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The initial-boundary value problem for the generalized double dispersion equation

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Abstract. We consider the initial-boundary value problem for the generalized double dispersion equation in all space dimension. Under the suitable assumptions on the initial data and the parameters in the equation, we establish several results concerning local existence, global existence, uniqueness, and finite time blowup property. The exponential decay rate of the energy is proved for global solutions. The sufficient and necessary conditions of global solutions and finite time blowup of solutions are given, respectively.

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1. Introduction

In this paper, we focus on the following initial-boundary value problem of the generalized double dispersion equation

$$u_{tt} - \Delta u_{tt} - \Delta u + \Delta^2 u - \alpha \Delta u_t = \Delta f(u), \quad (x,t) \in \Omega \times \mathbb{R}^+ \tag{1.1}$$

$$u \mid_{\partial\Omega} = 0, \qquad \Delta u \mid_{\partial\Omega} = 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \tag{1.3}$$

where $f(u) = \beta |u|^{p-1} u$, $\beta \in \mathbb{R} \setminus \{0\}$, $1 < p < \infty$, and $\alpha \geq 0$ is a constant. Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $u = u(x,t)$ denotes the unknown function, Δ is the n-dimensional Laplace operator, the subscript t indicates the partial derivative with respect to t, and u_0 and u_1 are the given initial value functions.

In the physical study of nonlinear wave propagation in waveguide, the interaction of the waveguides and the external medium and, therefore, the possibility of energy exchange through lateral surfaces of the waveguide cannot be neglected. To consider the model of interaction between the surface of a nonlinear elastic rod, whose material is hyperelastic (e.g., the Murnaghan material), and a medium, the longitudinal displacement $u(x,t)$ of the rod satisfies the following double dispersion equation (DDE)

$$u_{tt} - u_{xx} = \frac{1}{4}\left(6u^2 + a u_{tt} - b u_{xx}\right)_{xx}, \tag{1.4}$$

and the general cubic DDE

$$u_{tt} - u_{xx} = \frac{1}{4}\left(c u^3 + 6u^2 + a u_{tt} - b u_{xx} + c u_t\right)_{xx}, \tag{1.5}$$

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which can be obtained by meaning of the Hamiltonian principle (see [15,16]), where u is proportional to strain $\frac{\partial U}{\partial x}$, U is the longitudinal displacement, $a > 0$, $b > 0$ and $d \neq 0$ are some constants depending on the Young modulus. The multidimensional generalized forms of (1.4) and (1.5) are written as follows

$$u_{tt} - \Delta u - a\Delta u_{tt} + b\Delta^2 u = \Delta f(u), \tag{1.6}$$

$$u_{tt} - \Delta u - a\Delta u_{tt} + b\Delta^2 u - c\Delta u_t = \Delta f(u). \tag{1.7}$$

So far, there has been an extensive body of work on the equations (1.4), (1.5) and their generalized forms (1.6) and (1.7). For the Cauchy problem of these equations, the results are stated as follows. Wang and Chen [19] studied the Cauchy problem of (1.5) and proved the existence, uniqueness, and nonexistence of global solutions. By means of the potential well method, Xu and Liu [24] established the results on global solutions and finite time blowup of solutions of the Cauchy problem of (1.6) with the initial energy at low level and $f(u) = a|u|^p$ for $1 \leq p < \infty$ if $n = 1, 2$ and $1 + \frac{2}{n} \leq p < \frac{n+2}{n-2}$ if $n \geq 3$ without the local existence theory and uniqueness. Xu, Liu and Liu [25] studied the Cauchy problem of (1.6) with $f(u) = \pm a|u|^p$ or $-a|u|^{p-1}u$, $a > 0$, in which the authors established global solutions $u \in H^m(\mathbb{R}^n)$ with $m = 1$, for $n = 1$ and $m = 2, 3, 4$ for $n \leq 3$, and finite time blowup of solutions were established in the case of low initial energy. Kutev, Kolkovska, and Dimova [7] studied the Cauchy problem to (1.4) with combined power-type nonlinearities and proved the sign-preserving property of the Nehari functional to establish global solutions and finite time blowup of solutions with subcritical initial energy. Wang and Da [20] investigated the small amplitude solution to the Cauchy problem of (1.7). Wang and Chen [23] established asymptotic profile of solutions to the initial value problem of (1.7) and obtained the optimal decay estimate of solutions. Erbay and Erkip [2] considered the instability and stability properties of traveling wave solutions of (1.6) with $f(u) = -|u|^{p-1}u$, $p > 1$.

In [1,9], the initial-boundary value problem of (1.4) and (1.5) was investigated, respectively,

$$\begin{cases} u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} - du_{xxt} = f(u)_{xx}, \ x \in \Omega, t > 0 \\ u(0,t) = u(l,t) = 0, \quad u_{xx}(0,t) = u_{xx}(l,t) = 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \end{cases} \tag{1.8}$$

where $\Omega = (0, l)$, $d \geq 0$. The global classical solutions were obtained in [1], and global solutions and blowup of solutions with subcritical initial energy were studied in [9]. In [1] and [9], the authors studied (1.9) by considering the following auxiliary problem

$$\begin{cases} v_{tt} - v_{xx} - v_{xxtt} + v_{xxxx} - du_{xxt} = f(v_x)_x, \ x \in \Omega, t > 0 \\ v(0,t) = v(l,t) = 0, \quad v_{xxx}(0,t) = v_{xxx}(l,t) = 0, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x). \end{cases} \tag{1.9}$$

In [28], the authors studied the long-time behavior of solutions to the initial-boundary value problem of (1.7) and obtained the existence of global attractor of solutions.

It is well known that many multidimensional wave equations arise from physical models, so it is meaningful to study the multidimensional generalized double dispersion equations. However, until recently, very little work has been done on the initial-boundary problem for the multidimensional double dispersion equation. It is our purpose to study the behavior of solutions to the problem (1.1)–(1.3) in the phase space $H_0^1(\Omega) \times L^2(\Omega)$. Of course, it may be interesting to show how the nonlinear term affects the behavior of the solutions. When the coefficient of nonlinear term is positive ($\beta > 0$), we can obtain global *a prior* in the phase space without any other restriction on the initial data. However, the case $\beta < 0$ may lead to finite time blowup of solutions. To discuss the issue, the potential well theory is introduced by Sattinger in [17] to prove the global existence of solutions for nonlinear hyperbolic equations which have not necessary positive definite energy. As far as we know, most of the relatives works focus on the subcritical initial energy case $E(0) < d$. Sharp criteria has been obtained for the global solutions and finite time blowup solutions with subcritical initial energy. We refer the reader to [9,10,24,25] and the reference therein.

It may be a interesting work to study the global solutions and finite time blowup of solutions with supercritical initial energy. Based on a family of potential wells, Kutev, Kolkovska, and Dimova [6] established the global weak solution with supercritical initial energy for the Cauchy problem of (1.6) but without the result on finite time blowup of solutions. In [27], the authors provided the sufficient conditions of global solutions with arbitrary positive initial energy to the Cauchy problem of a class of the sixth-order "good" Boussinesq equation and also without the result about finite time blowup of solutions with high initial energy. Later, Kutev, Kolkovska, and Dimova [8] investigated the finite time blowup of solutions with $E(0) > 0$ for the Cauchy problem to Boussinesq equation with linear restoring and combined power nonlinearities, by means of the modification of the concave method. In [3,26], the authors studied semilinear parabolic equation and semilinear pseudo-parabolic equation, respectively. They obtained global solutions and finite time blowup of solutions with arbitrary positive initial energy by comparison principle.

Unfortunately, because of the absence of comparison principle for hyperbolic equation, this method cannot be applied to wave equations much less to (1.1). However, Gazzola and Squassina [4] considered the initial-boundary value problem of the following damped wave equation

$$u_{tt} - \Delta u + \alpha u_t - \gamma \Delta u_t = |u|^{p-2}u, \tag{1.10}$$

and presented the finite time blowup of solutions with arbitrarily positive initial energy under the case $\gamma = 0$. Recently, Wang and Su [22] studied the initial-boundary value problem for the Boussinesq-type equation and provided sufficient conditions of global solutions and finite time blowup of solutions with the initial energy at high level.

Now let us explain in some detail which are our main results. We first establish the local well-posedness by the standard Galërkin method (see Theorem 2.2). In the case of the negative initial energy, we prove the local solutions will blow up in finite time (see Theorem 2.5). By means of the potential well theory and the concavity method, we study the global solutions and finite time blowup of solutions with $E(0) \le d$ (see Theorem 2.6 and Theorem 2.7). At last, we consider the case of supercritical initial energy $E(0) > d$. Under some additional assumptions on the initial data, we establish global existence and finite time blowup of solutions with supercritical initial energy (see Theorem 2.9 and Theorem 2.11). Another interesting result of this paper is to show that under the appropriate assumptions on initial data and parameters p, if the solutions blow up in finite time $T_{\max}$, then there exists t_0 such that $u(t)$ belongs to the unstable set V, for all $[t_0, T_{\max})$. Comparing with Theorem 2.7, the necessary and sufficient conditions of blowup of solutions are presented (see Theorem 2.13). The last result is to construct explicitly arbitrary high energy initial data for which the solution of (1.1)–(1.3) blows up in finite time (see Theorem 2.11).

We conclude this section with several notations given. For $1 \le p \le \infty$, $L^p = L^p(\Omega)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$; in particular, $\|\cdot\| = \|\cdot\|_{L^2}$. $H^k = H^k(\Omega)$ stands for the classical Sobolev spaces. By $H_0^k = H_0^k(\Omega)$, we mean the closure in H^k of the smooth functions with compact support and we denote the H_0^1 norm by $\|\cdot\|_{H_0^1}^2 = \|\cdot\|^2 + \|\nabla \cdot\|^2$.

The powers $(-\Delta)^s$ of $-\Delta$ for $s \in \mathbb{R}$ on Ω are denoted as

$$(-\Delta)^s u(x,t) \triangleq \sum_{k=1}^{\infty} \lambda_k^s a_k(t) e_k(x), \quad s \in \mathbb{R}. \tag{1.11}$$

and the base functions $\{e_k(x)\}$ in H_0^1 with $e_k(x)$ being the eigenfunctions of the Laplace operator subject to the Dirichlet boundary condition:

$$\begin{cases} -\Delta e_k = \lambda_k e_k, & x \in \Omega, \\ e_k\big|_{\partial\Omega} = 0, \end{cases} \tag{1.12}$$

corresponding to eigenvalue λ_k $(k = 1, 2, \ldots)$ and $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots$. Using the definition (1.11), if $u \in L^2$, then $(-\Delta)^{-\frac{1}{2}} u \in L^2$. Define a Hilbert space

$$\mathcal{H} = \left(L^2, (u, v)_{\mathcal{H}}\right),$$

with the scalar product

$$(u, v)_{\mathcal{H}} = (u, v) + \left((-\Delta)^{-\frac{1}{2}} u, (-\Delta)^{-\frac{1}{2}} v\right),$$

and

$$\|u\|_{\mathcal{H}}^2 \triangleq \|u\|^2 + \left\|(-\Delta)^{-\frac{1}{2}} u\right\|^2,$$

which arising from scalar product and being an equivalent norm over L^2.

The notation $(\cdot, \cdot)$ stands for the L^2-inner product and $\langle \cdot, \cdot \rangle_{X,X'}$ is used for the notation of duality pairing between dual space. At last, $X \hookrightarrow Y$ means the inclusion map of X into Y continuous. we use the same letter C to denote a constant whose value may change from line to line and $C(\cdots)$ to denote positive constants depending on the quantities appearing in the parenthesis.

This paper is organized as follows.

1. In Sect. 2, we present the main results of the paper;
2. In Sect. 3, we provide the proofs of the results.

2. The Main Results

In discussing well-posedness of the problem (1.1)–(1.3), we shall impose another restriction on the parameter p

$$1 < p < \infty, \quad \text{if } n = 1, 2, \quad 1 < p \le \frac{n}{n-2}, \quad \text{if } n \ge 3. \tag{2.1}$$

In order to proceed with the presentation of our results, we shall introduce the appropriate definition of a weak solution, which satisfies a certain variational equality. Since $(-\Delta)^{-1}\Delta^2 u = -\Delta u$, for any

$$u \in \left\{u \in H^4 \cap H_0^1 : \Delta u \mid_{\partial\Omega} = 0\right\}$$

(see [5, Lemma 1.7]), then applying the operator $(-\Delta)^{-1}$ to Eq. (1.1), we have

$$(-\Delta)^{-1} u_{tt} + u_{tt} + u - \Delta u + \alpha u_t = -f(u). \tag{2.2}$$

For the reason, the weak solution of Eq. (2.2) with the initial data (1.2) and boundary value condition $u \mid_{\partial\Omega} = 0$ is said to be the weak solution of the problem (1.1)–(1.3). This leads to the following definition.

Definition 2.1. A function $u \in C\left([0, T]; H_0^1\right) \cap C^1([0, T]; L^2) \cap C^2([0, T]; H^{-1})$ is said to be a weak solution to the problem (1.1)–(1.3) over $[0, T]$, if and only if for any $t \in [0, T]$, it satisfies

$$\begin{aligned} &\left((-\Delta)^{-\frac{1}{2}} u_{tt}, (-\Delta)^{-\frac{1}{2}} \varphi\right) + \langle u_{tt}, \varphi \rangle_{H^{-1}, H_0^1} + (u, \varphi) + (\nabla u, \nabla \varphi) \\ &+ \alpha(u_t, \varphi) + (f(u), \varphi) = 0, \end{aligned} \tag{2.3}$$

for all test functions $\varphi \in C\left([0, T]; H_0^1\right)$, and

$$u(x, 0) = u_0(x) \quad \text{in } H_0^1, \quad u_t(x, 0) = u_1(x) \quad \text{in } L^2. \tag{2.4}$$

Moreover, if

$$T_{\max} = \sup\{T > 0 : u = u(x, t) \text{ exists on } [0, T]\} < \infty,$$

then u is called the local weak solution of the problem (1.1)–(1.3). If $T_{\max} = \infty$, then u is called the global weak solution of the problem (1.1)–(1.3).

The main result, which establishes local existence and uniqueness of weak solutions and conservation law of the energy, reads as follows.

Theorem 2.2 (Local well-posedness). *Let p satisfy (2.1), $\beta \in \mathbb{R}\setminus\{0\}$. If the initial data $u_0 \in H_0^1$, $u_1 \in L^2$, then there exists a maximal time $T_{\max}$ which depends only on u_0 and u_1 such that the problem (1.1)–(1.3) admits a unique weak solution u defined on $[0, T_{\max})$ and*

$$u \in C\left([0,T_{\max}); H_0^1\right), \quad u_t \in C\left([0,T_{\max}); L^2\right). \tag{2.5}$$

Moreover, if

$$\sup_{t\in[0,T_{\max})} \left(\|u_t\|_{\mathcal{H}} + \|u\|_{H_0^1}\right) < \infty, \tag{2.6}$$

then $T_{\max} = \infty$. Furthermore, the conservation law of the energy of u holds, i.e.,

$$E(t) + \alpha \int_0^t \|u_\tau\|^2 \mathrm{d}\tau = E(0) \tag{2.7}$$

where

$$E(t) = \frac{1}{2}\left(\|u_t\|_{\mathcal{H}}^2 + \|u\|_{H_0^1}^2\right) + \frac{\beta}{p+1}\|u\|_{L^{p+1}}^{p+1}.$$

Remark 2.3. It should be point out that if $\beta > 0$, a suitable modification to the proof of Theorem 2.2 allows us to get the global solution without uniqueness under the assumptions that

$$1 < p < \infty, \quad \text{if } n = 1, 2, \quad 1 < p \le \frac{n+2}{n-2}, \quad \text{if } n \ge 3. \tag{2.8}$$

Note that $\frac{n+2}{n-2}$ is the critical Sobolev exponent $p+1$ for the embedding $H_0^1 \hookrightarrow L^{p+1}$, but it is not known if the energy equality (2.7) holds for $\frac{n}{n-2} < p \le \frac{n+2}{n-2}$ and the uniqueness of solutions is also unknown.

If $\beta > 0$, the energy formula $E(t)$ is positive, and we can achieve global *a prior* estimates from the energy identity (2.7). If $\beta < 0$, the source term $\beta|u|^{p-1}u$ derives the solution to (1.1)–(1.3) to blow up in finite time. So we have the following two results.

Corollary 2.4. *Let p satisfy (2.1), and u be the unique local solution to (1.1)–(1.3). If $\beta > 0$, then u is global. Moreover, if $\alpha > 0$, then there exist constants $\delta > 0$ and $C > 0$ independent of t such that*

$$E(t) \le CE(0)e^{-\delta t}, \quad \forall t \in [0, \infty).$$

The proof of the energy decay estimate follows in a similar manner as in [22, Theorem 2.3].

Theorem 2.5. *Let $\beta < 0$, p satisfy (2.1), and u be the unique local solution to (1.1)–(1.3). If there exists $t_0 \in [0, T_{\max})$ such that $E(t_0) < 0$ or $E(t_0) = 0$, $u(t_0) \ne 0$, then the solution of the problem (1.1)–(1.3) blows up in finite time.*

Next, based on the potential well, we continue to address the issue of a *source* ($\beta < 0$) which leads to the fact that $E(t)$ is nonpositive. For this purpose, from the variational point of view two natural functionals associated with (1.1) need to be introduced. For $\beta < 0$, we define the continuous functional $J(u)$ as follows

$$J(u) = \frac{1}{2}\|u\|_{H_0^1}^2 - \frac{|\beta|}{p+1}\|u\|_{L^{p+1}}^{p+1}, \quad \forall\, u \in H_0^1,$$

which may be regarded as the potential energy functional. The second important functional $I(u)$ is the variation of $J(u)$ or may be called as the Nehari functional defined as

$$I(u) = \|u\|_{H_0^1}^2 - |\beta|\|u\|_{L^{p+1}}^{p+1}.$$

The *stable* set and the *unstable* set are defined by

$$W_1 = \left\{u \in H_0^1; J(u) < d\right\}$$
$$W = \left\{u \in H_0^1; I(u) > 0, J(u) < d\right\} \cup \{0\},$$
$$V = \left\{u \in H_0^1; I(u) < 0, J(u) < d\right\},$$

where the potential well depth d is defined as

$$d = \inf\left\{\sup_{\lambda \geq 0} J(\lambda u) : u \in H_0^1, u \neq 0\right\}. \tag{2.9}$$

Furthermore, the potential well depth d, the *stable*, and *unstable* sets are also, respectively, characterized by

$$d = \inf_{u \in \mathcal{N}} J(u) = \frac{p-1}{2(p+1)} |\beta|^{-\frac{2}{p-1}} C_*^{-\frac{2(p+1)}{p-1}} > 0, \tag{2.10}$$
$$W = \left\{u \in H_0^1; J(u) < d, \|u\|_{H_0^1} < |\beta|^{-\frac{1}{p-1}} C_*^{-\frac{p+1}{p-1}}\right\},$$
$$V = \left\{u \in H_0^1; J(u) < d, \|u\|_{H_0^1} > |\beta|^{-\frac{1}{p-1}} C_*^{-\frac{p+1}{p-1}}\right\};$$

see [22, Remark 8], where the Nehari manifold

$$\mathcal{N} = \left\{u \in H_0^1; I(u) = 0, u \neq 0\right\}.$$

Moreover, we have

$$W_1 = W \cup V, \quad \text{and} \quad W \cap V = \emptyset.$$

Here and in the sequel C_* is the optimal Sobolev constant of $H_0^1 \hookrightarrow L^{p+1}$, denoted by

$$C_* = \sup_{\substack{u \in H_0^1, \\ u \neq 0}} \frac{\|u\|_{L^{p+1}}}{\|u\|_{H_0^1}}.$$

The best constant C_* has been determined in [11].

The following several results are established to describe the global existence and nonexistence of solutions at three different levels: subcritical initial energy $E(0) < d$, critical initial energy $E(0) = d$, and supercritical initial energy $E(0) > d$.

Theorem 2.6 (Global existence in $\overline{W}$). *Let $\beta < 0$, p satisfy (2.1) and u be the unique local solution to (1.1)–(1.3). Assume that there exists $t_0 \in [0, T_{\max})$ such that*

$$E(t_0) \leq d, \qquad I(u(t_0)) \geq 0.$$

Then, u is global and $u \in \overline{W} = W \cup \partial W$, for all $t \in [t_0, \infty)$. Moreover, when $\alpha > 0$, $E(t_0) < d$, $I(u(t_0)) > 0$ or $u(t_0) = 0$, there exist positive constants ξ and C which depends on the initial data and independent of t such that

$$E(t) \leq Ce^{-\xi t}, \quad \forall\, t \in [t_0, \infty). \tag{2.11}$$

Theorem 2.7 (Blowup in $\overline{V}$). *Let $\beta < 0$, p satisfy (2.1) and u be the unique local solution to (1.1)–(1.3). Assume that there exists $t_0 \in [0, T_{\max})$ such that one of the following assumptions holds,*

1. *$0 < E(t_0) < d$ and $I(u(t_0)) < 0$,*
2. *$E(t_0) = d$, $I(u(t_0)) < 0$ and*
   $\left((-\Delta)^{-\frac{1}{2}} u(t_0), (-\Delta)^{-\frac{1}{2}} u_t(t_0)\right) + (u(t_0), u_t(t_0)) \geq 0.$

Then, problem (1.1)–(1.3) does not admit any global weak solution.

From Theorems 2.6 and 2.7, we can obtain the threshold result of global existence and nonexistence of solutions for the problem (1.1)–(1.3) as follows.

Corollary 2.8. *Let $\beta < 0$, p satisfy (2.1) and u be the unique local solution to (1.1)–(1.3). Assume that there exists $t_0 \in [0, T_{\max})$ such that $E(t_0) < d$. Then, when $I(u(t_0)) > 0$, the problem (1.1)–(1.3) admits a unique global weak solution, and when $I(u(t_0)) < 0$, the problem (1.1)–(1.3) blows up in finite time.*

Following the idea of [6,22], we shall establish the global existence and nonexistence results with arbitrary positive initial energy.

Theorem 2.9 (Global existence for $E(0) > 0$). *Let $\beta < 0$, p satisfy (2.1) and u be the unique local solution to (1.1)–(1.3). Assume that $E(0) > 0$ and the following two assumptions are true*

1. $\left((-\Delta)^{-\frac{1}{2}}u_0, (-\Delta)^{-\frac{1}{2}}u_1\right) + (u_0, u_1) + \frac{\alpha}{2}\|u_0\|^2 + \frac{\alpha^2(p+1)}{\sqrt{(p-1)(p+3)\lambda_1}}\left\|(-\Delta)^{-\frac{1}{2}}u_0\right\|^2$
   $+ \frac{p+1}{\sqrt{(p-1)(p+3)\lambda_1}}E(0) < 0,$
2. $K(u_0) \triangleq I(u_0) - \|u_1\|_{\mathcal{H}}^2 > 0.$

Then, u is global.

Remark 2.10. We note here that the second restriction on the initial data implies $I(u_0) > 0$. When $0 < E(0) < d$, we immediately get the existence of global solutions from Theorem 2.6. That is to say the first hypothesis turns out to be essential only when dealing with the case $E(0) > d$.

Theorem 2.11 (Blowup for $E(0) > 0$). *Let $\beta < 0$, p satisfy (2.1) and u be the unique local solution to (1.1)–(1.3). Assume that $E(0) > 0$, and*

$$\left((-\Delta)^{-\frac{1}{2}}u_0, (-\Delta)^{-\frac{1}{2}}u_1\right) + (u_0, u_1) + \frac{\alpha}{2}\|u_0\|^2 - \frac{p+1}{\kappa}E(0) > 0, \tag{2.12}$$

$$\kappa = \frac{1}{2(1+\lambda_1)}\left(\sqrt{A^2+B} - A\right), \tag{2.13}$$

where

$$A = \alpha(p+3)\lambda_1, \quad B = 4(p+3)(p-1)\lambda_1(1+\lambda_1)^2.$$

Then, the solution of the problem (1.1)–(1.3) blows up in finite time.

Noting that when $\alpha = 0$,

$$\kappa = \sqrt{(p+3)(p-1)\lambda_1}.$$

Next, we construct explicitly initial data with arbitrary high initial energy such that all assumptions of Theorem 2.11 are satisfied.

Theorem 2.12. *Let $\beta < 0$, p satisfy (2.1). Then, for every constant $M > 0$, there exist infinitely many initial data $u_0^M \in H_0^1$ and $u_1^M \in L^2$ such that $E(u_0^M, u_1^M) = M$ and $T_{\max} < \infty$ for the corresponding solution of (1.1)–(1.3).*

If we make a further assumption on p, $1 < p < 1 + \frac{4}{n}$, we can provide the necessary and sufficient conditions of blowup of the solutions for the problem (1.1)–(1.3).

Theorem 2.13. *In addition to condition (2.1), we also assume that $1 < p < 1 + \frac{4}{n}$. Let $\beta < 0$, $\alpha > 0$ and u be the local solution to (1.1)–(1.3). Then, u blows up in finite time, i.e., $T_{\max} < \infty$ if and only if there exists $t_0 \in [0, T_{\max})$ such that $E(t_0) < d$ and $u(t_0) \in V$.*

We conclude this section with a few words about our results. Our first result stated in Theorem 2.2 provides local well-posedness of the problem (1.1)–(1.3) in the energy space and establishes the energy identity. To obtain the global solutions, establishing global bounds on the solutions is the key steps. When $\beta > 0$, it is easy to get global *a priori* because of the energy identity. When the source term appears ($\beta < 0$), finite time blowup of solutions may happen. So, the next main works of this paper are to discuss the global solvability and finite time blowup of solutions for the problem (1.1)–(1.3) with $\beta < 0$. It was found that if the initial energy functional $E(0)$ is negative, then the weak solution always blows up for an appropriate finite time. When the initial energy is positive and smaller than d, then the global existence or blowup of solutions depends on the sign of the Nehari functional $I(u)$, see Theorem 2.6 and Theorem 2.7. The situation is, however, different when the initial energy is arbitrary positive especially lager than d, leading to the sign-preserving property of $I(u)$ which is no longer valid. Theorem 2.9 and Theorem 2.11 show that the global solvability or the finite time blowup of solutions of (1.1)–(1.3) with initial energy $E(0) > d$ depends not only on the initial profile u_0 but also on the initial velocity u_1. So, some additional assumptions on the initial data are necessary for global existence or finite time blowup of solutions with supercritical initial energy.

3. Proof of the Main Results

In this section, we present the proof of the main results. We establish the local well-posedness in Theorem 2.2 by applying an appropriate Faedo–Galërkin method which can be divided into four steps: constructing finite-dimensional Galërkin approximations to the problem (1.1)–(1.3), establishing *a priori* estimates, passing to limit, and verifying the conditions of initial data. The initial energy level plays a crucial role in dealing with the global solvability and finite time blowup of solutions, and these cases are, respectively, tackled with different tools.

3.1. Proof of Theorem 2.2

We divide into three steps to prove the theorem. Firstly, we prove the existence of local solutions $u \in L^\infty(0,T;H_0^1) \cap W^{1,\infty}(0,T;L^2)$. Secondly, we establish the energy identity (2.7). At last, we further its uniqueness and the continuity of solutions.

Let $\{e_k(x)\}$ be the base functions in H_0^1 as mentioned in (1.12), and we normalize e_k such that $\|e_k\| = 1$. Let

$$u_m(x,t) = \sum_{k=1}^{m} a_{km}(t) e_k(x) \tag{3.1}$$

be the Galërkin approximate solutions of the problem (1.1)–(1.3), satisfying

$$\left((-\Delta)^{-1} u_{mtt} + u_{mtt} + u_m - \Delta u_m + \alpha u_{mt}, e_k\right) = -\left(f(u_m), e_k\right), \tag{3.2}$$

$$u_m(x,0) = u_{m0}(x) = \sum_{k=1}^{m} \rho_k e_k \to u_0 \quad \text{in } H_0^1, \quad \text{as } m \to \infty, \tag{3.3}$$

$$u_{mt}(x,0) = u_{m1}(x) = \sum_{k=1}^{m} \xi_k e_k \to u_1 \quad \text{in } L^2, \quad \text{as } m \to \infty; \tag{3.4}$$

therefore, by Poincaré inequality, we have

$$(-\Delta)^{-\frac{1}{2}} u_{mt}(x,0) \to (-\Delta)^{-\frac{1}{2}} u_1 \quad \text{in } L^2, \quad \text{as } m \to \infty. \tag{3.5}$$

Substituting (3.1) into (3.2)–(3.4), it follows that

$$\lambda_k^{-1} a''_{km} + a''_{km} + a_{km} + \lambda_k a_{km} + \alpha a'_{km} = -(f(u_m), e_k), \tag{3.6}$$

$$a_{km}(0) = \rho_k, \quad a'_{km}(0) = \xi_k \quad (k = 1, 2, \ldots, m), \tag{3.7}$$

from here on the notation " $'$ " denotes the derivative with respect to time t. According to the standard ordinary differential equations theory, the problem (3.6), (3.7) admits a solution $a_{km} \in C^2$ on some interval $[0, t_m]$ for each m.

Multiplying both sides of (3.2) by $a'_{km}(t)$, summing up for $k = 1, 2, \ldots, m$ and integrating between 0 and t, one has

$$E_m(t) + \alpha \int_0^t \|u_{m\tau}\|^2 \mathrm{d}\tau = E_m(0), \quad \forall t \in [0, t_m], \tag{3.8}$$

where

$$E_m(t) = \frac{1}{2}\left(\|u_{mt}\|^2_{\mathcal{H}} + \|u_m\|^2_{H_0^1}\right) + \frac{\beta}{p+1}\|u_m\|^{p+1}_{L^{p+1}}.$$

Then, from (3.8), recalling $H_0^1 \hookrightarrow L^{2p}$, and thanks to the Hölder and Young inequalities, we have

$$\begin{aligned}
&\frac{1}{2}\left(\|u_{mt}\|^2_{\mathcal{H}} + \|u_m\|^2_{H_0^1}\right) + \alpha \int_0^t \|u_{m\tau}\|^2 \mathrm{d}\tau \\
=&\frac{1}{2}\left(\|u_{m1}\|^2_{\mathcal{H}} + \|u_{m0}\|^2_{H_0^1}\right) + \beta \int_0^t \int_\Omega |u_m|^{p-1} u_m u_{m\tau} \mathrm{d}x \mathrm{d}\tau \\
\le&\frac{1}{2}\left(\|u_{m1}\|^2_{\mathcal{H}} + \|u_{m0}\|^2_{H_0^1}\right) + |\beta| \int_0^t \|u_m\|^p_{L^{2p}} \|u_{m\tau}\|_{L^2} \mathrm{d}\tau \\
\le&\frac{1}{2}\left(\|u_{m1}\|^2_{\mathcal{H}} + \|u_{m0}\|^2_{H_0^1}\right) + C|\beta| \int_0^t \frac{1}{2}\left(\|u_m\|^2_{H_0^1} + \|u_{m\tau}\|^2\right)^{\frac{p+1}{2}} \mathrm{d}\tau.
\end{aligned}$$

According to the facts (3.3)–(3.5), for sufficient large m, we have

$$\begin{aligned}
&\frac{1}{2}\left(\|u_{mt}\|^2_{\mathcal{H}} + \|u_m\|^2_{H_0^1}\right) + \alpha \int_0^t \|u_{m\tau}\|^2 \mathrm{d}\tau \\
\le& \|u_1\|^2_{\mathcal{H}} + \|u_0\|^2_{H_0^1} + C|\beta| \int_0^t \frac{1}{2}\left(\|u_m\|^2_{H_0^1} + \|u_{m\tau}\|^2\right)^{\frac{p+1}{2}} \mathrm{d}\tau.
\end{aligned}$$

A simple computation entails

$$\|u_{mt}\|^2_{\mathcal{H}} + \|u_m\|^2_{H_0^1} + \alpha \int_0^t \|u_{m\tau}\|^2 \mathrm{d}\tau \le 2\left[A^{\frac{1-p}{2}} - \frac{p-1}{2} C|\beta| t\right]^{-\frac{2}{p-1}}, \tag{3.9}$$

where $A = \|u_1\|^2_{\mathcal{H}} + \|u_0\|^2_{H_0^1}$. We find the right-hand side of (3.9) will blow up, as $t \to \frac{2A^{\frac{1-p}{2}}}{C|\beta|(p-1)}$; however, for

$$T = \frac{A^{\frac{1-p}{2}}}{C|\beta|(p-1)}, \tag{3.10}$$

we reach

$$\|u_{mt}\|_{\mathcal{H}}^2 + \|u_m\|_{H_0^1}^2 + \alpha \int_0^t \|u_{m\tau}\|^2 \mathrm{d}\tau \le 2^{\frac{p+1}{p-1}} A, \quad \forall t \in [0,T]. \tag{3.11}$$

The uniformly boundedness of the norms $\|u_{mt}\|_{\mathcal{H}}^2$ and $\|u_m\|_{H_0^1}^2$ on $[0,T]$ is obtained. Using the Bessel inequality, we have $\sum_{k=1}^{m} a_{km}^2(t)$ and $\sum_{k=1}^{m} a_{km}'^2(t)$ are uniformly bounded over $[0,T]$ and $a_{km}(t)$ can be extended to $[0,T]$ for each m.

Now we pass to the limit. The estimate (3.11) implies that a subsequence of $\{u_m\}$, which we still denote by $\{u_m\}$, can be extracted such that

$$u_m \to u \quad \text{weakly-star in} \quad L^\infty(0,T;H_0^1), \tag{3.12}$$

$$u_{mt} \to v \quad \text{weakly-star in} \quad L^\infty(0,T;L^2), \tag{3.13}$$

and $u_t = v$ ([12]). It remains to show (2.3), since (2.4) is immediate. Since the injection $H_0^1 \hookrightarrow L^2$ is continuous and compact, there is a subsequence $\{u_m\}$ in H_0^1 such that

$$u_m \to u \quad \text{strongly in} \quad L^2,$$

extracting a further subsequence if necessary, $u_m \to u$ a.e. in Ω; hence, by the continuity of $f(u)$ we can get $f(u_m) \to f(u)$ a.e. in Ω. Combining with the boundedness of $f(u_m)$ in $L^\infty\left(0,T;L^2\right)$, we immediately get

$$f(u_m) \to f(u) \quad \text{weakly-star in} \quad L^\infty\left(0,T;L^2\right). \tag{3.14}$$

It follows from (3.2) that $\varphi_m \triangleq \left((-\Delta)^{-1} + I\right) u_{mtt} \in L^\infty(0,T;H^{-1})$ then

$$u_{mtt} = (I-\Delta)^{-1}(-\Delta)\varphi_m \in L^\infty(0,T;H^{-1}),$$

and

$$u_{mtt} \to u_{tt} \quad \text{weakly-star in} \quad L^\infty(0,T;H^{-1}), \tag{3.15}$$

Moreover, noting that $(-\Delta)^{-\frac{1}{2}} : H^{-1} \to L^2$ is a bounded linear operator,

$$(-\Delta)^{-\frac{1}{2}} u_{mtt} \to (-\Delta)^{-\frac{1}{2}} u_{tt} \quad \text{weakly-star in} \quad L^\infty\left(0,T;L^2\right). \tag{3.16}$$

Then, for $\psi \in C[0,T]$, $\varphi_k = \psi(t)e_k \in C\left([0,T];H_0^1\right)$, multiplying both sides of (3.2) by $\psi(t)$, and taking $m \to \infty$, for a.e. $t \in [0,T]$, we get

$$\begin{aligned}&\left((-\Delta)^{-\frac{1}{2}} u_{tt}, (-\Delta)^{-\frac{1}{2}} \varphi_k\right) + \langle u_{tt}, \varphi_k\rangle_{H^{-1},H_0^1} + (u,\varphi_k) + (\nabla u, \nabla \varphi_k)\\ &+ \alpha(u_t,\varphi_k) + (f(u),\varphi_k) = 0.\end{aligned}$$

The latter equation holds for all k, therefore, for any linear combination of the φ_k's. Then, (2.3) holds.

Having established the existence of local solutions, and now we further study the uniqueness, energy identity (2.7) and continuity of solutions.

Noting that (3.12), (3.13) and (3.15), we get

$$u_{tt} + (I-\Delta)u = -(-\Delta)^{-1}u_{tt} - f(u) - \alpha u_t \in L^\infty(0,T;L^2). \tag{3.17}$$

Then,

$$(u_{tt} + (I-\Delta)u, u_t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t\|^2 + \|u\|^2 + \|\nabla u\|^2\right), \tag{3.18}$$

which follows by Temam lemma [18, Section II, Lemma 4.1]. Then, multiplying both sides of (3.17) by u_t and then integrating over $\Omega \times [0,t]$, and using the fact (3.18), we can obtain then energy identity (2.7).

Let u, v be two solutions of the problem (1.1)–(1.3) which share the same initial data u_0, u_1. Let $w = u - v$, then $w(x,0) = 0$, $w_t(x,0) = 0$, and $w \in L^\infty(0,T;H_0^1)$, $w_t \in L^\infty(0,T;L^2)$, and w satisfies

$$(I + (-\Delta)^{-1})w_{tt} + (I-\Delta)w = -(f(u) - f(v)) - \alpha w_t \tag{3.19}$$

in the sense of distribution. Applying the bounded linear operator $A_1 = (I-\Delta)^{-1}(-\Delta) : L^2 \to L^2(H^{-1} \to H^{-1})$ to both sides of (3.19), we can get

$$w_{tt} - \Delta w = A_1\left(-(f(u) - f(v)) - \alpha w_t\right) \in L^\infty(0, T; L^2). \tag{3.20}$$

By Temam lemma again, we have

$$w \in C([0, T]; H_0^1), \quad w_t \in C([0, T]; L^2).$$

By applying standard energy estimates for equation (3.20), we obtain

$$\begin{aligned}
\|w_t\|^2 + \|\nabla w\|^2 &= \int_0^t \int_\Omega A_1\left(-(f(u) - f(v)) - \alpha w_t\right) w_t \mathrm{d}x\mathrm{d}t \\
&\le \int_0^t \left(\|A_1(f(u) - f(v))\| + \alpha \|A_1 w_t\|\right) \|w_t\| \mathrm{d}t \\
&\le C \int_0^t \|f(u) - f(v)\| \|w_t\| \mathrm{d}\tau + C \int_0^t \|w_t\|^2 \mathrm{d}\tau.
\end{aligned}$$

Noting the fact

$$|f(u) - f(v)| \le C(|u|^{p-1} + |v|^{p-1})|w|,$$

then by using the Hölder inequality, the Sobolev embedding $H_0^1 \hookrightarrow L^{2p}$, we can get

$$\|w_\tau\|^2 + \|\nabla w\|^2 \le C \int_0^t \left(\|w_\tau\|^2 + \|\nabla w\|^2\right) \mathrm{d}\tau.$$

By means of the Gronwall's inequality, we can get $w(x, s) = 0$ a.e. in Ω.

For extending the interval of existence, noting (3.10), we know T is uniformly upper bounded, if initial data and the coefficients are fixed; then, by standard argument, we can find a $T_{\max}$ such that (2.5) holds. Thus, we complete the proof of Theorem 2.2. □

3.2. Proof of Theorems 2.5–2.7

In the occasion of the proof of Theorem 2.6 and Theorem 2.7, we shall proceed our argument based on the following lemmas. The first one is the forward invariance of W and V under the flow of (1.1)–(1.3). The proof is quite similar to [21, Lemma 4.1]; in order to avoid redundancy, we omit it here.

Lemma 3.1. *Let $\beta < 0$, p satisfy (2.1), and u be the unique local solution to (1.1)–(1.3).*

1. *If $0 \le E(0) < d$, then for all $t \in [0, T_{\max})$, $u(t)$ belongs to W provided that u_0 belongs to W.*
2. *If either $0 < E(0) < d$ or $E(0) = d$, and*
$$\left((-\Delta)^{-\frac{1}{2}} u_0, (-\Delta)^{-\frac{1}{2}} u_1\right) + (u_0, u_1) \ge 0,$$
then for all $t \in [0, T_{\max})$, $u(t)$ belongs to V provided that u_0 belongs to V.
3. *If $E(0) < 0$ or $E(0) = 0$, $u_0 \ne 0$, then $u(t)$ belongs to V for all $t \in [0, T_{\max})$.*

Lemma 3.2. [13,14] *Suppose that for $t \ge t_0 \ge 0$, a positive, twice-differentiable function $\phi(t)$ satisfies the inequality*

$$\phi''\phi - (\alpha + 1)(\phi')^2 \ge 0$$

where $\alpha > 0$ is a constant. If $\phi(t_0) > 0$ and $\phi'(t_0) > 0$, then $\phi(t) \to \infty$ as $t \to t_1 \le T_0 = \phi(t_0)/(\alpha\phi'(t_0)) + t_0$.

The proof of Theorem 2.6 is divided into three steps. Firstly, we prove the global existence of solutions in the case of $E(t_0) < d$ and $I(u(t_0)) > 0$. Secondly, by the approximation method, we discuss the case $E(t_0) \leq d$ and $I(u(t_0)) \geq 0$. At last, we establish the energy decay estimates.

Proof of Theorem 2.6. Without loss of generality, we may assume that $t_0 = 0$ so that $u(t_0) = u_0$ and $u_t(t_0) = u_1$.

Step 1. When $E(0) < d$ and $I(u_0) > 0$ or $u_0 = 0$, it follows from the first statement of Lemma 3.1 that $I(u) > 0$ or $\|u\|_{H_0^1} = 0$ for all $t \in [0, T_{\max})$, which along with the equality (2.7) and the fact

$$E(t) = \frac{1}{2}\|u_t\|_{\mathcal{H}}^2 + \frac{p-1}{2(p+1)}\|u\|_{H_0^1}^2 + \frac{1}{p+1}I(u), \tag{3.21}$$

that

$$\frac{1}{2}\|u_t\|_{\mathcal{H}}^2 + \frac{p-1}{2(p+1)}\|u\|_{H_0^1}^2 + \alpha\int_0^t \|u_\tau\|^2 \mathrm{d}\tau \leq E(0) < d, \tag{3.22}$$

for all $t \in [0, T_{\max})$. Therefore, by virtue of Theorem 2.2 it yields $T_{\max} = \infty$.

Step 2. When $E(0) \leq d$ and $I(u_0) \geq 0$, picking a sequence $\{\lambda_m\}$ such that $0 < \lambda_m < 1$, $m = 1, 2, \ldots$ and $\lambda_m \to 1$ as $m \to \infty$. Let $u_{0m}(x) = \lambda_m u_0(x)$, $u_{1m}(x) = \lambda_m u_1(x)$. Then, $u_{0m} \in W$ and $E_{0m}(0) < d$. In fact, we know the condition $I(u_0) \geq 0$ ($I(u_0) > 0$) is equivalent to $\|u_0\|_{H_0^1}^2 \leq \frac{2(p+1)}{p-1}d$ ($\|u_0\|_{H_0^1}^2 < \frac{2(p+1)}{p-1}d$) for $E(0) \leq d$. If $u_0 \neq 0$, then $\|u_{0m}\|_{H_0^1}^2 < \frac{2(p+1)}{p-1}d$, so $I(u_{0m}) = I(\lambda_m u(0)) > 0$. By a direct calculation, we can show $J(u_{0m}) = J(\lambda_m u_0) < J(u_0)$ (see [21]). Moreover,

$$E_{0m}(0) = \frac{1}{2}\|u_{1m}\|_{\mathcal{H}}^2 + J(u_{0m}) < E(0) \leq d.$$

If $u_0 = 0$, then $u_{0m} = 0$, and

$$E_{0m}(0) = \frac{\lambda_m^2}{2}\|u_1\|_{\mathcal{H}}^2 < d.$$

Now considering Eq. (1.1) with the boundary condition (1.2) and the initial conditions,

$$u(x,0) = u_{0m}(x), \quad u_t(x,0) = u_{1m}(x). \tag{3.23}$$

Then, it follows from the argument of Step 1 that for each m the problem (1.1), (1.2), (3.23) admits a global weak solution $u_m(t) \in C\left([0,\infty); H_0^1\right)$ with $u_{mt} \in C\left([0,\infty); \mathcal{H}\right)$ and $u_m \in W$ for $0 \leq t < \infty$ satisfying

$$\left((-\Delta)^{-\frac{1}{2}}u_{mtt}, (-\Delta)^{-\frac{1}{2}}\varphi\right) + \langle u_{mtt}, \varphi\rangle_{H^{-1},H_0^1} + (u_m, \varphi) + (\nabla u_m, \nabla\varphi) + \alpha(u_{mt}, \varphi) + (f(u_m), \varphi) = 0,$$

for any $\varphi \in C\left([0,T]; H_0^1\right)$ and $T \in (0, \infty)$. Moreover, we have

$$\frac{1}{2}\|u_{mt}\|_{\mathcal{H}}^2 + \frac{p-1}{2(p+1)}\|u_m\|_{H_0^1} + \frac{1}{p+1}I(u_m) + \alpha\int_0^t \|u_{m\tau}\|^2 \mathrm{d}\tau = E_{0m}(0) < d,$$

which along with the fact $I(u_m) > 0$ yields

$$\frac{1}{2}\|u_{mt}\|_{\mathcal{H}}^2 + \frac{p-1}{2(p+1)}\|u_m\|_{H_0^1} + \alpha\int_0^t \|u_{m\tau}\|^2 \mathrm{d}\tau < d, \quad 0 \leq t < \infty.$$

The remainder of the proof is similar to that in the proof of Theorem 2.2. Therefore, by virtue of Theorem 2.2 it yields $T_{\max} = \infty$.

Step 3. Next, we prove the decay property of the solution. For $\alpha > 0$, we denote

$$l(t) = \left((-\Delta)^{-\frac{1}{2}}u, (-\Delta)^{-\frac{1}{2}}u_t\right) + (u, u_t) + \frac{\alpha}{2}\|u\|^2.$$

The standard approximation argument shows that $l'(t)$ exists and

$$l'(t) = -\|u\|_{H_0^1}^2 - \beta\|u\|_{L^{p+1}}^{p+1} + \left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|^2 + \|u_t\|^2.$$

Let $0 < \epsilon < 1$ be small enough that $\epsilon \le \frac{\lambda_1}{1+\lambda_1}$. Later, we may need to adjust ϵ again. Define

$$L(t) = E(t) + \epsilon l(t).$$

By the Cauchy inequality and the Poincaré inequality, it is readily seen that $L(t)$ and $E(t)$ are equivalent in the sense that

$$\beta_1 E(t) \le L(t) \le \beta_2 E(t), \quad \forall t \ge 0, \tag{3.24}$$

where

$$\beta_1 = 1 - \epsilon\left(1 + \frac{1}{\lambda_1}\right) + \epsilon\alpha > 0, \quad \beta_2 = 1 + \epsilon\left(1 + \frac{1}{\lambda_1} + \alpha\right) > 0.$$

Using the equality (2.7), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t) = \epsilon\left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|^2 + \epsilon\|u_t\|^2 - \alpha\|u_t\|^2 - \epsilon\|u\|_{H_0^1}^2 - \epsilon\beta\|u\|_{L^{p+1}}^{p+1}. \tag{3.25}$$

From (3.22), we see $\|u\|_{H_0^1}^2 \le \frac{2(p+1)}{p-1}E(0)$. Using the Sobolev embedding $H_0^1 \hookrightarrow L^{p+1}$, we have

$$\|u\|_{L^{p+1}}^{p+1} \le C_*^{p+1}\|u\|_{H_0^1}^{p+1} \le C_*^{p+1}\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{p-1}{2}}\|u\|_{H_0^1}^2.$$

Substituting the above inequality into (3.25), by the Poincaré inequality, and noting $\beta < 0$, it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L(t) \le& -\alpha\|u_t\|^2 + \epsilon\left(|\beta|C_*^{p+1}\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{p-1}{2}} - 1\right)\|u\|_{H_0^1}^2\\ &+ \epsilon\left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|^2 + \epsilon\|u_t\|^2\\ \le& \,\epsilon\left(|\beta|C_*^{p+1}\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{p-1}{2}} - 1\right)\|u\|_{H_0^1}^2 + (2\epsilon - \lambda_1\alpha)\|u_t\|_{\mathcal{H}}^2.\end{aligned}$$

Consequently, using the expression (3.21), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L(t) \le& -2\epsilon^2E(t) - \frac{2|\beta|\epsilon^2}{p+1}\|u\|_{L^{p+1}} + \left(2\epsilon + \epsilon^2 - \lambda_1\alpha\right)\|u_t\|_{\mathcal{H}}^2\\ &+ \epsilon\left(|\beta|C_*^{p+1}\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{p-1}{2}} + \epsilon - 1\right)\|u\|_{H_0^1}^2.\end{aligned} \tag{3.26}$$

Combing the fact $E(0) < d$ with the equality (2.10), it holds

$$|\beta|\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{p-1}{2}}C_*^{p+1} < 1.$$

Now, choosing

$$\epsilon < \min\left\{\frac{\lambda_1\alpha}{4}, 1 - |\beta|C_*^{p+1}\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{p-1}{2}}\right\},$$

inequality (3.26) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t) \le -2\epsilon^2 E(t), \quad \forall\, t \ge 0.$$

On the other hand, by virtue of (3.24), setting $\xi = \frac{2\epsilon^2}{\beta_2}$, and by Gronwall's inequality, the last inequality becomes

$$L(t) \le L(0)e^{-\xi t}, \quad \forall\, t \ge 0.$$

By using (3.24) again, we conclude

$$E(t) \le \frac{L(0)}{\beta_1}e^{-\xi t}, \quad \forall\, t \ge 0.$$

Consequently, (2.11) holds. Thus, we complete the proof of Theorem 2.6. □

Proof of Theorem 2.5 and Theorem 2.7. Without loss of generality, we may assume that $t_0 = 0$ so that $u(t_0) = u_0$ and $u_t(t_0) = u_1$. Arguing by contradiction, we suppose that $T_{\max} = \infty$; under the assumptions on the initial data, it follows from Lemma 3.1 that $u(t) \in V$, for any $t \in [0, \infty)$. For some $0 < T < \infty$, and for $t \in [0, T]$, we define

$$\phi(t) = \left\|(-\Delta)^{-\frac{1}{2}}u\right\|^2 + \|u\|^2 + \alpha\int_0^t \|u\|^2 \mathrm{d}\tau + \alpha(T-t)\|u_0\|^2. \tag{3.27}$$

Then,

$$\phi'(t) = 2\left((-\Delta)^{-\frac{1}{2}}u, (-\Delta)^{-\frac{1}{2}}u_t\right) + 2(u, u_t) + 2\alpha\int_0^t (u, u_\tau)\mathrm{d}\tau.$$

By using the Schwartz inequality, we have

$$\frac{1}{4}\left[\phi'(t)\right]^2 \le \phi(t)\left(\left\|(-\Delta)^{-\frac{1}{2}}u_t\right\|^2 + \|u_t\|^2 + \alpha\int_0^t \|u_\tau\|^2\mathrm{d}\tau\right). \tag{3.28}$$

Note that the standard approximation argument shows that $\phi''(t)$ exists and by the fact (2.7), we have

$$\begin{aligned}
\phi''(t) &= -2\|u\|^2 - 2\|\nabla u\|^2 - 2\beta\|u\|^{p+1}_{L^{p+1}} + 2\left\|u_t\right\|^2_{\mathcal{H}} \\
&= (p-1)\|u\|^2_{H^1_0} - 2(p+1)E(0) + (p+3)\left\|u_t\right\|^2_{\mathcal{H}} \\
&\quad + 2(p+1)\alpha\int_0^t \|u_\tau\|^2\mathrm{d}\tau.
\end{aligned}$$

When $E(0) \le d$ (including the case $E(0) < 0$ or $E(0) = 0$, $u_0 \ne 0$), we infer from the assumption $I(u_0) < 0$ and the second and the third result of Lemma 3.1 that $I(u) < 0$ for all $t \in [0, \infty)$. Moreover, it holds $\|u\|^2_{H^1_0} > \frac{2(p+1)}{p-1}d$, which implies

$$\phi''(t) > 2(p+1)(d - E(0)) + (p+3)\left\|u_t\right\|^2_{\mathcal{H}} + 2(p+1)\alpha\int_0^t \|u_\tau\|^2\mathrm{d}\tau. \tag{3.29}$$

Therefore,

$$\phi'(t) > \phi'(0) + 2(p+1)\left(d - E(0)\right)t.$$

Under the assumptions made in this theorem, it follows that for sufficiently large T, there is $t_1 > 0$ such that $\phi'(t) > 0$ and $\phi(t) > 0$ for any $t \in [t_1, T]$.

On the other hand, the fact (3.28) along with (3.29) yields

$$\begin{aligned}&\phi(t)\phi''(t)-\left(1+\frac{p-1}{4}\right)\phi'(t)^2\\ >&\phi(t)\left[2(p+1)(d-E(0)+(p-1)\alpha\int\limits_0^t\|u_\tau\|^2\mathrm{d}\tau\right]\geq 0.\end{aligned}$$

Applying Lemma 3.2, we can find $t_2\leq\frac{\phi(t_1)}{\frac{p-1}{4}\phi'(t_1)}+t_1$ such that

$$\lim_{t\uparrow t_2}\phi(t)=+\infty.$$

Thus, ϕ always becomes infinite at t_2 under the assumption made. This is a contradiction with $T_{\max}=\infty$. □

3.3. Proof of Theorem 2.9–Theorem 2.12

Proof of Theorem 2.9. Let

$$\psi(t)=\left\|(-\Delta)^{-\frac{1}{2}}u\right\|^2+\|u\|^2+\alpha\int\limits_0^t\|u\|^2\mathrm{d}\tau.$$

Comparing the expression of ψ with ϕ defined in (3.27), we reach

$$-\frac{1}{2}\psi''(t)=I(u)-\|u_t\|^2_{\mathcal{H}}\triangleq K(u(t))$$

Now we assert that if $K(u_0)>0$, then $K(u(t))>0$, for all $t\in[0,T_{\max})$. Suppose that there exists $t_0\in(0,T_{\max})$ such that $K(u(t_0))=0$, $K(u(t))>0$ for $t\in[0,t_0)$, and $K(u(t))<0$ for $t\in(t_0,T_{\max})$. Then,

$$E(0)=\frac{p+3}{2(p+1)}\|u_t(t_0)\|^2_{\mathcal{H}}+\frac{p-1}{2(p+1)}\|u(t_0)\|^2_{H_0^1}+\alpha\int\limits_0^{t_0}\|u_\tau\|^2\mathrm{d}\tau \tag{3.30}$$

By means of the Cauchy inequality, we obtain

$$\begin{aligned}&-2A\left((-\Delta)^{-\frac{1}{2}}u(t_0),(-\Delta)^{-\frac{1}{2}}u_t(t_0)\right)-2A\left(u(t_0),u_t(t_0)\right)\\ &\leq\frac{p+3}{2(p+1)}\|u_t(t_0)\|^2_{\mathcal{H}}+\frac{(p-1)\lambda_1}{2(p+1)}\|u(t_0)\|^2_{\mathcal{H}},\end{aligned} \tag{3.31}$$

where $A=\frac{\sqrt{(p-1)(p+3)\lambda_1}}{2(p+1)}$ and

$$-2\alpha(u,u_t)\leq\|u_t\|^2+\alpha^2\|u\|^2 \tag{3.32}$$

Substituting (3.31) and (3.32) into (3.30) and noting the fact

$$2\int\limits_0^{t_0}(u,u_\tau)\,\mathrm{d}\tau=\|u(t_0)\|^2-\|u_0\|^2,\quad\|u(t_0)\|_{H_0^1}\geq\lambda_1\|u(t_0)\|_{\mathcal{H}},$$

there holds

$$\begin{aligned}E(0) \geq & -2A\left((-\Delta)^{-\frac{1}{2}}u(t_0), (-\Delta)^{-\frac{1}{2}}u_t(t_0)\right) - 2A(u(t_0), u_t(t_0)) \\ & -\alpha^2\|u(t_0)\|_{\mathcal{H}}^2 - \alpha^3\int_0^{t_0}\|u\|^2\mathrm{d}\tau + \alpha^2\|u_0\|^2 \\ \geq & -A\psi'(t_0) - \alpha^2\psi(t_0) + \alpha^2\|u_0\|^2 \end{aligned} \tag{3.33}$$

By the continuity of $\psi(t)$ and $\psi'(t)$ on $[0,T]$, there are numbers ζ_1, $\zeta_2 \in (0,t_0)$ such that

$$\begin{aligned}&\psi(t_0) = \psi(0) + \psi'(0)t_0 + \psi''(\zeta_1)t_0^2 < \psi(0) + \psi'(0)t_0, \\ &\psi'(t_0) = \psi'(0) + \psi''(\zeta_2)t_0 < \psi'(0).\end{aligned}$$

Substituting the above two inequalities into (3.33) and noting that $\psi'(0) < 0$, we reach that

$$\begin{aligned}E(0) \geq & -A\psi'(0) - \alpha^2\psi(0) + \alpha^2\|u_0\|^2 \\ = & -2A\left((-\Delta)^{-\frac{1}{2}}u_0, (-\Delta)^{-\frac{1}{2}}u_1\right) - 2A\left(u_0, u_1\right) - \alpha A\|u_0\|^2 \\ & -\alpha^2\left\|(-\Delta)^{-\frac{1}{2}}u_0\right\|^2,\end{aligned}$$

which contradicts with the assumption 1.

In view of the hypothesis 2, we can obtain $K(u(t)) > 0$, for all $t \in [0, T_{\max})$. Thanks to the energy equality

$$\frac{p+3}{2(p+1)}\|u_t\|_{\mathcal{H}}^2 + \frac{p-1}{2(p+1)}\|u\|_{H_0^1}^2 + \frac{1}{p+1}K(u) + \alpha\int_0^t\|u_\tau\|^2\mathrm{d}\tau = E(0),$$

we have

$$\frac{p+3}{2(p+1)}\|u_t\|_{\mathcal{H}}^2 + \frac{p-1}{2(p+1)}\|u\|_{H_0^1}^2 < E(0). \tag{3.34}$$

Thus, for any $t \in [0, T_{\max})$, $\|u_t\|_{\mathcal{H}}^2 + \|u\|_{H_0^1}^2$ is uniformly bounded by a constant depending only on $E(0)$ and p. Then, according to Theorem 2.2, $T_{\max} = \infty$. We complete the proof of Theorem 2.9. □

Proof of Theorem 2.11. By contradiction, we assume $u(t)$ is a global solution for the problem (1.1)–(1.3). Denote

$$F(t) = \left\|(-\Delta)^{-\frac{1}{2}}u\right\|^2 + \|u\|^2$$

and set

$$H(t) = F'(t) + \alpha\|u\|^2 - \frac{2(p+1)}{\kappa}E(0).$$

The standard approximation argument shows that $F''(t)$ exists and

$$F''(t) = -2\|u\|^2 - 2\|\nabla u\|^2 - 2\beta\|u\|_{L^{p+1}}^{p+1} + 2\left\|u_t\right\|_{\mathcal{H}}^2 - 2\alpha(u, u_t). \tag{3.35}$$

In fact, for any $u \in C^2([0,T]; H_0^1)$, we have

$$F''(t) = 2\langle(-\Delta)^{-1}u_{tt} + u_{tt}, u\rangle_{H^{-1},H_0^1} + 2\|u_t\|_{\mathcal{H}}^2.$$

Integrating the above identity over $[0,t]$, we obtain

$$F'(t) = F'(0) + 2\int_0^t\langle(-\Delta)^{-1}u_{\tau\tau} + u_{\tau\tau}, u\rangle_{H^{-1},H_0^1} + 2\|u_\tau\|_{\mathcal{H}}^2\mathrm{d}\tau. \tag{3.36}$$

By a density argument, the fact (3.36) also holds for $u \in C([0,T];H_0^1)\cap C^1([0,T];L^2)\cap C^2([0,T];H^{-1})$. So (3.35) holds by using Eq. (2.2).

By means of the energy identity (2.7), we can get

$$\begin{aligned} H'(t) &= (p-1)\|u\|_{H_0^1}^2 + (p+3)\,\|u_t\|_{\mathcal{H}} - 2(p+1)E(0) + 2(p+1)\alpha\int_0^t \|u_\tau\|^2 \mathrm{d}\tau \\ &\geq (p-1)\xi_1\|u\|_{H_0^1}^2 + (p+3)\,\|u_t\|_{\mathcal{H}} - 2(p+1)E(0) + (p-1)(1-\xi_1)\|u\|_{H_0^1}^2. \end{aligned}$$

Let $0<\xi_1<1$ for $\alpha>0$ and $\xi_1=1$ for $\alpha=0$. By using the Poincaré inequality, it yields

$$\begin{aligned} H'(t) &\geq (p-1)\lambda_1\xi_1\|u\|_{\mathcal{H}}^2 + (p+3)\,\|u_t\|_{\mathcal{H}} - 2(p+1)E(0) \\ &\quad + (p-1)(1+\lambda_1)(1-\xi_1)\|u\|^2 \\ &\geq \sqrt{(p+3)(p-1)\lambda_1\xi_1}\,F'(t) - 2(p+1)E(0) \\ &\quad + (p-1)(1+\lambda_1)(1-\xi_1)\|u\|^2. \end{aligned} \tag{3.37}$$

When $\alpha>0$, choosing ξ_1 is the root of the following equation

$$(p-1)(1+\lambda_1)(1-\xi_1) = \alpha\sqrt{(p+3)(p-1)\xi_1\lambda_1}$$

and

$$\kappa \triangleq \sqrt{(p+3)(p-1)\xi_1\lambda_1} > 0.$$

A straightforward computation yields (2.13). Thus, (3.37) can be estimated as

$$H'(t) \geq \kappa\left(F'(t) + \alpha\|u\|^2 - \frac{2(p+1)}{\kappa}E(0)\right) = \kappa H(t),$$

which together with the assumption (2.12) yields that

$$H(t) \geq H(0)e^{\kappa t}, \quad \forall t\geq 0, \quad \lim_{t\to+\infty} H(t) = +\infty. \tag{3.38}$$

For $\alpha=0$, the rest proof follows from the concavity method and the idea of [22, Theorem 2.8]. To avoid redundancy, we omit it here.

For $\alpha>0$, on the one hand, as a corollary of Theorem 2.5, assume $t_0=0$; if $T_{\max}=\infty$, then

$$E(t)\geq 0, \quad \forall t\in[0,T_{\max}). \tag{3.39}$$

We infer from (3.38), (3.39), and the energy identity (2.7) that

$$\begin{aligned} F(t) &= \left\|(-\Delta)^{-\frac{1}{2}}u_0\right\|^2 + \|u_0\|^2 + \int_0^t \left(F'(\tau) + \alpha\|u\|^2\right)\mathrm{d}\tau - \alpha\int_0^t \|u\|^2\mathrm{d}\tau \\ &\geq \left\|(-\Delta)^{-\frac{1}{2}}u_0\right\|^2 + \|u_0\|^2 + \frac{H(0)}{\kappa}\left(e^{\kappa t}-1\right) - \alpha\int_0^t \|u\|^2\mathrm{d}\tau \end{aligned} \tag{3.40}$$

On the other hand, it follows from (3.39), the energy equality (2.7), and the Hölder inequality that

$$\begin{aligned} \|u\| &\leq \|u_0\| + \int_0^t \|u_\tau\|\mathrm{d}\tau \leq \|u_0\| + (\alpha^{-1}t)^{\frac{1}{2}}\left(\alpha\int_0^t \|u_\tau\|^2\mathrm{d}\tau\right)^{\frac{1}{2}} \\ &\leq \|u_0\| + (\alpha^{-1}t)^{\frac{1}{2}}E(0)^{\frac{1}{2}}. \end{aligned}$$

Then, (3.40) can be estimated as

$$F(t) \geq \left\|(-\Delta)^{-\frac{1}{2}}u_0\right\|^2 + \|u_0\|^2 + \frac{H(0)}{\kappa}\left(e^{\kappa t}-1\right) - 2t\left(\alpha\|u_0\|^2 + tE(0)\right). \tag{3.41}$$

Moreover, by using the Poincaré inequality, it follows that

$$F(t) \leq \left(1+\frac{1}{\lambda_1}\right)\|u\|^2 \leq 2\left(1+\frac{1}{\lambda_1}\right)\left(\|u_0\|^2 + \alpha^{-1}tE(0)\right).$$

From the assumption (2.12), the above inequality, and the fact (3.41), we have

$$\begin{aligned}&\left\|(-\Delta)^{-\frac{1}{2}}u_0\right\|^2 + \|u_0\|^2 + \frac{H(0)}{\kappa}\left(e^{\kappa t}-1\right)\\ &\quad\leq 2\left(1+\frac{1}{\lambda_1}\right)\left(\|u_0\|^2 + \alpha^{-1}tE(0)\right) + 2t\left(\alpha\|u_0\|^2 + tE(0)\right),\end{aligned}$$

which is a contradiction for sufficient large t. Thus, the proof of Theorem 2.11 is completed. □

Proof of Theorem 2.12. Since $H_0^1 \hookrightarrow \mathcal{H}$, for $w \in H_0^1$, $w \neq 0$, and $h \in \mathcal{H}\backslash H_0^1$, $h \neq 0$, taking $v = h - P_{H_0^1}h \in \mathcal{H}$, where $P_{H_0^1}h$ being the orthogonal projection of h onto H_0^1, then $(w,v)_{\mathcal{H}} = 0$. For any given arbitrary positive constant M, we construct initial data u_0^M, u_1^M in the following way:

$$u_0^M(x) = qw(x), \quad u_1^M(x) = qw(x) + sv(x), \tag{3.42}$$

where q and s are positive constants, which will be chosen later. By a simple computation, it holds

$$\left(u_0^M, u_1^M\right)_{\mathcal{H}} = q^2\|w\|_{\mathcal{H}}^2, \quad \|u_1^M\|_{\mathcal{H}}^2 = q^2\|w\|_{\mathcal{H}}^2 + s^2\|v\|_{\mathcal{H}}^2$$

and

$$E(0) = E(u_0^M, u_1^M) = Q(q) + \frac{1}{2}s^2\|v\|_{\mathcal{H}}^2,$$

where

$$Q(q) = \frac{1}{2}q^2\left(\|w\|_{\mathcal{H}}^2 + \|w\|_{H_0^1}^2\right) - \frac{|\beta|}{p+1}q^{p+1}\|w\|_{L^{p+1}}^{p+1}.$$

We aim to show u_0^M and u_1^M satisfy the assumptions (2.12) and $E(u_0^M, u_1^M) = M$ for appropriately chosen q and s. These assumptions are equivalent to

$$M = Q(q) + \frac{1}{2}s^2\|v\|_{\mathcal{H}}^2 < \frac{\kappa q^2}{p+1}\left(\|w\|_{\mathcal{H}}^2 + \frac{\alpha}{2}\|w\|^2\right).$$

Note that $Q(q) < 0$ for all sufficiently large q, we can choose $q = q_*$ sufficiently large so that

$$Q(q_*) < 0, \quad \text{and} \quad q_* > \left(\frac{\kappa}{(p+1)M}\left(\|w\|_{\mathcal{H}}^2 + \frac{\alpha}{2}\|w\|^2\right)\right)^{-\frac{1}{2}}.$$

We fix $s = s_*$ so that $M = Q(q_*) + \frac{1}{2}s_*^2\|v\|_{\mathcal{H}}^2$. Thus, initial data (3.42) with q_* and s_* satisfy the assumption (2.12). Moreover, these initial data have arbitrary high positive energy $E(0) = M$. In this way, if we take $M > d$, we find a wide class of initial data (3.42) with arbitrary high supercritical energy $E(0) = M > d$ for which the blowup result of Theorem 2.12 is valid. □

3.4. Proof of Theorem 2.13

The sufficiency immediately follows from Theorem 2.7, we only need to give the proof of necessity. Given $T_{\max} < \infty$, the assertion that there exists $t_0 \in [0, T_{\max})$ such that $u(t_0) \in V$ and $E(t_0) < d$ can be obtained provided we show

$$\lim_{t \to T_{\max}^-} \|u(t)\| = +\infty. \tag{3.43}$$

Indeed, if (3.43) holds, noting that

$$\begin{aligned} |\|u(t)\| - \|u_0\|| \leq & \|u(t) - u_0\| \leq \int_0^t \|u_\tau\| \mathrm{d}\tau \\ \leq & t^{\frac{1}{2}} \left(\int_0^t \|u_\tau\|^2 \mathrm{d}\tau \right)^{\frac{1}{2}} \end{aligned}$$

holds for all $t \in [0, T_{\max})$, which implies that

$$E(0) - E(t) = \alpha \int_0^t \|u_\tau\|^2 \mathrm{d}\tau \geq \alpha t^{-1} \left| \|u(t)\| - \|u_0\| \right|^2.$$

Then, it follows from the fact (3.43) that

$$\lim_{t \to T_{\max}^-} E(t) = -\infty. \tag{3.44}$$

On the other hand, from Theorem 2.2, we know that if $T_{\max} < \infty$, then

$$\lim_{t \to T_{\max}^-} \left(\|u_t\|_{\mathcal{H}} + \|u\|_{H_0^1} \right) = +\infty,$$

which along with the fact (3.21) yields

$$\lim_{t \to T_{\max}^-} I(u) = -\infty. \tag{3.45}$$

Then, the facts (3.44) and (3.45) imply that there exists $t_0 \in [0, T_{\max})$ such that

$$J(u(t_0)) \leq E(t_0) < d, \quad I(u(t_0)) < 0.$$

Now we turn to prove (3.43). Assume that for a constant $C_1 > 0$, there exists a monotonically increasing sequence $\{t_m\}$ such that $t_m \to T_{\max}$ as $m \to +\infty$ and

$$\|u(t_m)\| \leq C_1. \tag{3.46}$$

From the energy identity (2.7), we know

$$\|\nabla u\|^2 \leq 2E(0) + \frac{2|\beta|}{p+1} \|u\|_{L^{p+1}}^{p+1}, \quad \forall t \in [0, T_{\max}). \tag{3.47}$$

By means of the Gagliardo–Nirenber inequality and Young inequality, we have

$$\begin{aligned} \|u\|_{L^{p+1}}^{p+1} \leq & C_2 \|u\|^{\frac{2(p+1)-(p-1)n}{2}} \|\nabla u\|^{\frac{(p-1)n}{2}} \\ \leq & \frac{C_2}{q_1} \varepsilon^{q_1} \|\nabla u\|^{\frac{q_1(p-1)}{2} n} + \frac{C_2}{q_2 \varepsilon^{q_2}} \|u\|^{\frac{2(p+1)-(p-1)n}{2} q_2}, \end{aligned}$$

where C_2 is a positive constant, $\varepsilon > 0$ will be chosen later, and $q_1, q_2 > 1$ satisfy $\frac{1}{q_1} + \frac{1}{q_2} = 1$. Then,

$$\|\nabla u\|^2 \leq 2E(0) + \frac{2|\beta| C_2}{p+1} \left[\frac{\varepsilon^{q_1}}{q_1} \|\nabla u\|^{\frac{q_1(p-1)n}{2}} + \frac{1}{q_2 \varepsilon^{q_2}} \|u\|^{\frac{2(p+1)-(p-1)n}{2} q_2} \right]. \tag{3.48}$$

Taking $t = t_m$, $q_1 = \frac{4}{(p-1)n}$ in (3.48) and noting (3.46), we can obtain

$$\|\nabla u(t_m)\|^2 \leq C_3 + \frac{2|\beta|C_2}{(p+1)q_1}\varepsilon^{q_1}\|\nabla u(t_m)\|^2.$$

Choosing ε sufficiently small such that $\frac{2|\beta|C_2}{(p+1)q_1}\varepsilon^{q_1} < 1$. Thus,

$$\|\nabla u(t_m)\|^2 \leq C_4.$$

It follows from the Sobolev embedding $H_0^1 \hookrightarrow L^{p+1}$ that

$$\|u(t_m)\|_{L^{p+1}}^{p+1} \leq C_5,$$

in which together with (2.7), we have

$$\lim_{t \to T_{\max}^-} \left(\|u_t\|_{\mathcal{H}} + \|u\|_{H_0^1}\right) < \infty,$$

which contradicts with $T_{\max} < \infty$. Then, (3.43) follows and the desired assertion immediately follows. Theorem 2.13 is proved. □

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