Z. Angew. Math. Phys. (2017) 68:33

0044-2275/17/020001-16
published online February 9, 2017
DOI 10.1007/s00033-017-0778-8

Zeitschrift für angewandte Mathematik und Physik ZAMP

Inertial effect on frequency synchronization for the second-order Kuramoto model with local coupling

Rui Wang and Wen-Xin Qin

Abstract. In this paper, we study the influence of the inertial effect on frequency synchronization in an ensemble of Kuramoto oscillators with finite inertia and symmetric and connected interactions. We present sufficient conditions in terms of coupling strength, algebraic connectivity, natural frequencies, and the inertial term to guarantee the occurrence of frequency synchronization. We also make a comparison with the existing conditions proposed for the first-order Kuramoto model and conclude that the inertial effect, if appropriately small, has little influence on frequency synchronization as long as the initial phase configurations are distributed in a half circle.

Mathematics Subject Classification. 34C15, 34D06, 34D45.

Keywords. Kuramoto model, Coupled oscillators, Frequency synchronization, Inertial effect, Algebraic connectivity.

1. Introduction

The phenomenon of frequency synchronization is ubiquitous and is observed when the individual frequencies of oscillators converge to a common value via coupling despite the differences in natural frequencies. A typical model of coupled oscillators is the well-known Kuramoto model which captures various synchronization phenomena in natural systems [1,22,29] and has been studied in biology, sociology, physics, and chemistry. In this paper, we focus on the frequency synchronization of the Kuramoto model with inertia and local coupling.

We suppose that the oscillators are located on the vertices of a connected and undirected graph, and the interactions between vertices i and j are determined by the weight $a_{ij} \geq 0$. Then the dynamics of the second-order Kuramoto model with n oscillators is governed by the following equations:

$$m\,\ddot{\theta}_i + \dot{\theta}_i = \omega_i + K\sum_{j=1}^{n} a_{ij}\sin(\theta_j - \theta_i), \quad i = 1, 2, \ldots, n, \tag{1.1}$$

in which, θ_i denotes the phase of the ith oscillator whose natural frequency is ω_i, m the inertial coefficient, K the coupling strength, and $a_{ij} \geq 0$ the weight from vertex i to j.

The second-order Kuramoto model (1.1) has close connections with power network [7,8,17], granular superconductors [9], heat conduction [10,11], coupled rotators systems [30,31], and Josephson junctions [33]. See [2,16,18,19,32] for the study for the effects of inertial term on the dynamics of the system. For the research of the standard Kuramoto model, we refer to [1,6,24,25,29].

To obtain explicit and concise criteria of synchronization for a complex network consisting of coupled oscillators has always been an important and outstanding problem, as recognized in [17]. For the Kuramoto

Supported by the National Natural Science Foundation of China (11371270, 11071181).

oscillators with inertia and arbitrary interaction topology, the study for frequency synchronization has been, to our best knowledge, initiated by Dörfler and Bullo in [7] and Choi et al. in [4].

A significant progress was achieved by Dörfler and Bullo via using singular perturbation analysis [7]. Under the overdamped condition, i.e., the ratio of the inertial term and the damping coefficient is sufficiently small, the second-order Kuramoto model and the corresponding first-order Kuramoto model are equivalent, and one can derive sufficient conditions for second-order Kuramoto oscillators to synchronize by those for the first-order system. However, as is always the case in discussion of the singular perturbation theory, there is no explicit estimate for the parameter (the ratio of the inertial term and the damping coefficient) which should be small enough.

A breakthrough was made by Choi et al. in [4] on the estimates of parameters for frequency synchronization for the second-order Kuramoto model with local coupling. By the method developed in [4], one first establishes the boundedness of solutions and then applies the convergence result in [15] derived from the Łojasiewicz's gradient inequality, which requires the coupling function be analytic. The estimate for the inertia term is explicit (see Theorem 3.4 in [4]). However, the condition for the coupling strength K is implicitly given in the inequality (3.1) in [4].

Since we are only concerned with the phenomenon of frequency synchronization for the second-order Kuramoto model, we use in this paper LaSalle's invariance principle, instead of Łojasiewicz's inequality, so that we can extend our approach to more general smooth (not necessarily analytic) coupling functions other than the sine function.

Definition. System (1.1) is said to achieve frequency synchronization if there exists a positive measure set (so that the phenomenon is observable) of initial data such that a solution $\theta(t)$ starting from this set satisfies

$$\lim_{t\to\infty} |\dot{\theta}_i(t) - \dot{\theta}_j(t)| = 0 \quad \text{for} \quad i, j = 1, 2, \ldots, n.$$

In this paper, our aim is to combine the methods in [4,7,8] to give explicit estimates for both the coupling strength K and the inertia term m so that the second-order Kuramoto model (1.1) achieves frequency synchronization.

One novelty of this paper is that we extend the result of Dörfler and Bullo in [8] for the first-order Kuramoto model to the second-order system (see Theorem A), which is also an open question mentioned in [8]. The second novelty is that we improve the estimate on the coupling strength K so that the upper bound is explicit in contrast to the estimate in Theorem 3.4 in [4] (see Theorem B).

Let B_c denote the incidence matrix, of order $n \times n(n-1)/2$, of the complete graph with n vertices, in which each pair of vertices is connected by an edge. For each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $B_c^{\mathrm{T}} x \in \mathbb{R}^{n(n-1)/2}$ has components $x_i - x_j$ corresponding to the directed edge from j to i. Here the superscript "T" denotes transpose. In fact,

$$B_c^{\mathrm{T}} x = (x_1 - x_2, \ldots, x_1 - x_n, x_2 - x_3, \ldots, x_2 - x_n, \ldots, \ldots, x_{n-1} - x_n)^{\mathrm{T}}.$$

We remark that our results are independent of the orientation of the graph.

Let λ_2 and $\lambda_{\max}$ be the second smallest and the largest eigenvalues, respectively, of the Laplacian matrix of the graph associated with system (1.1), $\omega = (\omega_1, \ldots, \omega_n)^{\mathrm{T}}$ the vector of natural frequencies, and $\|\cdot\|$ denote the standard Euclidean norm in $\mathbb{R}^n$ or $\mathbb{R}^{n(n-1)/2}$, as the case may be.

Before presenting our main conclusions, we introduce an arbitrary constant $\delta \in (0, 1)$ to trade off the conditions upon the coupling strength K and the inertial term m, and set for the sake of convenience

$$\alpha = \frac{4\delta^2 + 2}{1 - \delta}, \quad \delta \in (0, 1). \tag{1.2}$$

Note that the range of α is $(2, +\infty)$ and α is an increasing function of δ.

First, we assume that

$$\text{(H1)} \qquad K \geq \frac{\alpha \left\| B_c^T \omega \right\|}{\lambda_2}.$$

Under the assumption (H1) in which we stress the condition on the coupling strength K, we may choose $\nu \in [\pi/2, \pi)$ such that $\sin\nu \geq \alpha \|B_c^T \omega\|/K\lambda_2$. Let $\rho = \sin\nu/\nu$. The second assumption is

$$\text{(H2)} \qquad m \leq \frac{\delta^2 \rho\, \lambda_2}{K\lambda_{\max}^2},$$

in which we emphasize that the inertial coefficient m needs to be small since K is probably large due to the distribution of natural frequencies or the algebraic connectivity λ_2 of the corresponding graph.

Theorem A. *Under the assumptions* ($H1$) *and* ($H2$)*, the second-order Kuramoto model* (1.1) *achieves frequency synchronization.*

We remark that for all-to-all coupling scheme in which $\lambda_2 = \lambda_{\max} = 1$, the conditions (H1) and (H2) reduce to $K \geq \alpha \|B_c^{\mathrm{T}} \omega\|$ and $m \leq \delta^2 \rho/K$.

If we consider in the assumption ($H1$) the quantity $\|\omega\|$, the total variance of natural frequencies, instead of $\|B_c^{\mathrm{T}} \omega\|$, then we have the following conclusion.

Theorem B. *Assume*

$$\text{(H1}'\text{)} \qquad K \geq \frac{\alpha\sqrt{2}\|\omega\|}{\lambda_2}, \ \textit{and} \ \text{(H2}'\text{)} \qquad m \leq \frac{\delta^2 \rho'\, \lambda_2}{K\lambda_{\max}^2},$$

in which $\rho' = \sin\nu'/\nu'$ *and* $\nu' \in [\pi/2, \pi)$ *satisfying* $\sin\nu' \geq \alpha\sqrt{2}\|\omega\|/K\lambda_2$. *Then the Kuramoto oscillators with inertia* (1.1) *achieve frequency synchronization.*

We remark that for system (1.1), the conditions (H1′) and (H2′) are easier to satisfy than (H1) and (H2). Indeed, since the coupling function in (1.1) is odd, then we may assume without loss of generality that $\omega_1 + \cdots + \omega_n = 0$ (see Sect. 2 for detailed discussions). As a consequence, we have by Lemma 2.3 that $\|B_c^{\mathrm{T}} \omega\| = \sqrt{n}\|\omega\| \geq \sqrt{2}\|\omega\|$. Nevertheless, we still present Theorem A since we believe the approach we used to prove Theorem A has independent interest.

We compare our results with the closely related previous studies in [4,7]. As mentioned before, the relationship between the second-order Kuramoto model and the first-order one has been investigated by Dörfler and Bullo in [6,7]. By means of singular perturbation analysis, it has been shown that if the ratio of inertial term and the damping coefficient is sufficiently small, then these two systems are equivalent (see [7] for more precise description). We remark that our approach is totally different, and moreover, the upper bound for m is explicitly given in Theorems A and B.

We remark that for the first-order Kuramoto model with non-complete coupling graph, a sufficient condition for phase cohesiveness and frequency synchronization has been given in [8] (in our notations): $K\lambda_2 > \|B_c^{\mathrm{T}} \omega\|$ (the coupling strength K is absorbed in $\lambda_2(L)$ in [8]). Here in this paper we extend the study to nonidentical oscillators with high-order dynamics, which is one of the open questions mentioned in [8].

Compared with the results presented by Choi et al. (see Theorem 3.4 in [4]), in which the condition for the coupling strength K is implicitly given (see the inequality (3.1) in [4]), we *explicitly* state the condition (H1′) for coupling strength K.

We stress that the constant $\delta \in (0, 1)$ is a trade-off parameter. If δ is chosen to be close to 0, then α is close to 2, and we have a better condition for the coupling strength K and a worse condition for the inertial term m. If δ is chosen to be close to 1, then α becomes large and the condition for m is better and that for K is more conservative. For example, if $K\lambda_2 > 2\|B_c^{\mathrm{T}} \omega\|$, then we choose δ sufficiently close to 0 such that α is close to 2 and hence (H1) is satisfied. In this case, for the frequency synchronization to occur, we need to require that the inertial coefficient m be small.

The region of initial data for which the solutions achieve frequency synchronization does not appear in the conditions and is discussed in Sect. 4.

2. Preliminaries

The network associated with the interactions of the Kuramoto model (1.1) is assumed to be a symmetric, weighted, and connected graph $G(\mathcal{V},\mathcal{E},A)$ with n vertices $\mathcal{V}=\{1,\dots,n\}$, e edges $\mathcal{E}\subset\mathcal{V}\times\mathcal{V}$, and positive weights $a_{ij}>0$ for each undirected edge $\{i,j\}\in\mathcal{E}$. The adjacency matrix of the graph $G(\mathcal{V},\mathcal{E},A)$ is $A=(a_{ij})$ where $a_{ij}=a_{ji}$ denotes the weight of edge $\{i,j\}$ for $i\neq j$ and $a_{ii}=0$. The Laplacian matrix L is defined as $L=D-A$, where A is the adjacency matrix and D is the diagonal matrix of vertex outdegrees, i.e., the diagonal element of D is $\sum_{j=1}^{n}a_{ij}$, $i=1,2,\dots$.

If a direction is assigned to graph $G(\mathcal{V},\mathcal{E},A)$, the incidence matrix $B=(B_{ij})$ is an $n\times e$ matrix such that $B_{ik}=1$ if the edge k ends at vertex i, $B_{ik}=-1$ if edge k starts at vertex i, and 0 otherwise. Let $\mathrm{diag}(\{a_{ij}\}_{\{i,j\}\in\mathcal{E}})$ denote the diagonal matrix of edge weights, then $L=B\mathrm{diag}(\{a_{ij}\}_{\{i,j\}\in\mathcal{E}})B^{\mathrm{T}}$ is the Laplacian matrix.

If we define $a_{ij}=0$ for $\{i,j\}\notin\mathcal{E}$, then

$$W=\mathrm{diag}(\{a_{ij}\}_{i<j}) \tag{2.1}$$

is a diagonal matrix of order $n(n-1)/2$ and

$$L=B_cWB_c^{\mathrm{T}},$$

where B_c is the incidence matrix of the complete graph with n vertices.

Since we assume the graph is symmetric and connected, the Laplacian matrix L is symmetric and positive semi-definite with eigenvalues denoted by

$$0=\lambda_1<\lambda_2\leq\cdots\leq\lambda_n=\lambda_{\max}.$$

Note that the eigenvalue $\lambda_1=0$ is simple since the graph is connected. The eigenvector associated with λ_1 is $\mathbf{1}=(1,\dots,1)^{\mathrm{T}}\in\mathbb{R}^n$. The second smallest eigenvalue λ_2 is called algebraic connectivity which plays an important role in synchronization study [34]. Moreover, we have the property

$$\langle\theta,L\theta\rangle\geq\lambda_2\|\theta\|^2 \quad \text{for all} \quad \theta\in\mathbb{R}^n \quad \text{if} \quad \langle\theta,\mathbf{1}\rangle=0.$$

We denote in this paper by $\langle\cdot,\cdot\rangle$ the standard Euclidean inner product in $\mathbb{R}^n$ or $\mathbb{R}^{n(n-1)/2}$, as the case may be.

By switching to a rotating frame, it is easy to show that we can assume without loss of generality the mean value of the natural frequencies of system (1.1)

$$\bar{\omega}=(\omega_1+\cdots+\omega_n)/n=0.$$

Indeed, let

$$\hat{\theta}_i=\theta_i-(\theta_1+\cdots+\theta_n)/n, \quad \text{and} \quad \hat{\omega}_i=\omega_i-\bar{\omega}, \quad i=1,\dots,n.$$

Due to the symmetry of the graph, i.e., $a_{ij}=a_{ji}$ for all i,j, and the fact that the coupling function is odd, we conclude from (1.1) that

$$m(\ddot{\theta}_1+\cdots+\ddot{\theta}_n)/n+(\dot{\theta}_1+\cdots+\dot{\theta}_n)/n=\bar{\omega},$$

and hence

$$m\ddot{\hat{\theta}}_i+\dot{\hat{\theta}}_i=\omega_i-\bar{\omega}+K\sum_{j=1}^{n}a_{ij}\sin(\theta_j-\theta_i)=\hat{\omega}_i+K\sum_{j=1}^{n}a_{ij}\sin(\hat{\theta}_j-\hat{\theta}_i).$$

Note that for the above system, we have

$$\sum_{i=1}^{n} \hat{\omega}_i = 0, \quad \sum_{i=1}^{n} \hat{\theta}_i = 0, \quad \text{and} \quad \sum_{i=1}^{n} \dot{\hat{\theta}}_i = 0.$$

Therefore, we always assume (see also [3,4,13,14]) in this paper for system (1.1) that

$$\sum_{i=1}^{n} \omega_i = 0, \quad \sum_{i=1}^{n} \theta_i = 0, \quad \text{and} \quad \sum_{i=1}^{n} \dot{\theta}_i = 0. \tag{2.2}$$

Our approach consists of two steps. First by introducing a new equivalent norm depending on the inertia m, we demonstrate the existence of a positively invariant set. Then we obtain the synchronization result by applying LaSalle's invariance principle [23], see also [12].

In what follows, we present some estimates which will be used in the proof of main conclusions.

Lemma 2.1. *Let $a > 0$, $b > 0$, and $\dot{y}(t) \leq -ay(t) + b$ for $t \geq 0$. Then we have*

$$y(t) \leq y(0)\mathrm{e}^{-at} + \frac{b}{a}(1 - \mathrm{e}^{-at}), \quad t \geq 0.$$

Lemma 2.2. *Assume $a > 0$ and $m > 0$. Then there exists a constant*

$$k = a + \frac{1}{2m} - \sqrt{a^2 + \frac{1}{4m^2}} > 0, \tag{2.3}$$

such that

$$a\|x\|^2 + m\|y\|^2 \geq k\left(\|x + my\|^2 + \|my\|^2\right), \quad \text{for all } \ x, y \in \mathbb{R}^n \ \text{ or } \ \mathbb{R}^{n(n-1)/2}.$$

Proof. Let $\xi = x + my$ and $\eta = my$. Then $x = \xi - \eta$ and $y = \eta/m$. Let I denote the identity matrix. It then follows that

$$a\|x\|^2 + m\|y\|^2 = a\|\xi\|^2 - 2a\langle \xi, \eta \rangle + a\|\eta\|^2 + \frac{1}{m}\|\eta\|^2 = \left(\xi^{\mathrm{T}} \ \eta^{\mathrm{T}}\right) J \begin{pmatrix} \xi \\ \eta \end{pmatrix},$$

in which

$$J = \begin{pmatrix} aI & -aI \\ -aI & (a + 1/m)I \end{pmatrix}$$

is a real symmetric matrix whose smaller eigenvalue is k. Therefore, we have the conclusion

$$a\|x\|^2 + m\|y\|^2 \geq k\left(\|\xi\|^2 + \|\eta\|^2\right) = k\left(\|x + my\|^2 + \|my\|^2\right).$$

□

Let B_c be the incidence matrix of the complete graph with n vertices.

Lemma 2.3. *Assume $\langle \theta, \mathbf{1} \rangle = 0$ for $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$. Then we have*

$$\|B_c^T \theta\|^2 = n\|\theta\|^2.$$

Proof. Since $(\theta_1 + \cdots + \theta_n)^2 = 0$, then

$$\theta_1^2 + \cdots + \theta_n^2 = -2(\theta_1\theta_2 + \cdots + \theta_1\theta_n + \theta_2\theta_3 + \cdots + \theta_2\theta_n + \cdots + \theta_{n-1}\theta_n).$$

As a consequence, it follows that

$$\begin{aligned}\|B_c^T \theta\|^2 = &(\theta_1 - \theta_2)^2 + (\theta_1 - \theta_3)^2 + \cdots + (\theta_1 - \theta_n)^2 \\ &+ (\theta_2 - \theta_3)^2 + \cdots + (\theta_2 - \theta_n)^2 \\ &+ \cdots + (\theta_{n-1} - \theta_n)^2\end{aligned}$$

$$
\begin{aligned}
&= (n-1)(\theta_1^2+\theta_2^2+\cdots+\theta_n^2)\\
&\quad -2(\theta_1\theta_2+\cdots+\theta_1\theta_n+\theta_2\theta_3+\cdots+\theta_2\theta_n+\cdots+\theta_{n-1}\theta_n)\\
&= n(\theta_1^2+\theta_2^2+\cdots+\theta_n^2) = n\|\theta\|^2.
\end{aligned}
$$

□

Remark. Similarly, if we assume $\dot\theta_1+\cdots+\dot\theta_n=0$, then we have

$$\|B_c^T\dot\theta\|^2 = n\|\dot\theta\|^2. \tag{2.4}$$

Let

$$\mathbf{sin}(x) = (\sin x_1, \sin x_2, \ldots)^{\mathrm{T}}$$

for a vector $x=(x_1,x_2,\ldots)^{\mathrm{T}}$.

Lemma 2.4. *Let $\theta=(\theta_1,\ldots,\theta_n)^{\mathrm{T}}\in\mathbb{R}^n$ and assume $|\theta_i-\theta_j|\le\nu$ for $i,j=1,\ldots,n$, in which $\nu\in[\pi/2,\pi)$. Then*

$$\left(B_c^T\theta\right)^T W\mathbf{sin}\left(B_c^T\theta\right) \ge \rho\langle\theta, L\theta\rangle, \quad \textit{where} \quad \rho=\sin\nu/\nu,$$

and W is defined in (2.1).

Proof. Note that $|\theta_i-\theta_j|\le\nu<\pi$. Then

$$(\theta_i-\theta_j)\sin(\theta_i-\theta_j) \ge \rho(\theta_i-\theta_j)^2.$$

As a consequence, it follows that

$$
\begin{aligned}
\left(B_c^T\theta\right)^T W\mathbf{sin}\left(B_c^T\theta\right) &= \sum_{i<j} a_{ij}\sin(\theta_i-\theta_j)(\theta_i-\theta_j)\\
&\ge \rho\sum_{i<j} a_{ij}(\theta_i-\theta_j)^2 = \rho(B_c^{\mathrm{T}}\theta)^{\mathrm{T}}W(B_c^{\mathrm{T}}\theta)\\
&= \rho\,\theta^{\mathrm{T}}B_cWB_c^{\mathrm{T}}\theta = \rho\langle\theta, L\theta\rangle.
\end{aligned}
$$

□

Lemma 2.5. *Let $\theta=(\theta_1,\ldots,\theta_n)$ and $\dot\theta=(\dot\theta_1,\ldots,\dot\theta_n)\in\mathbb{R}^n$. Then for each $\varepsilon>0$, we have*

$$\left|\left(B_c^T\dot\theta\right)^T W\mathbf{sin}\left(B_c^T\theta\right)\right| \le \frac{1}{2\varepsilon}\langle\theta, L\theta\rangle + \frac{\varepsilon}{2}\langle\dot\theta, L\dot\theta\rangle.$$

Proof. By Young's inequality, we have

$$
\begin{aligned}
\left|\left(B_c^T\dot\theta\right)^T W\mathbf{sin}\left(B_c^T\theta\right)\right| &= \left|\sum_{i<j} a_{ij}\sin(\theta_i-\theta_j)(\dot\theta_i-\dot\theta_j)\right|\\
&\le \sum_{i<j} a_{ij}|\theta_i-\theta_j||\dot\theta_i-\dot\theta_j|\\
&\le \frac{1}{2\varepsilon}\sum_{i<j} a_{ij}(\theta_i-\theta_j)^2 + \frac{\varepsilon}{2}\sum_{i<j} a_{ij}(\dot\theta_i-\dot\theta_j)^2\\
&= \frac{1}{2\varepsilon}\left(B_c^T\theta\right)^T W\left(B_c^T\theta\right) + \frac{\varepsilon}{2}\left(B_c^T\dot\theta\right)^T W(B_c^T\dot\theta)\\
&= \frac{1}{2\varepsilon}\langle\theta, L\theta\rangle + \frac{\varepsilon}{2}\langle\dot\theta, L\dot\theta\rangle.
\end{aligned}
$$

□

3. Proof of main results

In this section, we present the proof of the conclusions in Theorems A and B. We shall use LaSalle's invariance principle to obtain the conclusion of frequency synchronization. Therefore, the main aim in this section is to construct a compact positively invariant region with positive measure in the phase space of system (1.1). To this end, we introduce an equivalent norm [20] which is the key step to give more accurate estimates on the conditions for system parameters. For the construction of the equivalent norm, see [20,26] and the proof of Lemma 2.2.

Proof of Theorem A. Let $\theta = (\theta_1, \ldots, \theta_n)^{\mathrm{T}}$ and $\omega = (\omega_1, \ldots, \omega_n)^{\mathrm{T}}$. Then the second-order Kuramoto model (1.1) can be rewritten in a vector form

$$m\ddot{\theta} + \dot{\theta} = \omega - KB_cW\mathbf{sin}\left(B_c^T\theta\right), \tag{3.1}$$

in which B_c is the incidence matrix of the complete graph and W is defined in (2.1). Let

$$\xi = B_c^{\mathrm{T}}\theta.$$

Then ξ is a vector with dimension $n(n-1)/2$. Therefore, if $\theta(t)$ is a solution to (3.1), then $\xi(t) = B_c^{\mathrm{T}}\theta(t)$ is a solution of

$$m\ddot{\xi} + \dot{\xi} = B_c^T\omega - KB_c^TB_cW\mathbf{sin}(\xi). \tag{3.2}$$

Now the phase space of system (3.2) is $\mathbb{R}^{n(n-1)/2} \times \mathbb{R}^{n(n-1)/2}$. We define in this phase space an energy function $E(\xi, \dot{\xi})$ and show that $(E(\xi, \dot{\xi}))^{1/2}$ is a norm equivalent to the standard Euclidean norm $(\|\xi\|^2 + \|\dot{\xi}\|^2)^{1/2}$. □

We remark that the choice of the equivalent norm, which can also be regarded as a transformation of variables, is essential to our approach, and we refer to [20,26] for its use on the studies of other second-order systems.

Lemma 3.1. *Let ξ and $\dot{\xi} \in \mathbb{R}^{n(n-1)/2}$ and*

$$E(\xi, \dot{\xi}) = \frac{1}{2}\|\xi\|^2 + m\langle \xi, \dot{\xi}\rangle + m^2\|\dot{\xi}\|^2 = \frac{1}{2}\|\xi + m\dot{\xi}\|^2 + \frac{1}{2}\|m\dot{\xi}\|^2. \tag{3.3}$$

Then $\sqrt{E(\xi, \dot{\xi})}$ is a norm in $\mathbb{R}^{n(n-1)}$ equivalent to the norm $\sqrt{\|\xi\|^2 + \|\dot{\xi}\|^2}$. In particular, we have

$$E(\xi, \dot{\xi}) \geq \frac{1}{4}\|\xi\|^2. \tag{3.4}$$

Proof. It is easy to check that

$$\begin{aligned} E(\xi, \dot{\xi}) &= \frac{1}{6}\|\xi\|^2 + \frac{1}{3}\|\xi\|^2 + m\langle \xi, \dot{\xi}\rangle + \frac{3}{4}m^2\|\dot{\xi}\|^2 + \frac{1}{4}m^2\|\dot{\xi}\|^2 \\ &= \frac{1}{6}\|\xi\|^2 + \|\xi/\sqrt{3} + \sqrt{3}m\dot{\xi}/2\|^2 + \frac{1}{4}m^2\|\dot{\xi}\|^2 \\ &\geq \min\{1/6, m^2/4\}(\|\xi\|^2 + \|\dot{\xi}\|^2). \end{aligned}$$

Hence $\sqrt{E(\xi, \dot{\xi})}$ is a norm equivalent to the standard Euclidean norm. With a similar procedure, we have

$$E(\xi, \dot{\xi}) = \frac{1}{4}\|\xi\|^2 + \|\xi/2 + m\dot{\xi}\|^2 \geq \frac{1}{4}\|\xi\|^2.$$

□

Under the assumption (H1), we choose $\nu \in [\pi/2, \pi)$ such that

$$\sin\nu \geq \frac{\alpha \left\|B_c^T\omega\right\|}{K\lambda_2} \tag{3.5}$$

and set

$$\rho = \frac{\sin\nu}{\nu}.$$

Lemma 3.2. *Let*

$$a = K\rho\,\lambda_2 \quad \text{and} \quad \mu = k\left(1 - mK\lambda_{\max}/\sqrt{am}\right),$$

where $k > 0$ is a constant determined in (2.3). Then under the assumptions (H1) *and* (H2)*, we have*

$$\mu\,\nu \geq 2\left\|B_c^T\omega\right\|.$$

Proof.

$$\begin{aligned}(H2) &\Longleftrightarrow m^2K^2\lambda_{\max}^2 \leq \delta^2 mK\rho\lambda_2\\ &\Longleftrightarrow m^2K^2\lambda_{\max}^2 \leq \delta^2 am\\ &\Longleftrightarrow 1 - \frac{mK\lambda_{\max}}{\sqrt{am}} \geq 1-\delta.\end{aligned}$$

Meanwhile, note that

$$k = a + 1/2m - \sqrt{a^2 + 1/4m^2} \geq a/(2am+1).$$

Then it follows from (H2) and the facts $|\rho| \leq 1$ and $\lambda_2 \leq \lambda_{\max}$ that

$$2am = 2mK\rho\lambda_2 \leq \frac{2\delta^2\rho^2\lambda_2^2}{\lambda_{\max}^2} \leq 2\delta^2.$$

As a consequence, we derive by (1.2) that

$$\mu = k\left(1 - \frac{mK\lambda_{\max}}{\sqrt{am}}\right) \geq (1-\delta)\cdot\frac{a}{2\delta^2+1} = \frac{(1-\delta)K\rho\lambda_2}{2\delta^2+1} = \frac{2K\rho\,\lambda_2}{\alpha},$$

leading to the conclusion by (3.5) that

$$\mu\,\nu \geq \frac{2K\lambda_2\sin\nu}{\alpha} \geq 2\left\|B_c^T\omega\right\|.$$

□

The key step to apply LaSalle's invariance principle is to demonstrate the existence of a compact positively invariant set.

Due to the properties in (2.2) of solutions of (1.1), we may regard $\mathbb{R}^n/\langle\mathbf{1}\rangle \times \mathbb{R}^n/\langle\mathbf{1}\rangle$ as its phase space. Let

$$\mathcal{A} = \left\{(\theta,\dot\theta) \in \mathbb{R}^n/\langle\mathbf{1}\rangle \times \mathbb{R}^n/\langle\mathbf{1}\rangle \;\middle|\; E\left(B_c^{\mathrm{T}}\theta, B_c^{\mathrm{T}}\dot\theta\right) \leq \left\|B_c^T\omega\right\|^2/\mu^2\right\},$$

in which μ is a constant determined in Lemma 3.2.

Lemma 3.3. *$\mathcal{A}$ is a compact positively invariant set for system (3.1) if the conditions (H1) and (H2) hold true.*

Proof. Assume $(\theta(t), \dot{\theta}(t))$ is a solution to (3.1), that is, $(\xi(t), \dot{\xi}(t))$ is a solution of (3.2), with the initial data $(\theta(0), \dot{\theta}(0))$ satisfying

$$E\left(B_c^{\mathrm{T}}\theta(0), B_c^{\mathrm{T}}\dot{\theta}(0)\right) < \left\|B_c^T\omega\right\|^2/\mu^2. \tag{3.6}$$

Then there exists a largest $T > 0$ (or $T = +\infty$) such that for $t \in [0, T)$,

$$E\left(B_c^{\mathrm{T}}\theta(t), B_c^{\mathrm{T}}\dot{\theta}(t)\right) < \frac{\left\|B_c^T\omega\right\|^2}{\mu^2}, \text{ and } E\left(B_c^{\mathrm{T}}\theta(T), B_c^{\mathrm{T}}\dot{\theta}(T)\right) = \frac{\left\|B_c^T\omega\right\|^2}{\mu^2},$$

if T is finite. From Lemmas 3.1 and 3.2, we have for $t \in [0, T)$,

$$|\theta_j(t) - \theta_i(t)|^2 \le \|B_c^T\theta(t)\|^2 = \|\xi(t)\|^2 \le 4E(\xi(t), \quad \dot{\xi}(t)) < 4\left\|B_c^T\omega\right\|^2/\mu^2 \le \nu^2,$$

and hence

$$|\theta_j(t) - \theta_i(t)| \le \nu \text{ for } i, j \in \{1, 2, \ldots, n\} \text{ and } t \in [0, T).$$

Note that $B_cB_c^{\mathrm{T}} = nI_n - 1_{n\times n}$, where I_n denotes the identity matrix of order n and $1_{n\times n}$ the matrix of order n with each element being 1, and $\dot{\theta}^{\mathrm{T}}1_{n\times n} = 0$ due to the assumption (2.2). Multiplying $\dot{\xi}^T$ on both sides of (3.2), we obtain from Lemmas 2.3 and 2.5 that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m}{2}\|\dot{\xi}\|^2\right) + \|\dot{\xi}\|^2 \\
&= \dot{\xi}^TB_c^T\omega - K\dot{\xi}^TB_c^TB_cW\mathbf{sin}(B_c^T\theta) = \dot{\xi}^TB_c^T\omega - K\dot{\theta}^TB_cB_c^TB_cW\mathbf{sin}\left(B_c^T\theta\right) \\
&= \dot{\xi}^TB_c^T\omega - K\dot{\theta}^T(nI_n - 1_{n\times n})B_cW\mathbf{sin}\left(B_c^T\theta\right) \\
&= \dot{\xi}^TB_c^T\omega - nK\left(B_c^T\dot{\theta}\right)^TW\mathbf{sin}\left(B_c^T\theta\right) \\
&\le \dot{\xi}^TB_c^T\omega + \frac{nK}{2\varepsilon}\langle\theta, L\theta\rangle + \frac{nK\varepsilon}{2}\langle\dot{\theta}, L\dot{\theta}\rangle \\
&\le \dot{\xi}^TB_c^T\omega + \frac{nK\lambda_{\max}}{2\varepsilon}\|\theta\|^2 + \frac{\varepsilon nK\lambda_{\max}}{2}\|\dot{\theta}\|^2 \\
&= \dot{\xi}^TB_c^T\omega + \frac{K\lambda_{\max}}{2\varepsilon}\|B_c^T\theta\|^2 + \frac{\varepsilon K\lambda_{\max}}{2}\|B_c^T\dot{\theta}\|^2.
\end{aligned}$$

Consequently, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(m^2\|\dot{\xi}\|^2) \le -2m\|\dot{\xi}\|^2 + \frac{mK\lambda_{\max}}{\varepsilon}\|\xi\|^2 + \varepsilon mK\lambda_{\max}\|\dot{\xi}\|^2 + 2m\langle\dot{\xi}, B_c^T\omega\rangle. \tag{3.7}$$

Noting that

$$\langle\xi, \ddot{\xi}\rangle = \xi^T\ddot{\xi} = \frac{\mathrm{d}}{\mathrm{d}t}(\xi^T\dot{\xi}) - \|\dot{\xi}\|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\langle\xi, \dot{\xi}\rangle - \|\dot{\xi}\|^2,$$

and multiplying ξ^T on both sides of (3.2), we obtain by Lemmas 2.4 and 2.3 that for $t \in [0, T)$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(m\langle\xi, \dot{\xi}\rangle + \frac{1}{2}\|\xi\|^2\right) - m\|\dot{\xi}\|^2 \\
&= \langle\xi, B_c^T\omega\rangle - K\xi^TB_c^TB_cW\mathbf{sin}(\xi) \\
&= \langle\xi, B_c^T\omega\rangle - K\theta^TB_cB_c^TB_cW\mathbf{sin}\left(B_c^T\theta\right) \\
&= \langle\xi, B_c^T\omega\rangle - K\theta^T(nI_n - 1_{n\times n})B_cW\mathbf{sin}\left(B_c^T\theta\right) \\
&= \langle\xi, B_c^T\omega\rangle - nK\left(B_c^T\theta\right)^{\mathrm{T}}W\mathbf{sin}\left(B_c^T\theta\right) \\
&\le \langle\xi, B_c^T\omega\rangle - nK\rho\langle\theta, L\theta\rangle \le \langle\xi, B_c^T\omega\rangle - nK\rho\lambda_2\|\theta\|^2 \\
&= \langle\xi, B_c^T\omega\rangle - K\rho\lambda_2\|B_c^T\theta\|^2 = \langle\xi, B_c^T\omega\rangle - K\rho\lambda_2\|\xi\|^2,
\end{aligned}$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\langle\xi,\dot{\xi}\rangle+\frac{1}{2}\|\xi\|^2\right)\leq m\|\dot{\xi}\|^2+\langle\xi,B_c^T\omega\rangle-K\rho\lambda_2\|\xi\|^2. \tag{3.8}$$

We derive by (3.3), (3.7), and (3.8) that for $t\in[0,T)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(\xi(t),\dot{\xi}(t))&\leq-\rho K\lambda_2\|\xi\|^2-m\|\dot{\xi}\|^2+(mK\lambda_{\max}/\varepsilon)\|\xi\|^2\\&\quad+\varepsilon mK\lambda_{\max}\|\dot{\xi}\|^2+\langle\xi,B_c^T\omega\rangle+2m\langle\dot{\xi},B_c^T\omega\rangle\\&=-(a-b/\varepsilon)\|\xi\|^2-(m-\varepsilon b)\|\dot{\xi}\|^2+\langle\xi,B_c^T\omega\rangle+2m\langle\dot{\xi},B_c^T\omega\rangle,\end{aligned}$$

in which $a=\rho K\lambda_2$ and $b=mK\lambda_{\max}$.

Taking $\varepsilon=\sqrt{m/a}$, we obtain by Lemma 2.2 that

$$\begin{aligned}(a-b/\varepsilon)\|\xi\|^2+(m-\varepsilon b)\|\dot{\xi}\|^2&=(1-b/\sqrt{am})(a\|\xi\|^2+m\|\dot{\xi}\|^2)\\&\geq(1-b/\sqrt{am})\,k(\|\xi+m\dot{\xi}\|^2+\|m\dot{\xi}\|^2)\\&=\mu(\|\xi+m\dot{\xi}\|^2+\|m\dot{\xi}\|^2)\\&=2\mu E(\xi,\dot{\xi}).\end{aligned}$$

Meanwhile, we have the estimate by Young's inequality

$$\begin{aligned}\langle\xi,B_c^T\omega\rangle+2m\langle\dot{\xi},B_c^T\omega\rangle&=\langle\xi+m\dot{\xi},B_c^T\omega\rangle+\langle m\dot{\xi},B_c^T\omega\rangle\\&\leq\frac{1}{2\mu}\left\|B_c^T\omega\right\|^2+\frac{\mu}{2}\|\xi+m\dot{\xi}\|^2+\frac{1}{2\mu}\left\|B_c^T\omega\right\|^2+\frac{\mu}{2}\|m\dot{\xi}\|^2\\&=\frac{1}{\mu}\left\|B_c^T\omega\right\|^2+\mu E(\xi,\dot{\xi}).\end{aligned}$$

Therefore, we derive that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(\xi,\dot{\xi})&\leq-2\mu E(\xi,\dot{\xi})+\frac{1}{\mu}\left\|B_c^T\omega\right\|^2+\mu E(\xi,\dot{\xi})\\&=-\mu E(\xi,\dot{\xi})+\frac{1}{\mu}\left\|B_c^T\omega\right\|^2,\end{aligned}$$

and hence by Lemma 2.1

$$E(\xi(t),\quad\dot{\xi}(t))\leq E(\xi(0),\quad\dot{\xi}(0))\mathrm{e}^{-\mu t}+\frac{\left\|B_c^T\omega\right\|^2}{\mu^2}(1-\mathrm{e}^{-\mu t}),\quad t\in[0,T). \tag{3.9}$$

If T is finite, then $E(\xi(T),\dot{\xi}(T))=\left\|B_c^T\omega\right\|^2/\mu^2$. On the other hand, from (3.9) and (3.6) we deduce that

$$\begin{aligned}E(\xi(T),\dot{\xi}(T))&\leq\lim_{t\to T}\left[E(\xi(0),\dot{\xi}(0))\mathrm{e}^{-\mu t}+\frac{1}{\mu^2}\left\|B_c^T\omega\right\|^2(1-\mathrm{e}^{-\mu t})\right]\\&=E(\xi(0),\dot{\xi}(0))\mathrm{e}^{-\mu T}+\frac{1}{\mu^2}\left\|B_c^T\omega\right\|^2(1-\mathrm{e}^{-\mu T})\\&<\frac{1}{\mu^2}\left\|B_c^T\omega\right\|^2,\end{aligned}$$

which is a contradiction. As a result, $T=+\infty$, implying that the interior of $\mathcal{A}$ is positively invariant, and hence $\mathcal{A}$ is positively invariant for system (3.2).

It remains to verify that $\mathcal{A}$ is bounded since it is closed. Indeed, due to Lemma 2.3, we conclude that for $(\theta,\dot\theta)\in\mathcal{A}$,

$$\|\theta\|^2+\|\dot\theta\|^2=\frac{1}{n}\left(\left\|B_c^{\mathrm{T}}\theta\right\|^2+\left\|B_c^{\mathrm{T}}\dot\theta\right\|^2\right)\le\frac{C}{n}E\left(B_c^{\mathrm{T}}\theta,B_c^{\mathrm{T}}\dot\theta\right)\le\frac{C}{\mu^2 n}\left\|B_c^{\mathrm{T}}\omega\right\|^2,$$

in which we use Lemma 3.1 and C is a constant with $1/C=\min\{1/6,m^2/4\}$ (see the proof of Lemma 3.1). Therefore, $\mathcal{A}$ is bounded.

In order to apply LaSalle's invariance principle, we need to construct a continuously differentiable function $V:\mathbb{R}^{2n}\to\mathbb{R}$ such that $\dot V(\theta,\dot\theta)\le 0$ for $(\theta,\dot\theta)\in\mathcal{A}$. Indeed, let

$$V(\theta,\dot\theta)=(m/2)\|\dot\theta\|^2-\varphi(\theta),$$

where

$$\varphi(\theta)=\sum_{i=1}^n\omega_i\theta_i+\frac{K}{2}\sum_{i,j=1}^n a_{ij}\cos(\theta_i-\theta_j).$$

It is easy to verify that

$$\nabla\varphi(\theta)=\omega-KB_cW\mathbf{sin}(B_c^{\mathrm{T}}\theta)\quad\text{and hence}\quad m\ddot\theta+\dot\theta=\nabla\varphi(\theta).$$

Then we have

$$\begin{aligned}\dot V(\theta,\dot\theta)&=\left\langle\frac{\partial V}{\partial\theta},\frac{\mathrm{d}\theta}{\mathrm{d}t}\right\rangle+\left\langle\frac{\partial V}{\partial\dot\theta},\frac{\mathrm{d}\dot\theta}{\mathrm{d}t}\right\rangle\\&=-\langle\nabla\varphi(\theta),\dot\theta\rangle+m\langle\dot\theta,\ddot\theta\rangle\\&=\langle-m\ddot\theta-\dot\theta,\dot\theta\rangle+m\langle\ddot\theta,\dot\theta\rangle\\&=-\|\dot\theta\|^2\le 0.\end{aligned}$$

Let

$$E=\{(\theta,\dot\theta)\in\mathcal{A}\,|\,\dot V(\theta,\dot\theta)=0\}=\{(\theta,\dot\theta)\in\mathcal{A}\,|\,\dot\theta=0\}$$

and M the largest invariant set in E. Then we conclude by LaSalle's invariance principle that $(\theta(t),\dot\theta(t))\to M$, i.e., $\|\dot\theta(t)\|\to 0$ as $t\to+\infty$ if $(\theta(0),\dot\theta(0))\in\mathcal{A}$, implying that the solutions in $\mathcal{A}$ achieve frequency synchronization.

This completes the proof of Theorem A. □

Let $\nu_1\in(0,\pi/2]$ and $\nu_2\in[\pi/2,\pi)$ satisfying

$$\sin\nu_1=\sin\nu_2=\frac{\alpha\left\|B_c^{\mathrm{T}}\omega\right\|}{K\lambda_2}.$$

Then

$$\sin\nu\ge\frac{\alpha\left\|B_c^{\mathrm{T}}\omega\right\|}{K\lambda_2}$$

for all $\nu\in[\nu_1,\nu_2]$. Let

$$\rho_1=\frac{\sin\nu_1}{\nu_1},\quad\rho_2=\frac{\sin\nu_2}{\nu_2},\quad\text{and}\quad\rho=\frac{\sin\nu}{\nu}\quad\text{for}\quad\nu\in[\nu_1,\nu_2].$$

Then $\rho_2\le\rho_1$ and $\rho\in[\rho_2,\rho_1]$. Therefore, each value in $[\rho_2,\rho_1]$ can be chosen as the constant ρ in condition (H2).

If we choose the largest value $\rho=\rho_1$, then the condition for m becomes better. However, the set of admissible initial data $\mathcal{A}$ becomes smaller. Indeed, if we denote by μ_1 and μ_2 the constants determined in Lemma 3.2 and corresponding to ρ_1 and ρ_2, respectively, then it is easy to verify that $\mu_1\ge\mu_2$ and hence the diameter of $\mathcal{A}$ becomes smaller if we choose $\rho=\rho_1$ and $\nu=\nu_1$.

Proof of Theorem B. The idea for the proof of Theorem B is similar.

Under the assumptions (H1′) and (H2′), we have by a similar discussion to Lemma 3.2

$$\mu\,\nu' \geq 2\sqrt{2}\|\omega\|, \tag{3.10}$$

where $\mu = k(1 - mK\lambda_{\max}/\sqrt{am})$, $a = K\rho'\lambda_2$, and k is determined by (2.3).

Let

$$E(\theta,\dot\theta) = \frac{1}{2}\|\theta\|^2 + m\langle\theta,\dot\theta\rangle + m^2\|\dot\theta\|^2 = \frac{1}{2}\|\theta + m\dot\theta\|^2 + \frac{1}{2}\|m\dot\theta\|^2,\ \theta,\dot\theta \in \mathbb{R}^n.$$

Then $(E(\theta,\dot\theta))^{1/2}$ is a norm in $\mathbb{R}^{2n}$ equivalent to the Euclidean norm $(\|\theta\|^2 + \|\dot\theta\|^2)^{1/2}$. In particular, we have

$$E(\theta,\dot\theta) \geq (1/4)\|\theta\|^2. \tag{3.11}$$

Let

$$\mathcal{A}' = \{(\theta,\dot\theta) \in \mathbb{R}^n/\langle\mathbf{1}\rangle \times \mathbb{R}^n/\langle\mathbf{1}\rangle \,|\, E(\theta,\dot\theta) \leq \|\omega\|^2/\mu^2\}.$$

□

Lemma 3.4. *$\mathcal{A}'$ is a compact positively invariant set for system* (1.1) *if the assumptions* (H1′) *and* (H2′) *are satisfied.*

Proof. Assume $(\theta(t),\dot\theta(t))$ is a solution of (3.1) with the initial data satisfying

$$E(\theta(0),\dot\theta(0)) < \|\omega\|^2/\mu^2.$$

Then there exists a largest $T > 0$ (or $T = +\infty$) such that

$$E(\theta(t),\dot\theta(t)) < \frac{\|\omega\|^2}{\mu^2},\ t \in [0,T), \quad \text{and} \quad E(\theta(T),\dot\theta(T)) = \frac{\|\omega\|^2}{\mu^2}$$

if T is finite. It then follows from (3.11) and (3.10) that

$$|\theta_j(t) - \theta_i(t)|^2 \leq 2\|\theta(t)\|^2 \leq 8E(\theta(t), \quad \dot\theta(t)) < 8\|\omega\|^2/\mu^2 \leq (\nu')^2,$$

and hence

$$|\theta_j(t) - \theta_i(t)| \leq \nu' \ \text{ for } \ i,j \in \{1,2,\ldots,n\} \quad \text{and} \quad t \in [0,T). \tag{3.12}$$

Noting that $\theta(t)$ satisfies (3.1) and multiplying $\dot\theta^{\mathrm{T}}$ on both sides of (3.1), we obtain by Lemma 2.5 that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m}{2}\|\dot\theta\|^2\right) + \|\dot\theta\|^2 &= \dot\theta^T\omega - K\dot\theta^T B_c W\mathbf{sin}(B_c^{\mathrm{T}}\theta) = \dot\theta^T\omega - K\left(B_c^T\dot\theta\right)^T W\mathbf{sin}\left(B_c^T\theta\right)\\ &\leq \dot\theta^T\omega + \frac{K}{2\varepsilon}\langle\theta, L\theta\rangle + \frac{K\varepsilon}{2}\langle\dot\theta, L\dot\theta\rangle\\ &\leq \dot\theta^T\omega + \frac{K\lambda_{\max}}{2\varepsilon}\|\theta\|^2 + \frac{\varepsilon K\lambda_{\max}}{2}\|\dot\theta\|^2.\end{aligned}$$

Consequently, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(m^2\|\dot\theta\|^2) \leq -2m\|\dot\theta\|^2 + \frac{mK\lambda_{\max}}{\varepsilon}\|\theta\|^2 + \varepsilon mK\lambda_{\max}\|\dot\theta\|^2 + 2m\langle\omega,\dot\theta\rangle. \tag{3.13}$$

Since

$$\langle\theta,\ddot\theta\rangle = \theta^T\ddot\theta = \frac{\mathrm{d}}{\mathrm{d}t}(\theta^T\dot\theta) - \|\dot\theta\|^2,$$

multiplying θ^T on both sides of (3.1), we obtain by Lemma 2.4 together with (3.12) that for $t \in [0,T)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(m\langle\theta,\dot\theta\rangle + \frac{1}{2}\|\theta\|^2\right) - m\|\dot\theta\|^2 &= \langle\theta,\omega\rangle - K\theta^T B_c W\mathbf{sin}(B_c^{\mathrm{T}}\theta) = \langle\theta,\omega\rangle - K\left(B_c^T\theta\right)^{\mathrm{T}} W\mathbf{sin}\left(B_c^T\theta\right)\\ &\leq \langle\theta,\omega\rangle - K\rho'\,\langle\theta, L\theta\rangle \leq \langle\theta,\omega\rangle - K\rho'\,\lambda_2\|\theta\|^2,\end{aligned}$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\langle\theta,\dot\theta\rangle+\frac{1}{2}\|\theta\|^2\right)\le m\|\dot\theta\|^2-K\rho'\,\lambda_2\|\theta\|^2+\langle\omega,\theta\rangle. \tag{3.14}$$

We derive by (3.13) and (3.14) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(\theta(t),\dot\theta(t))&\le-\rho' K\lambda_2\|\theta\|^2-m\|\dot\theta\|^2+\frac{mK\lambda_{\max}}{\varepsilon}\|\theta\|^2\\&\quad+\ \varepsilon mK\lambda_{\max}\|\dot\theta\|^2+\langle\omega,\theta\rangle+2m\langle\omega,\dot\theta\rangle\\&=-(a-b/\varepsilon)\|\theta\|^2-(m-\varepsilon b)\|\dot\theta\|^2+\langle\omega,\theta\rangle+2m\langle\omega,\dot\theta\rangle,\end{aligned}$$

in which $a=\rho'\,K\lambda_2$ and $b=mK\lambda_{\max}$.

Taking $\varepsilon=\sqrt{m/a}$, we obtain by Lemma 2.2 that

$$\begin{aligned}(a-b/\varepsilon)\|\theta\|^2+(m-\varepsilon b)\|\dot\theta\|^2&=(1-b/\sqrt{am})(a\|\theta\|^2+m\|\dot\theta\|^2)\\&\ge(1-b/\sqrt{am})\,k\,(\|\theta+m\dot\theta\|^2+\|m\dot\theta\|^2)\\&=\mu(\|\theta+m\dot\theta\|^2+\|m\dot\theta\|^2)\\&=2\mu E(\theta,\dot\theta).\end{aligned}$$

Meanwhile, we deduce by Young's inequality that

$$\begin{aligned}\langle\omega,\theta\rangle+2m\langle\omega,\dot\theta\rangle&=\langle\omega,\theta+m\dot\theta\rangle+\langle\omega,m\dot\theta\rangle\\&\le\frac{1}{2\mu}\|\omega\|^2+\frac{\mu}{2}\|\theta+m\dot\theta\|^2+\frac{1}{2\mu}\|\omega\|^2+\frac{\mu}{2}\|m\dot\theta\|^2\\&=\frac{1}{\mu}\|\omega\|^2+\mu E(\theta,\dot\theta).\end{aligned}$$

Therefore, we derive that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(\theta,\dot\theta)&\le-2\mu E(\theta,\dot\theta)+\frac{1}{\mu}\|\omega\|^2+\mu E(\theta,\dot\theta)\\&=-\mu E(\theta,\dot\theta)+\frac{1}{\mu}\|\omega\|^2,\end{aligned}$$

and hence by Lemma 2.1

$$E(\theta(t),\quad\dot\theta(t))\le E(\theta(0),\quad\dot\theta(0))\mathrm{e}^{-\mu t}+\frac{\|\omega\|^2}{\mu^2}(1-\mathrm{e}^{-\mu t}),\quad t\in[0,T). \tag{3.15}$$

If T is finite, then $E(\theta(T),\dot\theta(T))=\|\omega\|^2/\mu^2$. On the other hand, from (3.15) we deduce that

$$\begin{aligned}E(\theta(T),\dot\theta(T))&\le\lim_{t\to T}\left(E(\theta(0),\dot\theta(0))\mathrm{e}^{-\mu t}+\frac{\|\omega\|^2}{\mu^2}(1-\mathrm{e}^{-\mu t})\right)\\&=E(\theta(0),\dot\theta(0))\mathrm{e}^{-\mu T}+\frac{\|\omega\|^2}{\mu^2}(1-\mathrm{e}^{-\mu T})\\&<\frac{\|\omega\|^2}{\mu^2},\end{aligned}$$

which is a contradiction. As a result, $T=+\infty$, implying that the interior of $\mathcal{A}'$ is positively invariant, and hence $\mathcal{A}'$ is positively invariant for system (1.1).

It is easy to check that $\mathcal{A}'$ is bounded and hence is compact. The Lyapunov function V is defined as in the proof of Theorem A. Then LaSalle's invariance principle implies that the solutions in $\mathcal{A}'$ achieve frequency synchronization. This completes the proof of Theorem B. □

4. Discussion and conclusion

We should remark that the inertial coefficient m cannot be arbitrarily chosen since it is closely related to the choice of the coupling strength K which should fulfill condition (H1) or (H1′).

Apart from synchronization phenomenon, the existence of rotating modes for system (1.1) was investigated in [28] by applying the topological degree theory.

We remark that it is also possible to use monotone dynamical systems approach (see [27]) to study frequency synchronization phenomenon for system (1.1): first establish the existence of a positively invariant set so that the system is monotone in this set and then apply the techniques in [27] to obtain the conclusions.

In the proof of our main results in Sect. 3, we demonstrate the existence of a compact set $\mathcal{A}$ (or $\mathcal{A}'$) such that system (1.1) is positively invariant in $\mathcal{A}$. The existence of such a set is closely related to the concept of phase cohesiveness [6]: for each solution $\theta(t)$ of (1.1) with initial data $(\theta(0), \dot{\theta}(0)) \in \mathcal{A}$, there exists an arc of length ν containing all angles $\theta_i(t)$ at each time $t \geq 0$.

The region of admissible initial data for frequency synchronization depends heavily on the coupling strength K and the inertial term m and is determined in the proof of the main conclusions. If the initial phase configurations are allowed to be distributed in a half circle, i.e., $\max_{i,j} |\theta_i(0) - \theta_j(0)| < \pi$, then we can choose $\nu \in [\pi/2, \pi)$ such that $(\theta(0), \dot{\theta}(0)) \in \mathcal{A}$ implying $\max_{i,j} |\theta_i(0) - \theta_j(0)| \leq \nu$, as long as the coupling strength K is large enough. As a consequence, $(\theta(t), \dot{\theta}(t)) \in \mathcal{A}$ implying $\max_{i,j} |\theta_i(t) - \theta_j(t)| \leq \nu$ for all $t \geq 0$ provided (H1) and (H2) hold true, and hence the solution $\theta(t)$ achieves frequency synchronization.

Note that if we take $\nu = \pi/2$, then $\rho = 2/\pi$ and the upper bound for m is more definite, but the set for allowable initial data becomes smaller.

Finally, we compare our results for the second-order Kuramoto model with those for the first-order system. The closely related result to our conclusion Theorem A was established by Dörfler and Bullo in [8] (see also [5,21]), in which it was shown that under the condition $K\lambda_2 > \|B_c^{\mathrm{T}}\omega\|$, the first-order Kuramoto oscillators achieve frequency synchronization. Our estimate for the second-order system is $K\lambda_2 \geq \alpha\|B_c^{\mathrm{T}}\omega\|$ in which α is arbitrarily chosen in $(2, \infty)$. In addition to this, if the inertia coefficient m is appropriately small, i.e., satisfies (H2), then frequency synchronization occurs. Note that the region for initial data for the first-order system and the second-order system is almost the same.

The related result for the first-order Kuramoto model corresponding to Theorem B was obtained by Ha, Li, and Xue in [14]. Since they did not use the information of eigenvalues of the Laplacian matrix, we restate the condition for K which can be derived by their method: $K\lambda_2 \geq \sqrt{2}\|\omega\|$. Our estimate is $K\lambda_2 \geq \alpha\sqrt{2}\|\omega\|$ for the second-order system. Then frequency synchronization occurs for the second-order system provided the inertia coefficient m is appropriately small, i.e., satisfies (H2′). Similarly, the region for initial data for the first-order system and the second-order system is almost the same.

Based on the above discussions, we draw a conclusion that the inertial effect in the second-order Kuramoto model (1.1), if appropriately small (with the upper bound given by (H2) or (H2′)), has little influence on the occurrence of frequency synchronization as long as the initial phase configurations are distributed in a half circle.

References

[1] Acebrón, J.A., et al.: The Kuramoto model: a simple paradigm for synchronization phenomena. Rev. Modern Phys. **77**, 137–185 (2005)

[2] Acebrón, J.A., Bonilla, L.L., Spigler, R.: Synchronization in populations of globally coupled oscillators with inertial effects. Phys. Rev. E **62**, 3437–3454 (2000)

[3] Choi, Y.P., Ha, S.Y., Yun, S.B.: Complete synchronization of Kuramoto oscillators with finite inertia. Phys. D **240**, 32–44 (2011)

[4] Choi, Y.P., et al.: Complete entrainment of Kuramoto oscillators with inertia on networks via gradient-like flow. J. Differ. Equ. **257**, 2591–2621 (2014)

[5] Chopra, N., Spong, M.W.: On exponential synchronization of Kuramoto oscillators. IEEE Trans. Autom. Control **54**, 353–357 (2009)

[6] Dörfler, F., Bullo, F.: On the critical coupling for Kuramoto oscillators. SIAM J. Appl. Dynam. Sys. **10**, 1070–1099 (2011)

[7] Dörfler, F., Bullo, F.: Synchronization and transient stability in power networks and nonuniform Kuramoto oscillators. SIAM J. Control Optim. **50**, 1616–1642 (2012)

[8] Dörfler, F., Bullo, F.: Exploring synchronization in complex oscillator networks. arXiv:1209.1335 [math.OC]

[9] Fishman, R.S., Stroud, D.: Role of long-range Coulomb interactions in granular superconductors. Phys. Rev. B **38**, 290–296 (1988)

[10] Gendelman, O.V., Savin, A.V.: Normal heat conductivity of the one-dimensional lattice with periodic potential of nearest-neighbor interaction. Phys. Rev. Lett. **84**, 2381–2384 (2000)

[11] Giardiná, C., et al.: Finite thermal conductivity in 1D lattices. Phys. Rev. Lett. **84**, 2144–2147 (2000)

[12] Gushchin, A., Mallada, E., Tang, A.: Synchronization of heterogeneous Kuramoto oscillators with arbitrary topology. arXiv:1410.7448v1 [math.DS]

[13] Ha, S.Y., Ha, T., Kim, J.H.: On the complete synchronization of the Kuramoto phase model. Phys. D **239**, 1692–1700 (2010)

[14] Ha, S.Y., Li, Z., Xue, X.: Formation of phase-locked states in a population of locally interacting Kuramoto oscillators. J. Differ. Equ. **255**, 3053–3070 (2013)

[15] Haraux, A., Jendoubi, M.A.: Convergence of solutions of second-order gradient-like systems with analytic nonlinearities. J. Differ. Equ. **144**, 313–320 (1998)

[16] Ha, S.Y., Kim, Y., Li, Z.: Large-time dynamics of Kuramoto oscillators under the effects of inertia and frustration. SIAM J. Appl. Dynam. Syst. **13**, 466–492 (2014)

[17] Hill, D.J., Chen, G.: Power systems as dynamic networks. In: Proceedings of the IEEE International Symposium on Circuits and systems, Kos, Greece, pp. 722–725 (2006)

[18] Hong, H., Choi, M.Y., Yi, J., Soh, K.S.: Inertia effects on periodic synchronization in a system of coupled oscillators. Phys. Rev. E **59**, 353–363 (1999)

[19] Hong, H., Jeon, G.S., Choi, M.Y.: Spontaneous phase oscillation induced by inertia and time delay. Phys. Rev. E **65**, 026208 (2002)

[20] Hu, B., Qin, W.-X., Zheng, Z.: Rotation number of the overdamped Frenkel–Kontorova model with ac-driving. Phys. D **208**, 172–190 (2005)

[21] Jadbabaie, A., Motee, N., Barahona, M.: On the stability of the Kuramoto model of coupled nonlinear oscillators. In: Proceedings of the American Control Conference, Boston, Massachusetts, pp. 4296–4301 (2004)

[22] Kuramoto, Y.: Self-entrainment of a population of coupled non-linear oscillators. In: Araki, H. (ed.) International Symposium on Mathematical Problems in Theoretical Physics. Lecture Notes in Physics, Vol. 39. Springer, Berlin, pp. 420–422 (1975)

[23] LaSalle, J.P.: Stability theory for ordinary differential equations. J. Differ. Equ. **4**, 57–65 (1968)

[24] Mirollo, R., Strogatz, S.H.: The spectrum of the partially locked state for the Kuramoto model. J. Nonlinear Sci. **17**, 309–347 (2007)

[25] Mirollo, R., Strogatz, S.H.: The spectrum of the locked state for the Kuramoto model of coupled oscillators. Phys. D **205**, 249–266 (2005)

[26] Qin, W.-X., Xu, C.-L., Ma, X.: Stability of single-wave-form solutions in the underdamped Frenkel–Kontorova model. SIAM J. Math. Anal. **40**, 952–967 (2008)

[27] Qin, W.-X.: Frequency synchronization in networks of coupled oscillators, a monotone dynamical systems approach. Int. J. Bifur. Chaos **19**, 4107–4116 (2009)

[28] Qin, W.-X.: Rotating modes in the Frenkel–Kontorova model with periodic interaction potential. Discrete Contin. Dynam. Syst. (Ser. A) **27**, 1147–1158 (2010)

[29] Strogatz, S.H.: From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators. Phys. D **143**, 1–20 (2000)

[30] Takeno, S., Peyrard, M.: Nonlinear modes in coupled rotators models. Phys. D **92**, 140–163 (1996)
[31] Takeno, S., Peyrard, M.: Nonlinear rotating modes: Green's-function solution. Phys. Rev. E **55**, 1922–1928 (1997)
[32] Tanaka, H.-A., Lichtenberg, A.J., Oishi, S.: First order phase transition resulting from finite inertia in coupled oscillator systems. Phys. Rev. Lett. **78**, 2104–2107 (1997)
[33] Wiesenfeld, K., Colet, P., Strogatz, S.H.: Frequency locking in Josephson arrays: connection with the Kuramoto model. Phys. Rev. E **57**, 1563–1569 (1998)
[34] Wu, C.W.: Synchronization in Complex Networks of Nonlinear Dynamical Systems. World Scientific, Singapore (2007)

Rui Wang and Wen-Xin Qin
Department of Mathematics
Soochow University
Suzhou 215006
China
e-mail: qinwx@suda.edu.cn

(Received: November 3, 2015; revised: November 3, 2016)