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Stability of non-constant steady-state solutions for bipolar non-isentropic Euler–Maxwell equations with damping terms

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Abstract. In this article, we consider the periodic problem for bipolar non-isentropic Euler–Maxwell equations with damping terms in plasmas. By means of an induction argument on the order of the time-space derivatives of solutions in energy estimates, the global smooth solution with small amplitude was established close to a non-constant steady-state solution with asymptotic stability property. Furthermore, we obtain the global stability of solutions with exponential decay in time near the non-constant steady-states for bipolar non-isentropic Euler–Poisson equations. This phenomenon on the charge transport shows the essential relation and difference between the bipolar non-isentropic and the bipolar isentropic Euler–Maxwell/Poisson equations.

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1. Introduction and main results

Recently, there have been many mathematical studies on the Euler–Maxwell equations which are used for modeling the motion of fluid plasmas (see [1,16,20,21,23] and the references theirin). In the following, we consider the period problem for the bipolar non-isentropic compressible Euler–Maxwell system with damping terms:

$$\begin{cases} \partial_t n^\nu + \nabla\cdot(n^\nu u^\nu) = 0, \\ \partial_t(n^\nu u^\nu) + \nabla\cdot(n^\nu u^\nu \otimes u^\nu) + \nabla p_\nu = q_\nu n^\nu(E + u^\nu \times B) - n^\nu u^\nu, \\ \partial_t(n^\nu \mathcal{E}_\nu) + \nabla\cdot(n^\nu \mathcal{E}_\nu u^\nu + p_\nu u^\nu) = \nabla\cdot(\kappa_\nu \nabla\theta^\nu) + q_\nu n^\nu u^\nu E - n^\nu |u^\nu|^2 - n^\nu(\theta^\nu - 1), \\ \partial_t E - \nabla\times B = n^e u^e - n^i u^i, \quad \nabla\cdot E = n^i - n^e + b(x), \\ \partial_t B + \nabla\times E = 0, \quad \nabla\cdot B = 0, \quad \nu = e, i, \quad (t,x)\in\mathbb{R}^+\times\mathbb{T}, \end{cases} \tag{1.1}$$

where $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^3$ denotes a three-dimensional torus and $q_e = -1$ $(q_i = 1)$ is the charge of electrons (ions). The variables are the density $n^\nu > 0$, the velocity $u^\nu = (u_1^\nu, u_2^\nu, u_3^\nu)$, the absolute temperature $\theta^\nu > 0$, the total energy, the electric field E and the magnetic field B. Functions $p_\nu = p_\nu(n^\nu, \theta^\nu), \mathcal{E}_\nu = e_\nu + \frac{1}{2}|u^\nu|^2, e_\nu = e_\nu(n^\nu, \theta^\nu), \kappa_\nu = \kappa_\nu(n^\nu, \theta^\nu)$ and $b(x) > 0$ denote, respectively, pressure, total energy, internal energy, coefficient of heat conduction and a doping term. Here, p_ν and e_ν satisfy the second principle of thermodynamics [28]:

$$p_\nu = (n^\nu)^2 \frac{\partial e_\nu}{\partial n^\nu} + \theta^\nu \frac{\partial p_\nu}{\partial \theta^\nu}, \quad \nu = e, i.$$

We set $e_\nu(n^\nu, \theta^\nu) = \theta^\nu$ for $\nu = e, i$, which implies that $p_\nu = n^\nu\theta^\nu$ is well defined. Moreover, we also set $\kappa_\nu(n^\nu, \theta^\nu) = \theta^\nu$ for the sake of simplicity. Then for $n^\nu > 0$, system (1.1) is written as:

$$\begin{cases} \partial_t n^\nu + \nabla\cdot(n^\nu u^\nu) = 0, \\ \partial_t u^\nu + (u^\nu\cdot\nabla)u^\nu + \nabla\theta^\nu + \theta^\nu\nabla\ln(n^\nu) + u^\nu = q_\nu(E + u^\nu\times B), \\ \partial_t\theta^\nu + u^\nu\cdot\nabla\theta^\nu + \theta^\nu\nabla\cdot u^\nu - \dfrac{1}{n^\nu}|\nabla\theta^\nu|^2 + (\theta^\nu - 1) = \dfrac{\theta^\nu}{n^\nu}\Delta\theta^\nu, \\ \partial_t E - \nabla\times B = n^e u^e - n^i u^i,\ \nabla\cdot E = n^i - n^e + b(x), \\ \partial_t B + \nabla\times E = 0,\ \nabla\cdot B = 0, \quad \nu = e, i, \quad (t,x)\in\mathbb{R}^+\times\mathbb{T}, \end{cases} \tag{1.2}$$

with the initial condition:

$$(n^\nu, u^\nu, \theta^\nu, E, B)\,|_{t=0} = \left(n^{\nu 0}, u^{\nu 0}, \theta^{\nu 0}, E^0, B^0\right), \quad x\in\mathbb{T}, \quad \nu = e, i, \tag{1.3}$$

which satisfies the compatibility condition:

$$\nabla\cdot E^0 = n^{i0} - n^{e0} + b(x),\ \nabla\cdot B^0 = 0, \quad x\in\mathbb{T}. \tag{1.4}$$

Now suppose $(n^\nu, u^\nu, \theta^\nu, E, B) = \left(\bar n^\nu(x), 0, 1, \bar E(x), \bar B(x)\right)$ to be a steady-state solution of (1.2). Then we get

$$\begin{cases} -\nabla\ln\left(\bar n^e\right) = \bar E = \nabla\ln\left(\bar n^i\right), \\ \nabla\times\bar B = 0,\ \nabla\cdot\bar E = \bar n^i - \bar n^e + b(x), \\ \nabla\times\bar E = 0,\ \nabla\cdot\bar B = 0, \quad x\in\mathbb{T}, \end{cases} \tag{1.5}$$

which implies that $\bar B$ is a constant vector in $\mathbb{R}^3$. Moreover, if we denote $\bar\phi = \ln(\bar n^e)$, then

$$-\nabla\ln(\bar n^i) = \nabla\bar\phi.$$

It follows that

$$\bar n^e = e^{\bar\phi}, \quad \bar n^i = e^{C_1 - \bar\phi},$$

where C_1 is a real constant. By the differential constraint of $\bar E$, namely

$$\nabla\cdot\bar E = e^{C_1-\bar\phi} - e^{\bar\phi} + b(x),$$

we obtain the equation of $\bar\phi$:

$$\Delta\bar\phi = e^{\bar\phi} - e^{C_1-\bar\phi} - b(x), \quad x\in\mathbb{T}. \tag{1.6}$$

It is easy to see that the function $f : \bar\phi \longmapsto e^{\bar\phi} - e^{C_1-\bar\phi}$ is strictly increasing on $\mathbb{R}^+$. Then the existence of smooth solutions to this semilinear monotone elliptic equation (1.6) can be obtained easily by a classical Schauder fixed point theorem or a minimization method [6]. The uniqueness follows from the strict monotonicity of f (see [12]). Furthermore, we obtain

Proposition 1.1. *Assume $b = b(x)$ is a smooth periodic function such that $b \geq$ const. > 0 in $\mathbb{T}$. Then the periodic problem (1.5) admits a unique smooth solution such that $\bar n^\nu \geq$ const. > 0 in $\mathbb{T}$, $\nu = e, i$.*

The bipolar non-isentropic Euler–Maxwell system (1.2) is nonlinear and symmetrizable hyperbolic–parabolic for $n^\nu, \theta^\nu > 0$ in the sense of Friedrichs. Therefore, it follows from the result of Kato [13] and the pioneering work of Matsumura–Nishida [17,18] that the periodic problem (1.2)–(1.3) has a unique local smooth solution when the initial data are smooth.

Proposition 1.2. *(Local existence of smooth solutions, see [13,15]) Assume integer $s \geq 3$ and (1.4) holds. Let $\bar B \in \mathbb{R}^3$ be any given constant vector and $(\bar n^\nu, 0, 1, \bar E, \bar B)$ be a steady-state solution of (1.2) satisfying $\bar n^\nu \geq$ const. > 0. Suppose $(n^{\nu 0} - \bar n^\nu, u^{\nu 0}, \theta^{\nu 0} - 1, E^0 - \bar E, B^0 - \bar B) \in H^s(\mathbb{T})$ with $n^{\nu 0}, \theta^{\nu 0} \geq 2\kappa$ for some given constant $\kappa > 0$. Then there exists $T > 0$ such that periodic problem (1.2)–(1.3) admits a unique smooth solution which satisfies $n^\nu, \theta^\nu \geq \kappa$ in $[0,T]\times\mathbb{T}$ and*

$$\theta^\nu - 1 \in C^1\big([0,T]; H^{s-2}(\mathbb{T})\big) \cap C\big([0,T]; H^s(\mathbb{T})\big), \quad \nu = e, i,$$

$$(n^\nu - \bar n^\nu, u^\nu, E - \bar E, B - \bar B) \in C^1\big([0,T]; H^{s-1}(\mathbb{T})\big) \cap C\big([0,T]; H^s(\mathbb{T})\big), \quad \nu = e, i.$$

There are mathematical investigations in numerical computations [3], the asymptotic limits with small parameters[19], the existence of solutions for Euler–Maxwell systems. Particularly, some of them are concerned with the global existence and asymptotic stability of small amplitude smooth solutions around constant steady-sates. For one-dimensional isentropic Euler–Maxwell system in which the energy equation is not contained, Chen–Jerome–Wang [2] proved the global existence of weak solutions with the help of the compensated compactness method and the fractional Godunov scheme. For the three-dimensional isentropic Euler–Maxwell equations, the existence of global smooth small solutions to the Cauchy problem in $\mathbb{R}^3$ is established by Ueda–Wang–Kawashima [25] when $s \geq 3$ and the asymptotic behaviors of solutions when $s \geq 4$. By using suitable choices of symmetrizers and energy estimates, Peng–Wang–Gu [20] and Peng [21] obtained the global existence and the longtime behaviors of smooth solutions to the periodic problem in $\mathbb{T}$ and to the initial value problem in $\mathbb{R}^3$ when $s \geq 3$. By using high- and low-frequency decomposition methods, Xu [29] constructs uniform (global) classical solutions to the initial value problem in Chemin–Lerner's spaces with critical regularity. When $s \geq 4$, by using the tools of Fourier analysis, Duan [4] and Duan–Liu–Zhu [5] obtained the decay rates of global smooth solutions in L^q with $2 \leq q \leq \infty$ when the time goes to infinity. And when $s \geq 6$, Ueda–Kawashima [24] also obtained the large time decay rates of global smooth solutions in $H^{s-2k}(\mathbb{R}^3)$ with $0 \leq k \leq \left[s/2\right]$. For the three-dimensional non-isentropic Euler–Maxwell equations, the existence of global smooth small solutions to the Cauchy problem in $\mathbb{R}^3$ is established by Feng–Wang–Kawashima [8] and Wang–Feng–Li [26]. For the Euler–Maxwell system without damping, an additional relation was made by Germain–Masmoudi [10] to establish such a global existence result for the unipolar case. And for the bipolar case without damping, Guo–Ionescu–Pausader [11] proved global stability of a constant neutral background, in the sense that irrotational, smooth and localized perturbations of a constant background with small amplitude lead to global smooth solutions in three space dimensions.

All these results above hold when the solutions are close to a constant steady-state of the Euler–Maxwell systems. In the last year, motivated by the Guo–Strauss's work on the study of the damped Euler–Poisson on the general bounded domain[12], with the help of an induction argument on the order of the derivatives of solutions, Peng [22], Feng–Peng–Wang [7] and Feng–Wang–Li [9] study the stabilities of non-constant steady-state solutions for the unipolar/bipolar isentropic and unipolar non-isentropic Euler–Maxwell systems, respectively. However, there is no result on the stability of non-constant steady-state solutions for the bipolar non-isentropic Euler–Maxwell system so far. The goal of this paper is to consider this problem.

Now we state the main results of this article.

Theorem 1.1. (Stability of smooth solutions for the Euler–Maxwell equations) *Let $s \geq 6$ and (1.4) hold. Let $\bar{B} \in \mathbb{R}^3$ be any given constant vector and $(\bar{n}^\nu, 0, 1, \bar{E}, \bar{B})$ be a steady-state solution of (1.2) satisfying $\bar{n}^\nu \geq const. > 0$. Assume initial data satisfy*

$$\int\limits_{\mathbb{T}} n^{\nu 0}(x)\,\mathrm{d}x = \int\limits_{\mathbb{T}} \bar{n}^\nu(x)\,\mathrm{d}x, \quad \nu = e, i. \tag{1.7}$$

Then there exist constants $\delta_0 > 0$ and $C > 0$, independent of any given time $t > 0$, such that if

$$\|(n^{\nu 0} - \bar{n}^\nu, u^{\nu 0}, \theta^{\nu 0} - 1, E^0 - \bar{E}, B^0 - \bar{B})\|_s \leq \delta_0, \quad \nu = e, i,$$

where $\|\cdot\|_m$ is the norm of usual Sobolev spaces $H^m(\mathbb{T})$, periodic problem (1.2)–(1.3) admits a unique global smooth solution $(n^\nu, u^\nu, \theta^\nu, E, B)$ satisfying

$$n^\nu - \bar{n}^\nu \in \bigcap_{k=0}^{2} C^k\left(\mathbb{R}^+, H^{s-k}(\mathbb{T})\right) \bigcap_{k=3}^{[s/2]} C^k\left(\mathbb{R}^+, H^{s-2k+2}(\mathbb{T})\right), \quad \nu = e, i, \tag{1.8}$$

$$u^\nu \in \bigcap_{k=0}^{1} C^k\left(\mathbb{R}^+, H^{s-k}(\mathbb{T})\right) \bigcap_{k=2}^{[s/2]} C^k\left(\mathbb{R}^+, H^{s-2k+1}(\mathbb{T})\right), \quad \nu = e, i, \tag{1.9}$$

$$\theta^\nu - 1 \in \bigcap_{k=0}^{[s/2]} C^k\left(\mathbb{R}^+, H^{s-2k}(\mathbb{T})\right), \quad \nu = e, i, \tag{1.10}$$

$$E - \bar{E} \in \bigcap_{k=0}^{3} C^k\left(\mathbb{R}^+, H^{s-k}(\mathbb{T})\right) \bigcap_{k=4}^{[s/2]} C^k\left(\mathbb{R}^+, H^{s-2k+3}(\mathbb{T})\right), \tag{1.11}$$

$$B - \bar{B} \in \bigcap_{k=0}^{4} C^k\left(\mathbb{R}^+, H^{s-k}(\mathbb{T})\right) \bigcap_{k=5}^{[s/2]} C^k\left(\mathbb{R}^+, H^{s-2k+4}(\mathbb{T})\right), \tag{1.12}$$

$$\begin{aligned} &\sup_{t\geq 0} |||(n^\nu(t) - \bar{n}^\nu, u^\nu(t), \theta^\nu(t) - 1, E(t) - \bar{E}, B(t) - \bar{B})|||_{[s/2]} \\ &\quad \leq C \sum_{\nu=e,i} \|(n^{\nu 0} - \bar{n}^\nu, u^{\nu 0}, \theta^{\nu 0} - 1, E^0 - \bar{E}, B^0 - \bar{B})\|_s, \end{aligned} \tag{1.13}$$

and

$$\begin{aligned} &\int_0^{+\infty} \left(\sum_{\nu=e,i} |||\,(n^\nu(t) - \bar{n}^\nu, u^\nu(t), \theta^\nu(t) - 1, \nabla\theta^\nu(t))\,|||_{[s/2]}^2 + |||E(t) - \bar{E}|||_{[s/2]-1}^2 \right. \\ &\quad \left. + |||\partial_t B(t)|||_{[s/2]-2}^2 + |||\nabla B(t)|||_{[s/2]-2}^2 \right) \mathrm{d}t \\ &\leq C \sum_{\nu=e,i} ||(n^{\nu 0} - \bar{n}^\nu, u^{\nu 0}, \theta^{\nu 0} - 1, E^0 - \bar{E}, B^0 - \bar{B})||_s^2, \end{aligned} \tag{1.14}$$

where $[\cdot]$ *denotes the integer part of the argument. Moreover, we have*

$$\lim_{t\to\infty} |||(n^\nu(t) - \bar{n}^\nu, u^\nu(t), \theta^\nu(t) - 1)|||_{[s/2]-1} = 0, \quad \nu = e, i, \tag{1.15}$$

$$\lim_{t\to\infty} |||E(t) - \bar{E}|||_{[s/2]-1} = 0, \tag{1.16}$$

and

$$\lim_{t\to+\infty} \left(|||\partial_t B(t)|||_{s-2} + |||\nabla B(t)|||_{[s/2]-2} \right) = 0, \tag{1.17}$$

where $|||\cdot|||_m$ *is defined in the end of this section.*

Remark 1.1. *Due to the second and third equations in system* (1.2)*, the regularities of solutions for the bipolar non-isentropic Euler–Maxwell system are different from that for the bipolar isentropic case* [7].

Remark 1.2. *Condition* (1.7) *allows us to apply the Poincaré inequality, since* $n^\nu - \bar{n}^\nu$ *is a conservative variable.*

Remark 1.3. *It should be emphasized that the temperature relaxation and viscosity terms of the considered bipolar non-isentropic Euler–Maxwell system here play a key role in the proof of Theorem 1.1. We shall study in the other forthcoming work the case of non-relaxation for which the proof is much more complicated to carry out.*

The proof of Theorem 1.1 is mainly based on the elaborate energy estimates and an induction argument on the order of the time-space derivatives of solutions. This argument, firstly used by Peng [22] in the unipolar isentropic case and then extended by Feng–Peng–Wang [7] to the bipolar isentropic case, can remove the difficulty due to the appearance of non-constant steady-state solutions. It should be pointed out that the bipolar non-isentropic Euler–Maxwell equations are much more complex than the bipolar isentropic Euler–Maxwell system because they contain two charged fluids energy equations besides the density and velocity equations.

Now let us explain the main difference of proofs in the bipolar isentropic and bipolar non-isentropic Euler–Maxwell systems. Different from the isentropic Euler–Maxwell systems, the pressure function p_ν

in system (1.1) depends on the absolute temperature θ^ν besides the density n^ν. Therefore, we have to make much more efforts to deal with the estimates on temperature. And new difficulties appear when we establish energy estimates for achieving a relation of recurrence [see (3.74)]. On the other hand, we also include the heat diffusion into the model, and indeed both relaxation and diffusion of temperature play a key role in the analysis. In fact, the dissipation of $\partial_t^k(u^\nu, \theta^\nu - 1, \nabla\theta^\nu)$ in $L^2(\mathbb{T})$ is straightforward for all $0 \le k \le [s/2]$. Furthermore, for $m \in \mathbb{N}$ with $k + m \le [s/2]$ and $m \ge 1$, the dissipation of $\|\partial_t^k(u^\nu, \theta^\nu - 1, \nabla\theta^\nu)\|_m$ depends on that of $\|\partial_t^k(n^\nu - \bar{n}^\nu)\|_m$ as well as $\|\big(\partial_t^k(u^\nu, \theta^\nu - 1, E - \bar{E}), \partial_t^{k+1}(\theta^\nu - 1)\big)\|_{m-1}$, while the dissipations of $\|\partial_t^k(n^\nu - \bar{n}^\nu)\|_m$ and $\|\partial_t^k(E - \bar{E})\|_{m-1}$ depend on both that of $\|\partial_t^k(n^\nu - \bar{n}^\nu, u^\nu, \theta^\nu - 1, \nabla\theta^\nu)\|_{m-1}$ and that of $\|\partial_t^{k+1}u^\nu\|_{m-1}$. This implies a recurrence relation which allows us to get the estimates by induction on (k, m) with k decreasing and m increasing. Then Theorem 1.1 follows in the way by combining these estimates above with Proposition 1.2 and the standard continuity argument.

In comparison with the Euler–Maxwell equations, Euler–Poisson systems are another important class of equations due to their applications in semiconductors and plasma physics [1,16]. On the one hand, the Euler–Maxwell system and the Euler–Poisson system are essentially different due to the coupling terms and to the difference between the Poisson equation and the Maxwell equations. On the other hand, the Euler–Poisson system can be regarded formally as a particular case of the Euler–Maxwell system with $E = -\nabla\phi$ and $B = 0$. Usually, the bipolar non-isentropic Euler–Poisson equations take the form:

$$\begin{cases} \partial_t n^\nu + \nabla\cdot(n^\nu u^\nu) = 0, \\ \partial_t u^\nu + (u^\nu\cdot\nabla)u^\nu + \nabla\theta^\nu + \dfrac{\theta^\nu}{n^\nu}\nabla n^\nu = -q_\nu\nabla\phi - u^\nu, \\ \partial_t\theta^\nu + u^\nu\cdot\nabla\theta^\nu + \theta^\nu\nabla\cdot u^\nu - \dfrac{1}{n^\nu}|\nabla\theta^\nu|^2 = -(\theta^\nu - 1) + \dfrac{\theta^\nu}{n^\nu}\Delta\theta^\nu, \\ -\Delta\phi = n^i - n^e + b(x), \quad (t,x)\in\mathbb{R}^+\times\mathbb{T}. \end{cases} \tag{1.18}$$

Initial conditions are also given as:

$$(n^\nu, u^\nu, \theta^\nu)|_{t=0} = (n^{\nu 0}, u^{\nu 0}, \theta^{\nu 0}), \quad x\in\mathbb{T}, \quad \nu = e, i. \tag{1.19}$$

Define ϕ^0 by

$$-\triangle\phi^0 = n^{i0} - n^{e0} + b(x), \quad x\in\mathbb{T}. \tag{1.20}$$

The steady-state solution $(n^\nu, u^\nu, \theta^\nu, \phi) = \big(\bar{n}^\nu(x), 0, 1, \bar{\phi}(x)\big)$ is still given by (1.5) and $\bar{\phi} = \ln(\bar{n}^e)$. In order that ϕ is uniquely determined, we add a restriction condition:

$$\int_{\mathbb{T}} \phi(\cdot, x)\,\mathrm{d}x = \int_{\mathbb{T}} \bar{\phi}(x)\,\mathrm{d}x. \tag{1.21}$$

By means of (1.6) and (1.18), we have

$$\Delta\left(\phi - \bar{\phi}\right) = (n^e - \bar{n}^e) - (n^i - \bar{n}^i).$$

It follows from Lemma 1.1 and (1.21) that for all integer $m \ge 0$,

$$\big\|\nabla\left(\phi - \bar{\phi}\right)\big\|_m \le C\big\|(n^e - \bar{n}^e, n^i - \bar{n}^i)\big\|_m. \tag{1.22}$$

Then regarding $\nabla\phi$ as a function of $n^\nu, \nu = e, i, (n^\nu, u^\nu, \theta^\nu)$ still satisfies a symmetrizable hyperbolic–parabolic system in which $\nabla\phi$ appearing on the right hand side of (1.18) is a low-order term of n^ν. Following results of Kato [13] and Matsumura–Nishida [17,18], this implies that periodic problem (1.18)–(1.19) admits a unique local smooth solution, provided that the initial data $(n^{\nu 0}, u^{\nu 0}, \theta^{\nu 0})$ are smooth. Furthermore, we obtain from (1.22) that $\phi \in C\big([0,T), H^{m+1}(\mathbb{T})\big)$ as soon as $n^\nu \in C\big([0,T), H^m(\mathbb{T})\big)$ for some constant $T > 0$ and integer $m \ge 0$.

As a byproduct, here we show that our treatment for the bipolar non-isentropic Euler–Maxwell system is still valid for the bipolar non-isentropic Euler–Poisson system. The stability results for periodic problem (1.18)–(1.19) are stated as follows.

Theorem 1.2. *(Stability of smooth solutions for the Euler–Poisson equations) Let $s \geq 6$ be an integer, $\left(n^{\nu 0}, u^{\nu 0}, \theta^{\nu 0}\right) \in H^s(\mathbb{T})$ and (1.7) holds for $n^{\nu 0}$. Then there exists a constant $\delta_0 > 0$ small enough such that if $\| \left(n^{\nu 0} - \bar{n}^\nu, u^{\nu 0}, \theta^{\nu 0} - 1\right) \|_s \leq \delta_0$, the periodic problem to the bipolar non-isentropic Euler–Poisson equations (1.18)–(1.19) admits a unique global smooth solution $(n^\nu, u^\nu, \theta^\nu, \phi)$ satisfying*

$$(n^\nu - \bar{n}^\nu, \nabla\phi - \nabla\bar{\phi}) \in \bigcap_{k=0}^{2} C^k\left(\mathbb{R}^+, H^{s-k}(\mathbb{T})\right) \bigcap_{k=3}^{[s/2]} C^k\left(\mathbb{R}^+, H^{s-2k+2}(\mathbb{T})\right), \quad \nu = e, i, \tag{1.23}$$

$$u^\nu \in C\left(\mathbb{R}^+, H^s(\mathbb{T})\right) \bigcap_{k=1}^{[s/2]} C^k\left(\mathbb{R}^+, H^{s-2k+1}(\mathbb{T})\right), \quad \nu = e, i, \tag{1.24}$$

$$\theta^\nu - 1 \in \bigcap_{k=0}^{[s/2]} C^k\left(\mathbb{R}^+, H^{s-2k}(\mathbb{T})\right), \quad \nu = e, i. \tag{1.25}$$

Moreover, there exists a constant $\eta > 0$ such that for all $t > 0$, it holds

$$\begin{aligned} &\sum_{\nu=e,i} ||| \left(n^\nu(t) - \bar{n}^\nu, u^\nu(t), \theta^\nu(t) - 1, \nabla\phi(t) - \nabla\bar{\phi}\right) |||_{[s/2]} \\ &\quad \leq Ce^{-\eta t} \sum_{\nu=e,i} \left\|(n^{\nu 0} - \bar{n}, u^{\nu 0}, \theta^{\nu 0} - \bar{\theta})\right\|_s. \end{aligned} \tag{1.26}$$

Let us introduce some notations for the use throughout this paper. The expression $f \sim g$ means $\gamma g \leq f \leq \frac{1}{\gamma} g$ for a constant $0 < \gamma < 1$. We denote by $\|\cdot\|_s$ the norm of the usual Sobolev space $H^s(\mathbb{T})$, and by $\|\cdot\|$ and $\|\cdot\|_{L^p}$ the norms of $L^2(\mathbb{T})$ and $L^p(\mathbb{T})$, respectively, where $2 < p \leq +\infty$. For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, we denote

$$\partial^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3} = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}, \ |\alpha| = \alpha_1 + \alpha_2 + \alpha_3.$$

For $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^3, \beta \leq \alpha$ stands for $\beta_j \leq \alpha_j$ for $j = 1, 2, 3$, and $\beta < \alpha$ stands for $\beta \leq \alpha$ and $\beta \neq \alpha$. For $T > 0$ and $m \geq 1$, we define the Banach space

$$B_{m,T}(\mathbb{T}) = \bigcap_{k=0}^{m} C^k\left([0, T], H^{m-k}(\mathbb{T})\right),$$

with the norm

$$|||f|||_m = \left(\sum_{k+|\alpha|\leq m} \|\partial_t^k \partial^\alpha f\|^2\right)^{\frac{1}{2}}, \quad \forall f \in B_{m,T}(\mathbb{T}).$$

It is easy to see that $\|\cdot\|_m \leq |||\cdot|||_m$.

Next, we introduce three lemmas which will be used in the sequel.

Lemma 1.1. *(Poincaré inequality, see [6].) Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^d$ be a bounded connected open domain with Lipschitz boundary. Then there exists a constant $C > 0$ depending only on p and Ω such that*

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\Omega),$$

where

$$u_\Omega = \frac{1}{|\Omega|}\int_\Omega u(x)\,\mathrm{d}x$$

is the average value of u over Ω.

Lemma 1.2. (see [22]) *Let $s \geq 3$ and $u, v \in B_{s,T}(\mathbb{T})$. It holds*

$$|||uv|||_s \leq C|||u|||_s\, |||v|||_s. \tag{1.27}$$

Lemma 1.3. *Let $s \geq 3$ and $v \in B_{s,T}(\mathbb{T})$ satisfying $\partial_t v = f(x, v, \partial_x v, \partial_{xx} v)$, with f being a smooth function such that $f(x, 0, 0, 0) = 0$. Then for all $t \in [0, T]$, we have*

$$\|\partial_t^k \partial^\alpha v(t,\cdot)\| \leq C\|v(t,\cdot)\|_s, \quad \forall\, k + |\alpha| \leq [s/2], \tag{1.28}$$

where the positive constant C may depend continuously on $\|v\|_s$.

Proof. It is similar to that of Lemma 2.8 in [22] and is omitted here for the sake of simplicity, where we have used the third equation of system (1.2). □

We conclude this section by stating the arrangement of the rest of this paper. In Sect. 2, we reformulate the periodic problem under consideration. In Sect. 3, detailed energy estimates are established. In Sect. 4, we complete the proof of Theorems 1.1 and 1.2 by combining the estimates above.

2. Reformulation of the problem

Suppose $(n^\nu, u^\nu, \theta^\nu, E, B)$ to be a local smooth solution to the periodic problem (1.2)–(1.4). Now, for $\nu = e, i$, set

$$n^\nu = \bar{n}^\nu + N^\nu,\ \theta^\nu = 1 + \Theta^\nu,\ E = \bar{E} + F,\ B = \bar{B} + G. \tag{2.29}$$

Thus, we can rewrite problem (1.2)–(1.4) as:

$$\begin{cases} \partial_t N^\nu + u^\nu \cdot \nabla N^\nu + (\bar{n}^\nu + N^\nu)\,\nabla \cdot u^\nu + u^\nu \cdot \nabla \bar{n}^\nu = 0, \\ \partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu + \nabla\Theta^\nu + \nabla(\ln(\bar{n}^\nu + N^\nu) - \ln \bar{n}^\nu) + \Theta^\nu \nabla(\ln(\bar{n}^\nu + N^\nu) + u^\nu \\ \ = q_\nu(F + u^\nu \times \bar{B} + u^\nu \times G), \\ \partial_t \Theta^\nu + u^\nu \cdot \nabla\Theta^\nu + (1 + \Theta^\nu)\nabla \cdot u^\nu - \dfrac{1}{n^\nu}|\nabla\theta^\nu|^2 + \Theta^\nu = \dfrac{1 + \Theta^\nu}{\bar{n}^\nu + N^\nu}\Delta\Theta^\nu, \\ \partial_t F - \nabla \times G = (\bar{n}^e + N^e)\, u^e - \left(\bar{n}^i + N^i\right) u^i, \quad \nabla \cdot F = N^i - N^e, \\ \partial_t G + \nabla \times F = 0, \quad \nabla \cdot G = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{T}, \end{cases} \tag{2.30}$$

with the initial condition:

$$(N^\nu, u^\nu, \Theta^\nu, F, G)\,|_{t=0} = \left(N^{\nu 0}, u^{\nu 0}, \Theta^{\nu 0}, F^0, G^0\right), \quad x \in \mathbb{T}, \tag{2.31}$$

satisfying the compatibility condition:

$$\nabla \cdot F^0 = N^{i0} - N^{e0},\ \nabla \cdot G^0 = 0, \quad x \in \mathbb{T}. \tag{2.32}$$

Here $N^{\nu 0} = n^{\nu 0} - \bar{n}^\nu, \Theta^{\nu 0} = \theta^{\nu 0} - 1, F^0 = E^0 - \bar{E}$ and $G^0 = B^0 - \bar{B}$.

A direct computation gives

$$\nabla(\ln(\bar{n}^\nu + N^\nu) = \frac{\nabla\bar{n}^\nu}{\bar{n}^\nu} + \frac{1}{n^\nu}\nabla N^\nu - \frac{\nabla\bar{n}^\nu}{n^\nu\bar{n}^\nu}N^\nu$$

and

$$\nabla(\ln(\bar{n}^\nu + N^\nu) - \ln \bar{n}^\nu) = \frac{1}{n^\nu}\nabla N^\nu - \frac{\nabla\bar{n}^\nu}{(\bar{n}^\nu)^2}N^\nu + \frac{1}{n^\nu(\bar{n}^\nu)^2}(N^\nu)^2.$$

Next, we also set

$$U^\nu = \begin{pmatrix} N^\nu \\ u^\nu \\ \Theta^\nu \end{pmatrix}, \qquad U = \begin{pmatrix} U^e \\ U^i \end{pmatrix}, \qquad W = \begin{pmatrix} U \\ F \\ G \end{pmatrix},$$
$$U^{\nu 0} = \begin{pmatrix} N^{\nu 0} \\ u^{\nu 0} \\ \Theta^{\nu 0} \end{pmatrix}, \quad U^0 = \begin{pmatrix} U^{e0} \\ U^{i0} \end{pmatrix}, \quad W^0 = \begin{pmatrix} U^0 \\ F^0 \\ G^0 \end{pmatrix}. \tag{2.33}$$

Then the Euler equations in (2.30) can be rewritten in the following matrix form:

$$\partial_t U^\nu + \sum_{j=1}^{3} A_j^\nu (n^\nu, u^\nu, \theta^\nu) \partial_j U^\nu + L^\nu(x) U^\nu + M^\nu(W) = f^\nu, \quad \nu = e, i, \tag{2.34}$$

with

$$A_j^\nu (n^\nu, u^\nu, \theta^\nu) = \begin{pmatrix} u_j^\nu & n^\nu e_j^T & 0 \\ \dfrac{\theta^\nu}{n^\nu} e_j & u_j^\nu I_3 & e_j \\ 0 & \theta^\nu e_j^T & u_j^\nu \end{pmatrix}, \; j = 1, 2, 3, \tag{2.35}$$

$$L^\nu(x) = \begin{pmatrix} 0 & (\nabla \bar{n}^\nu)^T & 0 \\ -\dfrac{\nabla \bar{n}^\nu}{(\bar{n}^\nu)^2} & 0 & \dfrac{\nabla \bar{n}^\nu}{\bar{n}^\nu} \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.36}$$

$$M^\nu(W) = \begin{pmatrix} 0 \\ u^\nu - q_\nu(F + u^\nu \times \bar{B}) \\ \Theta^\nu - \dfrac{\theta^\nu}{n^\nu} \Delta \Theta^\nu \end{pmatrix}, \tag{2.37}$$

$$f^\nu = \begin{pmatrix} 0 \\ \dfrac{\nabla \bar{n}^\nu}{n^\nu \bar{n}^\nu} N^\nu \Theta^\nu - \dfrac{(N^\nu)^2}{n^\nu (\bar{n}^\nu)^2} + q_\nu(u^\nu \times G) \\ -\dfrac{1}{n^\nu} |\nabla \Theta^\nu|^2 \end{pmatrix}, \tag{2.38}$$

where (e_1, e_2, e_3) denotes the canonical basis of $\mathbb{R}^3$, I_3 denotes the 3×3 unit matrix, and we use $[\cdot]^T$ to denote the transpose of a vector.

Obviously, system (2.34) for U^ν is symmetrizable hyperbolic–parabolic when $n^\nu = \bar{n}^\nu + N^\nu, \theta^\nu = 1 + \Theta^\nu > 0$. In fact, the symmetrizer can be chosen as

$$A_0^\nu(n^\nu, \theta^\nu) = \begin{pmatrix} \dfrac{\theta^\nu}{n^\nu} & 0 & 0 \\ 0 & n^\nu I_3 & 0 \\ 0 & 0 & \dfrac{n^\nu}{\theta^\nu} \end{pmatrix},$$

which implies that

$$\tilde{A}_j^\nu\left(n^\nu,u^\nu,\theta^\nu\right)=A_0^\nu\left(n^\nu,\theta^\nu\right)A_j^\nu\left(n^\nu,u^\nu,\theta^\nu\right)=\begin{pmatrix}\frac{\theta^\nu}{n^\nu}u_j^\nu & \theta^\nu e_j^T & 0\\ \theta^\nu e_j & n^\nu u_j^\nu I_3 & n^\nu e_j\\ 0 & n^\nu e_j^T & \frac{n^\nu}{\theta^\nu}u_j^\nu\end{pmatrix}$$

is a symmetric matrix.

From now on, let $T>0$ and W be a smooth solution of (1.2)–(1.3) defined on time interval $[0,T]$. We set

$$\omega_T=\sup_{0\le t\le T}|||W\left(t\right)|||_{[s/2]}. \tag{2.39}$$

From the continuous embedding $H^{m-1}(\mathbb{T})\hookrightarrow L^\infty(\mathbb{T})$ for $m\ge 3$, there is a constant $C>0$ such that

$$\|h\|_{L^\infty}\le C\|h\|_{m-1},\quad \forall\, h\in H^{m-1}(\mathbb{T}).$$

If ω_T is sufficiently small, it is easy to see from $\bar{n}^\nu\ge$ const. >0 that $n^\nu=\bar{n}^\nu+N^\nu,\theta^\nu=1+\Theta^\nu\ge$ const. >0.

3. Energy estimates

In this section, we establish the uniform estimates for proving Theorem 1.1. In the first subsection, we obtain L^2 energy estimates for any smooth solution. In the second subsection, we get the higher-order energy estimates.

3.1. L^2 energy estimates

Lemma 3.4. *Assume that the conditions of Theorem 1.1 hold and ω_T is sufficiently small independent of T, then it holds*

$$\begin{aligned}&\frac{d}{dt}\left(\sum_{\nu=e,i}\int_{\mathbb{T}}\left(A_0^\nu(n^\nu,\theta^\nu)U^\nu\right)U^\nu\mathrm{d}x+\|F\|^2+\|G\|^2\right)\\&\quad+2\sum_{\nu=e,i}\int_{\mathbb{T}}n^\nu|u^\nu|^2\mathrm{d}x+2\sum_{\nu=e,i}\int_{\mathbb{T}}\frac{n^\nu}{\theta^\nu}|\Theta^\nu|^2\mathrm{d}x+2\sum_{\nu=e,i}\|\nabla\Theta^\nu\|^2\le C|||U|||_{[s/2]}^2|||W|||_{[s/2]}.\end{aligned}\tag{3.40}$$

Proof. It is divided by two steps as follows.
Step 1. It holds that

$$\|\partial_t(N^\nu,\Theta^\nu)\|_{L^\infty}\le C|||U|||_{[s/2]},\quad \|\partial_t A_0^\nu(n^\nu,\theta^\nu)\|_{L^\infty}\le C|||U|||_{[s/2]} \tag{3.41}$$

and

$$\left|\int_{\mathbb{T}}\left(\sum_{j=1}^3\partial_j\tilde{A}_j^\nu(n^\nu,u^\nu,\theta^\nu)-2A_0^\nu(n^\nu,\theta^\nu)L^\nu\right)VV\mathrm{d}x\right|\le C||U||_{[s/2]}\|V\|^2,\ \forall\ V\in\mathbb{R}^3. \tag{3.42}$$

In fact, from the continuous imbedding $H^{[s/2]-1}(\mathbb{T})\hookrightarrow L^\infty(\mathbb{T})$, we have

$$\|\partial_t(N^\nu,\Theta^\nu)\|_{L^\infty}\le C\|\partial_t(N^\nu,\Theta^\nu)\|_{[s/2]-1}\le C|||U^\nu|||_{[s/2]}\le C|||U|||_{[s/2]}.$$

Since

$$\partial_t A_0^\nu(n^\nu,\theta^\nu) = \begin{pmatrix} \dfrac{\partial_t\Theta^\nu}{n^\nu} - \dfrac{\partial_t N^\nu}{(n^\nu)^2} & 0 & 0 \\ 0 & \partial_t N^\nu I_3 & 0 \\ 0 & 0 & \dfrac{\partial_t N^\nu}{\theta^\nu} - \dfrac{n^\nu\partial_t\Theta^\nu}{(\theta^\nu)^2} \end{pmatrix},$$

we obtain

$$\|\partial_t A_0^\nu(n^\nu,\theta^\nu)\|_{L^\infty} \le C\|\partial_t(N^\nu,\Theta^\nu)\|_{L^\infty} \le C|||U^\nu|||_{[s/2]} \le C|||U|||_{[s/2]}.$$

Finally,

$$\sum_{j=1}^3 \partial_j \tilde{A}_j^\nu(n^\nu,u^\nu,\theta^\nu) - 2A_0^\nu L^\nu = \begin{pmatrix} \nabla\cdot\left(\dfrac{\theta^\nu u^\nu}{n^\nu}\right) & \left(\nabla\Theta^\nu - 2\dfrac{\theta^\nu\nabla\bar{n}^\nu}{n^\nu}\right)^T & 0 \\ \nabla\Theta^\nu + 2\dfrac{n^\nu\nabla\bar{n}^\nu}{(\bar{n}^\nu)^2} & \nabla\cdot(n^\nu u^\nu)I_3 & \nabla n^\nu - 2\dfrac{n^\nu\nabla\bar{n}^\nu}{\bar{n}^\nu} \\ 0 & (\nabla\bar{n}^\nu)^T & \nabla\cdot\left(\dfrac{n^\nu u^\nu}{\theta^\nu}\right) \end{pmatrix},$$

which is an anti-symmetric matrix at $(n^\nu,u^\nu,\theta^\nu)=(\bar{n}^\nu,0,1)$, and then (3.42) follows.

Step 2. Now, following the step above, we are ready to prove (3.40). Multiplying (2.34) by $A_0^\nu(n^\nu,\theta^\nu)U^\nu$ and taking integrations in x over $\mathbb{T}$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\nu=e,i}\int_{\mathbb{T}} \left(A_0^\nu(n^\nu,\theta^\nu)U^\nu\right)U^\nu \mathrm{d}x \\
&\quad= \sum_{\nu=e,i}\int_{\mathbb{T}} \left(\partial_t A_0^\nu(n^\nu,\theta^\nu)U^\nu\right)U^\nu\mathrm{d}x + \sum_{\nu=e,i}\int_{\mathbb{T}}\left(\sum_{j=1}^3\partial_j\tilde{A}_j^\nu(n^\nu,u^\nu,\theta^\nu) - 2A_0^\nu(n^\nu,\theta^\nu)L^\nu\right)U^\nu U^\nu\mathrm{d}x \\
&\qquad + 2\sum_{\nu=e,i}\int_{\mathbb{T}} A_0^\nu(n^\nu,\theta^\nu)\left(f^\nu - M^\nu(W)\right)U^\nu\mathrm{d}x.
\end{aligned}$$

By (3.41)–(3.42), we get

$$\left|\int_{\mathbb{T}}\partial_t A_0^\nu(n^\nu,\theta^\nu)U^\nu U^\nu\mathrm{d}x\right| \le \|\partial_t A_0^\nu(n^\nu,\theta^\nu)\|_{L^\infty}\|U^\nu\|^2 \le C|||U|||_{[s/2]}^3$$

and

$$\left|\int_{\mathbb{T}}\left(\sum_{j=1}^3\partial_j\tilde{A}_j^\nu(n^\nu,u^\nu,\theta^\nu) - 2A_0^\nu(n^\nu,\theta^\nu)L^\nu\right)U^\nu U^\nu\mathrm{d}x\right| \le C|||U|||_{[s/2]}^3.$$

By (2.38), we also obtain

$$\begin{aligned}
&\sum_{\nu=e,i}\int_{\mathbb{T}} A_0^\nu(n^\nu,\theta^\nu)M^\nu(W)U^\nu\mathrm{d}x \\
&\quad= \int_{\mathbb{T}}(n^e u^e - n^i u^i)F\mathrm{d}x + \sum_{\nu=e,i}\int_{\mathbb{T}} n^\nu|u^\nu|^2\mathrm{d}x + \sum_{\nu=e,i}\int_{\mathbb{T}}\frac{n^\nu}{\theta^\nu}|\Theta^\nu|^2\mathrm{d}x + \sum_{\nu=e,i}\|\nabla\Theta^\nu\|^2
\end{aligned}$$

and

$$\left|\int_{\mathbb{T}} A_0^\nu(n^\nu,\theta^\nu) f^\nu U^\nu \mathrm{d}x\right| \le C|||U|||_{[s/2]}^2 |||W|||_{[s/2]}.$$

Therefore,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\nu=e,i}\int_{\mathbb{T}}(A_0^\nu(n^\nu,\theta^\nu)U^\nu)U^\nu \mathrm{d}x + 2\sum_{\nu=e,i}\int_{\mathbb{T}} n^\nu|u^\nu|^2\mathrm{d}x + 2\sum_{\nu=e,i}\int_{\mathbb{T}}\frac{n^\nu}{\theta^\nu}|\Theta^\nu|^2\mathrm{d}x + 2\sum_{\nu=e,i}\|\nabla\Theta^\nu\|^2\\ &\le -2\int_{\mathbb{T}}(n^e u^e - n^i u^i)F\mathrm{d}x + C|||U|||_{[s/2]}^2|||W|||_{[s/2]},\end{aligned}$$

provided that ω_T is sufficiently small. Moreover, for the Maxwell equations in (2.30), similar to the estimates in the second section of [26], we also have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|F\|^2+\|G\|^2\right) = 2\int_{\mathbb{T}}(n^e u^e - n^i u^i)F\mathrm{d}x. \tag{3.43}$$

These last two relations imply (3.40). □

Next, we are ready to establish higher energy estimates for W. Firstly, we denote by

$$D_m(t) = |||U(t)|||_m^2 + \sum_{\nu=e,i}|||\nabla\Theta^\nu(t)|||_m^2, \quad \forall\, m\in\mathbb{N}. \tag{3.44}$$

3.2. Higher-order energy estimates

Assume $k\in\mathbb{N}$ and $\alpha\in\mathbb{N}^3$ with $1\le k+|\alpha|\le[s/2]$. Applying $\partial_t^k\partial^\alpha$ to (2.34), we get

$$\partial_t U_{k,\alpha}^\nu + \sum_{j=1}^3 A_j^\nu(n^\nu,u^\nu,\theta^\nu)\partial_j U_{k,\alpha}^\nu + L^\nu U_{k,\alpha}^\nu + M_{k,\alpha}^\nu = f_{k,\alpha}^\nu + g_\nu^{k,\alpha}, \tag{3.45}$$

where

$$U_{k,\alpha}^\nu = \partial_t^k\partial^\alpha U^\nu, \quad M_{k,\alpha}^\nu = \partial_t^k\partial^\alpha M^\nu, \quad f_{k,\alpha}^\nu = \partial_t^k\partial^\alpha f^\nu,$$

and

$$g_\nu^{k,\alpha} = \sum_{j=1}^3\left(A_j^\nu(n^\nu,u^\nu,\theta^\nu)\partial_j U_{k,\alpha}^\nu - \partial_t^k\partial^\alpha\left(A_j^\nu(n^\nu,u^\nu,\theta^\nu)\partial_j U^\nu\right)\right) + L^\nu U_{k,\alpha}^\nu - \partial_t^k\partial^\alpha\left(L^\nu U^\nu\right). \tag{3.46}$$

For the Maxwell equations, we also have

$$\begin{cases}\partial_t F_{k,\alpha} - \nabla\times G_{k,\alpha} = \partial_t^k\partial^\alpha\left(n^e u^e - n^i u^i\right), \ \nabla\cdot F_{k,\alpha} = \partial_t^k\partial^\alpha\left(N^i - N^e\right),\\ \partial_t G_{k,\alpha} + \nabla\times F_{k,\alpha} = 0, \ \nabla\cdot G_{k,\alpha} = 0,\end{cases} \tag{3.47}$$

with $F_{k,\alpha} = \partial_t^k\partial^\alpha F$ and $G_{k,\alpha} = \partial_t^k\partial^\alpha G$.

Lemma 3.5. *Assume that the conditions of Theorem 1.1 hold and ω_T is sufficiently small independent of T, then there exists a positive constant c_0 such that, for all $k\in\mathbb{N}$ and $\alpha\in\mathbb{N}^3$ with $|\alpha|\ge 1$ and*

$k+|\alpha|\le[s/2]$, *it holds*

$$\begin{aligned}&\frac{d}{dt}\left(\sum_{\nu=e,i}\int_{\mathbb{T}}(A_0^\nu(n^\nu,\theta^\nu)U_{k,\alpha}^\nu)U_{k,\alpha}^\nu\mathrm{d}x+\|F_{k,\alpha}\|^2+\|G_{k,\alpha}\|^2\right)\\&\quad+c_0\sum_{\nu=e,i}\left\|\partial_t^k\partial^\alpha(u^\nu,\Theta^\nu,\nabla\Theta^\nu)\right\|^2\\&\le C\sum_{\nu=e,i}\left\|\partial_t^kN^\nu\right\|_{|\alpha|}^2+C\sum_{\nu=e,i}\left\|(\partial_t^ku^\nu,\partial_t^k\Theta^\nu,\partial_t^k\nabla\Theta^\nu,\partial_t^{k+1}\Theta^\nu,\partial_t^kF)\right\|_{|\alpha|-1}^2\\&\quad+CD_{[s/2]}(t)|||W|||_{[s/2]}.\end{aligned}\tag{3.48}$$

Proof. It is divided by three steps as follows.
Step 1. For all $k\in\mathbb{N}$ and $\alpha\in\mathbb{N}^3$ with $1\le k+|\alpha|\le[s/2]$, it holds that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{\nu=e,i}\int_{\mathbb{T}}A_0^\nu(n^\nu,\theta^\nu)U_{k,\alpha}^\nu U_{k,\alpha}^\nu\mathrm{d}x+\|F_{k,\alpha}\|^2+\|G_{k,\alpha}\|^2\right)\\&\quad\le-2\sum_{\nu=e,i}\int_{\mathbb{T}}A_0^\nu(n^\nu,\theta^\nu)M_{k,\alpha}^\nu U_{k,\alpha}^\nu\mathrm{d}x+2\int_{\mathbb{T}}\partial_t^k\partial^\alpha(n^eu^e-n^iu^i)F_{k,\alpha}\mathrm{d}x\\&\qquad+2\sum_{\nu=e,i}\int_{\mathbb{T}}A_0^\nu(n^\nu,\theta^\nu)g_\nu^{k,\alpha}U_{k,\alpha}^\nu\mathrm{d}x+CD_{[s/2]}(t)|||W|||_{[s/2]}.\end{aligned}\tag{3.49}$$

In fact, for $1\le k+|\alpha|\le[s/2]$, multiplying (3.45) by $A_0^\nu(n^\nu,\theta^\nu)U_{k,\alpha}^\nu$ and taking integrations in x over $\mathbb{T}$, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}}(A_0^\nu U_{k,\alpha}^\nu)U_{k,\alpha}^\nu\mathrm{d}x&=\int_{\mathbb{T}}\partial_t(A_0^\nu U_{k,\alpha}^\nu)U_{k,\alpha}^\nu\mathrm{d}x+\int_{\mathbb{T}}\left(\sum_{j=1}^3\partial_j\tilde{A}_j^\nu(n^\nu,u^\nu,\theta^\nu)-2A_0^\nu L^\nu\right)U_{k,\alpha}^\nu U_{k,\alpha}^\nu\mathrm{d}x\\&\quad+2\int_{\mathbb{T}}(A_0^\nu f_{k,\alpha}^\nu)U_{k,\alpha}^\nu\mathrm{d}x-2\int_{\mathbb{T}}(A_0^\nu M_{k,\alpha}^\nu)U_{k,\alpha}^\nu\mathrm{d}x+2\int_{\mathbb{T}}(A_0^\nu g_\nu^{k,\alpha})U_{k,\alpha}^\nu\mathrm{d}x.\end{aligned}$$

Moreover, a standard energy estimate for (3.47) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|F_{k,\alpha}\|^2+\|G_{k,\alpha}\|^2\right)=2\int_{\mathbb{T}}\partial_t^k\partial^\alpha(n^eu^e-n^iu^i)F_{k,\alpha}\mathrm{d}x.\tag{3.50}$$

Then we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{\nu=e,i}\int_{\mathbb{T}}(A_0^\nu U_{k,\alpha}^\nu)U_{k,\alpha}^\nu\mathrm{d}x+\|F_{k,\alpha}\|^2+\|G_{k,\alpha}\|^2\right)\\&\quad=\sum_{\nu=e,i}\int_{\mathbb{T}}(\partial_tA_0^\nu U_{k,\alpha}^\nu)U_{k,\alpha}^\nu\mathrm{d}x+\sum_{\nu=e,i}\int_{\mathbb{T}}(\sum_{j=1}^3\partial_j\tilde{A}_j^\nu-2A_0^\nu L^\nu)U_{k,\alpha}^\nu U_{k,\alpha}^\nu\mathrm{d}x+2\sum_{\nu=e,i}\int_{\mathbb{T}}(A_0^\nu f_{k,\alpha}^\nu)U_{k,\alpha}^\nu\mathrm{d}x\\&\qquad-2\sum_{\nu=e,i}\int_{\mathbb{T}}(A_0^\nu M_{k,\alpha}^\nu)U_{k,\alpha}^\nu\mathrm{d}x+2\sum_{\nu=e,i}\int_{\mathbb{T}}(A_0^\nu g_\nu^{k,\alpha})U_{k,\alpha}^\nu\mathrm{d}x+2\int_{\mathbb{T}}\partial_t^k\partial^\alpha(n^eu^e-n^iu^i)F_{k,\alpha}\mathrm{d}x.\end{aligned}$$

It is clear that the first three terms on the right hand side of the equality above are bounded by $CD_{[s/2]}(t)|||W|||_{[s/2]}$. This proves (3.49).

Step 2. For all $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^3$ with $|\alpha| \geq 1$ and $k + |\alpha| \leq [s/2]$, it holds

$$\begin{aligned}
&2\int_{\mathbb{T}} \partial_t^k \partial^\alpha (n^e u^e - n^i u^i) F_{k,\alpha} \mathrm{d}x - 2\sum_{\nu=e,i} \int_{\mathbb{T}} (A_0^\nu(n^\nu,\theta^\nu) M_{k,\alpha}^\nu) U_{k,\alpha}^\nu \mathrm{d}x \\
&\quad \leq - \sum_{\nu=e,i} \left(\int_{\mathbb{T}} n^\nu |u_{k,\alpha}^\nu|^2 \mathrm{d}x + \int_{\mathbb{T}} \frac{n^\nu}{\theta^\nu} |\Theta_{k,\alpha}^\nu|^2 \mathrm{d}x + \left\|\nabla \Theta_{k,\alpha}^\nu\right\|^2 \right) \\
&\qquad + C \sum_{\nu=e,i} \left\| \left(\partial_t^k u^\nu, \partial_t^k \Theta^\nu, \partial_t^{k+1}\Theta^\nu, \partial_t^k F\right) \right\|_{|\alpha|-1}^2 + C D_{[s/2]}(t) |||W|||_{[s/2]},
\end{aligned} \tag{3.51}$$

and

$$\begin{aligned}
\left| \int_{\mathbb{T}} (A_0^\nu(n^\nu,\theta^\nu) g_\nu^{k,\alpha}) U_{k,\alpha}^\nu \mathrm{d}x \right| &\leq \varepsilon \left\| \partial_t^k \partial^\alpha \left(u^\nu, \nabla\Theta^\nu\right) \right\|^2 + \left\| \partial_t^k \left(u^\nu, \Theta^\nu, \nabla\Theta^\nu\right) \right\|_{|\alpha|-1}^2 \\
&\quad + C \left\| \partial_t^k N^\nu \right\|_{|\alpha|}^2 + C |||U|||_{[s/2]}^3,
\end{aligned} \tag{3.52}$$

where $u_{k,\alpha}^\nu = \partial_t^k \partial^\alpha u^\nu, \Theta_{k,\alpha}^\nu = \partial_t^k \partial^\alpha \Theta^\nu$ and $\varepsilon > 0$ is a small constant to be chosen later.

In fact, from (2.37), we get

$$M_{k,\alpha}^\nu = \begin{pmatrix} 0 \\ u_{k,\alpha}^\nu - q_\nu (F_{k,\alpha} + u_{k,\alpha}^\nu \times \bar{B}) \\ \Theta_{k,\alpha}^\nu - \partial_t^k \partial^\alpha \left(\frac{\theta^\nu}{n^\nu} \Delta \Theta^\nu \right) \end{pmatrix}.$$

It follows from the fact $u_{k,\alpha}^\nu \cdot (u_{k,\alpha}^\nu \times \bar{B}) = 0$ that

$$\begin{aligned}
&- \int_{\mathbb{T}} (A_0^\nu(n^\nu,\theta^\nu) M_{k,\alpha}^\nu) U_{k,\alpha}^\nu \mathrm{d}x \\
&\quad = - \int_{\mathbb{T}} n^\nu |u_{k,\alpha}^\nu|^2 \mathrm{d}x - \int_{\mathbb{T}} \frac{n^\nu}{\theta^\nu} |\Theta_{k,\alpha}^\nu|^2 \mathrm{d}x + \int_{\mathbb{T}} \frac{n^\nu}{\theta^\nu} \partial_t^k \partial^\alpha \left(\frac{\theta^\nu}{n^\nu} \Delta\Theta^\nu \right) \Theta_{k,\alpha}^\nu \mathrm{d}x + q_\nu \int_{\mathbb{T}} n^\nu u_{k,\alpha}^\nu F_{k,\alpha} \mathrm{d}x,
\end{aligned}$$

which implies that

$$\begin{aligned}
&- \int_{\mathbb{T}} (A_0^\nu(n^\nu,\theta^\nu) M_{k,\alpha}^\nu) U_{k,\alpha}^\nu \mathrm{d}x - q_\nu \int_{\mathbb{T}} \partial_t^k \partial^\alpha (n^\nu u^\nu) F_{k,\alpha} \mathrm{d}x \\
&\quad = - \int_{\mathbb{T}} n^\nu |u_{k,\alpha}^\nu|^2 \mathrm{d}x - \int_{\mathbb{T}} \frac{n^\nu}{\theta^\nu} |\Theta_{k,\alpha}^\nu|^2 \mathrm{d}x + \int_{\mathbb{T}} \frac{n^\nu}{\theta^\nu} \partial_t^k \partial^\alpha \left(\frac{\theta^\nu}{n^\nu} \Delta\Theta^\nu \right) \Theta_{k,\alpha}^\nu \mathrm{d}x \\
&\qquad - q_\nu \int_{\mathbb{T}} \left(\partial_t^k \partial^\alpha (n^\nu u^\nu) - n^\nu u_{k,\alpha}^\nu \right) F_{k,\alpha} \mathrm{d}x.
\end{aligned} \tag{3.53}$$

For the third term on the right hand side of (3.53), by Leibniz formulas, we have

$$\int_{\mathbb{T}} \frac{n^\nu}{\theta^\nu} \partial_t^k \partial^\alpha \left(\frac{\theta^\nu}{n^\nu} \Delta\Theta^\nu \right) \Theta_{k,\alpha}^\nu \mathrm{d}x = I_1^\nu + \sum_{\substack{|\beta|<|\alpha| \\ l<k}} C_\alpha^\beta C_k^l I_{2l\beta}^\nu + \sum_{|\beta|<|\alpha|} C_\alpha^\beta I_{3\beta}^\nu + \sum_{l<k} C_k^l I_{4l}^\nu, \tag{3.54}$$

where

$$I_1^\nu = \int_{\mathbb{T}} \Delta\Theta_{k,\alpha}^\nu \Theta_{k,\alpha}^\nu \mathrm{d}x, \quad I_{2l\beta}^\nu = \int_{\mathbb{T}} \frac{n^\nu}{\theta^\nu} \partial_t^{k-l} \partial^{\alpha-\beta} \left(\frac{\theta^\nu}{n^\nu} \right) \partial_t^k \partial^\beta \Delta\Theta^\nu \Theta_{k,\alpha}^\nu \mathrm{d}x,$$

$$I_{3\beta}^{\nu}=\int\limits_{\mathbb{T}}\frac{n^{\nu}}{\theta^{\nu}}\partial^{\alpha-\beta}\left(\frac{\theta^{\nu}}{n^{\nu}}\right)\partial_t^k\partial^{\beta}\Delta\Theta^{\nu}\Theta_{k,\alpha}^{\nu}\mathrm{d}x,\quad I_{4l}^{\nu}=\int\limits_{\mathbb{T}}\frac{n^{\nu}}{\theta^{\nu}}\partial_t^{k-l}\left(\frac{\theta^{\nu}}{n^{\nu}}\right)\partial_t^l\partial^{\alpha}\Delta\Theta^{\nu}\Theta_{k,\alpha}^{\nu}\mathrm{d}x.$$

It is easy to see that $I_{2l\beta}^{\nu}=I_{4l}^{\nu}=0$ when $k=0$. Using an integration by parts, we get

$$I_1^{\nu}=\int\limits_{\mathbb{T}}\Delta\Theta_{k,\alpha}^{\nu}\Theta_{k,\alpha}^{\nu}\mathrm{d}x=-\left\|\nabla\Theta_{k,\alpha}^{\nu}\right\|^2. \tag{3.55}$$

By Sobolev embedding [6],Cauchy–Schwarz inequality and an integration by parts, we obtain

$$\begin{aligned}\left|I_{2l\beta}^{\nu}\right|&\le C\left\|\frac{n^{\nu}}{\theta^{\nu}}\right\|_{L^\infty}\left\|\partial_t^{k-l}\partial^{\alpha-\beta}\left(\frac{\theta^{\nu}}{n^{\nu}}\right)\right\|\left\|\partial_t^l\partial^{\beta}\Delta\Theta^{\nu}\right\|_{L^\infty}\left\|\Theta_{k,\alpha}^{\nu}\right\|\\&\le C\left\|\partial_t^k\partial^{\alpha}\left(\frac{\theta^{\nu}}{n^{\nu}}\right)\right\|\left\|\Delta\Theta^{\nu}\right\|_2\left\|\Theta_{k,\alpha}^{\nu}\right\|\le CD_{[s/2]}(t)\|\Theta^{\nu}\|_{[s/2]},\ \text{when } l=|\beta|=0,\end{aligned} \tag{3.56}$$

$$\begin{aligned}\left|I_{2l\beta}^{\nu}\right|&\le C\left\|\frac{n^{\nu}}{\theta^{\nu}}\right\|_{L^\infty}\left\|\partial_t^{k-l}\partial^{\alpha-\beta}\left(\frac{\theta^{\nu}}{n^{\nu}}\right)\right\|_{L^4}\left\|\partial_t^l\partial^{\beta}\Delta\Theta^{\nu}\right\|_{L^4}\left\|\Theta_{k,\alpha}^{\nu}\right\|\\&\le C\left\|\partial_t^{k-l}\partial^{\alpha-\beta}\left(\frac{\theta^{\nu}}{n^{\nu}}\right)\right\|_1\left\|\partial_t^l\partial^{\beta}\Delta\Theta^{\nu}\right\|_1\left\|\Theta_{k,\alpha}^{\nu}\right\|\le CD_{[s/2]}(t)\|\Theta^{\nu}\|_{[s/2]},\ \text{when } l+|\beta|=1,\end{aligned} \tag{3.57}$$

and

$$\begin{aligned}\left|I_{2l\beta}^{\nu}\right|&\le C\left\|\frac{n^{\nu}}{\theta^{\nu}}\right\|_{L^\infty}\left\|\partial_t^{k-l}\partial^{\alpha-\beta}\left(\frac{\theta^{\nu}}{n^{\nu}}\right)\right\|_{L^\infty}\left\|\partial_t^l\partial^{\beta}\Delta\Theta^{\nu}\right\|\left\|\Theta_{k,\alpha}^{\nu}\right\|\\&\le\left\|\partial_t^{k-l}\partial^{\alpha-\beta}\left(\frac{\theta^{\nu}}{n^{\nu}}\right)\right\|_2\left\|\partial_t^l\partial^{\beta}\Delta\Theta^{\nu}\right\|\left\|\Theta_{k,\alpha}^{\nu}\right\|\le CD_{[s/2]}(t)\|\Theta^{\nu}\|_{[s/2]},\ \text{when } l+|\beta|\ge 2.\end{aligned} \tag{3.58}$$

Moreover, by the third equation of (2.30) and an integration by parts, we have

$$\begin{aligned}I_{3\beta}&=\int\limits_{\mathbb{T}}\frac{n^{\nu}}{\theta^{\nu}}\partial^{\alpha-\beta}\left(\frac{\theta^{\nu}}{n^{\nu}}\right)\partial_t^k\partial^{\beta}\left(\frac{n^{\nu}}{\theta^{\nu}}\left(\partial_t\Theta^{\nu}+\nabla\cdot u^{\nu}+\Theta^{\nu}+u^{\nu}\cdot\nabla\Theta^{\nu}\right)\right)\Theta_{k,\alpha}^{\nu}\mathrm{d}x\\&\quad+\int\limits_{\mathbb{T}}\frac{n^{\nu}}{\theta^{\nu}}\partial^{\alpha-\beta}\left(\frac{\theta^{\nu}}{n^{\nu}}\right)\partial_t^k\partial^{\beta}\left(\frac{n^{\nu}}{\theta^{\nu}}\left(\Theta^{\nu}\nabla\cdot u^{\nu}-\frac{1}{n^{\nu}}|\nabla\Theta^{\nu}|^2\right)\right)\Theta_{k,\alpha}^{\nu}\mathrm{d}x\\&\le\varepsilon\left\|\left(\Theta_{k,\alpha}^{\nu},\nabla\Theta_{k,\alpha}^{\nu}\right)\right\|^2+C\left\|\left(\partial_t^k u^{\nu},\partial_t^k\Theta^{\nu},\partial_t^{k+1}\Theta^{\nu}\right)\right\|_{|\alpha|-1}^2,\end{aligned} \tag{3.59}$$

and

$$|I_{4l}^{\nu}|\le CD_{[s/2]}(t)\|\Theta^{\nu}\|_{[s/2]}. \tag{3.60}$$

For the last term on the right hand side of (3.53), recall that $|\alpha|\ge 1$ and for ω_T small $n^{\nu}=\bar{n}^{\nu}+N^{\nu}\ge$ const. >0. Then similarly to that in [22], an integration by parts to get

$$\begin{aligned}&\left|\int\limits_{\mathbb{T}}\left(\partial_t^k\partial^{\alpha}(n^{\nu}u^{\nu})-n^{\nu}u_{k,\alpha}^{\nu}\right)F_{k,\alpha}\mathrm{d}x\right|\\&\quad\le\varepsilon\int\limits_{\mathbb{T}}|n^{\nu}u_{k,\alpha}^{\nu}|^2\mathrm{d}x+C||\partial_t^k u^{\nu}||_{\alpha|-1}^2+C||\partial_t^k F||_{\alpha|-1}^2+C|||U|||_{[s/2]}^2|||W|||_{[s/2]},\end{aligned}$$

where $\alpha_1\in\mathbb{N}^3$ with $|\alpha_1|=|\alpha|-1$. Therefore, taking ε small enough, the above estimate together with (3.53)–(3.60), gives (3.51).

Next, by (3.46), we obtain

$$\int_{\mathbb{T}} (A_0^\nu(n^\nu,\theta^\nu) g_\nu^{k,\alpha}) U_{k,\alpha}^\nu \mathrm{d}x = \sum_{j=1}^{3} K_{1j}^\nu + K_2^\nu, \tag{3.61}$$

where

$$K_{1j}^\nu = \int_{\mathbb{T}} A_0^\nu(n^\nu,\theta^\nu) \left(A_j^\nu(n^\nu,u^\nu,\theta^\nu)\partial_j U_{k,\alpha}^\nu - \partial_t^k \partial^\alpha \left(A_j^\nu(n^\nu,u^\nu,\theta^\nu)\partial_j U^\nu \right)\right) U_{k,\alpha}^\nu \mathrm{d}x,$$

and

$$K_2^\nu = \int_{\mathbb{T}} A_0^\nu(n^\nu,\theta^\nu) \left(L^\nu U_{k,\alpha}^\nu - \partial_t^k\partial^\alpha \left(L^\nu U^\nu\right)\right) U_{k,\alpha}^\nu \mathrm{d}x.$$

By (2.35)–(2.36) and the definition of $A_0^\nu(n^\nu,\theta^\nu)$, we have

$$\begin{aligned} K_{1j}^\nu = & \int_{\mathbb{T}} \frac{\theta^\nu}{n^\nu} N_{k,\alpha}^\nu \left(u_j^\nu \partial_j N_{k,\alpha}^\nu - \partial_t^k\partial^\alpha \left(u_j^\nu \partial_j N^\nu\right)\right) \mathrm{d}x \\ & + \int_{\mathbb{T}} \frac{\theta^\nu}{n^\nu} N_{k,\alpha}^\nu \left(n^\nu \partial_j \partial_t^k \partial^\alpha u_j^\nu - \partial_t^k\partial^\alpha \left(n^\nu \partial_j u_j^\nu\right)\right) \mathrm{d}x \\ & + \int_{\mathbb{T}} n^\nu u_{k,\alpha}^\nu \left(\frac{\theta^\nu}{n^\nu} \partial_j N_{k,\alpha}^\nu - \partial_t^k\partial^\alpha \left(\frac{\theta^\nu}{n^\nu}\partial_j N^\nu\right)\right) \mathrm{d}x \\ & + \int_{\mathbb{T}} n^\nu u_{k,\alpha}^\nu \left(u_j^\nu \partial_j u_{k,\alpha}^\nu - \partial_t^k\partial^\alpha \left(u_j^\nu \partial_j u^\nu\right)\right) \mathrm{d}x \\ & + \int_{\mathbb{T}} \frac{n^\nu}{\theta^\nu} \Theta_{k,\alpha}^\nu \left(\theta^\nu \partial_j \partial_t^k \partial^\alpha u_j^\nu - \partial_t^k\partial^\alpha \left(\theta^\nu \partial_j u_j^\nu\right)\right) \mathrm{d}x \\ & + \int_{\mathbb{T}} \frac{n^\nu}{\theta^\nu} \Theta_{k,\alpha}^\nu \left(u_j^\nu \partial_j \Theta_{k,\alpha}^\nu - \partial_t^k\partial^\alpha \left(u_j^\nu \partial_j \Theta^\nu\right)\right) \mathrm{d}x \\ \le & \, \varepsilon \left\|\partial_t^k \partial^\alpha (u^\nu, \nabla\Theta^\nu)\right\|^2 + C\left\|\partial_t^k(u^\nu,\nabla\Theta^\nu)\right\|_{|\alpha|-1}^2 + C\left\|\partial_t^k N^\nu\right\|_{|\alpha|}^2 + C|||U|||_{[s/2]}^3, \end{aligned} \tag{3.62}$$

and

$$\begin{aligned} K_2^\nu = & \int_{\mathbb{T}} \frac{\theta^\nu}{n^\nu} N_{k,\alpha}^\nu \left(u_{k,\alpha}^\nu \cdot \nabla \bar{n}^\nu - \partial_t^k \partial^\alpha \left(u^\nu \cdot \nabla\bar{n}^\nu\right)\right) \mathrm{d}x \\ & + \int_{\mathbb{T}} n^\nu u_{k,\alpha}^\nu \left(\partial_t^k\partial^\alpha \left(\frac{N^\nu \nabla \bar{n}^\nu}{|\bar{n}^\nu|^2}\right) - \frac{\nabla\bar{n}^\nu}{|\bar{n}^\nu|^2} N_{k,\alpha}^\nu \right) \mathrm{d}x \\ & + \int_{\mathbb{T}} n^\nu u_{k,\alpha}^\nu \left(\frac{\nabla \bar{n}^\nu}{\bar{n}^\nu} \Theta_{k,\alpha}^\nu - \partial^\alpha \left(\frac{\nabla\bar{n}^\nu}{\bar{n}^\nu}\partial_t^k \Theta^\nu\right)\right) \mathrm{d}x \\ \le & \, \varepsilon \|u_{k,\alpha}^\nu\|^2 + C\left\|\partial_t^k N^\nu\right\|_{|\alpha|}^2 + C\left\|\partial_t^k (u^\nu,\Theta^\nu)\right\|_{|\alpha|-1}^2, \end{aligned} \tag{3.63}$$

where $N_{k,\alpha}^\nu = \partial_t^k \partial^\alpha N^\nu$. Therefore, it follows from (3.61)–(3.63) that (3.52).

Step 3. Now, following the two steps above, we are ready to prove (3.48). This inequality follows from (3.49) and (3.51)–(3.52) by taking $\varepsilon > 0$ small enough. □

Remark 3.4. *Lemma 3.5 is valid for $|\alpha| \ge 1$. The next result concerns the L^2 estimates for $\partial_t^k W$ (i.e., $\alpha = 0$), which is a starting point for applying the argument by induction.*

Lemma 3.6. *Assume that the conditions of Theorem 1.1 hold and ω_T is sufficiently small independent of T, then there exists a positive constant c_0 such that, for all $0 \le k \le [s/2]$, it holds*

$$\begin{aligned} &\frac{d}{dt}\left(\sum_{\nu=e,i}\int_{\mathbb{T}}\left(A_0^\nu(n^\nu,\theta^\nu)\partial_t^k U^\nu\right)\partial_t^k U^\nu \mathrm{d}x + \left\|\partial_t^k F\right\|^2 + \left\|\partial_t^k G\right\|^2\right) \\ &\quad + c_0\sum_{\nu=e,i}\left\|\partial_t^k(u^\nu,\Theta^\nu,\nabla\Theta^\nu)\right\|^2 \\ &\le CD_{[s/2]}(t)|||W|||_{[s/2]}. \end{aligned} \tag{3.64}$$

Proof. Recall that $n^\nu, \theta^\nu \ge$ const. > 0. For $k = 0$, estimate (3.64) is given by Lemma 3.4. For $1 \le k \le [s/2]$, (3.64) follows from (3.49) with $\alpha = 0$ and and the following two estimates:

$$\begin{aligned} &\int_{\mathbb{T}}\partial_t^k(n^e u^e - n^i u^i)\partial_t^k F \mathrm{d}x - \sum_{\nu=e,i}\int_{\mathbb{T}}(A_0^\nu(n^\nu,\theta^\nu)\partial_t^k M^\nu)\partial_t^k U^\nu \mathrm{d}x \\ &\le -\sum_{\nu=e,i}\left(\int_{\mathbb{T}} n^\nu|\partial_t^k u^\nu|^2\mathrm{d}x + \int_{\mathbb{T}}\frac{n^\nu}{\theta^\nu}|\partial_t^k\Theta^\nu|^2\mathrm{d}x + \left\|\nabla\partial_t^k\Theta^\nu\right\|^2\right) \\ &\quad + CD_{[s/2]}(t)|||W|||_{[s/2]}, \end{aligned} \tag{3.65}$$

and

$$\left|\int_{\mathbb{T}}\left(A_0^\nu(n^\nu,\theta^\nu)\partial_t^k g_\nu\right)\partial_t^k U^\nu \mathrm{d}x\right| \le C|||U|||^3_{[s/2]}. \tag{3.66}$$

In fact, for $\alpha = 0$, (3.53) becomes

$$\begin{aligned} &-\int_{\mathbb{T}}\left(A_0^\nu(n^\nu,\theta^\nu)\partial_t^k M^\nu\right)\partial_t^k U^\nu \mathrm{d}x - q_\nu\int_{\mathbb{T}}\partial_t^k(n^\nu u^\nu)\partial_t^k F\mathrm{d}x \\ &= -\int_{\mathbb{T}} n^\nu|\partial_t^k u^\nu|^2\mathrm{d}x - \int_{\mathbb{T}}\frac{n^\nu}{\theta^\nu}|\partial_t^k\Theta^\nu|^2\mathrm{d}x + \int_{\mathbb{T}}\frac{n^\nu}{\theta^\nu}\partial_t^k\left(\frac{\theta^\nu}{n^\nu}\Delta\Theta^\nu\right)\partial_t^k\Theta^\nu\mathrm{d}x \\ &\quad - q_\nu\int_{\mathbb{T}}\left(\partial_t^k(n^\nu u^\nu) - n^\nu\partial_t^k u^\nu\right)\partial_t^k F\mathrm{d}x. \end{aligned} \tag{3.67}$$

Using the Leibniz formula and an integration by parts, we get

$$\begin{aligned} &\int_{\mathbb{T}}\frac{n^\nu}{\theta^\nu}\partial_t^k\left(\frac{\theta^\nu}{n^\nu}\Delta\Theta^\nu\right)\partial_t^k\Theta^\nu\mathrm{d}x \\ &= \int_{\mathbb{T}}\Delta\partial_t^k\Theta^\nu\partial_t^k\Theta^\nu\mathrm{d}x + \sum_{l<k}C_k^l\int_{\mathbb{T}}\frac{n}{\theta}\partial_t^{k-l}\left(\frac{\theta}{n}\right)\Delta\partial_t^l\Theta^\nu\partial_t^k\Theta^\nu\mathrm{d}x \\ &\le -\left\|\nabla\partial_t^k\Theta^\nu\right\|^2 + CD_{[s/2]}(t)|||U|||_{[s/2]}, \end{aligned} \tag{3.68}$$

and

$$\begin{aligned} \int_{\mathbb{T}}\left(\partial_t^k(n^\nu u^\nu) - n^\nu\partial_t^k u^\nu\right)\partial_t^k F\mathrm{d}x &= \sum_{l<k}C_k^l\int_{\mathbb{T}}\partial_t^{k-l}N^\nu\partial_t^l u^\nu\partial_t^k F\mathrm{d}x \\ &\le C|||U|||^2_{[s/2]}|||W|||_{[s/2]}. \end{aligned} \tag{3.69}$$

Then (3.65) follows from (3.67)–(3.69).

From (3.61) with $\alpha = 0$, we obtain

$$\int_{\mathbb{T}} \left(A_0^\nu(n^\nu,\theta^\nu)\partial_t^k g_\nu\right)\partial_t^k U^\nu \mathrm{d}x = -\sum_{j=1}^{3}\sum_{l<k} C_k^l \int_{\mathbb{T}} A_0^\nu(n^\nu,\theta^\nu)\partial_t^{k-l}A_j^\nu \partial_t^l \partial_j U^\nu \partial_t^k U^\nu \mathrm{d}x$$
$$\le C|||U|||_{[s/2]}^3,$$

which implies (3.66). □

We conclude the result of this subsection as follows.

Proposition 3.3. *Assume that the conditions of Theorem 1.1 hold and ω_T is sufficiently small independent of T, then there exists a positive constant c_0 such that, for all $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^3$ with $|\alpha| \ge 1$ and $k + |\alpha| \le [s/2]$, it holds*

$$\begin{aligned}
&\frac{d}{dt}\sum_{\beta\le\alpha}\left(\sum_{\nu=e,i}\int_{\mathbb{T}}\left(A_0^\nu(n^\nu,\theta^\nu)U_{k,\beta}^\nu\right)U_{k,\beta}^\nu \mathrm{d}x + \|F_{k,\beta}\|^2 + \|G_{k,\beta}\|^2\right)\\
&\quad + c_0\sum_{\nu=e,i}\left\|\partial_t^k(u^\nu,\Theta^\nu,\nabla\Theta^\nu)\right\|_{|\alpha|}^2\\
&\le C\sum_{\nu=e,i}\left(\left\|\partial_t^k N^\nu\right\|_{|\alpha|}^2 + \left\|(\partial_t^k u^\nu,\partial_t^k\Theta^\nu,\partial_t^k\nabla\Theta^\nu,\partial_t^{k+1}\Theta^\nu)\right\|_{|\alpha|-1}^2\right) + C\|\partial_t^k F\|_{|\alpha|-1}^2\\
&\quad + CD_{[s/2]}(t)|||W|||_{[s/2]}.
\end{aligned} \tag{3.70}$$

Proof. Let $k \ge 0$ be fixed. Noting that

$$\left\|\partial_t^k(u^\nu,\Theta^\nu,\nabla\Theta^\nu)\right\|^2 + \sum_{1\le|\beta|,\beta\le\alpha}\left\|\partial_t^k\partial^\beta(u^\nu,\Theta^\nu,\nabla\Theta^\nu)\right\|^2 \sim \left\|\partial_t^k(u^\nu,\Theta^\nu,\nabla\Theta^\nu)\right\|_{|\alpha|}^2.$$

Hence, summing (3.48) for all indexes up to $|\alpha| \le [s/2] - k$ and combining the resulting inequality with Lemma 3.6 yields (3.70). □

3.3. Relation of recurrence

Lemma 3.7. *Assume that the conditions of Theorem 1.1 hold and ω_T is sufficiently small independent of T, then for all $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^3$ with $|\alpha| \ge 1$ and $k + |\alpha| \le [s/2]$, it holds*

$$\begin{aligned}
&\sum_{\nu=e,i}\left\|\partial_t^k N^\nu\right\|_{|\alpha|}^2 + \left\|\partial_t^k(N^e - N^i)\right\|_{|\alpha|-1}^2\\
&\quad\le C\sum_{\nu=e,i}\left\|(\partial_t^k N^\nu,\partial_t^k u^\nu,\partial_t^{k+1}u^\nu,\partial_t^k\Theta^\nu,\partial_t^k\nabla\Theta^\nu)\right\|_{|\alpha|-1}^2 + CD_{[s/2]}(t)|||W|||_{[s/2]},
\end{aligned} \tag{3.71}$$

and

$$\begin{aligned}
\left\|\partial_t^k F\right\|_{|\alpha|-1}^2 &\le C\sum_{\nu=e,i}\left\|(\partial_t^k N^\nu,\partial_t^k u^\nu,\partial_t^{k+1}u^\nu,\partial_t^k\Theta^\nu,\partial_t^k\nabla\Theta^\nu)\right\|_{|\alpha|-1}^2\\
&\quad + CD_{[s/2]}(t)|||W|||_{[s/2]}.
\end{aligned} \tag{3.72}$$

Proof. For $k \in \mathbb{N}$ and $\beta \in \mathbb{N}^3$ with $k + |\beta| \le s - 1$, applying $\partial_t^k\partial^\beta$ to the second equation of (2.30) and using

$$\nabla\left(\ln(\bar{n}^\nu + N^\nu) - \ln\bar{n}^\nu\right) = \nabla\left(\frac{N^\nu}{\bar{n}^\nu}\right) + \frac{(N^\nu)^2}{n^\nu(\bar{n}^\nu)^2} - \frac{N^\nu\nabla N^\nu}{n^\nu\bar{n}^\nu},$$

we get

$$\begin{aligned}&\partial^\beta\nabla\left(\frac{\partial_t^k N^\nu}{\bar n^\nu}\right)-q_\nu F_{k,\beta}\\&\quad=q_\nu u^\nu_{k,\beta}\times\bar B-u^\nu_{k,\beta}-u^\nu_{k+1,\beta}-\nabla\Theta^\nu_{k,\beta}+\partial^\beta\left(\frac{\nabla\bar n^\nu}{\bar n^\nu}\Theta_k\right)-H^{k,\beta}_{\nu1},\end{aligned}\tag{3.73}$$

where

$$H^{k,\beta}_{\nu1}=\partial_t^k\partial^\beta\left(\frac{(N^\nu)^2}{n^\nu(\bar n^\nu)^2}+(u^\nu\cdot\nabla)u^\nu+\frac{\Theta^\nu\nabla N^\nu}{n^\nu}-\frac{N^\nu\Theta^\nu}{n^\nu\bar n^\nu}-\frac{N^\nu\nabla N^\nu}{n^\nu\bar n^\nu}-q_\nu u^\nu\times G\right),$$

By Lemma 1.2, we have

$$\left\|H^{k,\beta}_{\nu1}\right\|\le C|||U|||_{[s/2]}|||W|||_{[s/2]}.$$

Now we write

$$\partial^\beta\nabla\left(\frac{\partial_t^k N^\nu}{\bar n^\nu}\right)=\frac{1}{\bar n^\nu}\nabla N^\nu_{k,\beta}+H^{k,\beta}_{\nu2},$$

where

$$H^{k,\beta}_{\nu2}=\sum_{\gamma\le\beta}h_{k\beta\gamma}(x)N^\nu_{k,\gamma},$$

where $h_{k\beta\gamma}(x)$ are given smooth functions. It follows that

$$\left\|H^{k,\beta}_{\nu2}\right\|\le C\left\|\partial_t^k N^\nu\right\|_{|\beta|}.$$

Taking the inner product of (3.73) with $\nabla N^\nu_{k,\beta}$ in $L^2(\mathbb{T})$ and noting that $\bar n^\nu\ge$ const. >0 yields

$$\begin{aligned}&\left\|(\bar n^\nu)^{-\frac12}\nabla N^\nu_{k,\beta}\right\|^2-q_\nu\int_{\mathbb{T}}F_{k,\beta}\cdot\nabla N^\nu_{k,\beta}\mathrm{d}x\\&=\int_{\mathbb{T}}\left(q_\nu u^\nu_{k,\beta}\times\bar B-u^\nu_{k,\beta}-u^\nu_{k+1,\beta}-\nabla\Theta^\nu_{k,\beta}-\partial^\beta\left(\frac{\nabla\bar n^\nu}{\bar n^\nu}\Theta^\nu_k-H^{k,\beta}_{\nu1}-H^{k,\beta}_{\nu2}\right)\right)\nabla N^\nu_{k,\beta}\mathrm{d}x\\&\le\varepsilon\left\|(\bar n^\nu)^{-\frac12}\nabla N^\nu_{k,\beta}\right\|^2+C\left\|(\partial_t^kN^\nu,\partial_t^ku^\nu,\partial_t^{k+1}u^\nu,\partial_t^k\Theta^\nu,\nabla\partial_t^k\Theta^\nu)\right\|^2_{|\beta|}+CD_{[s/2]}(t)|||W|||_{[s/2]},\end{aligned}$$

where $\varepsilon>0$ is small enough. Noting that

$$\begin{aligned}-\sum_{\nu=e,i}q_\nu\int_{\mathbb{T}}F_{k,\beta}\cdot\nabla N^\nu_{k,\beta}\mathrm{d}x&=-\int_{\mathbb{T}}F_{k,\beta}\cdot\nabla(N^i_{k,\beta}-N^e_{k,\beta})\mathrm{d}x\\&=\int_{\mathbb{T}}\left(N^i_{k,\beta}-N^e_{k,\beta}\right)\nabla\cdot F_{k,\beta}\mathrm{d}x\\&=\left\|\partial_t^k\partial^\beta(N^i-N^e)\right\|^2\ge0,\end{aligned}$$

we obtain

$$\begin{aligned}&\sum_{\nu=e,i}\left\|\nabla N^\nu_{k,\beta}\right\|^2+\left\|N^e_{k,\beta}-N^i_{k,\beta}\right\|^2\\&\quad\le C\sum_{\nu=e,i}\left\|(\partial_t^kN^\nu,\partial_t^ku^\nu,\partial_t^{k+1}u^\nu,\partial_t^k\Theta^\nu,\nabla\partial_t^k\Theta^\nu)\right\|^2_{|\beta|}+CD_{[s/2]}(t)|||W|||_{[s/2]}.\end{aligned}$$

Summing the inequalities for all indexes β yields

$$\sum_{\nu=e,i} \left\|\partial_t^k \nabla N^\nu\right\|_{|\beta|}^2 + \left\|\partial_t^k (N^e - N^i)\right\|_{|\beta|}^2 \leq C \sum_{\nu=e,i} \left\|(\partial_t^k N^\nu, \partial_t^k u^\nu, \partial_t^{k+1} u^\nu, \partial_t^k \Theta^\nu, \nabla \partial_t^k \Theta^\nu)\right\|_{|\beta|}^2 + CD_{[s/2]}(t)|||W|||_{[s/2]}.$$

This shows (3.71) by replacing β by α with $|\alpha| = |\beta| + 1$ and using Lemma 1.1.

Finally, from (3.73), we have

$$\left\|\partial_t^k \partial^\beta F\right\|^2 \leq C \sum_{\nu=e,i} \left(\left\|\partial_t^k N^\nu\right\|_{|\beta|+1}^2 + \left\|(\partial_t^k u^\nu, \partial_t^{k+1} u^\nu, \partial_t^k \Theta^\nu, \nabla \partial_t^k \Theta^\nu)\right\|_{|\beta|}^2 \right) + \left\|R_{1\nu}^{k,\beta}\right\|^2,$$

Summing the inequalities for all indexes β and combining the result with (3.71) yields (3.72). □

From Proposition 3.3 and Lemma 3.7, by noting $U^\nu = (N^\nu, u^\nu, \Theta^\nu)^T$ and $U = (U^e, U^i)^T$, taking $\varepsilon > 0$ sufficiently small, it is easy to obtain the following result.

Proposition 3.4. (Relation of recurrence) *Assume that the conditions of Theorem 1.1 hold and ω_T is sufficiently small independent of T, then there exists a positive constant c_0 such that, for all $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^3$ with $|\alpha| \geq 1$ and $k + |\alpha| \leq [s/2]$, it holds*

$$\begin{aligned} &\frac{d}{dt} \sum_{\beta \leq \alpha} \left(\sum_{\nu=e,i} \int_{\mathbb{T}} \left(A_0^\nu U_{k,\beta}^\nu\right) U_{k,\beta}^\nu \mathrm{d}x + \left\|F_{k,\beta}\right\|^2 + \left\|G_{k,\beta}\right\|^2 \right) \\ &\quad + C_0 \sum_{\nu=e,i} \left\|\partial_t^k (U^\nu, \nabla \Theta^\nu)\right\|_{|\alpha|}^2 \\ &\leq C \sum_{\nu=e,i} \left\|(\partial_t^k U^\nu, \partial_t^k \nabla \Theta^\nu, \partial_t^{k+1} u^\nu, \partial_t^{k+1} \Theta^\nu)\right\|_{|\alpha|-1}^2 + CD_{[s/2]}(t)|||W|||_{[s/2]}. \end{aligned} \tag{3.74}$$

Remark 3.5. *Proposition 3.4 is valid only for all $0 \leq k \leq [s/2] - 1$. Besides, we need an estimate on $\|\partial_t^k N^\nu\|$ as an initial value to use (3.74) by induction. The next result presents such an estimate and completes Proposition 3.4 for the case $k = [s/2]$.*

Proposition 3.5. *Assume that the conditions of Theorem 1.1 hold and ω_T is sufficiently small independent of T, then for all $0 \leq k \leq [s/2] - 1$, we have*

$$\left\|\partial_t^k N^\nu\right\|_1^2 \leq C \sum_{\nu=e,i} \left\|(\partial_t^k u^\nu, \partial_t^{k+1} u^\nu, \partial_t^k \Theta^\nu, \partial_t^k \nabla \Theta^\nu)\right\|^2 + CD_{[s/2]}(t)|||W|||_{[s/2]}, \tag{3.75}$$

and

$$\begin{aligned} &\frac{d}{dt} \left(\sum_{\nu=e,i} \int_{\mathbb{T}} \left(A_0^\nu \partial_t^{[s/2]} U^\nu\right) \partial_t^{[s/2]} U^\nu \mathrm{d}x + \left\|\partial_t^{[s/2]} F\right\|^2 + \left\|\partial_t^{[s/2]} G\right\|^2 \right) \\ &\quad + c_0 \sum_{\nu=e,i} \left\|\partial_t^{[s/2]} (U^\nu, \nabla \Theta^\nu)\right\|^2 \\ &\leq C \sum_{\nu=e,i} \left\|\partial_t^{[s/2]-1} u^\nu\right\|_1^2 + CD_{[s/2]}(t)|||W|||_{[s/2]}. \end{aligned} \tag{3.76}$$

Proof. For $k \in \mathbb{N}$ with $k \leq [s/2] - 1$, applying ∂_t^k to the second equation of (2.30), we get

$$\nabla \left(\frac{1}{\bar{n}^\nu} \partial_t^k N^\nu \right) - q_\nu \partial_t^k F = q_\nu u_k^\nu \times \bar{B} - \partial_t^k u^\nu - \partial_t^{k+1} u^\nu - \nabla \partial_t^k \Theta^\nu - \frac{\nabla \bar{n}^\nu}{\bar{n}^\nu} \partial_t^k \Theta^\nu - R_{\nu,1}^{k,0}, \tag{3.77}$$

where

$$H_{\nu,1}^{k,0}=\partial_t^k\left((u^\nu\cdot\nabla)\,u^\nu+\frac{|N^\nu|^2}{n^\nu(\bar n^\nu)^2}+\frac{\Theta^\nu\nabla N^\nu}{n^\nu}-\frac{N^\nu\Theta^\nu}{n^\nu\bar n^\nu}-\frac{N^\nu\nabla N^\nu}{n^\nu\bar n^\nu}-q_\nu u^\nu\times G\right).$$

Now, we define a potential function $\nabla\psi$ as

$$\nabla\cdot(\nabla\psi)=\Delta\psi=N^e-N^i,\quad \int\limits_{\mathbb{T}}\psi(t,x)\mathrm{d}x=0.$$

Then

$$\nabla\cdot(F+\nabla\psi)=-\left(N^e-N^i\right)+N^e-N^i=0,$$

and furthermore

$$\nabla\cdot\left(\partial_t^kF+\nabla\partial_t^k\psi\right)=0,\quad\forall\ 0\le k\le[s/2]-1.$$

From (3.77), we have

$$\begin{aligned}&\nabla\xi_k^\nu-q_\nu\left(\partial_t^kF+\nabla\partial_t^k\psi\right)\\&\quad=q_\nu\partial_t^ku^\nu\times\bar B-\partial_t^ku^\nu-\partial_t^{k+1}u^\nu-\nabla\partial_t^k\Theta^\nu-\frac{\nabla\bar n^\nu}{\bar n^\nu}\partial_t^k\Theta^\nu-H_{\nu,1}^{k,0},\end{aligned}\tag{3.78}$$

where

$$\xi_k^\nu=\frac{1}{\bar n^\nu}\partial_t^kN^\nu+q_\nu\partial_t^k\psi,\quad\nu=e,i.$$

Due to the fact that

$$\int\limits_{\mathbb{T}}\left(\partial_t^kF+\nabla\partial_t^k\psi\right)\cdot\nabla\xi_k^\nu\mathrm{d}x=-\int\limits_{\mathbb{T}}\xi_k^\nu\nabla\cdot\left(\partial_t^kF+\nabla\partial_t^k\psi\right)\mathrm{d}x=0,$$

we obtain

$$||\nabla\xi_k^\nu||^2\le C||\left(\partial_t^ku^\nu,\partial_t^{k+1}u^\nu,\partial_t^k\Theta^\nu,\partial_t^k\nabla\Theta^\nu\right)||^2+||H_{\nu1}^{k,0}||^2.\tag{3.79}$$

Since

$$\partial_t^kN^\nu=\bar n^\nu\xi_\nu^k-q_\nu\bar n^\nu\partial_t^k\psi,\quad\Delta\psi=N^e-N^i,$$

we have

$$-\Delta\partial_t^k\psi=\partial_t^k\left(N^i-N^e\right)=\bar n^i\xi_i^k-\bar n^i\partial_t^k\psi-\left(\bar n^e\xi_e^k+\bar n^e\partial_t^k\psi\right).$$

Thus,

$$-\Delta\partial_t^k\psi+\left(\bar n^e+\bar n^i\right)\partial_t^k\psi=\bar n^i\xi_i^k-\bar n^e\xi_e^k.$$

Since $\bar n^\nu\ge$ const. >0, taking the inner product of the previous equality with $\partial_t^k\psi$ in $L^2(\mathbb{T})$ and using an integration by parts, we get

$$\left\|\partial_t^k\nabla\psi\right\|^2+c_0\left\|\partial_t^k\psi\right\|^2\le\sum_{\nu=e,i}\left\|\xi_\nu^k\right\|^2\le\sum_{\nu=e,i}\left\|\nabla\xi_\nu^k\right\|^2,\tag{3.80}$$

where we have used Lemma 1.1.

From (3.77), (3.79)–(3.80) and the definition of ξ_ν^k, we have

$$\left\|\nabla\left(\frac{1}{\bar n^\nu}\partial_t^kN^\nu\right)\right\|^2\le||\nabla\xi_k^\nu||^2+||\partial_t^k\nabla\psi||^2\le C||\left(\partial_t^ku^\nu,\partial_t^{k+1}u^\nu,\partial_t^k\Theta^\nu,\partial_t^k\nabla\Theta^\nu\right)||^2+||H_{\nu1}^{k,0}||^2.$$

By means of Lemma 1.1 and noting that $\bar n^\nu\ge$ const. >0, we obtain

$$\left\|\nabla\left(\frac{1}{\bar n^\nu}\partial_t^kN^\nu\right)\right\|\sim\|\partial_t^kN^\nu\|_1.$$

This proves (3.75).

Next, from the first equation of system (2.30), we get

$$\partial_t^{[s/2]} N^\nu = -\nabla \cdot \left(\partial_t^{[s/2]-1}\left(N^\nu u^\nu\right)\right) - \nabla \cdot \left(\bar{n}^\nu \partial_t^{[s/2]-1} u^\nu\right).$$

It follows from Lemma 1.2 that

$$\|\partial_t^{[s/2]} N^\nu\|^2 \leq C\|\partial_t^{[s/2]-1} u^\nu\|_1^2 + C|||U|||_{[s/2]}^3,$$

together with (3.64) for $k = [s/2]$ implies (3.76). □

4. Proof of Theorems 1.1 and 1.2

4.1. Proof of Theorem 1.1

The proof of Theorem 1.1 is mainly based on the following a priori estimates which is a consequence on the estimates obtained in the previous section.

Proposition 4.6. (A priori estimates) *Assume that the conditions of Theorem 1.1 hold and ω_T is sufficiently small independent of T, then for all $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^3$ with $k + |\alpha| \leq [s/2]$, there exist positive constants $\lambda_{(k,|\alpha|)}$ such that*

$$\frac{d}{dt} \sum_{k+|\alpha|\leq s} \lambda_{(k,|\alpha|)} \left(\sum_{\nu=e,i} \int_{\mathbb{T}} (A_0^\nu(n^\nu,\theta^\nu) U_{k,\alpha}^\nu) U_{k,\alpha}^\nu \mathrm{d}x + \|F_{k,\alpha}\|^2 + \|G_{k,\alpha}\|^2 \right) + D_{[s/2]}(t) \leq 0. \tag{4.81}$$

Proof. First, applying Proposition 3.4 with $(k, |\alpha|) = ([s/2]-1, 1)$, we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \sum_{\beta\leq 1} \left(\sum_{\nu=e,i} \int_{\mathbb{T}} \left(A_0^\nu(n^\nu,\theta^\nu) U_{[s/2]-1,\beta}^\nu\right) U_{[s/2]-1,\beta}^\nu \mathrm{d}x + \left\|F_{[s/2]-1,\beta}\right\|^2 + \left\|G_{[s/2]-1,\beta}\right\|^2 \right) \\ &\quad + c_0 \sum_{\nu=e,i} \left\|\partial_t^{[s/2]-1}(U^\nu, \nabla\Theta^\nu)\right\|_1^2 \\ &\leq C \sum_{\nu=e,i} \left(\left\|\partial_t^{[s/2]-1}(U^\nu,\nabla\Theta^\nu)\right\|^2 + \left\|\partial_t^{[s/2]}(u^\nu,\Theta^\nu)\right\|^2 \right) + CD_{[s/2]}(t)|||W|||_{[s/2]}. \end{aligned} \tag{4.82}$$

We find that the dissipation term $\sum_{\nu=e,i} \|\partial_t^{[s/2]-1}(U^\nu, \nabla\Theta^\nu)\|_1^2$ on the left hand side of (4.82) can control the term $\sum_{\nu=e,i} \|\partial_t^{[s/2]-1} u^\nu\|_1^2$ on the right hand side of (3.76) through multiplying (3.76) by a small positive constant $\lambda_{(s,0)}^* \ll 1$.

Second, applying Proposition 3.4 with $(k, |\alpha|) = ([s/2]-2, 1)$ and $(k, |\alpha|) = ([s/2]-2, 2)$, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \sum_{\beta\leq 1} \left(\sum_{\nu=e,i} \int_{\mathbb{T}} \left(A_0^\nu(n^\nu,\theta^\nu) U_{[s/2]-2,\beta}^\nu\right) U_{[s/2]-2,\beta}^\nu \mathrm{d}x + \left\|F_{[s/2]-2,\beta}\right\|^2 + \left\|G_{[s/2]-2,\beta}\right\|^2 \right) \\ &\quad + c_0 \sum_{\nu=e,i} \left\|\partial_t^{[s/2]-2}(U^\nu, \nabla\Theta^\nu)\right\|_1^2 \\ &\leq C \sum_{\nu=e,i} \left(\left\|\partial_t^{[s/2]-2}(U^\nu,\nabla\Theta^\nu)\right\|^2 + \left\|\partial_t^{[s/2]-1}(u^\nu,\Theta^\nu)\right\|^2 \right) + CD_{[s/2]}(t)|||W|||_{[s/2]}, \end{aligned} \tag{4.83}$$

and

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\beta\le 2}\left(\sum_{\nu=e,i}\int_{\mathbb{T}}\left(A_0^\nu(n^\nu,\theta^\nu)U^\nu_{[s/2]-2,\beta}\right)U^\nu_{[s/2]-2,\beta}\mathrm{d}x+\left\|F_{[s/2]-2,\beta}\right\|^2+\left\|G_{[s/2]-2,\beta}\right\|^2\right)\\&\quad+c_0\sum_{\nu=e,i}\left\|\partial_t^{[s/2]-2}(U^\nu,\nabla\Theta^\nu)\right\|_2^2\\&\le C\sum_{\nu=e,i}\left(\left\|\partial_t^{[s/2]-2}(U^\nu,\nabla\Theta^\nu)\right\|_1^2+\left\|\partial_t^{[s/2]-1}(u^\nu,\Theta^\nu)\right\|_1^2\right)+CD_{[s/2]}(t)|||W|||_{[s/2]}.\end{aligned}\tag{4.84}$$

We also find that the terms $\sum_{\nu=e,i}\left\|\partial_t^{[s/2]-1}(u^\nu,\Theta^\nu)\right\|_1^2$ and $\sum_{\nu=e,i}\left\|\partial_t^{[s/2]-2}(U^\nu,\nabla\Theta^\nu)\right\|_1^2$ on the right side of (4.84) can be controlled by the dissipation term $\sum_{\nu=e,i}\left\|\partial_t^{[s/2]-1}(U^\nu,\nabla\Theta^\nu)\right\|_1^2$ on the left hand side of (4.82) and the term $\sum_{\nu=e,i}\left\|\partial_t^{[s/2]-2}(U^\nu,\nabla\Theta^\nu)\right\|_1^2$ on the left hand side of (4.83) through multiplying (4.84) by a small positive constant $\lambda^*_{([s/2]-2,2)}\ll 1$.

In this way and by induction on $(k,|\alpha|)$ with k decreasing and $|\alpha|$ increasing, in (3.74) both the term $\sum_{\nu=e,i}\|\partial_t^{k+1}(u^\nu,\Theta^\nu)\|^2_{|\alpha|-1}$ and the term $\sum_{\nu=e,i}\|\partial_t^{k}(U^\nu,\nabla\Theta^\nu)\|^2_{|\alpha|-1}$ can be controlled by $\sum_{\nu=e,i}\|\partial_t^{k}(U^\nu,\nabla\Theta^\nu)\|^2_{|\alpha|}$ in the proceeding steps.

Now, set:

$$\begin{cases}0<\lambda^*_{([s/2],\ 0)}\ll 1,\\ \lambda^*_{([s/2]-j,\ 1)}=1,\quad \text{as } j\le[s/2],\\ 0<\lambda^*_{([s/2]-j,\ l)}\ll\lambda^*_{([s/2]-j,\ l-1)}\ll 1,\quad \text{as } 1<l\le j\le[s/2],\\ 0<\lambda^*_{([s/2]-j,\ l_1)}\ll\lambda^*_{([s/2]-j+1,\ l_2)}\ll 1,\quad \text{as } 1<l_1,l_2\le j\le[s/2].\end{cases}$$

Then summing (3.76) $\times\lambda^*_{([s/2],0)}$, (4.82) $\times\lambda^*_{([s/2]-1,1)}$ and $\sum\limits_{2\le j\le[s/2]}\sum\limits_{l\le j}(3.74)_{([s/2]-j,l)}\times\lambda^*_{([s/2]-j,l)}$, we deduce that there are positive constants $\lambda_{(k,|\alpha|)}$ such that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k+|\alpha|\le[s/2]}\lambda_{(k,|\alpha|)}\left(\sum_{\nu=e,i}\int_{\mathbb{T}}\left(A_0^\nu(n^\nu,\theta^\nu)U^\nu_{k,\alpha}\right)U^\nu_{k,\alpha}\mathrm{d}x+\|F_{k,\alpha}\|^2+\|G_{k,\alpha}\|^2\right)\\&\quad+\sum_{k+|\alpha|\le[s/2]}\sum_{\nu=e,i}\left\|\partial_t^k(U^\nu,\nabla\Theta^\nu)\right\|^2_{|\alpha|}\\&\le C\sum_{k=0}^{[s/2]-1}\sum_{\nu=e,i}\left(\left\|\partial_t^k(U^\nu,\nabla\Theta^\nu)\right\|^2+\left\|\partial_t^{k+1}(u^\nu,\Theta^\nu)\right\|^2\right)+CD_{[s/2]}(t)|||W|||_{[s/2]}.\end{aligned}$$

By Proposition 3.5 and noting that

$$\sum_{k+|\alpha|\le[s/2]}\sum_{\nu=e,i}\left\|\partial_t^k(U^\nu,\nabla\Theta^\nu)\right\|^2_{|\alpha|}\sim D_{[s/2]}(t),$$

we have, after a modification of these constants (still denoted by $\lambda_{(k,|\alpha|)}$),

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k+|\alpha|\le[s/2]}\lambda_{(k,|\alpha|)}\left(\sum_{\nu=e,i}\int_{\mathbb{T}}\left(A_0^\nu(n^\nu,\theta^\nu)U_{k,\alpha}^\nu\right)U_{k,\alpha}^\nu\mathrm{d}x+\|F_{k,\alpha}\|^2+\|G_{k,\alpha}\|^2\right)+D_{[s/2]}(t)\\&\le C\sum_{k=0}^{[s/2]}\sum_{\nu=e,i}\left\|\partial_t^k(u^\nu,\Theta^\nu,\nabla\Theta^\nu)\right\|^2+CD_{[s/2]}(t)|||W|||_{[s/2]}.\end{aligned}$$

Next, utilizing Proposition 3.4 and modifying again these constants $\lambda_{(k,|\alpha|)}$, we further obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k+|\alpha|\le[s/2]}\lambda_{(k,|\alpha|)}\left(\sum_{\nu=e,i}\int_{\mathbb{T}}\left(A_0^\nu(n^\nu,\theta^\nu)U_{k,\alpha}^\nu\right)U_{k,\alpha}^\nu\mathrm{d}x+\|F_{k,\alpha}\|^2+\|G_{k,\alpha}\|^2\right)+D_{[s/2]}(t)\\&\le CD_{[s/2]}|||W|||_{[s/2]}.\end{aligned}$$

Then we obtain (4.81), provided that ω_T is small enough. □

The following result concerns the time dissipation of electromagnetic fields.

Lemma 4.8. *Assume that the conditions of Theorem 1.1 hold and ω_T is sufficiently small independent of T, then it holds*

$$|||F|||_{[s/2]-1}\le C|||U|||_{[s/2]},\tag{4.85}$$

and

$$|||\partial_t G|||_{[s/2]-2}+|||\nabla G|||_{[s/2]-2}\le C|||U|||_{[s/2]}.\tag{4.86}$$

Proof. It is similar to that of Lemma 4.6 in [9] and is omitted here for the sake of simplicity. □

Recall the following Lemma that is used in the following proof.

Lemma 4.9. *Let $f:\mathbb{R}^+\to\mathbb{R}$ be a uniformly continuous function such that $f\in L^1(\mathbb{R}^+)$. Then $\lim_{t\to+\infty}f(t)=0$. In particular, the conclusion holds when $f\in L^1(\mathbb{R}^+)\cap W^{1,+\infty}(\mathbb{R}^+)$.*

Proof of Theorem 1.1. Combining Proposition 4.6 and Lemma 4.8 and modifying again the constants $\lambda_{(k,|\alpha|)}$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k+|\alpha|\le[s/2]}\lambda_{(k,|\alpha|)}\left(\sum_{\nu=e,i}\int_{\mathbb{T}}\left(A_0^\nu(n^\nu,\theta^\nu)U_{k,\alpha}^\nu\right)U_{k,\alpha}^\nu\mathrm{d}x+\|F_{k,\alpha}\|^2+\|G_{k,\alpha}\|^2\right)+D_{[s/2]}(t)\\&+|||F|||_{[s/2]-1}^2+|||\partial_t G|||_{[s/2]-2}^2+|||\nabla G|||_{[s/2]-2}^2\le 0.\end{aligned}$$

Integrating the inequality over $[0,t]$ and noting

$$\sum_{k+|\alpha|\le[s/2]}\lambda_{(k,|\alpha|)}\left(\sum_{\nu=e,i}\int_{\mathbb{T}}\left(A_0^\nu(n^\nu,\theta^\nu)U_{k,\alpha}^\nu\right)U_{k,\alpha}^\nu\mathrm{d}x+\|F_{k,\alpha}\|^2+\|G_{k,\alpha}\|^2\right)\sim|||W|||_{[s/2]}^2,$$

we have

$$\begin{aligned}&|||W(t)|||_{[s/2]}^2+\int_0^t\left(D_{[s/2]}+|||F|||_{[s/2]-1}^2+|||\partial_\tau G|||_{[s/2]-2}^2+|||\nabla G|||_{[s/2]-2}^2\right)(\tau)\,\mathrm{d}\tau\\&\le|||W(0)|||_{[s/2]}^2,\quad\forall\, t\in[0,T].\end{aligned}\tag{4.87}$$

Since the Euler–Maxwell system (2.30) can be written as $\partial_t W=f(x,W,\partial_x W,\partial_{xx}W)$ with a smooth function f such that $f(x,0,0,0)=0$, it follows from Lemma 1.3 that

$$|||W(0)|||_{[s/2]}\le C\|W^0\|_s.$$

This implies the global existence of solutions and estimates (1.8)–(1.14), where the regularities of solutions are deduced from the equations in system (1.2).

Furthermore, (4.87) implies that, for all $k+|\beta|\le[s/2]-1$,

$$\partial_t^k\partial^\beta W\in L^\infty\big(\mathbb{R}^+,L^2(\mathbb{T})\big),\quad \partial_t(\partial_t^k\partial^\beta W)\in L^\infty\big(\mathbb{R}^+,L^2(\mathbb{T})\big).$$

Then we have

$$\partial_t^k\partial^\beta W\in W^{1,\infty}\big(\mathbb{R}^+,L^2(\mathbb{T})\big),\quad \forall\, k+|\beta|\le[s/2]-1.$$

Moreover, it also follows from (4.87) that

$$\partial_t^k\partial^\beta(n^\nu-\bar n^\nu,u^\nu,\theta^\nu-1,E-\bar E)\in L^2\big(\mathbb{R}^+,L^2(\mathbb{T})\big),\quad \forall\, k+|\beta|\le[s/2]-1.$$

Then for any $k+|\beta|\le[s/2]-1$, we obtain

$$\partial_t^k\partial^\beta(n^\nu-\bar n^\nu,u^\nu,\theta^\nu-1,E-\bar E)\in L^2\big(\mathbb{R}^+,L^2(\mathbb{T})\big)\cap W^{1,\infty}\big(\mathbb{R}^+,L^2(\mathbb{T})\big).$$

By Lemma 4.9, we get (1.15)–(1.16). Similarly, we have

$$\partial_t^k\partial^\beta B\in L^2\big(\mathbb{R}^+,L^2(\mathbb{T})\big)\cap W^{1,\infty}\big(\mathbb{R}^+,L^2(\mathbb{T})\big),\quad \forall\, 1\le k+|\beta|\le[s/2]-1,$$

which implies (1.17). We have completed the proof of Theorem 1.1. □

4.2. Proof of Theorem 1.2

Recall the following Lemma that is used in the following proof.

Lemma 4.10. *For all integer $m\ge0$, we have $|||\nabla\Phi|||_m\le C|||U|||_m$.*

Proof. By Lemma 1.1 and Poisson equation $\triangle\Phi=N^e-N^i$ yields this inequality. □

We still use notations in (2.29) and (2.33), with

$$\bar E=-\nabla\bar\phi,\quad \Phi=\phi-\bar\phi,\ F=-\nabla\Phi. \tag{4.88}$$

Then the bipolar non-isentropic Euler–Poisson system (1.18) can be written as:

$$\begin{cases}\partial_tN^\nu+u^\nu\cdot\nabla N^\nu+(\bar n^\nu+N^\nu)\nabla\cdot u^\nu+u^\nu\cdot\nabla\bar n^\nu=0,\\ \partial_tu^\nu+(u^\nu\cdot\nabla)u^\nu+\nabla\Theta^\nu+\nabla(\ln(\bar n^\nu+N^\nu)-\ln\bar n^\nu)+\Theta^\nu\nabla(\ln(\bar n^\nu+N^\nu))+u^\nu=-q_\nu\nabla\Phi,\\ \partial_t\Theta^\nu+u^\nu\cdot\nabla\Theta^\nu+(1+\Theta^\nu)\nabla\cdot u^\nu+\dfrac{1}{n^\nu}|\nabla\Theta^\nu|^2+\Theta^\nu=\dfrac{1+\Theta^\nu}{\bar n^\nu+N^\nu}\Delta\Theta^\nu,\\ \triangle\Phi=N^e-N^i,\quad \text{in}\quad \mathbb{R}^+\times\mathbb{T},\end{cases} \tag{4.89}$$

in which the Euler equations are the special case of those of (2.30) with $B=0$, while the Maxwell equations in (2.30) are replaced by $\triangle\Phi=N^e-N^i$.

For system (4.89), we start to establish a similar energy estimate to (4.81). By checking all the steps before (4.81), we find that the Maxwell equations are concerned in the proof of Lemmas 3.4–3.6. In fact, we only need to deal with the term $\int_{\mathbb{T}}\partial_t^k\partial^\alpha(n^eu^e-n^iu^i)F_{k,\alpha}\mathrm{d}x$ for all $k+|\alpha|\le s$, appeared in the proof due to the Maxwell equations [see (3.50)]. It can be treated as follows. By means of (4.88), the first and the last equations in (4.89), we have

$$2\int_{\mathbb{T}}\partial_t^k\partial^\alpha(n^eu^e-n^iu^i)F_{k,\alpha}\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}\big\|\partial_t^k\partial^\alpha\nabla\Phi\big\|^2.$$

This shows the validity of all the steps before (4.81). Consequently, there exist constants $\mu_{(k,|\alpha|)} > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{k+|\alpha| \le [s/2]} \mu_{(k,|\alpha|)} \left(\sum_{\nu=e,i} \left\langle A_0^\nu(n^\nu, \theta^\nu) U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \right\rangle + \left\| \partial_t^k \partial^\alpha \nabla \Phi \right\|^2 \right) + |||U|||_{[s/2]}^2 \le 0.$$

It follows from Lemma 4.10 that, after modifying the positive constants $\mu_{(k,|\alpha|)}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{k+|\alpha| \le [s/2]} \mu_{(k,|\alpha|)} \left(\sum_{\nu=e,i} \left\langle A_0^\nu(n^\nu, \theta^\nu) U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \right\rangle + \left\| \partial_t^k \partial^\alpha \nabla \Phi \right\|^2 \right) + |||U|||_{[s/2]}^2 + |||\nabla \Phi|||_{[s/2]}^2 \le 0.$$

Let us denote

$$\mathscr{E}(t) = \sum_{k+|\alpha| \le [s/2]} \mu_{(k,|\alpha|)} \left(\sum_{\nu=e,i} \left\langle A_0^\nu(n^\nu, \theta^\nu) U_{k,\alpha}^\nu, U_{k,\alpha}^\nu \right\rangle + \left\| \partial_t^k \partial^\alpha \nabla \Phi \right\|^2 \right).$$

Using Lemma 4.10 and noticing the fact that $\mu_{(k,|\alpha|)} > 0$ and $A_0^\nu(n^\nu, \theta^\nu)$ is positively definite, we have

$$\mathscr{E}(t) \sim |||U(t)|||_{[s/2]}^2.$$

Thus, there exists a constant $\gamma > 0$ such tat

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathscr{E}(t) + \gamma \mathscr{E}(t) \le 0, \quad \forall\, t \in [0, T],$$

which implies that

$$\mathscr{E}(t) \le \mathscr{E}(0) e^{-\gamma t}, \quad \forall\, t \in [0, T].$$

Using Lemma 1.3, we have

$$|||U(t)|||_{[s/2]}^2 + |||\nabla \Phi(t)|||_{[s/2]}^2 \le C e^{-\gamma t} \left\| U^0 \right\|_s^2, \quad \forall\, t \in [0, T].$$

This shows the global existence of smooth solutions with the exponential decay estimate of Theorem 1.2. □

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