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Asymptotic structure of viscous incompressible flow around a rotating body, with nonvanishing flow field at infinity

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Abstract. We consider weak ("Leray") solutions to the stationary Navier–Stokes system with Oseen and rotational terms, in an exterior domain. It is shown the velocity may be split into a constant times the first column of the fundamental solution of the Oseen system, plus a remainder term decaying pointwise near infinity at a rate which is higher than the decay rate of the Oseen tensor. This result improves the theory by Kyed (Q Appl Math 71:489–500, 2013).

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1. Introduction

Let $\mathcal{B} \subset \mathbb{R}^3$ be an open bounded set. Suppose this set describes a rigid body moving with constant nonzero translational and angular velocity in an incompressible viscous fluid. Further suppose the flow of this fluid is steady. Then it is natural to assume that the direction of translation and the axis of rotation of the rigid body are parallel. In order to describe the motion of the fluid in this situation, we choose a coordinate system which is attached to the body and whose negative x_1-axis points in the direction of the translation of the body. Then the flow in question is governed by the system of equations

$$-\Delta u + \tau\,\partial_1 u + \tau\,(u \cdot \nabla)u - \varrho\,(e_1 \times x) \cdot \nabla u + \varrho\, e_1 \times u + \nabla \pi = f, \quad \operatorname{div} u = 0, \tag{1.1}$$

in the exterior domain $\overline{\mathcal{B}}^c := \mathbb{R}^3 \backslash \overline{\mathcal{B}}$, supplemented by a decay condition at infinity,

$$u(x) \to 0 \quad \text{for} \quad |x| \to \infty, \tag{1.2}$$

and suitable boundary conditions on $\partial\mathcal{B}$. These latter conditions need not be specified here because they are not relevant in the context of the work at hand. In (1.1) and (1.2), the functions $u : \overline{\mathcal{B}}^c \mapsto \mathbb{R}^3$ and $\pi : \overline{\mathcal{B}}^c \mapsto \mathbb{R}$ represent the unknown velocity and pressure field of the fluid, respectively, whereas the function $f : \overline{\mathcal{B}}^c \mapsto \mathbb{R}^3$ stands for a prescribed volume force acting on the fluid. The parameter $\tau \in (0, \infty)$ is the Reynolds number, and $\varrho \in \mathbb{R}\backslash\{0\}$ the Taylor number. These quantities will be considered as fixed, like the domain $\mathcal{B}$. For a derivation of the model given by (1.1), (1.2), we refer to [26, pp. 665–669]. Note that u is not the velocity field of the fluid with respect to the coordinate system under consideration; it is the velocity relative to an observer at rest ("ground speed"). With respect to any frame which—like ours—adheres to a moving body, the velocity at infinity is nonvanishing, contrary to (1.2).

We are interested in "Leray solutions" of (1.1), (1.2), that is, weak solutions characterized by the conditions $u \in L^6(\overline{\mathcal{B}}^c)^3 \cap W^{1,1}_{\text{loc}}(\overline{\mathcal{B}}^c)^3$, $\nabla u \in L^2(\overline{\mathcal{B}}^c)^9$ and $\pi \in L^2_{\text{loc}}(\overline{\mathcal{B}}^c)$. The relation $u \in L^6(\overline{\mathcal{B}}^c)^3$ means that (1.2) is verified in a weak sense. Such solutions exist for data of arbitrary size if the velocity satisfies Dirichlet boundary conditions on the boundary $\partial\mathcal{B}$ of $\mathcal{B}$, some smoothness of $\partial\mathcal{B}$ is required, and suitable regularity conditions are imposed on f and the data on $\partial\mathcal{B}$ [28, Theorem IX.3.1]. It is known by [7,8,29]

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that the velocity part u of a Leray solution (u, π) to (1.1), (1.2) decays for $|x| \to \infty$ as expressed by the estimates

$$|u(x)| \le C\,\big(\,|x|\,s(x)\,\big)^{-1}, \quad |\nabla u(x)| \le C\,\big(\,|x|\,s(x)\,\big)^{-3/2} \tag{1.3}$$

for $x \in \mathbb{R}^3$ with $|x|$ sufficiently large, where $s(x) := 1 + |x| - x_1$ $(x \in \mathbb{R}^3)$ and $C > 0$ is a constant independent of x. The factor $s(x)$ may be considered as a mathematical manifestation of the wake extending downstream behind a body moving in a viscous fluid.

In view of (1.3), it is natural to look for a function $L : \mathbb{R}^3\backslash\{0\} \mapsto \mathbb{R}^3$ ("leading term") such that $|\partial^\alpha L(x)|$ decays with exactly the rate $\big(\,|x|\,s(x)\,\big)^{-1-|\alpha|/2}$ for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| := \alpha_1+\alpha_2+\alpha_3 \le 1$, whereas $u - L$ ("remainder") and $\nabla(u - L)$ decay pointwise with a rate which is higher than $\big(\,|x|\,s(x)\,\big)^{-1}$ and $\big(\,|x|\,s(x)\,\big)^{-3/2}$, respectively. Such a function L is interesting from a physical point of view because it gives a good idea of how the flow looks like at some distance from the rigid body. Also, a leading term may be useful in some mathematical applications. For example, if a numerical approximation of u is to be computed in a bounded domain around $\mathcal{B}$, knowledge of a leading term may help to determine an artificial boundary condition on the outer boundary of the computational domain in question, and to obtain error estimates in that situation.

We know of two articles dealing with leading terms of solutions to (1.1), (1.2). The first is due to Kyed [52], who showed that

$$u_j(x) = \gamma\,E_{j1}(x) + R_j(x), \quad \partial_l u_j(x) = \gamma\,\partial_l E_{j1}(x) + S_{jl}(x) \quad (x \in \overline{\mathcal{B}}^{\mathrm{c}},\ 1 \le j, l \le 3), \tag{1.4}$$

where $E : \mathbb{R}^3\backslash\{0\} \mapsto \mathbb{R}^4 \times \mathbb{R}^3$ denotes a fundamental solution to the Oseen system

$$-\Delta v + \tau\,\partial_1 v + \nabla\sigma = g, \quad \operatorname{div} v = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{1.5}$$

The definition of the function E is stated in Sect. 2. As becomes apparent from this definition, the term $E_{j1}(x)$ may be expressed explicitly in terms of elementary functions. The coefficient γ is also given explicitly, its definition involving the Cauchy stress tensor. The remainder terms R and S in (1.4) are characterized by the relations $R \in L^q(\overline{\mathcal{B}}^{\mathrm{c}})^3$ for $q \in (4/3, \infty)$ and $S \in L^q(\overline{\mathcal{B}}^{\mathrm{c}})^3$ for $q \in (1, \infty)$. It is known from [25, Section VII.3] that $E_{j1}|B_r^{\mathrm{c}} \notin L^q(B_r^{\mathrm{c}})$ for $q \in [1, 2]$, and $\partial_l E_{j1}|B_r^{\mathrm{c}} \notin L^q(B_r^{\mathrm{c}})$ for $q \in [1, 4/3]$, where $j, l \in \{1, 2, 3\}$ and $r > 0$. Therefore, the function R decays faster than E_{j1}, and S_{jl} faster than $\partial_l E_{j1}$, in the sense of L^q-integrability. Thus the equations in (1.4) may in fact be considered as asymptotic expansions of u and ∇u, respectively. However, the theory in [52] is valid only under the assumptions that u verifies the boundary conditions

$$u(x) = -\tau\,e_1 + \varrho\,(e_1 \times x) \quad \text{for} \quad x \in \partial\mathcal{B}, \tag{1.6}$$

and that f vanishes. Moreover, reference [52] does not deal with pointwise decay of R and S, but in [49], Kyed indicates that $|\mathfrak{R}(x)|$ behaves as $O(|x|^{-4/3+\epsilon})$ if $|x| \to \infty$, for some arbitrary but fixed $\epsilon > 0$.

The second article dealing with an asymptotic expansion of Leray solutions to (1.1), (1.2) is reference [9], which states that for $x \in \overline{B_{S_1}}^{\mathrm{c}}$, $1 \le j \le 3$,

$$u_j(x) = \sum_{k=1}^{3} \beta_k\,Z_{jk}(x, 0) + \left(\int_{\partial\mathcal{B}} u \cdot n\,\mathrm{d}o_x \right) x_j\,(4\,\pi\,|x|^3)^{-1} + \mathfrak{F}_j(x). \tag{1.7}$$

Here S_1 is a sufficiently large positive real number, $(Z_{jk})_{1\le j,k\le 3}$ is the velocity part of the fundamental solution constructed by Guenther, Thomann [60] for the linearization

$$-\Delta v + \tau\,\partial_1 v - \varrho\,(e_1 \times x)\cdot\nabla v + \varrho\,e_1 \times v + \nabla\sigma = g, \quad \operatorname{div} v = 0 \tag{1.8}$$

of (1.1) (see Sect. 2), β_1, β_2, β_3 are coefficients defined in terms of u, π and f (see Theorem 3.1 below), and $\mathfrak{F}$ is a function from $C^1(\overline{B_{S_1}}^{\mathrm{c}})^3$ given explicitly in terms of Z, u and π (again see Theorem 3.1 below).

As is shown in [9], this function $\mathfrak{F}$ decays pointwise, in the sense that

$$|\partial^\alpha \mathfrak{F}(x)| = O\Big((|x|\, s(x))^{-3/2-|\alpha|/2} \ln(2+|x|) \Big) \quad \text{for } |x| \to \infty \quad (\alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| \le 1). \tag{1.9}$$

It is known from [4, Theorem 2.19]—and restated below in Corollary 2.3—that

$$|\partial^\alpha Z(x,0)| = O\big((|x|\, s(x))^{-1-|\alpha|/2} \big) \quad \text{for } |x| \to \infty \quad (\alpha \quad \text{as in (1.9)}). \tag{1.10}$$

So, if the decay rate in (1.10) is sharp, Eq. (1.7) may be considered as an asymptotic expansion in the usual sense: The remainder exhibits a faster pointwise decay than the leading term. However, since the definition of the term $Z(x,0)$ involves an integral over $(0,\infty)$, the leading term $\sum_{k=1}^3 \beta_k\, Z_{jk}(x,0)$ in (1.7) is not as explicit as one would like it to be. This aspect and because it is not obvious whether the decay rate in (1.10) is sharp strongly suggests that the term $\sum_{k=1}^3 \beta_k\, Z_{jk}(x,0)$ should be studied more closely.

This is achieved in the work at hand, where we show that $Z_{j1}(x,0) = E_{j1}(x)$ for $x \in \mathbb{R}^3 \backslash \{0\}$, and $|\partial_x^\alpha Z_{jk}(x,0)| = O\big((|x|\, s(x))^{-3/2-|\alpha|/2} \big)$ for $|x| \to \infty$, where $1 \le j \le 3$ and $k \in \{2, 3\}$ (Corollary 4.5, Theorem 5.1).

These results and those in [9] taken together yield a satisfactory theory on a leading term for Leray solutions to (1.1), (1.2). In fact, by setting

$$\mathfrak{G}_j(x) := \sum_{k=2}^{3} \beta_k\, Z_{jk}(x,0) + \mathfrak{F}_j(x) \quad (x \in \overline{B_{S_1}}^{\mathrm{c}},\ 1 \le j \le 3), \tag{1.11}$$

we may deduce from (1.7) and (1.9) that

$$u_j(x) = \beta_1\, E_{j1}(x) + \Bigg(\int\limits_{\partial \mathcal{B}} u \cdot n \,\mathrm{d}o_x \Bigg)\, x_j\, (4\,\pi\, |x|^3)^{-1} + \mathfrak{G}_j(x) \quad (x \in \overline{B_{S_1}}^{\mathrm{c}},\ 1 \le j \le 3) \tag{1.12}$$

and

$$|\partial^\alpha \mathfrak{G}(x)| = O\big((|x|\, s(x))^{-3/2-|\alpha|/2} \ln(2+|x|) \big) \quad \text{for } |x| \to \infty \tag{1.13}$$

($\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$); see Theorem 3.2 and Corollary 3.1. The decay rate of $\partial^\alpha \mathfrak{G}(x)$ stated in (1.13) is optimal in the sense that derivatives of E_{j1} of order $|\alpha|+1$ exhibit the same rate (see [46]), except for the logarithmic factor $\ln(2+|x|)$. We are able to prove these results without imposing any boundary condition and without requiring that f vanishes. Our assumption that $\mathrm{supp}(f)$ is compact may be replaced by suitable decay conditions on f, but we do not elaborate this—very technical—aspect of our theory.

If we compare how the coefficient γ from (1.4) is defined in [52], and the coefficient β_1 from (1.12) in [9] (see Theorem 3.1 below), we see that γ and β_1 coincide, provided that u satisfies boundary condition (1.6) imposed in [52]. Moreover the function $\mathfrak{G}|B_{S_1}^{\mathrm{c}}$ belongs to $L^p(B_{S_1}^{\mathrm{c}})^3$ for $p > 4/3$, and $\partial_j \mathfrak{G}|B_{S_1}^{\mathrm{c}}$ to $L^p(B_{S_1}^{\mathrm{c}})^3$ for $p > 1,\ 1 \le j \le 3$ (Lemma 9). Thus our theory covers the one in [52] as a special case.

We remark that in the case of a rigid body which only rotates but does not translate, more detailed asymptotic expansions are available [17–19]. Any reader interested in further results on the asymptotic behavior of viscous incompressible flow around rotating bodies is referred to [3–6,8,13–16,20–24,27,28,30–45,47–51,54–56,60].

2. Notation, definition of fundamental solutions, auxiliary results

By $|\ |$ we denote the Euclidean norm in $\mathbb{R}^3$ and the length $\alpha_1+\alpha_2+\alpha_3$ of a multi-index $\alpha \in \mathbb{N}_0^3$. Put $e_1 := (1,0,0)$. For $r > 0$, we set $B_r := \{y \in \mathbb{R}^3 \ :\ |y| < r\}$. If $A \subset \mathbb{R}^3$, we put $A^{\mathrm{c}} := \mathbb{R}^3 \backslash A$. Recall the abbreviation $s(x) := 1 + |x| - x_1$ ($x \in \mathbb{R}^3$) introduced in Sect. 1.

If $A \subset \mathbb{R}^3$ is open, $p \in [1,\infty)$ and $k \in \mathbb{N}$, we write $W^{k,p}(A)$ for the usual Sobolev space of order k and exponent p. If $B \subset \mathbb{R}^3$ is again an open set, we define $L^p_{\mathrm{loc}}(B)$, $W^{k,p}_{\mathrm{loc}}(B)$ as the set of all functions

$v : B \mapsto \mathbb{R}$ such that $v|U \in L^p(U)$ and $v|U \in W^{k,p}(U)$, respectively, for any open bounded set $U \subset \mathbb{R}^3$ with $\overline{U} \subset B$. We write $\mathfrak{S}(\mathbb{R}^3)$ for the usual space of rapidly decreasing functions in $\mathbb{R}^3$; see [53, p. 138] for example. For the Fourier transform $\widehat{g}$ of a function $g \in L^1(\mathbb{R}^3)$, we choose the definition $\widehat{g}(\xi) := (2\pi)^{-3/2} \int_{\mathbb{R}^3} \mathrm{e}^{-i\,\xi\,x}\, g(x)\ \mathrm{d}x\ (\xi \in \mathbb{R}^3)$. This fixes the definition of the Fourier transform of a tempered distribution as well.

The numbers $\tau \in (0,\infty)$ and $\varrho \in \mathbb{R}\backslash\{0\}$ introduced in Sect. 1 will be kept fixed throughout. We introduce a matrix $\Omega \in \mathbb{R}^{3\times 3}$ by setting

$$\Omega := \varrho \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Note that $\varrho\, e_1 \times x = \Omega \cdot x$ for $x \in \mathbb{R}^3$. We write $\mathcal{C}$ for positive constants that may depend on τ or ϱ. Constants additionally depending on parameters $\sigma_1, \ldots, \sigma_n \in (0,\infty)$ for some $n \in \mathbb{N}$ are denoted by $\mathcal{C}(\sigma_1, \ldots, \sigma_n)$. We state some inequalities involving $s(x)$ or $x - \tau\, t\, e_1$.

Lemma 1. [2, Lemma 4.8] *$s(x-y)^{-1} \le \mathcal{C}\,(1+|y|)\, s(x)^{-1}$ for $x, y \in \mathbb{R}^3$.*

Lemma 2. [1, Lemma 2] *For $x \in \mathbb{R}^3$, $t \in (0,\infty)$, we have*

$$|x - \tau\, t\, e_1|^2 + t \ge \mathcal{C}\left[\, \chi_{[0,1]}(|x|)\,(|x|^2 + t) + \chi_{(1,\infty)}(|x|)\,\big(\,|x|\, s(x) + t\,\big)\,\right].$$

Lemma 3. [12, Lemma 2.3] *Let $\beta \in (1,\infty)$. Then $\int_{\partial B_r} s(x)^{-\beta}\, \mathrm{d}o_x \le \mathcal{C}(\beta)\, r$ for $r \in (0,\infty)$.*

Theorem 2.1. [4, Theorem 2.19] *Let $R_1, R_2 \in (0,\infty)$ with $R_1 < R_2$, $\nu \in (1,\infty)$. Then for $y \in B^{\mathrm{c}}_{R_2}$, $z \in B_{R_1}$,*

$$\int_0^\infty (|y - \tau\, t\, e_1 - \mathrm{e}^{-t\,\Omega} \cdot z|^2 + t)^{-\nu}\, \mathrm{d}t \le \mathcal{C}(R_1, R_2, \nu)\,\big(\,|y|\, s(y)\,\big)^{-\nu+1/2}.$$

Theorem 2.2. *Let $R \in (0,\infty)$. Then for $k \in \{0, 1\}$, $x, y \in B_R$ with $x \ne y$,*

$$\int_0^\infty \big(\, |x - \tau\, t\, e_1 - \mathrm{e}^{-t\cdot\Omega} \cdot y|^2 + t\big)^{-3/2-k/2}\, \mathrm{d}t \le C(R)\, |x-y|^{-1-k}.$$

Proof. See the last part of the proof of [3, Theorem 3.1]. Note that in [3, (3.7)] it should read $y + t\,U - \mathrm{e}^{-t\,\Omega} \cdot z$ instead of x. □

The next lemma is well known. It was already used in [19], for example. For the convenience of the reader, we give a proof.

Lemma 4. *Let $t \in \mathbb{R}$. Then* $\mathrm{e}^{t\,\Omega} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t\,\varrho) & -\sin(t\,\varrho) \\ 0 & \sin(t\,\varrho) & \cos(t\,\varrho) \end{pmatrix}$.

Proof. Put $\Omega' := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T := \begin{pmatrix} 1 & 1 \\ -i & -i \end{pmatrix}$, $A := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Obviously $\mathrm{e}^{t\,A} = \begin{pmatrix} \mathrm{e}^{t\,i} & 0 \\ 0 & \mathrm{e}^{-t\,i} \end{pmatrix}$. But $T \cdot A \cdot T^{-1} = \Omega'$, so $T \cdot \mathrm{e}^{t\,A} \cdot T^{-1} = \mathrm{e}^{t\,\Omega'}$. On computing the elements of the matrix on the left-hand side of the preceding equation, we obtain the lemma. □

Next we introduce some fundamental solutions. Put

$$N(x) := (4\,\pi\,|x|)^{-1} \quad \text{for} \quad x \in \mathbb{R}^3\backslash\{0\}$$

("Newton potential", fundamental solution of the Poisson equation in $\mathbb{R}^3$),

$$\mathfrak{O}(x) := (4\,\pi\,|x|)^{-1}\,\mathrm{e}^{-\tau\,(|x|-x_1)/2} \quad \text{for} \quad x \in \mathbb{R}^3\backslash\{0\}$$

(fundamental solution of the scalar Oseen equation $-\Delta v + \tau\,\partial_1 v = g$ in $\mathbb{R}^3$),

$$\mathfrak{O}^{(\lambda)}(x) := (4\,\pi\,|x|)^{-1}\,\mathrm{e}^{-\sqrt{\lambda+\tau^2/4}\,|x|+\tau\,x_1/2} \quad \text{for} \quad x \in \mathbb{R}^3\backslash\{0\}, \quad \lambda \in (0,\infty)$$

(fundamental solution of the scalar Oseen resolvent equation $-\Delta v + \tau\,\partial_1 v + \lambda\, v = g$ in $\mathbb{R}^3$),

$$K(x,t) := (4\,\pi\,t)^{-3/2}\,\mathrm{e}^{-|x|^2/(4\,t)} \quad \text{for} \quad x \in \mathbb{R}^3, \quad t \in (0,\infty)$$

(fundamental solution of the heat equation in $\mathbb{R}^3$),

$$\psi(r) := \int_0^r (1-\mathrm{e}^{-t})\,t^{-1}\,\mathrm{d}t \quad (r \in \mathbb{R}), \quad \Phi(x) := (4\,\pi\,\tau)^{-1}\,\psi\big(\tau\,(|x|-x_1)/2\big) \quad (x \in \mathbb{R}^3),$$

$$E_{jk}(x) := (\delta_{jk}\,\Delta - \partial_j\partial_k)\Phi(x), \quad E_{4k}(x) := x_k\,(4\,\pi\,|x|^3)^{-1} \quad (x \in \mathbb{R}^3\backslash\{0\},\ 1 \le j,k \le 3)$$

(fundamental solution of the Oseen system (1.5), with $(E_{jk})_{1\le j,k\le 3}$ the velocity part and $(E_{4k})_{1\le k\le 3}$ the pressure part). We further define

$$F^{(\lambda)}(\xi) := (2\,\pi)^{-3/2}\,(\lambda + |\xi|^2 + i\,\tau\,\xi_1)^{-1} \quad \text{for} \quad \xi \in \mathbb{R}^3, \quad \lambda \in (0,\infty)$$

(Fourier transform of $\mathfrak{O}^{(\lambda)}$; see Theorem 4.1).

We recall some basic properties of these functions, beginning with a classical result.

Lemma 5. *Let* $f \in \mathfrak{S}(\mathbb{R}^3)$ *and put* $\mathfrak{N}(f)(x) := \int_{\mathbb{R}^3} N(x-y)\,f(y)\,\mathrm{d}y$ *for* $x \in \mathbb{R}^3$. *Then* $\mathfrak{N}(f) \in C^\infty(\mathbb{R}^3)$ *and* $\partial^\alpha \mathfrak{N}(f)(x) = \int_{\mathbb{R}^3} N(x-y)\,\partial^\alpha f(y)\,\mathrm{d}y$ *for* $x \in \mathbb{R}^3, \alpha \in \mathbb{N}_0^3$.

Lemma 6. [11] $K \in C^\infty\big(\mathbb{R}^3 \times (0,\infty)\big)$ *and*

$$|\partial_t^l \partial_x^\alpha K(x,t)| \le \mathcal{C}(\alpha,l)\,(|x|^2+t)^{-3/2-|\alpha|/2-l}\,\mathrm{e}^{-|x|^2/(8\,t)}$$

for $x \in \mathbb{R}^3,\ t \in (0,\infty),\ \alpha \in \mathbb{N}_0^3,\ l \in \mathbb{N}_0$. *In particular* $K(\,\cdot\,,t) \in L^1(\mathbb{R}^3) \cap \mathfrak{S}(\mathbb{R}^3)$ *for* $t > 0$.

Theorem 2.3. [46] $E_{jk} \in C^\infty(\mathbb{R}^3\backslash\{0\})$ *and*

$$|\partial^\alpha E_{jk}(x)| \le \mathcal{C}\,\big(\,|x|\,s(x)\,\big)^{-1-|\alpha|/2}\,\max\{1,\,|x|^{-|\alpha|/2}\}$$

for $x \in \mathbb{R}^3\backslash\{0\},\ 1 \le j,k \le 3,\ \alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$.

As a consequence of Theorem 2.3, we have $E_{jk} \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, and $E_{jk}|B_1^{\mathrm{c}}$ is bounded $(1 \le j,k \le 3)$. Analogous properties are obvious for N, $\mathfrak{O}$ and $\mathfrak{O}^{(\lambda)}$. Moreover $|\Phi(x)| \le \mathcal{C}\,(1+|x|)$ $(x \in \mathbb{R}^3)$. In view of these observations, the Fourier transforms of these functions will be considered as tempered distributions (which, of course, will turn out to be represented by functions). Following Solonnikov [58, (40)], we use Lemmas 5 and 6 to introduce the velocity part $(T_{jk})_{1\le j,k\le 3}$ of a fundamental solution of the time-dependent Stokes system, setting

$$T_{jk}(x,t) := \delta_{jk}\,K(x,t) + \partial_j\partial_k\left(\int_{\mathbb{R}^3} N(x-y)\,K(y,t)\,\mathrm{d}y\right) \quad (x \in \mathbb{R}^3,\ t > 0,\ 1 \le j,k \le 3).$$

Lemma 7. [58, Lemma 13], [57] $T_{jk} \in C^\infty\big(\mathbb{R}^3 \times (0,\infty)\big)$ *and*

$$|\partial_t^l \partial_x^\alpha T_{jk}(x,t)| \le \mathcal{C}(\alpha,l)\,(|x|^2+t)^{-3/2-|\alpha|/2-l}$$

for $x \in \mathbb{R}^3,\ t \in (0,\infty),\ 1 \le j,k \le 3,\ \alpha \in \mathbb{N}_0^3,\ l \in \mathbb{N}_0$.

Lemma 7 yields that $T_{jk}(\,\cdot\,,t) \in L^2(\mathbb{R}^3)$, but does not imply $T_{jk}(\,\cdot\,,t) \in L^1(\mathbb{R}^3) \quad (t > 0)$. So the Fourier transform of this function should be understood either as a transform of an L^2-function or as a tempered distribution. For us it will be convenient to use the second possibility. Put

$$\Gamma(x,y,t) := T(x - \tau\, t\, e_1 - \mathrm{e}^{-t\,\Omega} \cdot y,\, t) \cdot \mathrm{e}^{-t\,\Omega} \quad \text{for} \quad x, y \in \mathbb{R}^3,\ t > 0. \tag{2.1}$$

The matrix-valued function Γ (not to confuse with the usual Gamma function) is the velocity part of a fundamental solution to the time-dependent variant of the linearization (1.8) of (1.1). This fundamental solution was constructed by Guenther, Thomann [60] via a procedure involving Kummer functions, an approach also used in [3–9]. However, Guenther, Thomann [60, (3.9)] showed that Γ is given by (2.1) as well, thus providing an access to this function which is more convenient in many respects. For example, from Lemma 7 and (2.1), we immediately obtain

Corollary 2.1. *Let $j, k \in \{1, 2, 3\}$. Then $\Gamma_{jk} \in C^\infty\big(\mathbb{R}^3 \times \mathbb{R}^3 \times (0,\infty)\big)$ and*

$$|\partial_x^\alpha \Gamma_{jk}(x,y,t)| \le \mathcal{C}(\alpha)\,(|x - \tau\,t\,e_1 - \mathrm{e}^{-t\,\Omega}\cdot y|^2 + t)^{-3/2-|\alpha|/2}$$

for $x, y \in \mathbb{R}^3,\ t \in (0,\infty),\ \alpha \in \mathbb{N}_0^3$.

By Theorem 2.2 and Corollary 2.1, we have $\int_0^\infty |\Gamma_{jk}(x,y,t)|\,\mathrm{d}t < \infty$ for $x, y \in \mathbb{R}^3$ with $x \ne y,\ 1 \le j, k \le 3$, so we may define

$$Z(x,y) := \int_0^\infty \Gamma(x,y,t)\,\mathrm{d}t \quad \text{for} \quad x, y \in \mathbb{R}^3 \text{ with } x \ne y.$$

This function Z was introduced on [60, p. 96] as the velocity part of a fundamental solution to (1.8). We collect the properties of Z that will be needed in what follows.

Lemma 8. [4, Lemma 2.15] $Z \in C^1\big((\mathbb{R}^3 \times \mathbb{R}^3)\backslash \mathrm{diag}\,(\mathbb{R}^3 \times \mathbb{R}^3)\big)^{3\times 3}, \quad \partial x_l Z(x,y) = \int_0^\infty \partial x_l \Gamma(x,y,t)\,\mathrm{d}t$ *for $x, y \in \mathbb{R}^3$ with $x \ne y,\ 1 \le l \le 3$.*

Note that due to Theorem 2.2 and Corollary 2.1, we have $\int_0^\infty |\partial x_l \Gamma(x,y,t)|\,\mathrm{d}t < \infty$ for x, y, l as in Lemma 8.

Corollary 2.2. *Let $R_1,\ R_2 \in (0,\infty)$ with $R_1 < R_2$. Then*

$$|\partial_x^\alpha Z(x,y)| \le \mathcal{C}(R_1,R_2)\,\big(|x|\,s(x)\big)^{-1-|\alpha|/2} \quad \textit{for} \quad x \in B_{R_2}^{\mathrm{c}},\ y \in B_{R_1},\ \alpha \in \mathbb{N}_0^3 \textit{ with } |\alpha| \le 1.$$

Proof. Lemma 8, Corollary 2.1, Theorem 2.1. □

Corollary 2.3. *The function $Z(\,\cdot\,,0)$ belongs to $C^1(\mathbb{R}^3\backslash\{0\})^{3\times 3}$.*
Let $S \in (0,\infty)$. Then $|\partial_x^\alpha Z(x,0)| \le \mathcal{C}(S)\,\big(|x|\,s(x)\big)^{-1-|\alpha|/2}$ for $x \in B_S^{\mathrm{c}},\ \alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$.
Moreover $|Z(x,0)| \le \mathcal{C}\,|x|^{-1}$ for $x \in B_1\backslash\{0\}$.

Proof. The first two claims of the corollary follow from Lemma 8 and Corollary 2.2. The last estimate is a consequence of Corollary 2.1 and Theorem 2.2. □

Corollary 2.3 justifies to introduce the Fourier transform of $Z(\,\cdot\,,0)$ in the sense of a tempered distribution.

3. Statement of our main result

It will be convenient to first recall the main result from [9].

Theorem 3.1. [9, Theorem 3.1] *Let $\mathcal{B} \subset \mathbb{R}^3$ be open, $p \in (1,\infty)$, $f \in L^p(\mathbb{R}^3)^3$ with supp(f) compact. Let $S_1 \in (0,\infty)$ with $\overline{\mathcal{B}} \cup supp(f) \subset B_{S_1}$.*

Let $u \in L^6(\overline{\mathcal{B}}^c)^3 \cap W^{1,1}_{\mathrm{loc}}(\overline{\mathcal{B}}^c)^3$, $\pi \in L^2_{\mathrm{loc}}(\overline{\mathcal{B}}^c)$ with $\nabla u \in L^2(\overline{\mathcal{B}}^c)^9$, $div\, u = 0$ and

$$\int_{\overline{\mathcal{B}}^c} \Big[\nabla u \cdot \nabla \varphi \;+\; \big(\tau\,\partial_1 u + \tau\,(u\cdot\nabla)u - \varrho\,(e_1 \times z)\cdot \nabla u + \varrho\, e_1 \times u \big)\cdot \varphi - \pi\; div\; \varphi \Big]\,\mathrm{d}z$$
$$= \int_{\overline{\mathcal{B}}^c} f\cdot\varphi\,\mathrm{d}z \quad for \quad \varphi \in C_0^\infty(\overline{\mathcal{B}}^c)^3.$$

(This means the pair (u,π) is a Leray solution to (1.1), (1.2).) Suppose in addition that

$$\mathcal{B} \text{ is } C^2\text{-bounded}, \quad u|\partial\mathcal{B} \in W^{2-1/p,\,p}(\partial\mathcal{B})^3, \quad \pi|B_{S_1}\backslash\overline{\mathcal{B}} \in L^p(B_{S_1}\backslash\overline{\mathcal{B}}). \tag{3.1}$$

Let n denote the outward unit normal to $\mathcal{B}$, and define

$$\beta_k := \int_{\overline{\mathcal{B}}^c} f_k(y)\,\mathrm{d}y + \int_{\partial\mathcal{B}} \sum_{l=1}^{3} \big(-\partial_l u_k(y) + \delta_{kl}\,\pi(y) + (\tau\, e_1 - \varrho\, e_1 \times y)_l\, u_k(y) + \tau\,(u_l\, u_k)(y)\big)\, n_l(y)\,\mathrm{d}o_y$$

for $1 \le k \le 3$,

$$\begin{aligned}\mathfrak{F}_j(x) := &\int_{\overline{\mathcal{B}}^c} \Big[\sum_{k=1}^{3} \big(Z_{jk}(x,y) - Z_{jk}(x,0)\big)\, f_k(y) - \tau\cdot \sum_{k,l=1}^{3} Z_{jk}(x,y)\,(u_l\,\partial_l u_k)(y)\Big]\,\mathrm{d}y\\ &+ \int_{\partial\mathcal{B}} \sum_{k=1}^{3} \Big[\big(Z_{jk}(x,y) - Z_{jk}(x,0)\big) \sum_{l=1}^{3} \big(-\partial_l u_k(y) + \delta_{kl}\,\pi(y)\\ &\quad + (\tau\, e_1 - \varrho\, e_1 \times y)_l\, u_k(y)\big)\, n_l(y) \;+\; \big(E_{4j}(x-y) - E_{4j}(x)\big)\, u_k(y)\, n_k(y)\\ &\quad + \sum_{l=1}^{3} \big(\partial y_l Z_{jk}(x,y)\,(u_k\, n_l)(y) - \tau Z_{jk}(x,0)\,(u_l\, u_k\, n_l)(y)\big)\Big]\,\mathrm{d}o_y\end{aligned}$$

for $x \in \overline{B_{S_1}}^c$, $1 \le j \le 3$. The preceding integrals are absolutely convergent. Moreover $\mathfrak{F} \in C^1(\overline{B_{S_1}}^c)^3$ and Eq. (1.7) holds. In addition, for any $S \in (S_1,\infty)$, there is a constant $C > 0$ which depends on τ, ϱ, S_1, S, f, u and π, and which is such that

$$|\partial^\alpha \mathfrak{F}(x)| \le C\,\big(|x|\, s(x)\big)^{-3/2-|\alpha|/2}\, \ln(2+|x|) \quad for \quad x \in \overline{B_S}^c,\ \alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| \le 1.$$

In the preceding theorem, the coefficients β_1, β_2, β_3 and the function $\mathfrak{F}$ are defined in terms of integrals on $\partial\mathcal{B}$ and $\overline{\mathcal{B}}^c$. The integral over $\partial\mathcal{B}$ may allow to exploit boundary conditions verified by u or π. However, this way of introducing β_1, β_2, β_3 and $\mathfrak{F}$ requires the additional assumptions imposed on $\mathcal{B}$, u and π in (3.1). If boundary conditions on $\partial\mathcal{B}$ do not matter, we may drop (3.1) and consider $(u|\overline{B_{S_0}}^c, \pi|\overline{B_{S_0}}^c)$ instead of (u,π), where S_0 may be any number from $(0,S_1)$ with $\overline{\mathcal{B}} \cup \mathrm{supp}(f) \subset B_{S_0}$. In view of interior regularity of u and π, we may then define the coefficients β_k and the functions $\mathfrak{F}$ in terms of integrals over ∂B_{S_0} and $\overline{B_{S_0}}^c$, obtaining an analogous result as the one in Theorem 3.1, but with B_{S_0} in the role of $\mathcal{B}$. Below we will present a variant of this idea which takes account of the results in the work at hand (Corollary 3.1).

The principal aim of this article consists in improving Theorem 3.1 in the way specified in

Theorem 3.2. *Let $\mathcal{B}$, p, f, S_1, u, π satisfy the assumptions of* Theorem 3.1*, including* (3.1)*. Let β_1, β_2, β_3 and $\mathfrak{F}$ be defined as in* Theorem 3.1*. Define the function $\mathfrak{G}$ as in* (1.11)*.*

Then $\mathfrak{G} \in C^1(\overline{B_{S_1}}^c)^3$, Eq. (1.12) *holds, and for any $S \in (S_1, \infty)$, there is a constant $C > 0$ which depends on τ, ϱ, S_1, S, f, u and π, and which is such that*

$$|\partial^\alpha \mathfrak{G}(x)| \le C \left(|x|\, s(x) \right)^{-3/2-|\alpha|/2} \ln(2+|x|) \quad \textit{for} \quad x \in \overline{B_S}^c,\ \alpha \in \mathbb{N}_0^3 \quad \textit{with} \quad |\alpha| \le 1.$$

We recall that the asymptotic behavior of the function E appearing in the leading term in (1.12) is described in Theorem 2.3. As explained above, we may drop the assumptions in (3.1) if we replace (u, π) by $(u|\overline{B_{S_0}}^c, \pi|\overline{B_{S_0}}^c)$, with some suitably chosen number S_0. Here are the details.

Corollary 3.1. *Take $\mathcal{B}$, p, f, S_1, u, π as in* Theorem 3.1*, but without requiring* (3.1)*. (This means that (u, π) is only assumed to be a Leray solution of* (1.1), (1.2)*). Put $\widetilde{p} := \min\{3/2, p\}$.*

Then $u \in W^{2,\widetilde{p}}_{\mathrm{loc}}(\overline{\mathcal{B}}^c)^3$ and $\pi \in W^{1,\widetilde{p}}_{\mathrm{loc}}(\overline{\mathcal{B}}^c)$.

Fix some number $S_0 \in (0, S_1)$ with $\overline{\mathcal{B}} \cup supp(f) \subset B_{S_0}$, and define β_1, β_2, β_3 and $\mathfrak{F}$ as in Theorem 3.1*, but with $\mathcal{B}$ replaced by B_{S_0}, and $n(x)$ by $S_0^{-1} x$, for $x \in \partial B_{S_0}$. Moreover, define $\mathfrak{G}$ as in* (1.11)*.*

Then all the conclusions of Theorem 3.2 *are valid.*

4. Some Fourier transforms

In this section we show that $Z_{j1}(\,\cdot\,, 0) = E_{j1}$. To this end, we prove that the Fourier transforms of these two functions coincide. To begin with, we recall some well-known facts about the Fourier transforms of some of the fundamental solutions introduced in Sect. 2. Other intermediate results in this section may also be well known (Corollary 4.2 for example), but since their proofs are very short, we present them for completeness.

Theorem 4.1. *For $\xi \in \mathbb{R}^3 \backslash \{0\}$, we have $\widehat{N}(\xi) = (2\,\pi)^{-3/2}\, |\xi|^{-2}$. If $f \in \mathfrak{S}(\mathbb{R}^3)$ and $\mathfrak{N}(f)(x) := \int_{\mathbb{R}^3} N(x - y)\, f(y)\, \mathrm{d}y$ for $x \in \mathbb{R}^3$, then $\widehat{\mathfrak{N}(f)}(\xi) = |\xi|^{-2}\, \widehat{f}(\xi)$ for ξ as above.*

Moreover $\left[K(\,\cdot\,, t) \right]^\wedge(\xi) = (2\,\pi)^{-3/2}\, \mathrm{e}^{-t\,|\xi|^2}$ and $\widehat{\mathfrak{O}^{(\lambda)}}(\xi) = F^{(\lambda)}(\xi)$ for $t \in (0, \infty)$, $\xi \in \mathbb{R}^3$ and $\lambda \in (0, \infty)$.

Proof. For the first formula, the reader may consult [53, Proposition 2.1.1] and its proof. The second equation follows from the first by a well-known formula for the Fourier transform of a convolution. As a direct reference we mention [59, Lemma V.1.1]. The third equation is well known, and as concerns the forth, we refer to [10, Theorem 2.1]. □

Corollary 4.1. $\widehat{\mathfrak{O}}(z) = (2\,\pi)^{-3/2}\, (i\,\tau\, z_1 + |z|^2)^{-1}$ *and* $\int_0^\infty K(z - \tau\, t\, e_1,\, t)\, \mathrm{d}t = \mathfrak{O}(z)$ *for* $z \in \mathbb{R}^3 \backslash \{0\}$.

Proof. Let $\varphi \in \mathfrak{S}(\mathbb{R}^3)$. For $n \in \mathbb{N}$, $\xi \in \mathbb{R}^3$, we have $|F^{(1/n)}(\xi)\, \varphi(\xi)| \le \mathcal{C}\, |\xi|^{-2}\, |\varphi(\xi)|$. But $\int_{\mathbb{R}^3} |\xi|^{-2}\, |\varphi(\xi)|\, d\xi < \infty$, because φ is rapidly decreasing. Thus we get from Lebesgue's theorem

$$\mathfrak{A} := (2\,\pi)^{-3/2} \int_{\mathbb{R}^3} (i\,\tau\,\xi_1 + |\xi|^2)^{-1}\, \varphi(\xi)\, \mathrm{d}\xi = \lim_{n\to\infty} \int_{\mathbb{R}^3} F^{(1/n)}(\xi)\, \varphi(\xi)\, \mathrm{d}\xi.$$

Due to the last equation in Theorem 4.1, we may conclude

$$\mathfrak{A} = \lim_{n\to\infty} \int_{\mathbb{R}^3} \mathfrak{O}^{(1/n)}(x)\,\widehat{\varphi}(x)\,\mathrm{d}x. \tag{4.1}$$

But $|\mathfrak{O}^{(1/n)}(x)\,\widehat{\varphi}(x)| \le \mathcal{C}\,|x|^{-1}\,|\widehat{\varphi}(x)|$ for $n \in \mathbb{N},\ x \in \mathbb{R}^3\backslash\{0\}$, with $\int_{\mathbb{R}^3} |x|^{-1}\,|\widehat{\varphi}(x)|\,\mathrm{d}x < \infty$ because φ hence $\widehat{\varphi}$ is rapidly decreasing. Thus Eq. (4.1) and Lebesgue's theorem yield $\mathfrak{A} = \int_{\mathbb{R}^3} \mathfrak{O}(x)\,\widehat{\varphi}(x)\,\mathrm{d}x$. Since this is true for any $\varphi \in \mathfrak{S}(\mathbb{R}^3)$, the first equation in the corollary follows. The second is a consequence of the first and the formula for $\big[\,K(\,\cdot\,,t)\,\big]^\wedge$ in Theorem 4.1. □

Corollary 4.2. *Let $t \in (0,\infty),\ j,k \in \{1,\,2,\,3\}$. Then*

$$[T_{jk}(\,\cdot\,,t)]^\wedge(\xi) = (2\,\pi)^{-3/2}\,(\delta_{jk} - \xi_j\,\xi_k\,|\xi|^{-2})\,\mathrm{e}^{-t\,|\xi|^2} \quad \textit{for} \quad \xi \in \mathbb{R}^3\backslash\{0\}.$$

Proof. We have $K(\,\cdot\,,t) \in \mathfrak{S}(\mathbb{R}^3)$ (Lemma 6). Therefore by Lemma 5,

$$T_{jk}(x,t) = \delta_{jk}\,K(x,t) + \int_{\mathbb{R}^3} N(x-y)\,\partial_j\partial_k K(y,t)\,\mathrm{d}y \quad (x \in \mathbb{R}^3).$$

Since $K(\,\cdot\,,t)$ belongs to $\mathfrak{S}(\mathbb{R}^3)$ hence $\partial_j\partial_k K(\,\cdot\,,t)$ does, too, Corollary 4.2 follows from Theorem 4.1. □

Corollary 4.3. *Let $j \in \{1,\,2,\,3\},\ t \in (0,\infty)$. Then*

$$[\Gamma_{j1}(\,\cdot\,,0,t)]^\wedge(\xi) = (2\,\pi)^{-3/2}\,(\delta_{j1} - \xi_j\,\xi_1\,|\xi|^{-2})\,\mathrm{e}^{-t\,(i\,\tau\,\xi_1 + |\xi|^2)} \quad \textit{for} \quad \xi \in \mathbb{R}^3\backslash\{0\}.$$

Proof. By Lemma 4, we have $\Gamma_{j1}(x,0,t) = \big(T(x - \tau\,t\,e_1,\,t)\,\mathrm{e}^{-t\,\Omega}\big)_{j1} = T_{j1}(x - \tau\,t\,e_1,\,t)$, so Corollary 4.3 follows from Corollary 4.2. □

Corollary 4.4. *Let $j \in \{1,\,2,\,3\},\ t \in (0,\infty)$. Then*

$$[Z_{j1}(\,\cdot\,,0)]^\wedge(\xi) = (2\,\pi)^{-3/2}\,(\delta_{j1} - \xi_j\,\xi_1\,|\xi|^{-2})\,(i\,\tau\,\xi_1 + |\xi|^2)^{-1} \quad \textit{for} \quad \xi \in \mathbb{R}^3\backslash\{0\}.$$

Proof. Let $\varphi \in \mathfrak{S}(\mathbb{R}^3)$. With Corollary 2.1, we get

$$A := \int_{\mathbb{R}^3}\int_0^\infty |\Gamma_{j1}(x,0,t)\,\widehat{\varphi}(x)|\,\mathrm{d}t\,\mathrm{d}x \le \mathcal{C}\int_{\mathbb{R}^3}\int_0^\infty (|x - \tau\,t\,e_1|^2 + t)^{-3/2}\,|\widehat{\varphi}(x)|\,\mathrm{d}t\,\mathrm{d}x.$$

By Lemma 2 and because $\hat{\varphi}$ belongs to $\mathfrak{S}(\mathbb{R}^3)$, we get that A is bounded by

$$\mathcal{C}\left(\int_{\mathbb{R}^3}\int_1^\infty t^{-3/2}\,|\widehat{\varphi}(x)|\,\mathrm{d}t\,\mathrm{d}x + \int_{B_1}\int_0^1 |x|^{-3/2}\,t^{-3/4}|\widehat{\varphi}(x)|\,\mathrm{d}t\,\mathrm{d}x + \int_{B_1^c}\int_0^1 |\widehat{\varphi}(x)|\,\mathrm{d}t\,\mathrm{d}x\right),$$

and hence $A < \infty$. Therefore, we may apply Fubini's theorem, to obtain

$$\begin{aligned}\int_{\mathbb{R}^3} Z_{j1}(x,0)\,\widehat{\varphi}(x)\,\mathrm{d}x &= \int_0^\infty\int_{\mathbb{R}^3} \Gamma_{j1}(x,0,t)\,\widehat{\varphi}(x)\,\mathrm{d}x\,\mathrm{d}t,\\ &= \int_0^\infty\int_{\mathbb{R}^3} (2\,\pi)^{-3/2}\,(\delta_{j1} - \xi_j\,\xi_1\,|\xi|^{-2})\,\mathrm{e}^{-t\,(i\,\tau\,\xi_1 + |\xi|^2)}\,\varphi(\xi)\,\mathrm{d}\xi\,\mathrm{d}t,\end{aligned}$$

where the last equation follows from Corollary 4.3. But

$$\int_0^\infty\int_{\mathbb{R}^3} |(\delta_{j1} - \xi_j\,\xi_1|\xi|^{-2})\,\mathrm{e}^{-t\,(i\,\tau\,\xi_1 + |\xi|^2)}\varphi(\xi)|\,\mathrm{d}\xi\,\mathrm{d}t \le \mathcal{C}\int_0^\infty\int_{\mathbb{R}^3} \mathrm{e}^{-t\,|\xi|^2}|\varphi(\xi)|\,\mathrm{d}\xi\,\mathrm{d}t < \infty.$$

Thus we may use Fubini's theorem again, arriving at the equation

$$\int_{\mathbb{R}^3} Z_{j1}(x,0)\,\widehat{\varphi}(x)\,\mathrm{d}x = \int_{\mathbb{R}^3} (2\,\pi)^{-3/2}\,(\delta_{j1} - \xi_j\,\xi_1\,|\xi|^{-2})\,(i\,\tau\,\xi_1 + |\xi|^2)^{-1}\,\varphi(\xi)\,\mathrm{d}\xi.$$

This proves Corollary 4.4. □

Theorem 4.2. *Let $j,k \in \{1,\,2,\,3\}$. Then for $\xi \in \mathbb{R}^3\backslash\{0\}$,*

$$\widehat{E}_{jk}(\xi) = (2\,\pi)^{-3/2}\,(\delta_{jk} - \xi_j\,\xi_j\,|\xi|^{-2})\,(i\,\tau\,\xi_1 + |\xi|^2)^{-1}.$$

Proof. For $x \in \mathbb{R}^3\backslash\{0\}$, we find

$$\begin{aligned}\partial_1\Phi(x) &= (4\,\pi\,\tau)^{-1}\,\psi'\big(\,\tau\,(|x| - x_1)/2\,\big)\,\tau\,(x_1/|x| - 1)/2 = (4\,\pi\,\tau\,|x|)^{-1}\,(\mathrm{e}^{-\tau\,(|x|-x_1)/2} - 1)\\ &= \tau^{-1}\big(\,\mathfrak{O}(x) - N(x)\,\big).\end{aligned}$$

Hence with Corollary 4.1 and Theorem 4.1, for $\xi \in \mathbb{R}^3\backslash\{0\}$,

$$\begin{aligned} i\,\xi_1\,\widehat{\Phi}(\xi) = \widehat{\partial_1\Phi}(\xi) &= \tau^{-1}\,(2\,\pi)^{-3/2}\,\big(\,(i\,\tau\,\xi_1 + |\xi|^2)^{-1} - |\xi|^{-2}\,\big)\\ &= -i\,(2\,\pi)^{-3/2}\,\xi_1\,\big(\,(i\,\tau\,\xi_1 + |\xi|^2)\,|\xi|^2\,\big)^{-1}.\end{aligned}$$

As a consequence $\widehat{\Phi}(\xi) = -(2\,\pi)^{-3/2}\,\big(\,(i\,\tau\,\xi_1 + |\xi|^2)\,|\xi|^2\,\big)^{-1}$, so the theorem follows by the definition of E_{jk}. □

Theorem 4.2 may be deduced also from the results in [25, Chapter VII]. In fact, it is shown in [25, Section VII.3] that the convolution $\mathfrak{O} * f$, for $f \in C_0^\infty(\mathbb{R}^3)^3$, belongs to $C^\infty(\mathbb{R}^3)^3$ and is the velocity part of a solution to the Oseen system (1.5) in $\mathbb{R}^3$. On the other hand, by [25, Section VII.4], the inverse Fourier transform of the function $(2\,\pi)^{-3/2}\,(i\,\tau\,\xi_1 + |\xi|^2)^{-1}\,\widehat{f}(\xi)\,(\delta_{jk} - \xi_j\,\xi_k\,|\xi|^{-2})_{1\le j,k\le 3}$ also solves (1.5) in $\mathbb{R}^3$, and belongs to certain Sobolev spaces. A uniqueness result would yield that the two solutions coincide, implying Theorem 4.2. However, we prefer to carry out a direct proof of this theorem, instead of relying on the rather lengthy theory in [25, Chapter VII], which in fact yields much stronger results, not needed here, than Theorem 4.2.

Combining Theorem 4.2 and Corollary 4.4, we arrive at the main result of this section.

Corollary 4.5. $Z_{j1}(\,\cdot\,,0) = E_{j1}$ *for* $1 \le j \le 3$.

5. Proof of Theorem 3.2 and Corollary 3.1

We first show that in the case $k \in \{2,\,3\}$, the function $\partial^\alpha_{jk} Z(\cdot,0)$ decays faster for $|x| \to \infty$ than indicated by Corollary 2.3.

Theorem 5.1. *Let $S \in [2\,\tau\,\pi/|\varrho|,\,\infty)$. Then $|\partial^\alpha_x Z_{jk}(x,0)| \le \mathcal{C}(S)\,\big(\,|x|\,s(x)\,\big)^{-3/2-|\alpha|/2}$ for $x \in B^{\mathrm{c}}_{S+\tau\,\pi/|\varrho|}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1,\ j \in \{1,\,2,\,3\},\ k \in \{2,\,3\}$.*

Proof. Take x, α, j, k as in the theorem. We get with Lemma 8 that

$$\partial^\alpha_x Z_{jk}(x,0) = \int_0^\infty \partial^\alpha_x \Gamma_{jk}(x,0,t)\,\mathrm{d}t = \int_0^\infty \big[\,\partial^\alpha_x T(x - \tau\,t\,e_1,\,t)\cdot \mathrm{e}^{-t\,\Omega}\,\big]_{jk}\,\mathrm{d}t,$$

so with Lemma 4 in the case $k = 2$,

$$\partial^\alpha_x Z_{jk}(x,0) = \int_0^\infty \big(\,\partial^\alpha_x T_{j2}(x - \tau\,t\,e_1,\,t)\,\cos(\varrho\,t) - \partial^\alpha_x T_{j3}(x - \tau\,t\,e_1,\,t)\,\sin(\varrho\,t)\,\big)\,\mathrm{d}t, \tag{5.1}$$

with a similar formula in the case $k = 3$. Let $\phi : \mathbb{R} \mapsto \mathbb{R}$ be defined by either $\phi(t) := \cos(\varrho\, t)$ for $t \in \mathbb{R}$, or by $\phi(t) := \sin(\varrho\, t)$ for $t \in \mathbb{R}$. Let $m \in \{1, 2, 3\}$. Then, since $\phi(t + \pi/|\varrho|) = -\phi(t)$ for $t \in \mathbb{R}$,

$$\begin{aligned}\int_0^\infty \partial_x^\alpha T_{jm}(x - \tau\, t\, e_1,\, t)\, \phi(t)\, \mathrm{d}t &= \sum_{n=0}^\infty \int_{2\,n\,\pi/|\varrho|}^{2\,(n+1)\,\pi/|\varrho|} \partial_x^\alpha T_{jm}(x - \tau\, t\, e_1,\, t)\, \phi(t)\, \mathrm{d}t \\ &= \sum_{n=0}^\infty \int_{2\,n\,\pi/|\varrho|}^{(2\,n+1)\,\pi/|\varrho|} \Big(\partial_x^\alpha T_{jm}(x - \tau\, t\, e_1,\, t) \\ &\qquad -\partial_x^\alpha T_{jm}(x - \tau\,(t + \pi/|\varrho|)\, e_1,\, t + \pi/|\varrho|)\Big)\, \phi(t)\, \mathrm{d}t \\ &= \sum_{n=0}^\infty \int_{2\,n\,\pi/|\varrho|}^{(2\,n+1)\,\pi/|\varrho|} \int_0^1 (-\tau\, \partial_x^{\alpha+e_1} + \partial_x^\alpha \partial_4) T_{jm}(x - \tau\,(t + \vartheta\,\pi/|\varrho|)\, e_1,\, t + \vartheta\,\pi/|\varrho|) \\ &\qquad \times (-\pi/|\varrho|)\, \phi(t)\, \mathrm{d}\vartheta\, \mathrm{d}t.\end{aligned}$$

Therefore by Lemma 7,

$$\begin{aligned}A &:= \Big|\int_0^\infty \partial_x^\alpha T_{jm}(x - \tau\, t\, e_1,\, t)\, \phi(t)\, \mathrm{d}t\Big| \\ &\le \mathcal{C} \sum_{n=0}^\infty \sum_{m=1}^2 \int_{2\,n\,\pi/|\varrho|}^{(2\,n+1)\,\pi/|\varrho|} \int_0^1 \big(\, |x - \tau\,(t + \vartheta\,\pi/|\varrho|)\, e_1|^2 + t + \vartheta\,\pi/|\varrho|\,\big)^{-3/2-|\alpha|/2-m/2}\, \mathrm{d}\vartheta\, \mathrm{d}t \\ &\le \mathcal{C} \sum_{m=1}^2 \int_0^1 \int_0^\infty \big(\, |x - (\tau\,\vartheta\,\pi/|\varrho|)\, e_1 - \tau\, t\, e_1|^2 + t\,\big)^{-3/2-|\alpha|/2-m/2}\, \mathrm{d}t\, \mathrm{d}\vartheta.\end{aligned}$$

Since $x \in B^c_{S+\tau\,\pi/|\varrho|}$, we have $|x - (\tau\,\vartheta\,\pi/|\varrho|)\, e_1| \ge S$ for $\vartheta \in [0, 1]$, so we may apply Theorem 2.1 with $z = 0,\ R_2 = S,\ R_1 = S/2,\ y = x - (\tau\,\vartheta\,\pi/|\varrho|)\, e_1,\ \nu = 3/2 + |\alpha|/2 + l/2$, to obtain

$$A \le \mathcal{C}(S) \sum_{m=1}^2 \int_0^1 \big[\, |x - (\tau\,\vartheta\,\pi/|\varrho|)\, e_1|\, s\big(\, x - (\tau\,\vartheta\,\pi/|\varrho|)\, e_1\,\big)\,\big]^{-1-|\alpha|/2-m/2}\, \mathrm{d}\vartheta. \tag{5.2}$$

But for $\vartheta \in [0, 1]$, we have $|x - (\tau\,\vartheta\,\pi/|\varrho|)\, e_1| \ge |x|/2 + S/2 - \tau\,\vartheta\,\pi/|\varrho| \ge |x|/2$, where the last inequality holds because $S \ge 2\,\tau\,\pi/|\varrho|$. Moreover, we get from Lemma 1 that $s\big(\, x - (\tau\,\vartheta\,\pi/|\varrho|)\, e_1\,\big)^{-1} \le \mathcal{C}\, s(x)^{-1}$ for $\vartheta \in [0, 1]$. Therefore from (5.2),

$$A \le \mathcal{C}(S) \sum_{m=1}^2 \big(\, |x|\, s(x)\,\big)^{-1-|\alpha|/2-m/2} \le \mathcal{C}(S)\, \big(\, |x|\, s(x)\,\big)^{-3/2-|\alpha|/2}.$$

Theorem 5.1 follows with Eq. (5.1) and its analogue for $k = 3$. □

Corollary 5.1. *Let $S \in (0, \infty)$. Then $|\partial_x^\alpha Z_{jk}(x, 0)| \le \mathcal{C}(S)\, \big(\, |x|\, s(x)\,\big)^{-3/2-|\alpha|/2}$ for $x \in B^c_S$ and for α, j, k as in Theorem 5.1.*

Proof. Let $x \in B^c_S$, and take α, j, k as in Theorem 5.1. By Corollary 2.3, we have $|\partial_x^\alpha Z_{jk}(x, 0)| \le \mathcal{C}(S)\, \big(\, |x|\, s(x)\,\big)^{-1-|\alpha|/2}$.

Suppose that $S \ge 2\,\tau\,\pi/|\varrho|$. Then we distinguish the cases $x \in B^c_{S+\tau\,\pi/|\varrho|}$ and $x \in B_{S+\tau\,\pi/|\varrho|} \backslash B_S$. If $x \in B^c_{S+\tau\,\pi/|\varrho|}$, the inequality stated in the corollary follows from Theorem 5.1. In the second case, we observe that $1 \le (S + \tau\,\pi/|\varrho|)\,|x|^{-1} \le \mathcal{C}(S)\,\big(\,|x|\,s(x)\,\big)^{-1/2}$, so the inequality claimed in Corollary 5.1 may be deduced from the estimate stated at the beginning of this proof.

Now suppose that $S < 2\,\tau\,\pi/|\varrho|$, Then we use that either $x \in B^{c}_{3\,\tau\,\pi/|\varrho|}$ or $x \in B_{3\,\tau\,\pi/|\varrho|}\backslash B_S$. If $x \in B^{c}_{3\,\tau\,\pi/|\varrho|}$, the inequality we want to show follows from Theorem 5.1 with $2\,\tau\,\pi/|\varrho|$ in the place of S. In the case $x \in B_{3\,\tau\,\pi/|\varrho|}\backslash B_S$, we use the relation $1 \le (3\,\tau\,\pi/|\varrho|)\,|x|^{-1} \le \mathcal{C}(S)\,\big(\,|x|\,s(x)\,\big)^{-1/2}$ and again the estimate from the beginning of the proof, once more obtaining an upper bound $\mathcal{C}(S)\,\big(\,|x|\,s(x)\,\big)^{-3/2-|\alpha|/2}$ for $|\partial^{\alpha}_{x} Z_{jk}(x,0)|$, as stated in Corollary 5.1. □

The proofs of Theorem 3.2 and Corollary 3.1 are now obvious.

Proof of Theorem 3.2. Combine Theorem 3.1, Corollary 4.5 and 5.1. □

Proof of Corollary 3.1. From interior regularity of solutions to the Stokes system [25, Theorem IV.4.1] and the assumption $f \in L^p(\mathbb{R}^3)^3$, we may conclude that $u \in W^{2,\widetilde{p}}_{\text{loc}}(\overline{\mathcal{B}}^{c})^3$ and $\pi \in W^{1,\widetilde{p}}_{\text{loc}}(\overline{\mathcal{B}}^{c})$, with $\widetilde{p}$ from Corollary 3.1. More details about this conclusion may be found in the proof of [4, Theorem 5.5]. It follows that $u|\partial B_{S_0} \in W^{2-1/\widetilde{p},\,\widetilde{p}}(\partial B_{S_0})^3$ and $\pi|B_R\backslash\overline{B_{S_0}} \in L^{\widetilde{p}}(B_R\backslash\overline{B_{S_0}})$ for any $R \in (S_0,\infty)$. Now we may apply Theorem 3.2 with $\mathcal{B}$, f, u, π replaced by B_{S_0}, $f|\overline{B_{S_0}}^{c}$, $u|\overline{B_{S_0}}^{c}$ and $\pi|\overline{B_{S_0}}^{c}$, respectively. Corollary 3.1 then follows from Theorem 3.2. □

We add a lemma which shows that the pointwise decay properties of our remainder imply L^p-integrability as derived by Kyed [52] and restated in Sect. 1.

Lemma 9. *$\mathfrak{G}|B^{c}_{S_1} \in L^p(B^{c}_{S_1})^3$ for $p \in (4/3,\,\infty]$ and $\partial_j\mathfrak{G}|B^{c}_{S_1} \in L^p(B^{c}_{S_1})^3$ for $p \in (1,\,\infty]$, $1 \le j \le 3$.*

Proof. Take $p \in (1,\infty)$ and $j \in \{1,\,2,\,3\}$. We show that $\partial_j\mathfrak{G}|B^{c}_{S_1} \in L^p(B^{c}_{S_1})^3$. The same type of argument yields $\mathfrak{G}|B^{c}_{S_1} \in L^p(B^{c}_{S_1})^3$ if $p > 4/3$. Since $2 - 2/p > 0$, we may fix some $\epsilon \in (0,\,2 - 2/p)$. Take $\epsilon := (2 - 2/p)/2$ in order to specify how this parameter depends on p. The relation $\epsilon > 0$ implies the term $|x|^{-\epsilon}\,\ln(2 + |x|)$ is bounded uniformly in $x \in B^{c}_{S_1}$. Therefore, with Theorem 3.2,

$$\int\limits_{B^{c}_{S_1}} |\partial_i G(x)|^p\,\mathrm{d}x \le C^p \int\limits_{B^{c}_{S_1}} \Big(\big(\,|x|\,s(x)\,\big)^{-2}\,\ln(2 + |x|)\Big)^p\,\mathrm{d}x$$
$$\le C^p\,\mathcal{C}(p) \int\limits_{B^{c}_{S_1}} \big(\,|x|\,s(x)\,\big)^{-2\,p+\epsilon\,p}\,\mathrm{d}x,$$

where the constant C was introduced in Theorem 3.2. Since $\epsilon < 2 - 2/p$, we have $-2\,p + \epsilon\,p < -2$, so we get with Lemma 3

$$\int\limits_{B^{c}_{S_1}} |\partial_i G(x)|^p\,\mathrm{d}x \le C^p\,\mathcal{C}(p) \int\limits_{S_1}^{\infty} r^{-2\,p+\epsilon\,p} \int\limits_{\partial B_r} s(x)^{-2\,p+\epsilon\,p}\,\mathrm{d}o_x\,\mathrm{d}r$$
$$\le C^p\,\mathcal{C}(p) \int\limits_{S_1}^{\infty} r^{-2\,p+\epsilon\,p+1}\,\mathrm{d}r \le C^p\,\mathcal{C}(p).$$

□

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