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Multiple solutions for nonhomogeneous Choquard equation involving Hardy–Littlewood–Sobolev critical exponent

Zifei Shen, Fashun Gao and Minbo Yang

Abstract. We consider the following critical nonhomogeneous Choquard equation

$$-\Delta u = \left(\int_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu} \mathrm{d}y\right) |u|^{2_\mu^*-2}u + \lambda u + f(x) \quad \text{in} \quad \Omega,$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, 0 in interior of Ω, $\lambda \in \mathbb{R}$, $N \geq 7$, $0 < \mu < N$, $2_\mu^* = (2N-\mu)/(N-2)$ is the upper critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality, and $f(x)$ is a given function. Using variational methods, we obtain the existence of multiple solutions for the above problem when $0 < \lambda < \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

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1. Introduction and main results

In this paper we are going to consider the existence of multiple solutions for the following nonlocal equation:

$$\begin{cases} -\Delta u = \left(\displaystyle\int_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu} \mathrm{d}y\right) |u|^{2_\mu^*-2}u + \lambda u + f(x) \quad \text{in} \quad \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{1.1}$$

Here Ω is a smooth bounded domain of $\mathbb{R}^N$ $(N \geq 7)$, 0 in interior of Ω, $0 < \lambda < \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, $0 < \mu < N$ and $2_\mu^* = (2N-\mu)/(N-2)$ is the upper critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality (see [14]). $f(x) \in L^\infty(\Omega)$ and $f(x) \not\equiv 0$.

Recently, people have paid much attention to the nonlinear Choquard–Pekar equation [18,29]

$$-\Delta u + V(x)u = \left(\frac{1}{|x|^\mu} * |u|^p\right)|u|^{p-2}u \quad \text{in} \quad \mathbb{R}^3, \tag{1.2}$$

also known as the stationary Hartree equation or the Newton–Schrödinger equation, see [22]. In subcritical case, the existence and qualitative properties of solutions of (1.2) have been widely studied. In 1976/77, Lieb [18] proved the existence and uniqueness, up to translations, of the ground state. Lions [20] obtained the existence of a sequence of radially symmetric solutions in 1980. Moreover, in [9,23,24] the authors showed the regularity, positivity and radial symmetry of the ground states and derived decay property at infinity as well. For periodic potential V that changes sign and 0 lies in the gap of the spectrum of

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the Schrödinger operator $-\Delta + V$, the problem is strongly indefinite, and the existence of solution for $p = 2$ was considered in [5] by reduction arguments. For other related results, we refer the readers to [2] for the existence of multibump shaped solution for the equation with deepening potential well, [8,12] for the existence of sign-changing solutions, [25] for the existence of ground states under the assumptions of Berestycki–Lions type. The semiclassical regime of standing wave solutions of Choquard equations has also attracted a lot of interest, the first result in this direction seems to be [11] where the authors considered a nonlocal Schrödinger equation with magnetic field and Hartree-type nonlinearities by penalization techniques. We may also refer the readers to [3,4,10,26,30,32] for other existence and concentration results of the semiclassical solutions. For a complete review of recent progress of the literature of (1.2), we refer the readers to [28] and references therein.

In critical case, the authors [1] considered the case of critical growth in the sense of Trudinger–Moser inequality and studied the existence and concentration of the ground states. A recent paper [27] by Moroz and Van Schaftingen, the authors considered the nonlinear Choquard equation (1.2) in $\mathbb{R}^N$ with lower critical exponent $\frac{2N-\mu}{N}$. In [14], Gao and Yang considered the existence and nonexistence of solutions for the Brezis–Nirenberg type problem of the nonlinear Choquard equation, that is (1.1) with $f = 0$. In [15], Gao and Yang study the existence and multiplicity results for the critical nonlocal equation.

The starting point of the variational approach to the problem (1.1) is the following well-known Hardy–Littlewood–Sobolev inequality.

Proposition 1.1. (Hardy–Littlewood–Sobolev inequality). (See [19]) *Let $t, r > 1$ and $0 < \mu < N$ with $1/t+\mu/N+1/r = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(t, N, \mu, r)$, independent of f, h, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \le C(t, N, \mu, r)|f|_t|h|_r. \tag{1.3}$$

If $t = r = 2N/(2N - \mu)$, then

$$C(t, N, \mu, r) = C(N, \mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma\left(\frac{N}{2} - \frac{\mu}{2}\right)}{\Gamma\left(N - \frac{\mu}{2}\right)} \left\{\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)}\right\}^{-1+\frac{\mu}{N}}.$$

In this case there is equality in (1.3) if and only if $f \equiv (const.)h$ and

$$h(x) = A(\gamma^2 + |x - a|^2)^{-(2N-\mu)/2}$$

for some $A \in \mathbb{C}$, $0 \ne \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^N$.

As in [14], we also use $S_{H,L}$ to denote best constant defined by

$$S_{H,L} := \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)^{\frac{N-2}{2N-\mu}}}. \tag{1.4}$$

In [14], the authors showed:

Proposition 1.2. *The constant $S_{H,L}$ defined in (1.4) is achieved if and only if*

$$u = C\left(\frac{b}{b^2 + |x - a|^2}\right)^{\frac{N-2}{2}},$$

where $C > 0$ is a fixed constant, $a \in \mathbb{R}^N$ and $b \in (0, \infty)$ are parameters. What's more,

$$S_{H,L} = \frac{S}{C(N,\mu)^{\frac{N-2}{2N-\mu}}},$$

where S is the best Sobolev constant.

In order to study the problem by variational methods, we introduce the energy functional associated with equation (1.1) by

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{1}{22^*_\mu}\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y - \frac{\lambda}{2}\int_\Omega |u|^2\mathrm{d}x - \int_\Omega fu\mathrm{d}x.$$

Then the Hardy–Littlewood–Sobolev inequality implies J_λ belongs to $C^1(H_0^1(\Omega), \mathbb{R})$ with

$$\langle J'_\lambda(u), \varphi\rangle = \int_\Omega \nabla u \nabla\varphi \mathrm{d}x - \int_\Omega\int_\Omega \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu-2}u(y)\varphi(y)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y - \lambda\int_\Omega u\varphi\mathrm{d}x - \int_\Omega f\varphi\mathrm{d}x$$

for all $\varphi \in C_0^\infty(\Omega)$. And so u is a weak solution of (1.1) if and only if u is a critical point of functional J_λ.

Throughout this paper we denote the norm $\|u\| := \left(\int_\Omega |\nabla u|^2\mathrm{d}x\right)^{\frac{1}{2}}$ on $H_0^1(\Omega)$ and write $|\cdot|_q$ for the $L^q(\Omega)$-norm for $q \in [1, \infty]$, always assume $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain containing 0 in its interior. We denote positive constants by $C, C_1, C_2, C_3, \cdots$.

The main results of this paper are stated in the following two theorems.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^N (N \geq 7)$ is a smooth bounded domain containing 0 in its interior, $0 < \lambda < \lambda_1$, $0 < \mu < N$, $f(x) \in L^\infty(\Omega)$ and $f(x) \not\equiv 0$. If, for any $u \in H_0^1(\Omega)$ with $\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y = 1$,*

$$\int_\Omega fu\mathrm{d}x \leq C_{N,\mu}(\|u\|^2 - \lambda|u|_2^2)^{\frac{2\cdot 2^*_\mu-1}{2\cdot 2^*_\mu-2}}, \tag{1.5}$$

where $C_{N,\mu} := (\frac{1}{2\cdot 2^*_\mu-1})^{\frac{2\cdot 2^*_\mu-1}{2\cdot 2^*_\mu-2}}(2\cdot 2^*_\mu - 2)$, then problem (1.1) exists at least one weak solution u in $H_0^1(\Omega)$. Moreover, $u \geq 0$ for $f \geq 0$.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^N (N \geq 7)$ is a smooth bounded domain containing 0 in its interior, $0 < \lambda < \lambda_1$, $0 < \mu < N$, $f(x) \in L^\infty(\Omega)$, $f \geq 0$ and $f(x) \not\equiv 0$. If, for any $u \in H_0^1(\Omega)$ with $\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y = 1$,*

$$\int_\Omega fu\mathrm{d}x < C_{N,\mu}(\|u\|^2 - \lambda|u|_2^2)^{\frac{2\cdot 2^*_\mu-1}{2\cdot 2^*_\mu-2}}, \tag{1.6}$$

then problem (1.1) exists at least two nonnegative solutions in $H_0^1(\Omega)$.

In 2001, Zhang [37] proved that there are at least two solutions for the following equation

$$-\Delta u + u = \left(\frac{1}{|x|^\mu} * |u|^2\right)u + g(x) \quad \text{in} \quad \mathbb{R}^3, \tag{1.7}$$

where $g(x) \geq 0$, $g(x) \not\equiv 0$ and $g(x) \in H^{-1}(\mathbb{R}^3)$. Later, Küpper et al. [17] studied the existence of multiple positive solutions for the problem (1.7) with $g(x)$ replaced by $\lambda g(x)$ and proved that there are positive constants λ^* and λ^{**} such that the above equation possesses at least two positive solutions for $\lambda \in (0, \lambda^*)$, no positive solution for $\lambda > \lambda^{**}$ and $\lambda = \lambda^*$ is a bifurcation point for the equation under study. The interest and motivation of the present paper goes back to the results for the local nonlinear Schrödinger equation

$$-\Delta u = |u|^{2^*-2}u + f(x), \quad \text{in } \Omega. \tag{1.8}$$

Tarantello [31] proved the existence of multiple solutions for problem (1.8) by using the Ekeland's variational principle and the Mountain Pass Theorem. Analogously, Kang and Deng [16] proved the existence of multiple solutions for the singular critical inhomogeneous elliptic problems involving critical Sobolev–Hardy exponents. Besides, we also want to mention [21,35,36] for the existence of solutions of the critical Schrödinger-Poisson equations. In this paper, we are going to study the multiplicity results for solutions of (1.1) in the spirit of [31] or [16]. In fact, we will use the Ekeland's variational principle and the Mountain Pass Theorem to prove the existence of the first solution for problem (1.1) and the existence of the second solution for problem (1.1), in Sects. 2 and 3, respectively. Since problem (1.1) includes the nonlocal term, our proof is different to [31] or [16]. In the proof of existence for the second solution, we will show that J_λ satisfies a compactness property, has suitable geometrical features and prove a local Palais–Smale (PS for short) condition for $c \in \mathbb{R}$ under a critical level related with $S_{H,L}$ defined in (1.4).

An outline of the paper is as follows: we establish the existence of the first solution for problem (1.1) under condition (1.5) in Sect. 2; then we prove the existence of the second solution for problem (1.1) under condition (1.6) in Sect. 3. In Sect. 4, we prove that assumed condition (2.9) holds by considering a minimization problem.

2. Existence of the first solution

We devote this section to prove the existence of the first solution for problem (1.1) by the Ekeland's variational principle under assumption (1.5) and first begin with a standard method as well as some ideas given in [31]. Let

$$\begin{aligned} I_\lambda(u) \ &= \ \|u\|^2 - \lambda|u|_2^2 - (2\cdot 2_\mu^* - 1)\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y, \quad u \in H_0^1(\Omega),\\ \mathcal{N} &:= \{u \in H_0^1(\Omega) : \langle J_\lambda'(u), u\rangle = 0\},\\ \mathcal{N}^+ &:= \{u \in \mathcal{N} : I_\lambda(u) > 0\},\\ \mathcal{N}^0 &:= \{u \in \mathcal{N} : I_\lambda(u) = 0\},\\ \mathcal{N}^- &:= \{u \in \mathcal{N} : I_\lambda(u) < 0\},\\ c_0 &:= \inf_{u\in\mathcal{N}} J_\lambda(u). \end{aligned} \tag{2.1}$$

We will prove that there exists $u_0 \in \mathcal{N}$ such that u_0 is a minimizer for the minimizing problem (2.1) and u_0 is a solution of problem (1.1).

Lemma 2.1. *Assume that* $0 < \lambda < \lambda_1$, $f(x) \in L^\infty(\Omega)$ *and* $f(x) \not\equiv 0$ *satisfies condition* (1.6). *Then for any* $u \in H_0^1(\Omega)$ *with* $u \not\equiv 0$, *there exists a unique* $t^+ = t^+(u) > 0$ *such that* $t^+(u)u \in \mathcal{N}^-$,

$$t^+ > \left(\frac{\|u\|^2 - \lambda|u|_2^2}{(2\cdot 2_\mu^* - 1)\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y}\right)^{\frac{1}{2\cdot 2_\mu^*-2}} := t_{\max}$$

and $J_\lambda(t^+u) = \max_{t\ge t_{\max}} J_\lambda(tu)$. *Moreover, if* f *satisfies* (1.6) *and* $\int_\Omega fu\mathrm{d}x > 0$, *then there exists a unique* $t^- = t^-(u) > 0$ *such that* $t^-(u)u \in \mathcal{N}^+$, $t^- < t_{\max}$ *and* $J_\lambda(t^-u) = \inf_{0\le t\le t_{\max}} J_\lambda(tu)$.

Proof. We set, for any $u \in H_0^1(\Omega)$

$$\Phi(t) = t(\|u\|^2 - \lambda|u|_2^2) - t^{2\cdot 2_\mu^*-1} \int_\Omega \int_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y.$$

By direct computation, we know Φ achieves its maximum at $t_{\max}$ and

$$\Phi(t_{\max}) = \left(\frac{1}{2\cdot 2_\mu^*-1}\right)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}} (2\cdot 2_\mu^*-2) \left(\frac{(\|u\|^2-\lambda|u|_2^2)^{2\cdot 2_\mu^*-1}}{\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y}\right)^{\frac{1}{2\cdot 2_\mu^*-2}},$$

that is

$$\Phi(t_{\max}) = C_{N,\mu} \left(\frac{(\|u\|^2-\lambda|u|_2^2)^{2\cdot 2_\mu^*-1}}{\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y}\right)^{\frac{1}{2\cdot 2_\mu^*-2}},$$

where $C_{N,\mu}$ is given in (1.5). If $\int_\Omega fu\mathrm{d}x \le 0$ then there exists a unique $t^+ = t^+(u) > 0$ such that $\Phi(t^+) = \int_\Omega fu\mathrm{d}x$ and $\Phi'(t^+) < 0$. Thus $t^+(u)u \in \mathcal{N}^-$ and

$$J_\lambda(t^+u) \ge J_\lambda(tu), \quad \forall t \ge t_{\max}.$$

If $\int_\Omega fu\mathrm{d}x > 0$ then by assumption (1.6)

$$\int_\Omega fu\mathrm{d}x < C_{N,\mu} \left(\frac{(\|u\|^2-\lambda|u|_2^2)^{2\cdot 2_\mu^*-1}}{\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y}\right)^{\frac{1}{2\cdot 2_\mu^*-2}} = \Phi(t_{\max}).$$

Therefore, we have unique $0 < t^- < t_{\max} < t^+$ such that $t^-(u)u \in \mathcal{N}^+$, $t^+(u)u \in \mathcal{N}^-$,

$$\Phi(t^+) = \int_\Omega fu\mathrm{d}x = \Phi(t^-)$$

and

$$\Phi'(t^+) < 0 < \Phi'(t^-).$$

We also have

$$J_\lambda(t^-u) \le J_\lambda(tu), \quad \forall t \in [0, t_{\max}]$$

and

$$J_\lambda(t^+u) \ge J_\lambda(tu), \quad \forall t \ge t_{\max}.$$

□

Lemma 2.2. *Assume that* $0 < \lambda < \lambda_1$, $f(x) \in L^\infty(\Omega)$ *and* $f(x) \not\equiv 0$ *satisfies condition* (1.6). *Then for any* $u \in \mathcal{N}$ *with* $u \not\equiv 0$, *we have*

$$\|u\|^2 - \lambda|u|_2^2 - (2\cdot 2_\mu^* - 1)\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \neq 0. \tag{2.2}$$

Proof. Now, we argue by contradiction and we suppose that there exists some $u_0 \in \mathcal{N}$ with $u_0 \not\equiv 0$ such that

$$\|u_0\|^2 - \lambda|u_0|_2^2 - (2\cdot 2_\mu^* - 1)\int_\Omega\int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y = 0. \tag{2.3}$$

Since $u_0 \in \mathcal{N}$, we get

$$\|u_0\|^2 - \lambda|u_0|_2^2 - \int_\Omega\int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y = \int_\Omega f u_0 \mathrm{d}x. \tag{2.4}$$

Gathering (2.3) and (2.4), we have

$$(2\cdot 2_\mu^* - 2)\int_\Omega\int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y = \int_\Omega f u_0 \mathrm{d}x \tag{2.5}$$

and

$$(2\cdot 2_\mu^* - 2)(\|u_0\|^2 - \lambda|u_0|_2^2) = (2\cdot 2_\mu^* - 1)\int_\Omega f u_0 \mathrm{d}x. \tag{2.6}$$

We set

$$v_0 = u_0 \Big/ \left(\int_\Omega\int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2\cdot 2_\mu^*}}$$

and so

$$\left(\int_\Omega\int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*}} = \frac{1}{2\cdot 2_\mu^* - 2}\int_\Omega f v_0 \mathrm{d}x \tag{2.7}$$

and

$$\left(\int_\Omega\int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2\cdot 2_\mu^*}} (\|v_0\|^2 - \lambda|v_0|_2^2) = \frac{2\cdot 2_\mu^* - 1}{2\cdot 2_\mu^* - 2}\int_\Omega f v_0 \mathrm{d}x. \tag{2.8}$$

From (2.7) and (2.8), we obtain

$$\frac{2\cdot 2_\mu^* - 2}{(2\cdot 2_\mu^* - 1)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}}}(\|v_0\|^2 - \lambda|v_0|_2^2)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}} = \int_\Omega f v_0 \mathrm{d}x$$

and

$$\int_\Omega\int_\Omega \frac{|v_0(x)|^{2_\mu^*}|v_0(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y = 1,$$

which contradicts assumption (1.6) and the result follows. □

Lemma 2.3. *Assume that* $0<\lambda<\lambda_1$, $f(x)\in L^\infty(\Omega)$ *and* $f(x)\not\equiv 0$ *satisfies condition* (1.6). *Then for any* $u\in\mathcal{N}$, $I_\lambda(u)\neq 0$, *there exist an* $\varepsilon>0$ *and a differentiable function* $t=t(v)>0$, $v\in H_0^1(\Omega)$, $\|v\|<\varepsilon$ *such that*

$$t(0)=1,\quad t(v)(u-v)\in\mathcal{N}$$

and

$$\langle t'(0),v\rangle=\frac{2\int\limits_\Omega(\nabla u\nabla v-\lambda uv)\mathrm{d}x-2\cdot 2_\mu^*\int\limits_\Omega\int\limits_\Omega\frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*-2}u(y)v(y)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y-\int\limits_\Omega fv\mathrm{d}x}{\|u\|^2-\lambda|u|_2^2-(2\cdot 2_\mu^*-1)\int\limits_\Omega\int\limits_\Omega\frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y}.$$

Proof. Define the map $F:\mathbb{R}\times H_0^1(\Omega)\to\mathbb{R}$,

$$F(t,v)=t(\|u-v\|^2-\lambda|u-v|_2^2)-t^{2\cdot 2_\mu^*-1}\int\limits_\Omega\int\limits_\Omega\frac{|(u-v)(x)|^{2_\mu^*}|(u-v)(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y-\int\limits_\Omega f(u-v)\mathrm{d}x.$$

Since Lemma 2.2, we have

$$\frac{\partial F}{\partial t}(1,0)=\|u\|^2-\lambda|u|_2^2-(2\cdot 2_\mu^*-1)\int\limits_\Omega\int\limits_\Omega\frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\neq 0.$$

Combining with the fact that $F(1,0)=0$, we can get the result of this lemma by applying the implicit function theorem at the point $(1,0)$. □

Lemma 2.4. *Assume that* $0<\lambda<\lambda_1$, $f(x)\in L^\infty(\Omega)$ *and* $f(x)\not\equiv 0$ *satisfies condition* (1.6). *Then there exist a minimizing sequence* $\{u_n\}\subset\mathcal{N}$ *for* (2.1) *such that*

(1) $J_\lambda(u_n)<c_0+\frac{1}{n}$.
(2) $J_\lambda(v)\geq J_\lambda(u_n)-\frac{1}{n}\|v-u_n\|$, $\forall v\in\mathcal{N}$.

Proof. We need to prove that J_λ is bounded from below in $\mathcal{N}$. Firstly, for $u\in\mathcal{N}$ we have

$$\begin{aligned}J_\lambda(u)&=\left(\frac{1}{2}-\frac{1}{2\cdot 2_\mu^*}\right)(\|u\|^2-\lambda|u|_2^2)-\left(1-\frac{1}{2\cdot 2_\mu^*}\right)\int\limits_\Omega fu\mathrm{d}x\\&\geq\frac{N+2-\mu}{4N-2\mu}(\|u\|^2-\lambda|u|_2^2)-\frac{3N+2-2\mu}{4N-2\mu}\|f\|_{H^{-1}}\|u\|\\&\geq\frac{N+2-\mu}{4N-2\mu}\sigma\|u\|^2-\frac{3N+2-2\mu}{4N-2\mu}\|f\|_{H^{-1}}\|u\|\\&\geq-\frac{(3N+2-2\mu)^2}{8(2N-\mu)(N+2-\mu)\sigma}\|f\|_{H^{-1}}^2,\end{aligned}$$

where $\sigma>0$ is some constant satisfying:

$$\|u\|^2-\lambda|u|_2^2\geq\sigma\|u\|^2.$$

Therefore,

$$c_0=\inf_{u\in\mathcal{N}}J_\lambda(u)\geq-\frac{(3N+2-2\mu)^2}{8(2N-\mu)(N+2-\mu)\sigma}\|f\|_{H^{-1}}^2.$$

In order to find an upper bound for c_0, let $v_0\in H_0^1(\Omega)$ be the weak solution of the following problem:

$$-\Delta v_0=f\quad\text{in}\ \ \Omega.$$

By $f(x) \not\equiv 0$, we have $\int\limits_\Omega f v_0 \mathrm{d}x = \|v_0\| > 0$. From Lemma 2.1, we can get a $t_0 = t_0(v_0)$ such that $t_0 v_0 \in \mathcal{N}$ and $I_\lambda(t_0 v_0) > 0$. Moreover,

$$\begin{aligned}
J_\lambda(t_0v_0) &= \frac{t_0^2}{2}(\|v_0\|^2 - \lambda|v_0|_2^2) - \frac{t_0^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*}\int\limits_\Omega\int\limits_\Omega \frac{|v_0(x)|^{2_\mu^*}|v_0(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y - t_0\int\limits_\Omega f v_0\mathrm{d}x \\
&= \frac{t_0^2}{2}(\|v_0\|^2 - \lambda|v_0|_2^2) - \frac{t_0^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*}\int\limits_\Omega\int\limits_\Omega \frac{|v_0(x)|^{2_\mu^*}|v_0(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \\
&\quad - \left(t_0^2(\|v_0\|^2 - \lambda|v_0|_2^2) - t_0^{2\cdot 2_\mu^*}\int\limits_\Omega\int\limits_\Omega \frac{|v_0(x)|^{2_\mu^*}|v_0(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right) \\
&= -\frac{t_0^2}{2}(\|v_0\|^2 - \lambda|v_0|_2^2) + \left(1 - \frac{1}{2\cdot 2_\mu^*}\right)t_0^{2\cdot 2_\mu^*}\int\limits_\Omega\int\limits_\Omega \frac{|v_0(x)|^{2_\mu^*}|v_0(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \\
&\le -\frac{t_0^2}{2}(\|v_0\|^2 - \lambda|v_0|_2^2) + \frac{t_0^2}{2\cdot 2_\mu^*}(\|v_0\|^2 - \lambda|v_0|_2^2) \\
&= \left(\frac{1}{2\cdot 2_\mu^*} - \frac{1}{2}\right)t_0^2(\|v_0\|^2 - \lambda|v_0|_2^2) \\
&< -\frac{N+2-\mu}{4N-2\mu}t_0^2\sigma\|v_0\|^2 < 0.
\end{aligned}$$

Thus, $c_0 \le J_\lambda(t_0v_0) < 0$. We can get a minimizing sequence $\{u_n\} \subset \mathcal{N}$ satisfying conditions (1) and (2) of the lemma by applying the Ekeland's variational principle to the minimization problem (2.1). □

Lemma 2.5. *Assume that $0 < \lambda < \lambda_1$, $f(x) \in L^\infty(\Omega)$ and $f(x) \not\equiv 0$ satisfies condition (1.6), $\{u_n\} \subset \mathcal{N}$ is the minimizing sequence obtained by Lemma 2.4 and*

$$E = \left\{u \in H_0^1(\Omega) : \int\limits_\Omega\int\limits_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y = 1\right\}.$$

If the following minimization problem:

$$\inf_{u\in E}\left\{C_{N,\mu}(\|u\|^2 - \lambda|u|_2^2)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}} - \int\limits_\Omega f u\mathrm{d}x\right\} := \rho_0 \tag{2.9}$$

can be achieved, then we have

$$\|J_\lambda'(u_n)\|_{H^{-1}} \to 0$$

as $n \to \infty$.

Proof. Taking n large enough, from Lemma 2.4 we get

$$\begin{aligned}
-\frac{N+2-\mu}{4N-2\mu}t_0^2\sigma\|v_0\|^2 &> c_0 + \frac{1}{n} > J_\lambda(u_n) \\
&= \frac{1}{2}(\|u_n\|^2 - \lambda|u_n|_2^2) - \frac{1}{2\cdot 2_\mu^*}\int\limits_\Omega\int\limits_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y - \int\limits_\Omega f u_n\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \frac{N+2-\mu}{4N-2\mu}(\|u_n\|^2 - \lambda|u_n|_2^2) - \frac{3N+2-2\mu}{4N-2\mu}\int_\Omega f u_n \mathrm{d}x \\
&\geq -\frac{3N+2-2\mu}{4N-2\mu}\int_\Omega f u_n \mathrm{d}x, \qquad (2.10)
\end{aligned}
$$

that is

$$
\|f\|_{H^{-1}}\|u_n\| \geq \int_\Omega f u_n \mathrm{d}x > \frac{N+2-\mu}{3N+2-2\mu} t_0^2 \sigma \|v_0\|^2 > 0. \qquad (2.11)
$$

So we have $u_n \neq 0$. By (2.10), we have

$$
\begin{aligned}
\frac{N+2-\mu}{4N-2\mu}\sigma\|u_n\|^2 &\leq \frac{N+2-\mu}{4N-2\mu}(\|u_n\|^2 - \lambda|u_n|_2^2) \\
&\leq \frac{3N+2-2\mu}{4N-2\mu}\int_\Omega f u_n \mathrm{d}x \\
&\leq \frac{3N+2-2\mu}{4N-2\mu}\|f\|_{H^{-1}}\|u_n\|,
\end{aligned}
$$

that is

$$
\|u_n\| \leq \frac{3N+2-2\mu}{(N+2-\mu)\sigma}\|f\|_{H^{-1}}.
$$

Thus,

$$
\frac{N+2-\mu}{3N+2-2\mu} t_0^2 \sigma \|v_0\|^2 \|f\|_{H^{-1}}^{-1} \leq \|u_n\| \leq \frac{3N+2-2\mu}{(N+2-\mu)\sigma}\|f\|_{H^{-1}} \qquad (2.12)
$$

by (2.11).

In order to prove

$$
\|J'_\lambda(u_n)\|_{H^{-1}} \to 0
$$

as $n \to \infty$, arguing by contradiction, assume that $\|J'_\lambda(u_n)\|_{H^{-1}} > 0$ as n large enough. For $u = u_n$ and $v = \varepsilon \frac{J'_\lambda(u_n)}{\|J'_\lambda(u_n)\|_{H^{-1}}}$ with $\varepsilon > 0$ small enough, applying Lemma 2.3, we can get $t_n(\varepsilon) := t(v)$ such that $v_\varepsilon := t_n(\varepsilon)(u_n - v) \in \mathcal{N}$. By Lemma 2.4 (2) we have

$$
\begin{aligned}
\frac{1}{n}\|v_\varepsilon - u_n\| &\geq J_\lambda(u_n) - J_\lambda(v_\varepsilon) \\
&= (1 - t_n(\varepsilon))\langle J'_\lambda(v_\varepsilon), u_n\rangle + \varepsilon t_n(\varepsilon)\left\langle J'_\lambda(v_\varepsilon), \frac{J'_\lambda(u_n)}{\|J'_\lambda(u_n)\|_{H^{-1}}}\right\rangle + o(\varepsilon). \qquad (2.13)
\end{aligned}
$$

We can derive as $\varepsilon \to 0$,

$$
\begin{aligned}
\frac{1}{\varepsilon}\|v_\varepsilon - u_n\| &= \frac{1}{\varepsilon}\|(t_n(\varepsilon) - 1)u_n - t_n(\varepsilon)v\| \\
&= \left\|\frac{1}{\varepsilon}(t_n(\varepsilon) - 1)u_n - t_n(\varepsilon)\frac{J'_\lambda(u_n)}{\|J'_\lambda(u_n)\|_{H^{-1}}}\right\| \\
&\leq |t'_n(0)| \cdot \|u_n\| + 1,
\end{aligned}
$$

where

$$
t'_n(0) := \left(t'(0) \cdot \frac{J'_\lambda(u_n)}{\|J'_\lambda(u_n)\|_{H^{-1}}}\right).
$$

Thus, dividing by $\varepsilon > 0$ in (2.13) and passing to the limit as $\varepsilon \to 0$, we derive

$$\frac{1}{n}(1+|t_n'(0)|\cdot\|u_n\|) \geq -t_n'(0)\langle J_\lambda'(u_n),u_n\rangle + \|J_\lambda'(u_n)\|_{H^{-1}} = \|J_\lambda'(u_n)\|_{H^{-1}}.$$

From (2.12), for some constant $C>0$ we also have

$$\|J_\lambda'(u_n)\|_{H^{-1}} \leq \frac{1}{n}(1+|t_n'(0)|).$$

Finally, we prove that $\{|t_n'(0)|\}$ is bounded uniformly in n. In view of (2.12) and Lemma 2.3, we conclude that for some constant $C_1>0$,

$$|t_n'(0)| \leq \frac{C_1}{\left|\|u_n\|^2-\lambda|u_n|_2^2-(2\cdot 2_\mu^*-1)\int_\Omega\int_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right|}.$$

Therefore, we only need to verify the following inequality:

$$|I_\lambda(u_n)| \geq C_2 > 0.$$

Arguing by contradiction, for a subsequence, assume that

$$I_\lambda(u_n) = o(1) \tag{2.14}$$

as $n\to\infty$. According to (2.12) and (2.14), there exists a constant $C_3>0$ such that

$$\int_\Omega\int_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \geq C_3$$

and

$$\left(\frac{\|u_n\|^2-\lambda|u_n|_2^2}{2\cdot 2_\mu^*-1}\right)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}} - \left(\int_\Omega\int_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}} = o(1). \tag{2.15}$$

In addition, from (2.14) and by the fact that $u_n\in\mathcal{N}$, we get

$$\int_\Omega f u_n\mathrm{d}x = (2\cdot 2_\mu^*-2)\int_\Omega\int_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y + o(1). \tag{2.16}$$

By (1.6) and (2.9) we can easily deduce that $\rho_0>0$. Now let

$$\Phi(u_n) := C_{N,\mu}\left(\|u_n\|^2-\lambda|u_n|_2^2\right)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}} - \left(\int_\Omega\int_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)^{\frac{2N-\mu}{2\cdot 2_\mu^*(N+2-\mu)}}\int_\Omega f u_n\mathrm{d}x$$

and

$$v_n := u_n\Big/\left(\int_\Omega\int_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2\cdot 2_\mu^*}}.$$

then we have

$$\begin{aligned}\Phi(u_n) &= \left(\int\limits_\Omega\int\limits_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)^{\frac{3N+2-2\mu}{2\cdot 2_\mu^*(N+2-\mu)}}\left(C_{N,\mu}\left(\|v_n\|^2-\lambda|v_n|_2^2\right)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}}-\int\limits_\Omega fv_n\mathrm{d}x\right)\\ &\geq \rho_0\left(\int\limits_\Omega\int\limits_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)^{\frac{3N+2-2\mu}{2\cdot 2_\mu^*(N+2-\mu)}}\\ &\geq \rho_0 C_3^{\frac{3N+2-2\mu}{2\cdot 2_\mu^*(N+2-\mu)}}>0.\end{aligned}$$

On the other hand, from (2.15) and (2.16) we derive

$$\begin{aligned}\Phi(u_n) &= C_{N,\mu}\Big(\|u_n\|^2-\lambda|u_n|_2^2\Big)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}}-(2\cdot 2_\mu^*-2)\left(\int\limits_\Omega\int\limits_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}}+o(1)\\ &= (2\cdot 2_\mu^*-2)\left(\Big(\frac{\|u_n\|^2-\lambda|u_n|_2^2}{2\cdot 2_\mu^*-1}\Big)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}}-\Big(\int\limits_\Omega\int\limits_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\Big)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}}\right)=o(1).\end{aligned}$$

This is a contradiction, and we conclude that $\{|t_n'(0)|\}$ is bounded uniformly in n. The result follows. □

The proof of verification (2.9) will be given in Sect. 4. Firstly, we prove Theorem 1.3 under assumption (2.9).

Proof of Theorem 1.3. By Lemmas 2.4 and 2.5, we obtain a minimizing sequence $\{u_n\}\subset\mathcal{N}$ for (2.1) such that

(1) $\lim_{n\to\infty} J_\lambda(u_n)=c_0$,
(2) $\lim_{n\to\infty}\|J_\lambda'(u_n)\|_{H^{-1}}=0$.

Let $u_0\in H_0^1(\Omega)$ be the weak limit of $\{u_n\}$ in $H_0^1(\Omega)$. From (2.11), we have $\int\limits_\Omega fu_0\mathrm{d}x>0$. We have

$$\langle J_\lambda'(u_0),v\rangle=0,\quad \forall v\in H_0^1(\Omega)$$

by Lemma 2.4. It means that u_0 is a weak solution of (1.1) and $u_0\in\mathcal{N}$. Therefore,

$$\begin{aligned}c_0\leq J_\lambda(u_0) &= \frac{N+2-\mu}{4N-2\mu}(\|u_0\|^2-\lambda|u_0|_2^2)-\frac{3N+2-2\mu}{4N-2\mu}\int\limits_\Omega fu_0\mathrm{d}x\\ &\leq \lim_{n\to\infty} J_\lambda(u_n)=c_0.\end{aligned}$$

So we deduce that $u_n\to u_0$ strongly in $H_0^1(\Omega)$ and

$$c_0=J_\lambda(u_0)=\inf_{u\in\mathcal{N}} J_\lambda(u).$$

From Lemma 2.1 and $\|J_\lambda'(u_n)\|_{H^{-1}}\to 0$, we have $u_0\in\mathcal{N}^+$.

Moreover, if $f\geq 0$, take, $t_0=t^-(|u_0|)$ with $t_0|u_0|\in\mathcal{N}^+$. Since $t_0<t_{\max}$ and $t_{\max}>1$, and so

$$J_\lambda(t_0|u_0|)\leq J_\lambda(|u_0|)\leq J_\lambda(u_0).$$

Then we can take $u_0\geq 0$.

Next, we prove the case where f satisfying

$$\int\limits_\Omega fu\mathrm{d}x=C_{N,\mu}(\|u\|^2-\lambda|u|_2^2)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}}.$$

As a consequence, we have $f_\varepsilon := (1-\varepsilon)f$ satisfies (1.6) for any $\varepsilon \in (0,1)$. Let

$$J_{\lambda,\varepsilon}(u) = \frac{1}{2}(\|u\|^2 - \lambda|u|_2^2) - \frac{1}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y - \int_\Omega f_\varepsilon u \mathrm{d}x, \quad \forall u \in H_0^1(\Omega)$$

and

$$u_\varepsilon \in \mathcal{N}_\varepsilon^+ := \{u \in H_0^1(\Omega) : \langle J'_{\lambda,\varepsilon}(u), u\rangle = 0, I_\lambda(u) > 0\}$$

satisfies

$$J_{\lambda,\varepsilon}(u_\varepsilon) = \inf_{u\in\mathcal{N}_\varepsilon} J_{\lambda,\varepsilon}(u) = c_\varepsilon$$

and

$$\langle J'_{\lambda,\varepsilon}(u_\varepsilon), v\rangle = 0, \quad \forall v \in H_0^1(\Omega). \tag{2.17}$$

Obviously, there exists a constant $C_3 > 0$ such that $\|u_\varepsilon\| \le C_3$ for any $\varepsilon \in (0,1)$. For all $u \in \mathcal{N}^+$, we have that $\int_\Omega f u \mathrm{d}x > 0$, which implies $(1-\varepsilon)\int_\Omega f u \mathrm{d}x > 0$ for any $\varepsilon \in (0,1)$. Applying Lemma 2.1 with $f = f_\varepsilon$, we can get $t_\varepsilon^- \in (0, t_{\max})$ such that $t_\varepsilon^- u \in \mathcal{N}_\varepsilon^+$. By the fact that $t_{\max} > 1$ we deduce that

$$J_{\lambda,\varepsilon}(t_\varepsilon^- u) \le J_{\lambda,\varepsilon}(u),$$

and so

$$c_\varepsilon \le J_{\lambda,\varepsilon}(t_\varepsilon^- u) \le J_{\lambda,\varepsilon}(u) \le J_\lambda(u) + \varepsilon\|f\|_{H^{-1}}\|u\| \le J_\lambda(u) + \varepsilon C_4,$$

where C_4 is a positive constant. Applying the proof of Lemma 2.4 with $f = f_\varepsilon$, from the above inequality we get

$$-\frac{(3N+2-2\mu)^2}{8(2N-\mu)(N+2-\mu)\sigma}\|f\|_{H^{-1}}^2 \le -\frac{(3N+2-2\mu)^2}{8(2N-\mu)(N+2-\mu)\sigma}\|f_\varepsilon\|_{H^{-1}}^2 \le c_\varepsilon \le c_0 + \varepsilon C_4.$$

As $n \to \infty$, taking $\varepsilon_n \to 0$ such that for some $u_0 \in H_0^1(\Omega)$ we have

$$c_{\varepsilon_n} \to \overline{c} \le c_0$$

and $u_{\varepsilon_n} \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$. From (2.17) we obtain

$$\langle J'_\lambda(u_0), v\rangle = 0, \quad \forall v \in H_0^1(\Omega),$$

furthermore, $J_\lambda(u_0) \le c_0$ and $u_0 \in \mathcal{N}$. This implies $J_\lambda(u_0) = c_0$ Thus, $u_{\varepsilon_n} \to u_0$ strongly in $H_0^1(\Omega)$. The proof of this theorem is now complete. □

3. Existence of the second solution

In this section, we shall prove the existence of the second solution for problem (1.1) under assumption (1.6) by the mountain pass lemma. Firstly, we give a Brézis–Lieb type lemma about the nonlocal term $\int_{\mathbb{R}^N} (|x|^{-\mu} * |u|^{2_\mu^*})|u|^{2_\mu^*}\mathrm{d}x$ proved in [14]:

Lemma 3.1. (See [14]) *Let $N \ge 3$ and $0 < \mu < N$. If $\{u_n\}$ is a bounded sequence in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ such that $u_n \to u$ almost everywhere in $\mathbb{R}^N$ as $n \to \infty$, then the following hold,*

$$\int_{\mathbb{R}^N} (|x|^{-\mu} * |u_n|^{2_\mu^*})|u_n|^{2_\mu^*}\mathrm{d}x - \int_{\mathbb{R}^N} (|x|^{-\mu} * |u_n - u|^{2_\mu^*})|u_n - u|^{2_\mu^*}\mathrm{d}x \to \int_{\mathbb{R}^N} (|x|^{-\mu} * |u|^{2_\mu^*})|u|^{2_\mu^*}\mathrm{d}x$$

as $n \to \infty$.

Lemma 3.2. *Assume that $0 < \lambda < \lambda_1$, $f(x) \in L^\infty(\Omega)$ and $f(x) \not\equiv 0$ satisfies condition (1.6). Then u_0, the solution of problem (1.1) obtained in Sect. 2, is a local minimizer of J_λ.*

Proof. From Lemma 2.1, for any $u \in H_0^1(\Omega)$ satisfies $\int_\Omega fu\mathrm{d}x > 0$, there exists a unique $t^-(u) \in (0, t_{\max}(u))$ such that $t^-(u)u \in \mathcal{N}^+$ and

$$J_\lambda(tu) \geq J_\lambda(t^-u), \quad \forall t \in (0, t_{\max}(u)).$$

For any $u_0 \in \mathcal{N}^+$ we have

$$t^-(u_0) = 1 < \left(\frac{\|u_0\|^2 - \lambda |u_0|_2^2}{(2 \cdot 2_\mu^* - 1) \int\limits_\Omega \int\limits_\Omega \frac{|u_0(x)|^{2_\mu^*} |u_0(y)|^{2_\mu^*}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y} \right)^{\frac{1}{2 \cdot 2_\mu^* - 2}} := A(u_0).$$

Choosing $\varepsilon > 0$ small enough such that the following inequality holds for any $\|v\| < \varepsilon$:

$$1 < A(u_0 - v). \tag{3.1}$$

By Lemma 2.3, for all $\|v\| < \varepsilon$ we have $t(v)(u_0 - v) \in \mathcal{N}$. Since $t(v) \to 1$ as $\|v\| \to 0$, we can assume that the following inequality

$$t(v) < A(u_0 - v).$$

holds for all $\|v\| < \varepsilon$, $v \in H_0^1(\Omega)$. Then $t(v)(u_0 - v) \in \mathcal{N}^+$ and for any t:

$$0 < t < A(u_0 - v)$$

we have

$$J_\lambda(t(u_0 - v)) \geq J_\lambda(t(v)(u_0 - v)) \geq J_\lambda(u_0).$$

From (3.1), we can take $t = 1$ in the above inequality, and then for all $v \in H_0^1(\Omega)$ with $\|v\| < \varepsilon$, we get

$$J_\lambda(u_0 - v) \geq J_\lambda(u_0).$$

The proof is completed. □

Lemma 3.3. *Assume that $0 < \lambda < \lambda_1$, $f(x) \in L^\infty(\Omega)$ and $f(x) \not\equiv 0$ satisfies condition (1.6). If $\{u_n\}$ is a $(PS)_c$ sequence of J_λ, then $\{u_n\}$ is bounded. Let $u_0 \in H_0^1(\Omega)$ be the weak limit of $\{u_n\}$, then u_0 is a weak solution of problem (1.1).*

Proof. It is easy to see that there exists $C_1 > 0$ such that

$$|J_\lambda(u_n)| \leq C_1, \quad \left| \left\langle J'_\lambda(u_n), \frac{u_n}{\|u_n\|} \right\rangle \right| \leq C_1.$$

By direct computation, we have

$$\begin{aligned} J_\lambda(u_n) - \frac{1}{2}\langle J'_\lambda(u_n), u_n \rangle &= \frac{N+2-\mu}{4N-2\mu} \int\limits_\Omega \int\limits_\Omega \frac{|u_n(x)|^{2_\mu^*} |u_n(y)|^{2_\mu^*}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y - \frac{1}{2} \int\limits_\Omega f u_n \mathrm{d}x \\ &\leq C_1(1 + \|u_n\|). \end{aligned}$$

According to the Hölder inequality we get

$$\frac{N+2-\mu}{4N-2\mu} \int\limits_\Omega \int\limits_\Omega \frac{|u_n(x)|^{2_\mu^*} |u_n(y)|^{2_\mu^*}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \leq C_1(1 + \|u_n\|) + \frac{1}{2} \|f\|_{H^{-1}} \|u_n\|$$

and then, we can obtain

$$\begin{aligned}\rho_0\|u_n\|^2 &\leq \|u_n\|^2-\lambda|u_n|^2\\ &= \int_\Omega\int_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y+\int_\Omega fu_n\mathrm{d}x+\langle J_\lambda'(u_n),u_n\rangle\\ &\leq \frac{4N-2\mu}{N+2-\mu}C_1(1+\|u_n\|)+\left(\frac{2N-\mu}{N+2-\mu}+1\right)\|f\|_{H^{-1}}\|u_n\|+C_1\|u_n\|.\end{aligned}$$

So $\{u_n\}$ is bounded in $H_0^1(\Omega)$.

Since $H_0^1(\Omega)$ is reflexive, up to a subsequence, still denoted by u_n, there exists $u_0\in H_0^1(\Omega)$ such that $u_n\rightharpoonup u_0$ in $H_0^1(\Omega)$ and $u_n\rightharpoonup u_0$ in $L^{2^*}(\Omega)$ as $n\to+\infty$. Then

$$|u_n|^{2_\mu^*}\rightharpoonup |u_0|^{2_\mu^*}\quad\text{in}\quad L^{\frac{2N}{2N-\mu}}(\Omega)$$

as $n\to+\infty$. By the Hardy–Littlewood–Sobolev inequality, the Riesz potential defines a linear continuous map from $L^{\frac{2N}{2N-\mu}}(\Omega)$ to $L^{\frac{2N}{\mu}}(\Omega)$, we know that

$$|x|^{-\mu}*|u_n|^{2_\mu^*}\rightharpoonup |x|^{-\mu}*|u_0|^{2_\mu^*}\quad\text{in}\quad L^{\frac{2N}{\mu}}(\Omega)$$

as $n\to+\infty$. Combining with the fact that

$$|u_n|^{2_\mu^*-2}u_n\rightharpoonup |u_0|^{2_\mu^*-2}u_0\quad\text{in}\quad L^{\frac{2N}{N-\mu+2}}(\Omega),$$

as $n\to+\infty$, we have

$$(|x|^{-\mu}*|u_n|^{2_\mu^*})|u_n|^{2_\mu^*-2}u_n\rightharpoonup (|x|^{-\mu}*|u_0|^{2_\mu^*})|u_0|^{2_\mu^*-2}u_0\quad\text{in}\quad L^{\frac{2N}{N+2}}(\Omega)$$

as $n\to+\infty$. Since, for any $\varphi\in H_0^1(\Omega)$,

$$0\leftarrow\langle J_\lambda'(u_n),\varphi\rangle=\int_\Omega\nabla u_n\nabla\varphi\mathrm{d}x-\lambda\int_\Omega u_n\varphi\mathrm{d}x-\int_\Omega\int_\Omega\frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*-2}u_n(y)\varphi(y)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y-\int_\Omega f\varphi\mathrm{d}x.$$

Passing to the limit as $n\to+\infty$ we obtain

$$\int_\Omega\nabla u_0\nabla\varphi\mathrm{d}x-\lambda\int_\Omega u_0^q\varphi\mathrm{d}x-\int_\Omega\int_\Omega\frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*-2}u_0(y)\varphi(y)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y-\int_\Omega f\varphi\mathrm{d}x=0$$

for any $\varphi\in H_0^1(\Omega)$, which means u_0 is a weak solution of problem (1.1). □

Lemma 3.4. *Assume that* $0<\lambda<\lambda_1$, $f(x)\in L^\infty(\Omega)$ *and* $f(x)\not\equiv 0$ *satisfies condition* (1.6). *If* $\{u_n\}$ *is a* $(PS)_c$ *sequence with*

$$c<c_0+\frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}. \tag{3.2}$$

Then $\{u_n\}$ *has a convergent subsequence.*

Proof. Let u_0 be the weak limit of $\{u_n\}$ obtained in Lemma 3.3 and define $v_n:=u_n-u_0$, then we know $v_n\rightharpoonup 0$ in $H_0^1(\Omega)$ and $v_n\to 0$ a.e. in Ω. Moreover, by the Brézis–Lieb Lemma in [6] and Lemma 3.1, we know

$$\int_\Omega|\nabla u_n|^2\mathrm{d}x=\int_\Omega|\nabla v_n|^2\mathrm{d}x+\int_\Omega|\nabla u_0|^2\mathrm{d}x+o_n(1),$$

$$\int_\Omega|u_n|^2\mathrm{d}x=\int_\Omega|v_n|^2\mathrm{d}x+\int_\Omega|u_0|^2\mathrm{d}x+o_n(1),$$

and

$$\int_\Omega\int_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y = \int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y + \int_\Omega\int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y + o_n(1).$$

Combining with the fact that

$$\int_\Omega fu_n\mathrm{d}x = \int_\Omega fv_n\mathrm{d}x + \int_\Omega fu_0\mathrm{d}x,$$

we have

$$\begin{aligned} c \leftarrow &\, J_\lambda(u_n) \\ &= \frac{1}{2}(\|v_n\|^2 - \lambda|v_n|_2^2) + \frac{1}{2}(\|u_0\|^2 - \lambda|u_0|_2^2) - \frac{1}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \\ &\quad -\frac{1}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y - \int_\Omega fv_n\mathrm{d}x - \int_\Omega fu_0\mathrm{d}x + o_n(1) \\ &= J_\lambda(u_0) + \frac{1}{2}\|v_n\|^2 - \frac{1}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y + o_n(1), \end{aligned} \tag{3.3}$$

since $\int_\Omega v_n^2 \to 0$ and $\int_\Omega fv_n\mathrm{d}x \to 0$, as $n \to +\infty$. Similarly, since $\langle J'_\lambda(u_0), u_0\rangle = 0$, we have

$$\begin{aligned} o_n(1) &= \langle J'_\lambda(u_n), u_n\rangle \\ &= \langle J'_\lambda(u_0), u_0\rangle + \|v_n\|^2 - \lambda|v_n|_2^2 - \int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y - \int_\Omega fv_n\mathrm{d}x + o_n(1) \\ &= \|v_n\|^2 - \int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y + o_n(1). \end{aligned} \tag{3.4}$$

From (3.4), we know there exists a nonnegative constant b such that

$$\|v_n\|^2 \to b$$

and

$$\int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \to b$$

as $n \to +\infty$. From (3.3), we obtain

$$c \geq \frac{N+2-\mu}{4N-2\mu}b + J_\lambda(u_0).$$

On the other hand, by Lemma 3.3, we know $u_0 \in \mathcal{N}$ and so

$$c_0 \leq J_\lambda(u_0).$$

Thus,

$$c \geq \frac{N+2-\mu}{4N-2\mu}b + c_0. \tag{3.5}$$

By the definition of the best constant $S_{H,L}$ in (1.4), we have

$$S_{H,L}\left(\int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)^{\frac{N-2}{2N-\mu}} \leq \int_\Omega |\nabla v_n|^2\mathrm{d}x,$$

which yields $b \geq S_{H,L} b^{\frac{N-2}{2N-\mu}}$. Thus, we have either $b = 0$ or $b \geq S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$. If $b = 0$, the proof is complete. Otherwise $b \geq S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$, then we obtain from (3.5),

$$\frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} \leq \frac{N+2-\mu}{4N-2\mu} b \leq c - c_0,$$

which contradicts with the fact that $c < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}} + c_0$. Thus $b = 0$, and

$$\|u_n - u_0\| \to 0$$

as $n \to +\infty$. This ends the proof of Lemma 3.4.

□

Lemma 3.5. *Let $0 < \lambda < \lambda_1$, $f(x) \in L^\infty(\Omega)$, $f(x) \geq 0$ and $f(x) \not\equiv 0$ satisfies condition (1.6). Then, there exists $v_0 \in H_0^1(\Omega)$, $v_0 \geq 0$, $v_0 \not\equiv 0$ such that*

$$\sup_{t\geq 0} J_\lambda(u_0 + tv_0) < c_0 + \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}. \tag{3.6}$$

Proof. From Lemma 3.1 of [14], we know if $N \geq 4$ and $\lambda > 0$, then, there exists $v_0 \in H_0^1(\Omega)\backslash\{0\}$ with $v_0 \geq 0$ and $v_0 \not\equiv 0$ such that

$$\frac{\int_\Omega |\nabla v_0|^2\mathrm{d}x - \lambda\int_\Omega |v_0|^2\mathrm{d}x}{\left(\int_\Omega\int_\Omega \frac{|v_0(x)|^{2_\mu^*}|v_0(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)^{\frac{N-2}{2N-\mu}}} < S_{H,L}.$$

From the definition of J_λ, we can get a constant $t_0 > 0$ such that

$$J_\lambda(u_0 + t_0v_0) = \sup_{t>0} J_\lambda(u_0 + tv_0).$$

From $f \geq 0$, we have $u_0 \geq 0$. Since $(a+b)^p \geq a^p + b^p + a^{p-1}b$ for every $a, b \geq 0$ and $p \geq 1$, then

$$\begin{aligned}\int_\Omega\int_\Omega \frac{|(t_0v_0+u_0)(x)|^{2_\mu^*}|(t_0v_0+u_0)(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y &\geq \int_\Omega\int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \\ &+ \int_\Omega\int_\Omega \frac{|t_0v_0(x)|^{2_\mu^*}|t_0v_0(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y + \int_\Omega\int_\Omega \frac{|u_0(y)|^{2_\mu^*}|u_0(x)|^{2_\mu^*-1}|t_0v_0(x)|}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
&J_\lambda(u_0+t_0v_0)\\
&=\frac{1}{2}\|u_0+t_0v_0\|^2-\frac{\lambda}{2}|u_0+t_0v_0|_2^2-\frac{1}{2\cdot 2_\mu^*}\int\limits_\Omega\int\limits_\Omega\frac{|(u_0+t_0v_0)(x)|^{2_\mu^*}|(u_0+t_0v_0)(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\\
&\quad-\int\limits_\Omega f(u_0+t_0v_0)\mathrm{d}x\\
&=\frac{1}{2}\|u_0\|^2+\frac{t_0^2}{2}\|v_0\|^2+t_0\int\limits_\Omega\nabla v_0\nabla u_0\mathrm{d}x-\frac{\lambda}{2}|u_0|_2^2-\frac{\lambda t_0^2}{2}|v_0|_2^2-\lambda t_0\int\limits_\Omega u_0v_0\mathrm{d}x\\
&\quad-\frac{1}{2\cdot 2_\mu^*}\int\limits_\Omega\int\limits_\Omega\frac{|(u_0+t_0v_0)(x)|^{2_\mu^*}|(u_0+t_0v_0)(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y-\int\limits_\Omega fu_0\mathrm{d}x-t_0\int\limits_\Omega fv_0\mathrm{d}x\\
&\le\frac{1}{2}\|u_0\|^2+\frac{t_0^2}{2}\|v_0\|^2-\frac{\lambda}{2}|u_0|_2^2-\frac{\lambda t_0^2}{2}|v_0|_2^2-\frac{1}{2\cdot 2_\mu^*}\int\limits_\Omega\int\limits_\Omega\frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\\
&\quad-\frac{t_0^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*}\int\limits_\Omega\int\limits_\Omega\frac{|v_0(x)|^{2_\mu^*}|v_0(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y-\int\limits_\Omega fu_0\mathrm{d}x\\
&=J_\lambda(u_0)+\frac{t_0^2}{2}(\|v_0\|^2-\lambda|v_0|_2^2)-\frac{t_0^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*}\int\limits_\Omega\int\limits_\Omega\frac{|v_0(x)|^{2_\mu^*}|v_0(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Since u_0 is a solution of (1.1), we have

$$
\begin{aligned}
J_\lambda(u_0+t_0v_0)&\le c_0+\frac{N+2-\mu}{4N-2\mu}\left(\frac{\int\limits_\Omega|\nabla v_0|^2\mathrm{d}x-\lambda\int\limits_\Omega v_0^2\mathrm{d}x}{\left(\int\limits_\Omega\int\limits_\Omega\frac{|v_0(x)|^{2_\mu^*}|v_0(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)^{\frac{N-2}{2N-\mu}}}\right)^{\frac{2N-\mu}{N+2-\mu}}\\
&<c_0+\frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}.
\end{aligned}
$$

The result follows. □

Proof of Theorem 1.4. By Lemma 3.2, we know there exist $\alpha,\rho>0$ such that

$$J_\lambda(u_0+u)-J_\lambda(u_0)\ge\alpha$$

for any $u\in H_0^1(\Omega)$ with $\|u\|=\rho$. From the definition of J_λ, we can get a constant w_0 such that

$$J_\lambda(u_0+w_0)-J_\lambda(u_0)<0.$$

Thus, by the mountain pass theorem without (PS) condition (cf. [33]), there exist a (PS) sequence $\{u_n\}$ such that $J_\lambda(u_n)\to c$ and $J_\lambda'(u_n)\to 0$ in $H_0^1(\Omega)^{-1}$ at the minimax level

$$c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}J_\lambda(\gamma(t))>0,$$

where

$$\Gamma := \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, J_\lambda(\gamma(1)) < 0\}.$$

From Lemma 3.5 and the definition of c, we know $c_0 < c < c_0 + \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$. Applying Lemma 3.4, we know $\{u_n\}$ contains a convergent subsequence. And so, we have J_λ has a critical value $c \in (c_0, c_0 + \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}})$ and problem (1.1) has a nontrivial solution u_1 and moreover, $u_1 \neq u_0$. By the Proof of Theorem 1.3, we know $u_1 \geq 0$. □

4. A minimization problem

In this section, we denote to prove that (2.9) holds under assumptions of Theorem 1.3.

Lemma 4.1. *Assume that* $0 < \lambda < \lambda_1$, $f(x) \in L^\infty(\Omega)$, $f(x) \not\equiv 0$ *satisfies condition* (1.6) *and*

$$E = \left\{ u \in H_0^1(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^*}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y = 1 \right\}.$$

Then the following minimization problem:

$$\inf_{u \in E} \left\{ C_{N,\mu} (\|u\|^2 - \lambda |u|_2^2)^{\frac{2\cdot 2_\mu^* - 1}{2\cdot 2_\mu^* - 2}} - \int_\Omega f u \mathrm{d}x \right\} := \rho_0 \tag{4.1}$$

can be achieved at some function $u_\star \in H_0^1(\Omega)$.

Proof. We denote

$$J(u) = C_{N,\mu} (\|u\|^2 - \lambda |u|_2^2)^{\frac{2\cdot 2_\mu^* - 1}{2\cdot 2_\mu^* - 2}} - \int_\Omega f u \mathrm{d}x$$

and

$$\|u\|_{NL}^{2\cdot 2_\mu^*} = \int_\Omega \int_\Omega \frac{|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^*}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y$$

for all $u \in H_0^1(\Omega)$. Now, let $\{u_n\}$ be a minimizing sequence of (4.1), u_0 be the weak limit of $\{u_n\}$ and define $v_n := u_n - u_0$, then we know $v_n \rightharpoonup 0$ in $H_0^1(\Omega)$ and $v_n \to 0$ a.e. in Ω. We have $\|u_0\|_{NL} \leq 1$ and only need to prove $\|u_0\|_{NL} = 1$. Arguing by contradiction, we assume that $\|u_0\|_{NL} < 1$. By (4.1), we have

$$\begin{aligned} \rho_0 + o_n(1) &= C_{N,\mu} (\|u_n\|^2 - \lambda |u_n|_2^2)^{\frac{2\cdot 2_\mu^* - 1}{2\cdot 2_\mu^* - 2}} - \int_\Omega f u_n \mathrm{d}x \\ &= C_{N,\mu} (\|u_0\|^2 + \|v_n\|^2 - \lambda |u_0|_2^2)^{\frac{2\cdot 2_\mu^* - 1}{2\cdot 2_\mu^* - 2}} - \int_\Omega f u_0 \mathrm{d}x + o_n(1). \end{aligned}$$

On the other hand, from Lemma 3.1, we get

$$1 = \|u_n\|_{NL}^{2\cdot 2_\mu^*} = \|u_0\|_{NL}^{2\cdot 2_\mu^*} + \|v_n\|_{NL}^{2\cdot 2_\mu^*} + o_n(1),$$

which implies

$$(\|v_n\|_{NL}^{2\cdot 2_\mu^*})^{\frac{1}{2_\mu^*}} = (1 - \|u_0\|_{NL}^{2\cdot 2_\mu^*})^{\frac{1}{2_\mu^*}} + o_n(1).$$

So, by the definition of $S_{H,L}$, we have

$$\rho_0 + o_n(1) \geq C_{N,\mu}\left(\|u_0\|^2 + S_{H,L}(1-\|u_0\|_{NL}^{2\cdot 2_\mu^*})^{\frac{1}{2_\mu^*}} - \lambda|u_0|_2^2\right)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}} - \int\limits_\Omega f u_0 \mathrm{d}x + o_n(1),$$

that is

$$C_{N,\mu}\left(\|u_0\|^2 + S_{H,L}(1-\|u_0\|_{NL}^{2\cdot 2_\mu^*})^{\frac{1}{2_\mu^*}} - \lambda|u_0|_2^2\right)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}} - \int\limits_\Omega f u_0 \mathrm{d}x \leq \rho_0. \tag{4.2}$$

As is known to all, $U(x) = \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}$ is a minimizer for S, the best Sobolev constant. By Proposition 1.2, we know that $U(x)$ is also a minimizer for $S_{H,L}$. We assume that $B_\delta \subset \Omega \subset B_{k\delta}$ for some positive constant k. Let $\psi \in C_0^\infty(\Omega)$ such that

$$\begin{cases} \psi(x) = \begin{cases} 1 & \text{if} \quad x \in B_\delta, \\ 0 & \text{if} \quad x \in \mathbb{R}^N \setminus \Omega, \end{cases} \\ 0 \leq \psi(x) \leq 1 & \forall x \in \mathbb{R}^N, \\ |D\psi(x)| \leq C = const & \forall x \in \mathbb{R}^N. \end{cases}$$

We define, for $\varepsilon > 0$,

$$U_\varepsilon(x) := \varepsilon^{\frac{2-N}{2}} U\left(\frac{x}{\varepsilon}\right),$$
$$u_\varepsilon(x) := \psi(x) U_\varepsilon(x). \tag{4.3}$$

From [14], we know that as $\varepsilon \to 0^+$,

$$\int\limits_{\mathbb{R}^N} |u_\varepsilon|^{2^*} \mathrm{d}x = C(N,\mu))^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}} S_{H,L}^{\frac{N}{2}} + O(\varepsilon^N), \tag{4.4}$$

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u_\varepsilon(x)|^{2_\mu^*}|u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \leq C(N,\mu)^{\frac{N}{2}} S_{H,L}^{\frac{2N-\mu}{2}} + O\left(\varepsilon^{N-\frac{\mu}{2}}\right) \tag{4.5}$$

and

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u_\varepsilon(x)|^{2_\mu^*}|u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \geq C(N,\mu)^{\frac{N}{2}} S_{H,L}^{\frac{2N-\mu}{2}} - O\left(\varepsilon^{N-\frac{\mu}{2}}\right), \tag{4.6}$$

where $C(N,\mu)$ is defined in Proposition 1.1.

Set $v_\varepsilon(x) = \frac{u_\varepsilon(x)}{\|u_\varepsilon\|_{NL}}$. Gathering Lemma 11.1 in [13] and Lemma 11.2 in [13] with the fact (4.4) to (4.6), we have that as $\varepsilon \to 0^+$

$$\|v_\varepsilon\|_{NL} = 1, \tag{4.7}$$

$$\|v_\varepsilon\|^2 = S_{H,L} + O\left(\varepsilon^{\frac{N-2}{2}}\right), \tag{4.8}$$

$$|v_\varepsilon|_1 = O\left(\varepsilon^{\frac{N-2}{4}}\right) \tag{4.9}$$

and

$$\int\limits_\Omega |v_\varepsilon|^2 \mathrm{d}x = \begin{cases} O\left(\varepsilon^{\frac{1}{2}}\right) & \text{if} \quad N = 3, \\ O(\varepsilon|ln\varepsilon|) & \text{if} \quad N = 4, \\ O(\varepsilon) & \text{if} \quad N \geq 5. \end{cases} \tag{4.10}$$

For any $C \geq 0$ and $u \in H_0^1(\Omega)$, it is easy to show that v_ε is bounded in $H_0^1(\Omega)$ for $\varepsilon > 0$ and $v_\varepsilon \to 0$ almost everywhere in Ω as $\varepsilon \to 0^+$. By Proposition 5.4.7 in [34], we know for every sequence $\{\varepsilon_n\}_{n=1}$ with $\varepsilon_n \to 0^+$,

$$\int\limits_\Omega |u|^{\frac{N-\mu+2}{N-2}\cdot\frac{2N}{2N-\mu}} v_{\varepsilon_n}^{\frac{2N}{2N-\mu}} \mathrm{d}x \to 0$$

and

$$\int\limits_\Omega |u|^{\frac{2N}{2N-\mu}} v_{\varepsilon_n}^{\frac{N-\mu+2}{N-2}\cdot\frac{2N}{2N-\mu}} \mathrm{d}x \to 0.$$

So we have as $\varepsilon \to 0^+$,

$$\int\limits_\Omega |u|^{\frac{N-\mu+2}{N-2}\cdot\frac{2N}{2N-\mu}} v_{\varepsilon}^{\frac{2N}{2N-\mu}} \mathrm{d}x \to 0 \tag{4.11}$$

and

$$\int\limits_\Omega |u|^{\frac{2N}{2N-\mu}} v_{\varepsilon}^{\frac{N-\mu+2}{N-2}\cdot\frac{2N}{2N-\mu}} \mathrm{d}x \to 0. \tag{4.12}$$

By the Hardy–Littlewood–Sobolev inequality, the Riesz potential defines a linear continuous map from $L^{\frac{2N}{2N-\mu}}(\Omega)$ to $L^{\frac{2N}{\mu}}(\Omega)$, we know that as $\varepsilon \to 0^+$

$$|x|^{-\mu} * |v_\varepsilon|^{2_\mu^*} \rightharpoonup 0 \quad \text{in} \quad L^{\frac{2N}{\mu}}(\Omega),$$

that is,

$$\int\limits_\Omega\int\limits_\Omega \frac{|u(x)|^{2_\mu^*}|v_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \to 0. \tag{4.13}$$

For any $u \in H_0^1(\Omega)$ with $\|u\|_{NL} < 1$, we have that there exists $C_\varepsilon > 0$ such that $\|u + C_\varepsilon v_\varepsilon\|_{NL}^{2\cdot 2_\mu^*} = 1$. It follows from [7] and (4.8) that as $\varepsilon \to 0^+$

$$\|u + C_\varepsilon v_\varepsilon\|^2 = \|u\|^2 + C_\varepsilon^2\|v_\varepsilon\|^2 + o_n(1) = \|u\|^2 + C_\varepsilon^2 S_{H,L} + o_n(1). \tag{4.14}$$

On the other hand, gathering (4.11), (4.12), (4.13), the Hardy–Littlewood–Sobolev inequality and the fact that as $\varepsilon \to 0^+$,

$$u + C_\varepsilon v_\varepsilon = |u|^{2_\mu^*} + C_\varepsilon|v_\varepsilon|^{2_\mu^*} + C_\varepsilon 2_\mu^*|u|^{2_\mu^*-2}uv_\varepsilon + C_\varepsilon^{2_\mu^*-1}2_\mu^*|v_\varepsilon|^{2_\mu^*-2}v_\varepsilon u + \omega_\varepsilon,$$

where $\omega_\varepsilon = o(C_\varepsilon 2_\mu^*|u|^{2_\mu^*-2}uv_\varepsilon + C_\varepsilon^{2_\mu^*-1}2_\mu^*|v_\varepsilon|^{2_\mu^*-2}v_\varepsilon u)$, we have

$$\|u + C_\varepsilon v_\varepsilon\|_{NL}^{2\cdot 2_\mu^*} = \|u\|_{NL}^{2\cdot 2_\mu^*} + C_\varepsilon^{2\cdot 2_\mu^*}\|v_\varepsilon\|_{NL}^{2\cdot 2_\mu^*} + o_n(1).$$

Thus,

$$C_\varepsilon^2 = \left(1 - \|u\|_{NL}^{2\cdot 2_\mu^*}\right)^{\frac{1}{2_\mu^*}} + o_n(1). \tag{4.15}$$

From (4.15) and (4.14) we have

$$\|u + C_\varepsilon v_\varepsilon\|^2 = \|u\|^2 + S_{H,L}\left(1 - \|u\|_{NL}^{2\cdot 2_\mu^*}\right)^{\frac{1}{2_\mu^*}} + o_n(1).$$

which implies

$$\rho_0 \le C_{N,\mu}(\|u+C_\varepsilon v_\varepsilon\|^2-\lambda|u+C_\varepsilon v_\varepsilon|_2^2)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}}-\int\limits_\Omega f(u+C_\varepsilon v_\varepsilon)\mathrm{d}x$$

$$=C_{N,\mu}\left(\|u\|^2+S_{H,L}\left(1-\|u\|_{NL}^{2\cdot 2_\mu^*}\right)^{\frac{1}{2_\mu^*}}-\lambda|u|_2^2\right)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}}-\int\limits_\Omega fu\mathrm{d}x+o_n(1).$$

Taking the limit as $\varepsilon\to 0$, for any $u\in H_0^1(\Omega)$ with $\|u\|_{NL}\le 1$ we get

$$\rho_0 \le C_{N,\mu}\left(\|u\|^2+S_{H,L}\left(1-\|u\|_{NL}^{2\cdot 2_\mu^*}\right)^{\frac{1}{2_\mu^*}}-\lambda|u|_2^2\right)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}}-\int\limits_\Omega fu\mathrm{d}x.$$

On the other hand, from (4.2) we conclude that

$$C_{N,\mu}\left(\|u\|^2+S_{H,L}\left(1-\|u\|_{NL}^{2\cdot 2_\mu^*}\right)^{\frac{1}{2_\mu^*}}-\lambda|u|_2^2\right)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}}-\int\limits_\Omega fu\mathrm{d}x=\rho_0. \tag{4.16}$$

Hence for any $v\in H_0^1(\Omega)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[C_{N,\mu}\left(\|u_0+tv\|^2+S_{H,L}\left(1-\|u_0+tv\|_{NL}^{2\cdot 2_\mu^*}\right)^{\frac{1}{2_\mu^*}}-\lambda|u_0+tv|_2^2\right)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}}-\int\limits_\Omega f(u_0+tv)\mathrm{d}x\right]_{t=0}=0,$$

which implies that $u_0\in H_0^1(\Omega)$ is a weak solution of the following equation

$$-\Delta u=\lambda u+\theta_1\left(\int\limits_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}y\right)|u|^{2_\mu^*-2}u+\frac{1}{\theta_2}f,\quad \text{in } \Omega, \tag{4.17}$$

where

$$\theta_1:=S_{H,L}\left(1-\|u_0\|_{NL}^{2\cdot 2_\mu^*}\right)^{\frac{1}{2_\mu^*}-1}>0$$

and

$$\theta_2:=\frac{2\cdot 2_\mu^*-1}{2_\mu^*-1}C_{N,\mu}\left(\|u_0\|^2+S_{H,L}\left(1-\|u_0\|_{NL}^{2\cdot 2_\mu^*}\right)^{\frac{1}{2_\mu^*}-1}-\lambda|u_0|_2^2\right)^{\frac{1}{2\cdot 2_\mu^*-2}}>0.$$

Since $f\in L^\infty(\Omega)$, we have

$$\left|\lambda u+\theta_1\left(\int\limits_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}y\right)|u|^{2_\mu^*-2}u+\frac{1}{\theta_2}f\right|\le C_1(1+|u|)+\theta_1\left(\int\limits_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}y\right)|u|^{2_\mu^*-2}u,$$

and so by Lemma 4.4 [15], we can obtain $u_0\in L^\infty(\Omega)$.

From (4.10), we have, if $N\ge 4$,

$$O\left(\varepsilon^{\frac{N-2}{2}}\right)=o(|v_\varepsilon|_2^2) \tag{4.18}$$

and if $N\ge 7$,

$$O\left(\varepsilon^{\frac{N-2}{4}}\right)=o(|v_\varepsilon|_2^2). \tag{4.19}$$

By $u_0 \in L^\infty(\Omega)$ and (4.9), we have

$$\left|\int\limits_\Omega \int\limits_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*-2}u_0(y)v_\varepsilon(y)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right| \le C_2 \int\limits_{\Omega_1} \frac{1}{|z|^\mu}dz \int\limits_\Omega |v_\varepsilon|\mathrm{d}x \le C_3 \int\limits_\Omega |v_\varepsilon|\mathrm{d}x = O\left(\varepsilon^{\frac{N-2}{4}}\right),$$

where $\Omega_1 := \{z \in \mathbb{R}^N : z = x-y, \forall x, y \in \Omega\}$ and so, we have

$$\left|\int\limits_\Omega \int\limits_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*-2}u_0(y)v_\varepsilon(y)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right| = o(|v_\varepsilon|_2^2) \tag{4.20}$$

when $N \ge 7$. Analogously, we have

$$\left|\int\limits_\Omega f v_\varepsilon \mathrm{d}x\right| = o(|v_\varepsilon|_2^2) \tag{4.21}$$

when $N \ge 7$. Since $f \neq 0$, we get $u_0 \not\equiv 0$. Noting that C_ε satisfies $\|u_0 + C_\varepsilon v_\varepsilon\|_{NL} = 1$ for $\varepsilon > 0$ small enough, we shall get a contradiction by showing

$$J(u_0 + C_\varepsilon v_\varepsilon) < \rho_0.$$

Let $C_0^2 := (1 - \|u_0\|_{NL}^{2\cdot 2_\mu^*})^{\frac{1}{2_\mu^*}}$, it follows from (4.15) that $C_\varepsilon \to C_0$ as $\varepsilon \to 0$. Denote $C_\varepsilon = C_0(1-\delta_\varepsilon)$, where $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$, then we have

$$\begin{aligned}
&J(u_0 + C_\varepsilon v_\varepsilon)\\
&= C_{N,\mu}(\|u_0 + C_\varepsilon v_\varepsilon\|^2 - \lambda|u_0 + C_\varepsilon v_\varepsilon|_2^2)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}} - \int\limits_\Omega f(u_0 + C_\varepsilon v_\varepsilon)\mathrm{d}x\\
&= C_{N,\mu}\left(\|u_0\|^2 + C_\varepsilon^2\|v_\varepsilon\|^2 + 2C_\varepsilon \int\limits_\Omega \nabla u_0 \nabla v_\varepsilon \mathrm{d}x - \lambda|u_0|_2^2 - \lambda C_\varepsilon^2|v_\varepsilon|_2^2 - 2\lambda C_\varepsilon \int\limits_\Omega u_0 v_\varepsilon \mathrm{d}x\right)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}}\\
&\quad - \int\limits_\Omega f(u_0 + C_\varepsilon v_\varepsilon)\mathrm{d}x\\
&= C_{N,\mu}\Bigg(\|u_0\|^2 + C_0^2(1-\delta_\varepsilon)^2 S + O\left(\varepsilon^{\frac{N-2}{2}}\right) + 2C_\varepsilon \int\limits_\Omega \nabla u_0 \nabla v_\varepsilon \mathrm{d}x - 2\lambda C_\varepsilon \int\limits_\Omega u_0 v_\varepsilon \mathrm{d}x\\
&\quad -\lambda|u_0|_2^2 - \lambda C_\varepsilon^2 |v_\varepsilon|_2^2\Bigg)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}} - \int\limits_\Omega f(u_0 + C_\varepsilon v_\varepsilon)\mathrm{d}x\\
&= C_{N,\mu}\Bigg((\|u_0\|^2 + C_0^2 S - \lambda|u_0|_2^2) - 2C_0^2\delta_\varepsilon S + C_0^2\delta_\varepsilon^2 S + O\left(\varepsilon^{\frac{N-2}{2}}\right) - \lambda C_\varepsilon^2|v_\varepsilon|_2^2\\
&\quad +2C_\varepsilon \int\limits_\Omega \nabla u_0 \nabla v_\varepsilon \mathrm{d}x - 2\lambda C_\varepsilon \int\limits_\Omega u_0 v_\varepsilon \mathrm{d}x\Bigg)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}} - \int\limits_\Omega f(u_0 + C_\varepsilon v_\varepsilon)\mathrm{d}x\\
&= C_{N,\mu}((\|u_0\|^2 + C_0^2 S - \lambda|u_0|_2^2)^{\frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}} - \int\limits_\Omega f u_0 \mathrm{d}x + \frac{2\cdot 2_\mu^*-1}{2\cdot 2_\mu^*-2}(\|u_0\|^2 + C_0^2 S - \lambda|u_0|_2^2)^{\frac{1}{2\cdot 2_\mu^*-2}}
\end{aligned}$$

$$\times\Big(\left(-2C_0^2\delta_\varepsilon S+C_0^2\delta_\varepsilon^2 S+O\left(\varepsilon^{\frac{N-2}{2}}\right)-\lambda C_\varepsilon^2|v_\varepsilon|_2^2\right)+\Big(2C_\varepsilon\int\limits_\Omega\nabla u_0\nabla v_\varepsilon\mathrm{d}x-2\lambda C_\varepsilon\int\limits_\Omega u_0v_\varepsilon\mathrm{d}x\Big)\Big)$$

$$+A_\varepsilon-C_\varepsilon\int\limits_\Omega fv_\varepsilon\mathrm{d}x$$

$$=\rho_0+\frac{\theta_2}{2}\left(-2C_0^2\delta_\varepsilon S+C_0^2\delta_\varepsilon^2S+O\left(\varepsilon^{\frac{N-2}{2}}\right)-\lambda C_\varepsilon^2|v_\varepsilon|_2^2\right)+C_\varepsilon\theta_2\left(\int\limits_\Omega\nabla u_0\nabla v_\varepsilon\mathrm{d}x\right.$$

$$\left.-\lambda\int\limits_\Omega u_0v_\varepsilon\mathrm{d}x-\int\limits_\Omega\frac{f}{\theta_2}v_\varepsilon\mathrm{d}x\right)+A_\varepsilon$$

$$=\rho_0+\frac{\theta_2}{2}\left(-2C_0^2\delta_\varepsilon S+C_0^2\delta_\varepsilon^2S+O\left(\varepsilon^{\frac{N-2}{2}}\right)-\lambda C_\varepsilon^2|v_\varepsilon|_2^2\right)+C_\varepsilon\theta_2\theta_1\int\limits_\Omega|u_0|^{2^*-2}u_0v_\varepsilon\mathrm{d}x+A_\varepsilon,$$

where

$$A_\varepsilon=o\Big((-2C_0^2\delta_\varepsilon S+C_0^2\delta_\varepsilon^2S+O\left(\varepsilon^{\frac{N-2}{2}}\right)-\lambda C_\varepsilon^2|v_\varepsilon|_2^2)+\Big(2C_\varepsilon\int\limits_\Omega\nabla u_0\nabla v_\varepsilon\mathrm{d}x-2\lambda C_\varepsilon\int\limits_\Omega u_0v_\varepsilon\mathrm{d}x\Big)\Big)$$

$$=o\Big((-2C_0^2\delta_\varepsilon S+C_0^2\delta_\varepsilon^2S+O\left(\varepsilon^{\frac{N-2}{2}}\right)-\lambda C_\varepsilon^2|v_\varepsilon|_2^2)$$

$$+\Big(2C_\varepsilon\theta_1\int\limits_\Omega\int\limits_\Omega\frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*-2}u_0(y)v_\varepsilon(y)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y+2C_\varepsilon\int\limits_\Omega\frac{f}{\theta_2}v_\varepsilon\mathrm{d}x\Big)\Big)$$

$$=o(\delta_\varepsilon)+o(|v_\varepsilon|_2^2),$$

since (4.18), (4.20) and (4.21). Then we have

$$\begin{aligned}J(u_0+C_\varepsilon v_\varepsilon)&=\rho_0+\frac{\theta_2}{2}\left(-2C_0^2\delta_\varepsilon S+C_0^2\delta_\varepsilon^2S+O\left(\varepsilon^{\frac{N-2}{2}}\right)-\lambda C_\varepsilon^2|v_\varepsilon|_2^2\right)\\&\quad+C_\varepsilon\theta_2\theta_1\int\limits_\Omega\int\limits_\Omega\frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*-2}u_0(y)v_\varepsilon(y)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y+o(\delta_\varepsilon)+o(|v_\varepsilon|_2^2)\\&=\rho_0-\theta_2C_0^2S\delta_\varepsilon+o(\delta_\varepsilon)-\lambda C_\varepsilon^2|v_\varepsilon|_2^2+o(|v_\varepsilon|_2^2).\end{aligned}$$

Hence $J(u_0+C_\varepsilon v_\varepsilon)<\rho_0$ as ε small enough, this is a contradiction. So minimization problem (4.1) can be achieved. The result follows. □

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Zifei Shen, Fashun Gao and Minbo Yang
Department of Mathematics
Zhejiang Normal University
Jinhua 321004 Zhejiang
People's Republic of China
e-mail: mbyang@zjnu.edu.cn

Zifei Shen
e-mail: szf@zjnu.cn

Fashun Gao
e-mail: fsgao@zjnu.edu.cn

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