_0\|_2^{-(q-1)}}{(q-1)^2\left[1 - \left(\frac{\alpha_1}{\alpha_2}\right)^{q+1}\right]} \\ &\leq \frac{(q+1)|\Omega|^{\frac{q-1}{2}}\|u_0\|_2^{-(q-1)}}{(q-1)^2\left[1 - \left((q+1)\left(\frac{1}{2} - \frac{J_0(u_0)}{\alpha_1^2}\right)\right)^{-\frac{q+1}{q-1}}\right]}. \end{aligned}$$

Integrating (3.15) from t to T, note that $\lim_{t\to T} H(t) = +\infty$, we get

$$\begin{aligned} H(t) &\leq \left[\frac{(q-1)C}{2}\right]^{-\frac{2}{q-1}}(T-t)^{-\frac{2}{q-1}} \\ &\leq \frac{|\Omega|(q+1)^{\frac{2}{q-1}}}{2(q-1)^{\frac{4}{q-1}}}\left[1 - \left(\frac{1}{(q+1)\left(\frac{1}{2} - \frac{J_0(u_0)}{\alpha_1^2}\right)}\right)^{\frac{q+1}{q-1}}\right]^{-\frac{2}{q-1}}(T-t)^{-\frac{2}{q-1}}. \end{aligned}$$

□

Proof of Theorem 7. We denote by $T(u_0)$ or T the maximal existence time of the solution to the problem (1.1) with initial value u_0. If $T(u_0) = \infty$, we denote by

$$\omega(u_0) \triangleq \bigcap_{t \geq 0} \overline{\{u(s) : \ s \geq t\}}^{H}$$

the ω-limit set of u_0.

Now, we prove the first conclusion. Assume that $u_0 \in \mathcal{N}_+$ satisfies $\|u_0\|_2 \leq \lambda_{J(u_0)}$. We claim that $u(t) \in \mathcal{N}_+$ for all $t \in [0, T)$. By contradiction, if there exist $s \in (0, T)$ such that $u(t) \in \mathcal{N}_+$ for $0 \leq t < s$ and $u(s) \in \mathcal{N}$, then by (3.1), we have $J(u(s)) \leq J(u_0)$, i.e., $u(s) \in J^{J(u_0)}$, hence $u(s) \in \mathcal{N}_{J(u_0)}$. Furthermore, according to the definition of $\lambda_{J(u_0)}$, we get

$$\|u(s)\|_2 \geq \lambda_{J(u_0)}. \tag{3.17}$$

Note that $I(u(t)) > 0$ for $t \in [0, s)$, it follows from (2.3) that

$$\|u(s)\|_2 < \|u_0\|_2 \leq \lambda_{J(u_0)}.$$

This contradicts (3.17). So $u(t) \in \mathcal{N}_+$ and then $u(t) \in J^{J(u_0)}$ for all $t \in [0, T)$. By (2.5), we obtain

$$\|u_{xx}(t)\|_2^2 < \frac{2(q+1)}{q-1} J(u_0), \quad \forall t \in [0, T),$$

which shows that the orbit $\{u(t)\}$ remains bounded in H for $t \in [0, T)$ so that $T = \infty$. Now for any $\omega \in \omega(u_0)$, by (2.3) and (2.4), we have

$$\|\omega\|_2 < \lambda_{J(u_0)}, \quad J(\omega) \leq J(u_0).$$

Note the definition of $\lambda_{J(u_0)}$, we can get $\omega(u_0) \cap \mathcal{N} = \emptyset$. Then, $\omega(u_0) = \{0\}$, i.e., $u_0 \in \mathcal{G}_0$.

Next, we prove the second conclusion. Assume that $u_0 \in \mathcal{N}_-$ satisfies $\|u_0\|_2 \geq \Lambda_{J(u_0)}$. We claim that $u(t) \in \mathcal{N}_-$ for all $t \in [0, T)$. By contradiction, if there exist $s \in (0, T)$ such that $u(t) \in \mathcal{N}_-$ for $0 \leq t < s$ and $u(s) \in \mathcal{N}$, then by (3.1), we have $J(u(s)) \leq J(u_0)$, i.e., $u(s) \in J^{J(u_0)}$, hence $u(s) \in \mathcal{N}_{J(u_0)}$. Furthermore, according to the definition of $\Lambda_{J(u_0)}$, we get

$$\|u(s)\|_2 \leq \Lambda_{J(u_0)}. \tag{3.18}$$

Note that $I(u(t)) < 0$ for $t \in [0, s)$, it follows from (2.3) that

$$\|u(s)\|_2 > \|u_0\|_2 \geq \Lambda_{J(u_0)}.$$

This contradicts (3.18). Assume $T(u_0) = +\infty$, then for every $\omega \in \omega(u_0)$, (2.3) and (2.4) imply that

$$\|\omega\|_2 > \Lambda_{J(u_0)}, \quad J(\omega) \leq J(u_0).$$

Note the definition of $\Lambda_{J(u_0)}$, we can get $\omega(u_0) \cap \mathcal{N} = \emptyset$. Then, $\omega(u_0) = \{0\}$. However, it follows from (1.13) that $\mathrm{dist}(0, \mathcal{N}_-) > 0$, we also have $0 \notin \omega(u_0)$. That means $\omega(u_0) = \varnothing$, which contradicts to $\omega(u_0) = \{0\}$. Hence, we conclude that $T(u_0) < \infty$, and the proof of Theorem 7 is complete. □

Proof of Corollary 1. Since $J(u_0) > d$ and $u_0 \in H$, one can easily prove $\|u_{0xx}\|_2 > 0$ and $\|u_{0x}\|_p > 0$.

Case (i): $1 < p < 2$. By using (1.25) and Hölder's inequality, we have

$$|\Omega|^{\frac{q-1}{2}} \frac{2(q+1)}{q-1} J(u_0) \leq \|u_0\|_2^{q+1} \leq |\Omega|^{\frac{q-1}{2}} \|u_0\|_{q+1}^{q+1}. \tag{3.19}$$

Then, it follows from (3.19) and the definition of $I(u_0)$ and $J(u_0)$ that

$$\begin{aligned} J(u_0) &= \frac{1}{2}\|u_{0xx}\|_2^2 + \frac{1}{p}\|u_{0x}\|_p^p - \frac{1}{q+1}\|u_0\|_{q+1}^{q+1} \\ &> \frac{1}{2}\|u_{0xx}\|_2^2 + \frac{1}{2}\|u_{0x}\|_p^p - \frac{1}{q+1}\|u_0\|_{q+1}^{q+1} \\ &= \frac{1}{2}I(u_0) + \left(\frac{1}{2} - \frac{1}{q+1}\right)\|u_0\|_{q+1}^{q+1} \\ &= \frac{1}{2}I(u_0) + \frac{q-1}{2(q+1)}\|u_0\|_{q+1}^{q+1} \\ &\geq \frac{1}{2}I(u_0) + J(u_0), \end{aligned} \tag{3.20}$$

which means $I(u_0) < 0$, thus we have $u_0 \in \mathcal{N}_-$.

For any $u \in \mathcal{N}_{J(u_0)}$, by (1.12), we have

$$\begin{aligned} \|u\|_2^{q+1} &\leq |\Omega|^{\frac{q-1}{2}}\|u\|_{q+1}^{q+1} \\ &= |\Omega|^{\frac{q-1}{2}}\left(\|u_{xx}\|_2^2 + \|u_x\|_p^p\right) \\ &= |\Omega|^{\frac{q-1}{2}}\frac{1}{\frac{1}{2}-\frac{1}{q+1}}\left[\left(\frac{1}{2} - \frac{1}{q+1}\right)\|u_{xx}\|_2^2 + \left(\frac{1}{2} - \frac{1}{q+1}\right)\|u_x\|_p^p\right] \\ &\leq |\Omega|^{\frac{q-1}{2}}\frac{2(q+1)}{q-1}\left[\left(\frac{1}{2} - \frac{1}{q+1}\right)\|u_{xx}\|_2^2 + \left(\frac{1}{p} - \frac{1}{q+1}\right)\|u_x\|_p^p\right] \\ &< |\Omega|^{\frac{q-1}{2}}\frac{2(q+1)}{q-1}J(u_0). \end{aligned}$$

Therefore, taking the supremum of above inequality over $\mathcal{N}_{J(u_0)}$, we can obtain

$$\Lambda_{J(u_0)}^{q+1} \leq |\Omega|^{\frac{q-1}{2}}\frac{2(q+1)}{q-1}J(u_0) \leq \|u_0\|_2^{q+1},$$

i.e., $\|u_0\|_2 \geq \Lambda_{J(u_0)}$. Then, Theorem 7 shows that $u_0 \in \mathcal{B}$.

Case (ii): $p = 2$. Since the first inequality of (3.19) is strict, we can also get $I(u_0) < 0$ by changing the "$>$"with "$=$"and the "$\geq$"with "$>$"in the second line and last line of (3.20), respectively, the remaining proof is the same as case(i).

Case (iii): $p > 2$. By using (1.26) and Hölder's inequality, we have

$$|\Omega|^{\frac{q-1}{2}}\frac{p(q+1)}{q+1-p}J(u_0) \leq \|u_0\|_2^{q+1} \leq |\Omega|^{\frac{q-1}{2}}\|u_0\|_{q+1}^{q+1}. \tag{3.21}$$

By (3.21) and the definition of $I(u_0)$ and $J(u_0)$, we get

$$\begin{aligned} J(u_0) &= \frac{1}{2}\|u_{0xx}\|_2^2 + \frac{1}{p}\|u_{0x}\|_p^p - \frac{1}{q+1}\|u_0\|_{q+1}^{q+1} \\ &> \frac{1}{p}\|u_{0xx}\|_2^2 + \frac{1}{p}\|u_{0x}\|_p^p - \frac{1}{q+1}\|u_0\|_{q+1}^{q+1} \\ &= \frac{1}{p}I(u_0) + \left(\frac{1}{p} - \frac{1}{q+1}\right)\|u_0\|_{q+1}^{q+1} \\ &= \frac{1}{p}I(u_0) + \frac{q+1-p}{p(q+1)}\|u_0\|_{q+1}^{q+1} \\ &\geq \frac{1}{p}I(u_0) + J(u_0), \end{aligned}$$

which means that $I(u_0) < 0$, thus we have $u_0 \in \mathcal{N}_-$.

For any $u \in \mathcal{N}_{J(u_0)}$, by (1.12), we have

$$\begin{aligned}
\|u\|_2^{q+1} &\le |\Omega|^{\frac{q-1}{2}} \|u\|_{q+1}^{q+1} \\
&= |\Omega|^{\frac{q-1}{2}} \left(\|u_{xx}\|_2^2 + \|u_x\|_p^p\right) \\
&= |\Omega|^{\frac{q-1}{2}} \frac{1}{\frac{1}{p} - \frac{1}{q+1}} \left[\left(\frac{1}{p} - \frac{1}{q+1}\right)\|u_{xx}\|_2^2 + \left(\frac{1}{p} - \frac{1}{q+1}\right)\|u_x\|_p^p\right] \\
&\le |\Omega|^{\frac{q-1}{2}} \frac{p(q+1)}{q+1-p} \left[\left(\frac{1}{2} - \frac{1}{q+1}\right)\|u_{xx}\|_2^2 + \left(\frac{1}{p} - \frac{1}{q+1}\right)\|u_x\|_p^p\right] \\
&< |\Omega|^{\frac{q-1}{2}} \frac{p(q+1)}{q+1-p} J(u_0).
\end{aligned}$$

Therefore, taking the supremum of above inequality over $\mathcal{N}_{J(u_0)}$, we can obtain

$$\Lambda_{J(u_0)}^{q+1} \le |\Omega|^{\frac{q-1}{2}} \frac{p(q+1)}{q+1-p} J(u_0) \le \|u_0\|_2^{q+1}.$$

i.e., $\|u_0\|_2 \ge \Lambda_{J(u_0)}$. Then, Theorem 7 shows that $u_0 \in \mathcal{B}$. □

Proof of Corollary 2. We assume $M > d$ and Ω_1, Ω_2 be two arbitrary disjoint open subdomains of Ω. Furthermore, we assume $v \in H \cap H_0^2(\Omega_1)$ be an arbitrary nonzero function, then we take α large enough such that $J(\alpha v) \le 0$ (since $p > 1, q > \max\{1, p-1\}$) and

$$\|\alpha v\|_2^{q+1} > \begin{cases} |\Omega|^{\frac{q-1}{2}} \dfrac{2(q+1)}{q-1} M, & \text{if } 1 < p \le 2; \\ |\Omega|^{\frac{q-1}{2}} \dfrac{p(q+1)}{q+1-p} M, & \text{if } p > 2. \end{cases}$$

Next, we fix such a number $\alpha > 0$ and choose a function $\omega \in H \cap H_0^2(\Omega_2)$ satisfying $M = J(\omega) + J(\alpha v)$. Then, $u_M = \alpha v + \omega$ satisfies $J(u_M) = J(\alpha v) + J(\omega) = M$ and

$$\|u_M\|_2^{q+1} \ge \|\alpha v\|_2^{q+1} > \begin{cases} |\Omega|^{\frac{q-1}{2}} \dfrac{2(q+1)}{q-1} J(u_M), & \text{if } 1 < p \le 2; \\ |\Omega|^{\frac{q-1}{2}} \dfrac{p(q+1)}{q+1-p} J(u_M), & \text{if } p > 2. \end{cases}$$

Hence, it shows $u_M \in \mathcal{N}_- \cap \mathcal{B}$ by Corollary 1. □

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Zhihua Dong and Jun Zhou
School of Mathematics and Statistics
Southwest University
Chongqing 400715
People's Republic of China
e-mail: jzhouwm@163.com

Zhihua Dong
e-mail: dxmDong@163.com

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