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An explicit solution for the dynamics of a taut string of finite length carrying a traveling mass: the subsonic case

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Abstract. The authors investigate the linear vibrations induced in an elastic string by a loading point-like mass constrained to moving on it with constant horizontal velocity. Exact solutions are shown in the case of subsonic regime. The displacement is explicitly provided in terms of a power series determined by iteration, which is shown to converge to the solution of the problem. The presence of a discontinuity in the right extremum of the considered space interval is also shown both analytically and numerically.

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1. Introduction

The present paper deals with the analysis of linear vibrations induced in an elastic string by a loading point-like mass moving on it having a constant component of the velocity parallel to the reference configuration of the string. Problems of this kind are common to a wide field of engineering applications and significant literature exist on this topic (see e.g., [29,30,34,37,53]; a useful introduction to the problem is [57], while experimental results are provided in [8]). Despite the great amount of studies devoted to this theme, there are still some related mathematical questions not completely defined in the available literature. This state of affairs might perhaps be due to scarcity of concrete examples of exact solutions to which compare the available numerical results.

The aim of this work is to exhibit an example of exact solutions in the case of the so-called subsonic regime; more precisely, the situation when the speed of the traveling mass is less than the string celerity.

In [30], the authors study a similar problem by means of Fourier transforms, the paradox cited in the title of their paper is related to the fact that the problem presented in the paper is not well posed, since the solution cannot respect all the imposed conditions; in fact, the authors find a solution with a discontinuity at the end of the string. Also in [34] the authors, obtaining the equations governing the system by means of variational methods, discuss about the nonwellposedness of the problem and find a numerical solution.

In [37], the authors consider the same problem, starting from a different point of view, i.e., considering explicitly in the motion equation of the mass, a term related to the moving contact force $p(t)$ between the mass and the string. This force, being an internal force of the system, is not explicitly present in the variational formulation proposed in [34]. The authors of [37] consider strings of finite, semi-infinite and infinite length and devote most of the paper to the determination of the explicit form of $p(t)$, obtaining the expressions for the displacements only as a consequence of the ones for $p(t)$.

In this paper, we will directly address the problem of finding the expression of the displacement, given in terms of a power series (see [35] and also the classical reference [69]), which is shown to converge to

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the solution of the problem. Moreover, we also show analytically and numerically the presence of the discontinuity in the right extremum of the considered space interval.

The physical model and the governing equations which are here assumed are those described in [34]. The key points and notations of that paper are shortly recalled for convenience of the reader.

The taut string has length L, mass per-unit length m, is subjected to an axial prestress force N and is traveled by a single-point mass M, bilaterally constrained to the string, and moving at a constant horizontal velocity U. Denoting $\bar{x}(t)$ the known instantaneous abscissa of M at time t, the resulting vertical string displacement at the same time t is thereafter denoted by $v^+(x,t)$ $(v^-(x,t))$ for points having abscissa x greater (less) than $\bar{x}(t)$, respectively. The orientation of $z = v^{\pm}(x,t)$ is assumed to be in accordance with the convention that positive displacements correspond to displacements in the upward direction. The governing equations are obtained by the authors by means of variational techniques, i.e., imposing that the action functional of the system vanishes and deriving the Euler–Lagrange equations (see Eqs. (1–5) in [34]). The final strong formulation leads to the following initial boundary value problem (see 1 below) which also includes matching and jump conditions, for the displacement and its spatial derivative, respectively, along the straight line $x = \bar{x}(t) = Ut$.

$$\Pi_1 = \begin{cases} mv^-_{tt}(x,t) - Nv^-_{xx}(x,t) = 0, & \text{in } D^- \\ mv^+_{tt}(x,t) - Nv^+_{xx}(x,t) = 0, & \text{in } D^+ \\ (N - mU^2)[\![v_x]\!] = Mg + M(v^-_{tt} + 2Uv^-_{xt} + U^2 v^-_{xx}), & \text{for } x = \bar{x}(t), \quad t \in \left(0, \frac{L}{U}\right) \\ [\![v]\!] = 0, & \text{in } x = \bar{x}(t), \quad \forall t \in \left(0, \frac{L}{U}\right) \\ v^-(0,t) = v^+(L,t) = 0, & \forall t \in \left[0, \frac{L}{U}\right] \\ v^+(x,0) = 0, & \forall x \in [0, L] \\ v^+_t(x,0) = 0, & \forall x \in [0, L] \end{cases} \tag{1}$$

where the sets D^- and D^+ are defined by $D^- = \{(x,t) : t \in \left(0, \frac{L}{U}\right),\ x \in (0, \bar{x}(t))\}$ and $D^+ = \{(x,t) : t \in \left(0, \frac{L}{U}\right),\ x \in (\bar{x}(t), L)\}$, respectively. Moreover, for any function $\varphi(x,t)$ the jump along the line $x = \bar{x}(t) = Ut$ is given by

$$[\![\varphi(\bar{x}(t),t)]\!] = \varphi^+(Ut,t) - \varphi^-(Ut,t). \tag{2}$$

The assumption of null initial conditions $v^+(x,0)$, $v^+_t(x,0)$ is not a real limitation since, by addition of regular solutions to the one found out here, one could treat the case of arbitrary initial data.

Following [34], the equations of system (1) are first of all recast in a dimensionless form

$$\tilde{x} = \frac{x}{L}; \quad \tilde{t} = \frac{1}{L}\sqrt{\frac{N}{m}}\, t; \quad \tilde{v} = \frac{v}{L}; \quad \tilde{U} = \sqrt{\frac{m}{N}}\, U; \quad \mu = \frac{M}{mL}; \quad p = \frac{Mg}{N}; \tag{3}$$

where $\dfrac{N}{m}$ is the celerity of the mass, μ is the mass ratio, $\tilde{U}$ is the dimensionless velocity of the mass M, equal to the ratio between the mass velocity and the celerity of the taut string, p is the dimensionless load, equal to the ratio of the weight of the traveling mass and the prestress force N.

However, for the sake of simplicity, we will discard the symbol $\tilde{}$ on the variables; thus, the system takes the following final form

$$\Pi_2 = \begin{cases} v^-_{tt}(x,t) - v^-_{xx}(x,t) = 0, & \text{in } D^- \\ v^+_{tt}(x,t) - v^+_{xx}(x,t) = 0, & \text{in } D^+ \\ (1 - U^2)[\![v_x]\!] = p + \mu(v^-_{tt} + 2Uv^-_{xt} + U^2 v^-_{xx}), & \text{in } x = Ut \\ [\![v]\!] = 0, & \text{in } x = Ut \\ v^-(0,t) = v^+(1,t) = 0, & \forall t \in \left[0, \frac{1}{U}\right] \\ v^+(x,0) = 0, & \forall x \in [0, 1] \\ v^+_t(x,0) = 0, & \forall x \in [0, 1] \end{cases} \tag{4}$$

where $D^- = \{(x,t) : t \in \left(0, \frac{1}{U}\right),\ x \in (0, \bar{x}(t))\}$ and $D^+ = \{(x,t) : t \in \left(0, \frac{1}{U}\right),\ x \in (\bar{x}(t), 1)\}$.

According to these last formulas, the condition characterizing the subsonic case is expressed by the inequality $U < 1$. In [34] the supersonic case is considered, too, but it will not be discussed here, because it is less interesting, from a mathematical point of view, since the mass always precedes the string movements.

In our paper, for problem (1), we will exhibit a solution only in the subsonic case $U = 0.5$, because, as shown in Fig. 1, in this case, differently from the construction of the solution in D^+, which in any case needs an infinite partition in subdomains, the construction of the solution in D^- needs a much simpler geometry. Changing the value of U implies just a more complex geometry of D^-, which however does not imply any theoretical complication concerning the technique used in this paper.

The paper has the following structure: In Sect. 2, we approach the problem dividing the domain $D = D^+ \cup D^-$ into several subdomains, depending on the characteristic lines of the equation governing the system. We determine a function H, which describes the behavior of the solution along the line $x = Ut$, and we expand H in series with respect to the time t. Since the line $x = Ut$ is divided by the characteristic lines into infinite segments, whose endpoins tend to the limit point $P_{lim} = (1, 1/U)$, in Sect. 3, we build H by means of the solutions of an infinite sequence of problems along the line $x = Ut$. This approach implies that the solution, as expected, cannot *a priori* satisfy any condition in P_{lim}. In Sect. 4, we will show some numerical simulations which describe the shape of the solution and confirm the presence of a discontinuity at the endpoint P_{lim}. In Sect. 5, some remarks are added, and the potential relevance of the presented results for related fields is discussed.

2. Setting of the problem with $U = 0.5$ (a special subsonic case)

In order to find the solution to problem (4), it is convenient to introduce a decomposition of the domains D^+ and D^- in collections of sub-domains $D_j^{\pm}$, which share parts of their boundaries with the boundary of D. This partition is obtained as illustrated in Fig. 1. Other parts of their boundaries consist either of pieces of the characteristic blue straight lines $x - t =$ constant, of the characteristic green straight lines $x + t =$ constant, or of pieces of the red straight line $x = Ut$.

We begin by considering, on the straight line $x = Ut$, the sequence of points $\{\Omega_j\}_{j=0,1,2...}$ given by:

$$\Omega_0 = (0,0), \quad \Omega_j = \left(\frac{2U(1-\lambda^j)}{(1+U)(1-\lambda)}, \frac{2(1-\lambda^j)}{(1+U)(1-\lambda)}\right) \quad \forall j \in N, \quad \lambda = \frac{1-U}{1+U} \tag{5}$$

whose time coordinates are, respectively, denoted by

$$t_0 = t(\Omega_0) = 0, \quad t_j = t(\Omega_j) = \frac{2(1-\lambda^j)}{(1+U)(1-\lambda)} \quad \text{for } j = 1, 2 \ldots \tag{6}$$

Let us observe that, in the subsonic case, $\lambda < 1$.

The points Ω_j can also be characterized in the following way

$$\begin{aligned}\Omega_j = \left((1-\lambda)\sum_{k=0}^{j-1}\lambda^k\ ,\ (1+\lambda)\sum_{k=0}^{j-1}\lambda^k\right) &= \left((1-\lambda^j)\ ,\ \frac{1+\lambda}{1-\lambda}(1-\lambda^j)\right)\\ &= \left((1-\lambda^j), \frac{1}{U}(1-\lambda^j)\right)\end{aligned} \tag{7}$$

which implies the recursive formula $t_{j-1} = \frac{t_j(1+U)-2}{1-U}$, recalling that

$$\frac{1-\lambda}{1+\lambda} = U; \quad 1-\lambda = \frac{2U}{1+U}; \quad 1+\lambda = \frac{2}{1+U}. \tag{8}$$

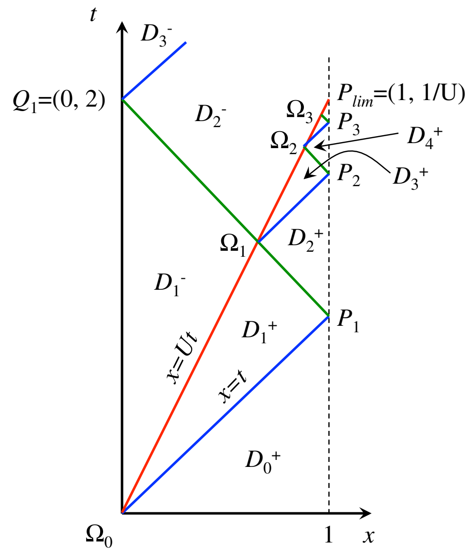

FIG. 1. Scheme of the division of the space-time domain D in subdomains. The subset D^+ is divided in infinitely many subdomains (in figure 5 of them are shown)

It is easily seen that

$$\left[0, \frac{1}{U}\right) = \bigcup_{j=0}^{\infty} [t_j, t_{j+1}] \tag{9}$$

where any couple of intervals $I_j = [t_j, t_{j+1}]$, $I_k = [t_k, t_{k+1}]$ has non-void intersection only in the case of consecutive intervals, in which case, the intersection reduces to the point where the preceding interval ends and the successive begins. It is also convenient to consider the sequence of the points P_j of the vertical boundary $x = 1$, given by

$$P_1 = (1, 1); \quad P_j = (1, 1 + t_{j-1}(1 - U)) = \left(1, 1 + 2\lambda \sum_{k=0}^{j-2} \lambda^k\right) \quad j \geq 2. \tag{10}$$

Here, the generic point P_j is the intersection of the vertical line $x = 1$ with the characteristic line $x - Ut_{j-1} - t + t_{j-1} = 0$ issuing from Ω_{j-1}, while the generic point Ω_j is the intersection of $x = Ut$ with the characteristic line $x - 1 + t - t(P_j) = 0$ issuing from P_j.

By means of these sets of points, we partition the domains D^+ and D^- according to the following formulas

$$D^+ = \bigcup_{j=0}^{\infty} D_j^+ \tag{11}$$

with

$$D_0^+ = \{(x,t) : \ t \leqslant x \leqslant 1, \ 0 \leqslant t \leqslant 1\}$$
$$D_1^+ = \{(x,t) : \ Ut \leqslant x \leqslant t, \ 0 \leqslant t \leqslant t(P_1)\} \cup \{(x,t) : \ Ut \leqslant x \leqslant 1 + t(P_1) - t, \ t(P_1) \leqslant t \leqslant t_1\}$$
$$D_2^+ = \{(x,t) : \ Ut_1 \leqslant x \leqslant 1, \ 1 + t(P_1) - x \leqslant t \leqslant x + t_1(1 - U)\}$$
$$D_{2j-1}^+ = \{(x,t) : \ Ut \leqslant x \leqslant t + 1 - t(P_j), \ t_{j-1} \leqslant t \leqslant t(P_j)\}$$
$$\cup \{(x,t) : \ Ut \leqslant x \leqslant 1 + t(P_j) - t, \ t(P_j) \leqslant t \leqslant t_j\}, \ \text{for } j \geqslant 2$$

$$D_{2j}^{+} = \{(x,t): \; Ut_j \leqslant x \leqslant 1, \; t(P_j) + 1 - x \leqslant t \leqslant t_j(1-U) + x\}, \quad \text{for } j \geqslant 2$$

$$D^{-} = \bigcup_{j=1}^{\infty} D_j^{-} \tag{12}$$

with

$$D_1^{-} = \{(x,t) \in D^{-}: \quad t + x \leqslant 1 + t(P_1)\}$$
$$D_j^{-} = \{(x,t) \in D^{-}: \quad 1 + t(P_j) \leqslant x + t \leqslant 1 + t(P_{j+1})\} \text{ for } j = 2, 3, \ldots$$

It must be remarked that, in Fig. 1, the characteristic line $P_1\Omega_1$, with equation $t + x = 2$, intersects the line $x = 0$ in the point $Q_1 = (0, 2)$.When we consider the case $U = 0.5$, since the line $x = Ut$ ends in the point $P_\infty = (1, 2)$, in order to study the problem, it is sufficient to divide the domain D^- into two parts, D_1^- and D_2^-. Thus, the study of the case $U = 0.5$ simplifies the analysis of the problem. Nothing would substantially change in the geometry of D^+ and D^- if we would consider $0.5 < U < 1$, while values of $U < 0.5$ would imply a more complicated partition of D^-, just from a technical point of view, without adding any new theoretical issue about the technique for finding the solution on D^-. This justifies our choice of considering only the case $U = 0.5$, in order to show the mathematical approach to the explicit solution.

Finally, on the line $x = Ut$, we introduce the notations:

$$H(t) = v^{+}(Ut,t) = v^{-}(Ut,t), \quad t \in [t_0, t_\infty) = \left[0, \frac{1}{U}\right) \tag{13}$$

$$H_0(t) = H(t) \text{ if } t \in [0, t_1] \tag{14}$$

$$H_j(t) = H(t) \text{ if } t \in [t_j, t_{j+1}] \tag{15}$$

It is also useful to introduce the two functions

$$V(x,t) = v_t(x,t) - v_x(x,t) \quad \text{and} \quad W(x,t) = v_t(x,t) + v_x(x,t) \tag{16}$$

which are constant along the lines $x - t = \text{const}$, $x + t = \text{const}$, respectively. Concerning their restrictions to the sets D^+ and D^-, we use the notations

$$V^{\pm}(x,t) = v_t^{\pm}(x,t) - v_x^{\pm}(x,t) \tag{17}$$

$$W^{\pm}(x,t) = v_t^{\pm}(x,t) + v_x^{\pm}(x,t) \tag{18}$$

Further distinctions can be introduced, when necessary, such as for instance

$$V_j^{\pm}(x,t) = \text{ restriction of } V^{\pm}(x,t) \text{ to the set } D_j^{\pm} \tag{19}$$

$$W_j^{\pm}(x,t) = \text{ restriction of } W^{\pm}(x,t) \text{ to the set } D_j^{\pm} \tag{20}$$

It is easy to show that

$$\begin{aligned} \dot{H}(t) &= v_x^{+}(Ut,t) + v_t^{+}(Ut,t) = v_x^{-}(Ut,t) + v_t^{-}(Ut,t) \\ \ddot{H}(t) &= U^2 v_{xx}^{+}(Ut,t) + 2U v_{xt}^{-}(Ut,t) + v_{xx}^{-}(Ut,t) \end{aligned} \tag{21}$$

where $\dot{}$ denotes the total derivative of H with respect to t.

Thanks to Eq. (13) and the Hadamard conditions in (1), the equations expressing the jump of the partial derivatives and, consequently, of the functions $V^{\pm}(Ut,t)$, $W^{\pm}(Ut,t)$ can be written as follows:

$$v_x^{-}(Ut,t) = v_x^{+}(Ut,t) - \left[\frac{p}{1-U^2} + \frac{\mu}{1-U^2}\ddot{H}(t)\right] \tag{22}$$

$$v_t^{-}(Ut,t) = v_t^{+}(Ut,t) + U\left[\frac{p}{1-U^2} + \frac{\mu}{1-U^2}\ddot{H}(t)\right] \tag{23}$$

or, equivalently:

$$V^-(Ut,t) = V^+(Ut,t) + \frac{p}{1-U} + \frac{\mu}{1-U}\ddot{H}(t) \tag{24}$$

$$W^-(Ut,t) = W^+(Ut,t) - \frac{p}{1+U} - \frac{\mu}{1+U}\ddot{H}(t). \tag{25}$$

It is evident that the solution of our problem is achieved once one finds the function $H(t) = v(Ut,t)$. In order to determine the function $H(t)$, we now state and prove the following lemma.

Lemma 2.1. *Let us consider the sequence of real numbers b_k recursively defined by*

$$b_2 = -\frac{p}{2\mu}, \quad b_{k+1} = -\frac{2b_k}{\mu(k+1)}\left[1-\left(\frac{1-U}{1+U}\right)^k\right]^{-1} \quad \text{for } k \geqslant 2 \tag{26}$$

Then the series $\sum_{k=2}^{\infty} b_k t^k$ has radius of convergence $+\infty$, and the function

$$G(t) = \sum_{k=2}^{\infty} b_k t^k \tag{27}$$

which belongs to $\mathcal{C}^\infty(0,+\infty)$, solves, all over this interval, the following retarded initial value differential problem:

$$\frac{U\,p\,t}{1-U^2} + G\left(\frac{t}{1-U}\right) - \frac{\mu}{2}\dot{G}\left(\frac{t}{1+U}\right) + \frac{\mu}{2}\dot{G}\left(\frac{t}{1-U}\right) = 0 \tag{28}$$

$$G(0) = 0, \quad \dot{G}(0) = 0 \tag{29}$$

Moreover, $G(t)$ is the unique solution, analytic in a right neighborhood of $t = 0$.

Proof. From the recursive formula, it is easily seen that $\lim_{k\to\infty} \frac{|b_{k+1}|}{|b_k|} = 0$ and thus, our series has infinite radius of convergence. Furthermore, the left hand side of (28), with $G(t)$ given by (27), takes the form:

$$\begin{aligned} &\frac{U\,p\,t}{1-U^2} + b_2\frac{t^2}{(1-U)^2} + b_3\frac{t^3}{(1-U)^3} + b_4\frac{t^4}{(1-U)^4} + \dots \\ &\quad + \left[-\frac{\mu}{2}\left(2b_2\frac{t}{1+U} + 3b_3\frac{t^2}{(1+U)^2} + 4b_4\frac{t^3}{(1+U)^3} + \dots\right)\right. \\ &\quad \left. +\frac{\mu}{2}\left(2b_2\frac{t}{1-U} + 3b_3\frac{t^2}{(1-U)^2} + 4b_4\frac{t^3}{(1-U)^3} + \dots\right)\right] \end{aligned} \tag{30}$$

that means

$$\begin{aligned} &t\left(\frac{U\,p}{1-U^2} + \mu\frac{2U\,b_2}{1-U^2}\right) + t^2\left(\frac{b_2}{(1-U)^2} - \frac{\mu}{2}\frac{3b_3}{(1+U)^2} + \frac{\mu}{2}\frac{3b_3}{(1-U)^2}\right) + \dots \\ &\quad + t^k\left(\frac{b_k}{(1-U)^k} - \frac{\mu}{2}\frac{(k+1)b_{k+1}}{(1+U)^k} + \frac{\mu}{2}\frac{(k+1)b_{k+1}}{(1-U)^k}\right) + \dots \end{aligned} \tag{31}$$

All the coefficients of this power series are equal to zero if b_k satisfy the recursive formula (26), while the conditions $G(0) = \dot{G}(0) = 0$ are evidently fulfilled by the function (27). Thus, the proof that $G(t)$ solves problem (28) and (29) is achieved. The uniqueness is easily checked by showing that for any solution of problem (28) and (29) which is the sum of a series $\sum_{k=2}^{\infty} c_k t^k$, necessarily the c_k coincide with the b_k of (26). □

Remark 2.1. Let us remark that in (28) G is computed in two different times, before and after t, respectively. This will be further clarified in the following Section, where the solution is built step-by-step by means of the functions $V_j^{\pm}(x,t)$ and $W_j^{\pm}(x,t)$.

3. Construction of the approximating power series

Let us now go to our main goal and try to define the solution $v_j^{\pm}(x,t)$ for each of the domains $D_j^{\pm}$. At first, we will find the corresponding expressions of the functions $V_j^{\pm}(x,t)$ and $W_j^{\pm}(x,t)$ which will in turn give the gradient of $v_j^{\pm}(x,t)$. Due to the fact that all the domains $D_j^{\pm}$ are simply connected, the further knowledge of the values of $v_j^{\pm}(x,t)$ provided by the initial (boundary) conditions will determine the solution in a unique manner wherever needed.

To begin with, let us remark that the homogeneous initial conditions uniquely determine the functions $V_0^{+}(x,t)$ and $W_0^{+}(x,t)$ inside the domain D_0^{+}. Thus

In D_0^{+}

$$V_0^{+}(x,t) = 0, \quad W_0^{+}(x,t) = 0. \tag{32}$$

Furthermore, recalling the meaning of these two functions (see Eq. (16)), we obtain:

In D_1^{+}

$$V_1^{+}(x,t) = \frac{2}{1-U}\dot{H}_0\left(\frac{t-x}{1-U}\right), \quad W_1^{+}(x,t) = 0 \tag{33}$$

With the aid of this formula, system (22), (23) assumes the following form, in which the first equation is unchanged:

$$v_x^{-}(Ut,t) = v_x^{+}(Ut,t) - \left[\frac{p}{1-U^2} + \frac{\mu}{1-U^2}\ddot{H}(t)\right] \tag{34}$$

$$v_t^{-}(Ut,t) = \frac{1}{1-U}\dot{H}(t) + U\left[\frac{p}{1-U^2} + \frac{\mu}{1-U^2}\ddot{H}(t)\right]. \tag{35}$$

Recalling the identities

$$V^{-}(x,t) = V^{-}\left(U\frac{t-x}{1-U}\ ,\ \frac{t-x}{1-U}\right); \quad W^{-}(x,t) = W^{-}\left(U\frac{t+x}{1+U}\ ,\ \frac{t+x}{1+U}\right) \tag{36}$$

and formulas (17), (18), one deduces that

In D_1^{-}

$$\begin{aligned} v_t^{-}(x,t) &= \frac{Up}{1-U^2} + \frac{1}{1-U}\dot{H}_0\left(\frac{t-x}{1-U}\right) - \frac{\mu}{2(1+U)}\ddot{H}_0\left(\frac{x+t}{1+U}\right) \\ &\quad + \frac{\mu}{2(1-U)}\ddot{H}_0\left(\frac{t-x}{1-U}\right) \\ v_x^{-}(x,t) &= -\frac{p}{1-U^2} - \frac{1}{1-U}\dot{H}_0\left(\frac{t-x}{1-U}\right) - \frac{\mu}{2(1+U)}\ddot{H}_0\left(\frac{x+t}{1+U}\right) \\ &\quad - \frac{\mu}{2(1-U)}\ddot{H}_0\left(\frac{t-x}{1-U}\right) \end{aligned} \tag{37}$$

Identities (37) are due to the fact that V^{-} (resp. W^{-}) is constant along the lines $t-x = constant$ (resp. $t+x = constant$). By integrating the linear differential form naturally associated with these last two equations and recalling that $v(Ut,t) = H_0(t),\ \forall t \in [0,t_1]$, we obtain:

$$v^{-}(x,t) = p\left(\frac{Ut-x}{1-U^2}\right) + H_0\left(\frac{t-x}{1-U}\right) - \frac{\mu}{2}\dot{H}_0\left(\frac{x+t}{1+U}\right) + \frac{\mu}{2}\dot{H}_0\left(\frac{t-x}{1-U}\right) \tag{38}$$

Finally, thanks to the boundary condition $v(0,t) = v^-(0,t) = 0$, and together with Lemma 2.1, it is seen that $H_0(t)$ is the solution, all over $[0,\infty)$, of the problem

$$\frac{p\,Ut}{1-U^2} + H_0\left(\frac{t}{1-U}\right) - \frac{\mu}{2}\,\dot{H}_0\left(\frac{t}{1+U}\right) + \frac{\mu}{2}\,\dot{H}_0\left(\frac{t}{1-U}\right) = 0 \tag{39}$$

$$H_0(0) = 0, \quad \dot{H}_0(0) = 0 \tag{40}$$

Thus, (see Lemma 2.1) (assuming analyticity for H_0) one has

$$H_0(t) = G(t) = \sum_{k=0}^{\infty} b_k t^k, \quad \forall t \in [0, t_1] \tag{41}$$

where the b_k are provided by (26).

The passage from (38) to (39) explains the meaning of the retarded Eq. (28): the values of G (or, equivalently, H_0) for all times in $[0,t_1]$ are strictly related to $v^-(x,t)$, which is completely determined on D_1^-.

Similar considerations will be made for what concerns the following intervals $[t_{j-1}, t_j]$

We now have to find out, for each j, the solution $H_j(t)$ relative to the interval $[t_{j-1}, t_j]$. To this end, we proceed step-by-step for the various domains D_j^+. In this analysis, the following Lemmas 3.1 and 3.2 will turn out of some utility.

Lemma 3.1. *The function $H_0(t)$, solution of (39), satisfies the identity:*

$$\begin{aligned}\frac{2\,p\,U^2}{1-U^2} &+ \frac{\mu}{2}\left(\frac{1-U}{1+U}\right)\ddot{H}_0\left(\frac{t(1-U)}{1+U}\right) - \frac{\mu}{2}\ddot{H}_0\left(\frac{t(1+U)-2}{1+U}\right) + \frac{\mu}{2}\left(\frac{1+U}{1-U}\right)\ddot{H}_0\left(\frac{t(1+U)-2}{1-U}\right)\\ &= -\frac{p\,U}{1+U} + \frac{\mu}{2}\left(\frac{1-U}{1+U}\right)\ddot{H}_0\left(\frac{t(1-U)}{1+U}\right) - \left(\frac{1+U}{1-U}\right)\dot{H}_0\left(\frac{t(1+U)-2}{1-U}\right)\end{aligned}$$

Proof. This identity is evidently equivalent to

$$\begin{aligned}\frac{p\,U}{1-U} - \frac{\mu}{2}\ddot{H}_0\left(\frac{t(1+U)-2}{1+U}\right) + \frac{\mu}{2}\left(\frac{1+U}{1-U}\right)\ddot{H}_0\left(\frac{t(1+U)-2}{1-U}\right)\\ + \left(\frac{1+U}{1-U}\right)\dot{H}_0\left(\frac{t(1+U)-2}{1-U}\right) = 0\end{aligned} \tag{42}$$

The left hand side of Eq. (42) is obtained by differentiating the left hand side of Eq. (39) and then substituting t with $t(1+U)-2$. Therefore, Eqs. (42) is fulfilled and the theorem is proven. □

Lemma 3.2. *The function $V^-(Ut,t)$ is completely determined all over $\left[0, \frac{1}{U}\right)$ by the knowledge of $H_0(t)$, $\forall t \in [0, t_1]$.*

Proof. Thanks to Eqs. (33) and (41) it is enough to consider the 'remaining' time interval $\left[t_1, \frac{1}{U}\right)$. Due to the properties of the function $V(x,t)$ and the condition $v^-(0,t) = 0$, we have:

$$V^-(Ut,t) = V^-(0,(1-U)t) = -W^-(0,(1-U)t), \qquad \forall t \in \left[0, \frac{1}{U}\right) \tag{43}$$

Moreover, the jump condition (25) with the second of Eqs. (33) and (20) give

$$V^-(Ut,t) = -W^-(0,(1-U)t) = \frac{p}{1+U} + \frac{\mu}{1+U}\,\ddot{H}_0\left(\frac{t(1-U)}{1+U}\right) \tag{44}$$

where, recalling that $U = 0.5$,

$$0 < \frac{t(1-U)}{1+U} < \frac{2}{1+U} = t_1, \qquad \forall t \in \left[t_1, \frac{1}{U}\right) \tag{45}$$

□

At this point, we are in the position of computing, for each j, the functions $V_j^+(Ut,t)$ and $W_j^+(Ut,t)$, which, together with the jump and boundary conditions, lead to a set of ordinary differential equations whose solution determines the function $H_j(t)$.

Here are the steps of the needed algorithm:

$$V_{2j-1}^+(Ut,t) = -\frac{2\,U\,p}{1-U^2} + \frac{\mu}{1+U}\,\ddot{H}_0\left(\frac{t(1-U)}{1+U}\right) - \frac{\mu}{1-U}\ddot{H}_j(t), \quad \text{for } j = 1,2,3,\ldots \tag{46}$$

$$V_0^+(1,t) = -\frac{2\,U\,p}{1-U^2} + \frac{\mu}{1+U}\,\ddot{H}_0\left(\frac{t-1}{1+U}\right) - \frac{\mu}{1-U}\ddot{H}_0\left(\frac{t-1}{1-U}\right) \tag{47}$$

$$V_{2j}^+(1,t) = -\frac{2\,U\,p}{1-U^2} + \frac{\mu}{1+U}\,\ddot{H}_0\left(\frac{t-1}{1+U}\right) - \frac{\mu}{1-U}\ddot{H}_j\left(\frac{t-1}{1-U}\right), \quad \text{for } j = 1,2,3,\ldots \tag{48}$$

$$W_1^+(1,t(P_1)) = -V_0^+(1,t(P_1)) \tag{49}$$

$$W_{2j-1}^+(1,t(P_j)) = -V_{2j-2}^+(1,t(P_j)) \tag{50}$$

$$W_{2j-1}^+(x,t) = \frac{2\,U\,p}{1-U^2} - \frac{\mu}{1+U}\,\ddot{H}_0\left(\frac{t+x-2}{1+U}\right) + \frac{\mu}{1-U}\ddot{H}_{j-1}\left(\frac{t+x-2}{1-U}\right), \quad \text{for } j = 1,2,3,\ldots \tag{51}$$

$$W_1^+(Ut,t) = \frac{2\,U\,p}{1-U^2} - \frac{\mu}{1+U}\,\ddot{H}_0\left(\frac{t(1+U)-2}{1+U}\right) + \frac{\mu}{1-U}\ddot{H}_0\left(\frac{t(1+U)-2}{1-U}\right) \tag{52}$$

$$W_{2j-1}^+(Ut,t) = \frac{2\,U\,p}{1-U^2} - \frac{\mu}{1+U}\,\ddot{H}_0\left(\frac{t(1+U)-2}{1+U}\right) + \frac{\mu}{1-U}\ddot{H}_{j-1}\left(\frac{t(1+U)-2}{1-U}\right), \quad \text{for } j = 2,3,\ldots \tag{53}$$

The functions $V_j^+(Ut,t)$, $W_j^+(Ut,t)$ (see Eqs. (46) and (53)) and the following identities:

$$\dot{H}_j(t) = \left[\frac{\partial}{\partial t}v_{2j-1}^+(x,t) + U\frac{\partial}{\partial x}v_{2j-1}^+(x,t)\right]_{x=Ut} \tag{54}$$

$$\frac{\partial}{\partial t}v_j^+(x,t) = \frac{1}{2}\left[V_j^+(x,t) + W_j^+(x,t)\right]$$

$$\frac{\partial}{\partial x}v_j^+(x,t) = \frac{1}{2}\left[W_j^+(x,t) - V_j^+(x,t)\right] \tag{55}$$

lead to

$$\dot{H}_j(t) = \left(\frac{1-U}{2}\right)V_{2j-1}^+(Ut,t) + \left(\frac{1+U}{2}\right)W_{2j-1}^+(Ut,t) \tag{56}$$

and then to the linear second order ordinary differential equation

$$\frac{\mu}{2}\,\ddot{H}_j(t) + \dot{H}_j(t) = \frac{2\,p\,U^2}{1-U^2} + \frac{\mu}{2}\left(\frac{1-U}{1+U}\right)\ddot{H}_0\left(\frac{t(1-U)}{1+U}\right) \tag{57}$$

$$-\frac{\mu}{2}\ddot{H}_0\left(\frac{t(1+U)-2}{1+U}\right) + \frac{\mu}{2}\left(\frac{1+U}{1-U}\right)\ddot{H}_{j-1}\left(\frac{t(1+U)-2}{1-U}\right). \tag{58}$$

Now, we seek the solution of Eq. (57) which satisfies the Cauchy conditions

$$H_j(t_j) = H_{j-1}(t_j), \quad \dot{H}_j(t_j) = \dot{H}_{j-1}(t_j) \tag{59}$$

Note that $\forall t \in [t_j, t_{j+1}]$ the right hand side of Eq. (57) is a known quantity.

Thus, thanks to a notation of the kind:

$$\begin{aligned} F^0_{j-1}(t) = \frac{2\,p\,U^2}{1-U^2} + \frac{\mu}{2}\left(\frac{1-U}{1+U}\right)\ddot{H}_0\left(\frac{t(1-U)}{1+U}\right) - \frac{\mu}{2}\ddot{H}_0\left(\frac{t(1+U)-2}{1+U}\right) \\ + \frac{\mu}{2}\left(\frac{1+U}{1-U}\right)\ddot{H}_{j-1}\left(\frac{t(1+U)-2}{1-U}\right) \end{aligned} \tag{60}$$

the explicit form of the solution to (57) and (59) can be written as

$$H_j(t) = H_{j-1}(t_j) + \int_{t_j}^{t}\int_{t_j}^{s} e^{\frac{\mu}{2}(s-t)}\left[F^0_{j-1}(s) + H_{j-1}(t_j) + \dot{H}_{j-1}(t_j)\right]\mathrm{d}s\mathrm{d}t. \tag{61}$$

However, a more relevant point is provided by the following lemma.

Lemma 3.3. *The function $H_j(t)$ which solves the Cauchy problem (57) and (59) satisfies the further condition*

$$\ddot{H}_j(t_j) = \ddot{H}_{j-1}(t_j) \tag{62}$$

Proof. The proof is obtained via mathematical induction.

In the case $j = 1$ the equality (62) has the form:

$$\ddot{H}_1(t_1) = \ddot{H}_0(t_1) \tag{63}$$

and Eq. (57) is

$$\begin{aligned} \frac{\mu}{2}\ddot{H}_1(t) + \dot{H}_1(t) = \frac{2\,p\,U^2}{1-U^2} + \frac{\mu}{2}\left(\frac{1-U}{1+U}\right)\ddot{H}_0\left(\frac{t(1-U)}{1+U}\right) \\ - \frac{\mu}{2}\ddot{H}_0\left(\frac{t(1+U)-2}{1+U}\right) + \frac{\mu}{2}\left(\frac{1+U}{1-U}\right)\ddot{H}_0\left(\frac{t(1+U)-2}{1-U}\right) \end{aligned} \tag{64}$$

which, thanks to Lemma 3.1, has the equivalent form:

$$\frac{\mu}{2}\ddot{H}_1(t) + \dot{H}_1(t) = -\frac{p\,U}{1+U} + \frac{\mu}{2}\left(\frac{1-U}{1+U}\right)\ddot{H}_0\left(\frac{t(1-U)}{1+U}\right) - \left(\frac{1+U}{1-U}\right)\dot{H}_0\left(\frac{t(1+U)-2}{1-U}\right) \tag{65}$$

Furthermore, differentiating Eq. (39) and multiplying by $(1-U)$ one obtains:

$$\frac{p\,U}{1+U} + \dot{H}_0\left(\frac{t}{1-U}\right) - \frac{\mu}{2}\left(\frac{1-U}{1+U}\right)\ddot{H}_0\left(\frac{t}{1+U}\right) + \frac{\mu}{2}\ddot{H}_0\left(\frac{t}{1-U}\right) = 0 \tag{66}$$

which in turn, for $t = 2\lambda = 2\left(\frac{1-U}{1+U}\right)$ gives

$$\frac{\mu}{2}\ddot{H}_0\left(\frac{2}{1+U}\right) = -\frac{p\,U}{1+U} - \dot{H}_0\left(\frac{2}{1+U}\right) + \frac{\mu}{2}\left(\frac{1-U}{1+U}\right)\ddot{H}_0\left(\frac{2}{1+U}\lambda\right) \tag{67}$$

Furthermore, by computing Eq. (65) in $t = t_1 = \frac{2}{1+U}$, and recalling that $\dot{H}_0(0) = 0$, one obtains:

$$\frac{\mu}{2}\ddot{H}_1\left(\frac{2}{1+U}\right) = -\frac{p\,U}{1+U} - \dot{H}_1\left(\frac{2}{1+U}\right) + \frac{\mu}{2}\left(\frac{1-U}{1+U}\right)\ddot{H}_0\left(\frac{2}{1+U}\lambda\right) \tag{68}$$

From (67) and (68), thanks to the hypothesis $\dot{H}_1\left(\frac{2}{1+U}\right) = \dot{H}_0\left(\frac{2}{1+U}\right)$, we find that

$$\ddot{H}_1\left(\frac{2}{1+U}\right) = \ddot{H}_0\left(\frac{2}{1+U}\right) \tag{69}$$

that is (62) with $t = t_1$. Now, we have to show that, if the equality $\ddot{H}_{j-1}(t_{j-1}) = \ddot{H}_{j-2}(t_{j-1})$ is satisfied by the corresponding solutions $H_{j-2}(t)$ and $H_{j-1}(t)$, then the equality $\ddot{H}_j(t_j) = \ddot{H}_{j-1}(t_j)$ is satisfied by the solutions $H_{j-1}(t)$ and $H_j(t)$.

This is achieved by considering a pair of formulas of the type (57), (59) corresponding to the pair of consecutive indices $j-1$, j evaluated at the point $t = t_j$, and then subtracting the first from the second one (concerning the correctness of this procedure see Remark 3.1 below):

$$\begin{aligned}&\frac{\mu}{2}\left(\ddot{H}_j(t_j) - \ddot{H}_{j-1}(t_j)\right) + \left(\dot{H}_j(t_j) - \dot{H}_{j-1}(t_j)\right)\\ &\quad = \frac{\mu}{2}\left(\frac{1+U}{1-U}\right)\left[\ddot{H}_{j-1}\left(\frac{t_j(1+U)-2}{1-U}\right) - \ddot{H}_{j-2}\left(\frac{t_j(1+U)-2}{1-U}\right)\right]\end{aligned} \tag{70}$$

From (70), observing that $\frac{t_j(1+U)-2}{1-U} = t_{j-1}$ (see (5), (6) and (7)) and recalling the induction hypothesis, one sees that

$$\frac{\mu}{2}\left(\ddot{H}_j(t_j) - \ddot{H}_{j-1}(t_j)\right) + \left(\dot{H}_j(t_j) - \dot{H}_{j-1}(t_j)\right) = 0 \tag{71}$$

Therefore, since the (Cauchy) condition $\dot{H}_j(t_j) = \dot{H}_{j-1}(t_j)$ is also satisfied, we finally obtain

$$\ddot{H}_j(t_j) = \ddot{H}_{j-1}(t_j) \tag{72}$$

□

Remark 3.1. It is worth noting that each function $H_j(t)$ is a global solution, all over $[0, \infty)$, of an initial value problem, for an ordinary differential equation which is utilized to write down (70), even though the identity

$$H(t) = v(Ut, t) \tag{73}$$

holds true only for $t \in [t_{j-1}, t_j]$.

The function $H(t)$ obtained step-by-step with the method just described is a function of the class $\mathcal{C}^2([0, \infty))$ such that $H(t) = v(Ut, t)$ if $v(x, t)$ is a solution of Problem (1), thus, $v^+(x, t)$ (resp. $v^-(x, t)$) is a function of the class $\mathcal{C}^2_{\text{loc}}$ in the right (resp. left) neighborhood of the open segment $x = Ut$, $t \in \left[0, \frac{1}{U}\right)$.

Once one has determined the function $H(t)$, the two formulas (46), (53) and the boundary conditions along $x = 1$ determine $V_j^+(x, t)$ and $W_j^+(x, t)$ in D_j^+ for any $j > 1$. Afterward, one determines $V_j^-(x, t)$ and $W_j^-(x, t)$ by means of the jump conditions along $x = Ut$. The functions V_0^+ and W_0^+ are determined by the boundary conditions along $t = 0$ and $x = 1$. The functions $V_1^-(x, t)$ and $W_1^-(x, t)$ are determined by means of the jump conditions along $x = Ut$ and the boundary conditions along $x = 0$.

4. Numerical simulations

The iterative technique used to find the exact solution of the problem does not allow us to arrive in a finite number of steps to the point $\left(1, \frac{1}{U}\right)$. This is why we cannot expect the solution can respect the boundary condition in this point. Here, we want to show some numerical simulations which confirm the presence of a discontinuity of the solution in the point $\left(1, \frac{1}{U}\right)$.

As already remarked, the choice of a constant speed $U = 0.5$ is arbitrary and is related to the fact that, in this case, the structure of D^+ and D^- is simple. Considering subsonic cases with different speeds does not affect the scheme of our approach, but only complicates the geometry of D^+ and D^-.

Equation (57) has been solved with MATLAB (employing a Runge–Kutta-based ODE solver), suitably selecting the integer j and the last term of the power series in order to have convergence of the solution with an error comfortably controlled by the accuracy of the numerical procedure.

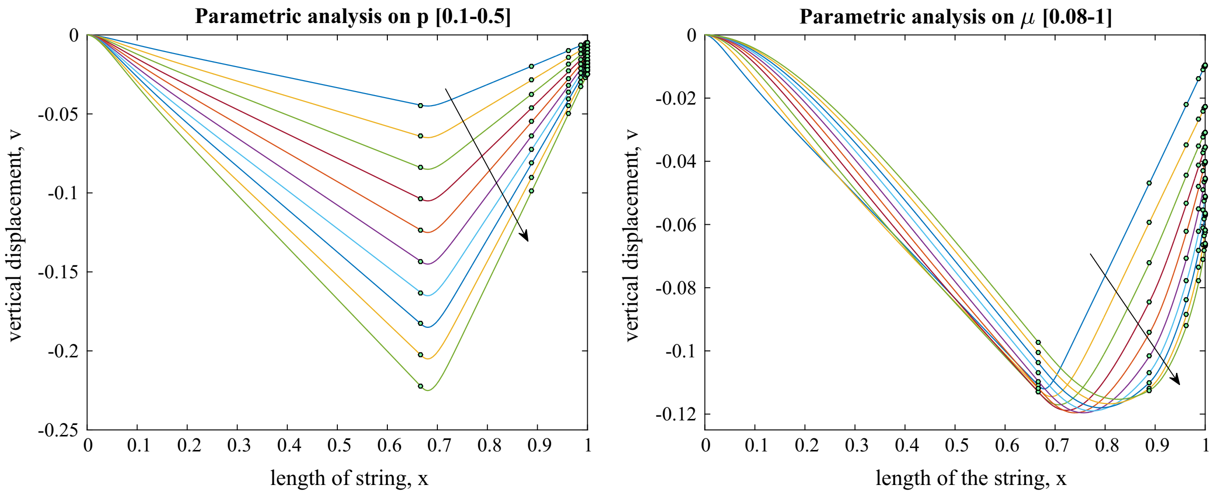

FIG. 2. *Left* Parametric analysis of the deformed shape of the string when varying p between 0.1 and 0.5. *Right* Parametric analysis of the deformed shape of the string when varying μ between 0.08 and 1

We performed a parametric analysis in order to check the sensitivity of the system to changes in the two parameters that characterize it, i.e., p and μ. In the first simulation, we set the parameter $\mu = 0.1$ and vary p in a set equally spaced between 0.1 and 0.5. The results are shown in Fig. 2-left. It can be seen that the deflection increases with the parameter p. In the second case, we set $p = 0.25$ and vary μ in a set equally spaced between 0.08 and 1. Figure 2-right exhibits the effect of the traveling mass. In these plots, we can see that increasing the value of μ, the minimum of the vertical string displacement moves toward the right end.

Circlets in the plot mark the extrema of the subintervals on which the solution is constructed. As anticipated, in both cases, the solution shows a jump discontinuity in the right extremum.

To allow a comparison, in Fig. 3, left panel, the simulation shown in Fig. 2 (left panel) has been performed by means of a standard Galerkin method.

A discretization consisting of 1000 points, with 40 Galerkin modes, was employed. The result is consistent with what shown in the previous figure based on the analytic method for what concerns the overall shape of the graph and the maximum amplitude. However, some differences concern: (i) the limit value at the right extremum, which in the case of the Galerkin simulation appears to be always positive independently of the value of p (in the right panel of Fig. 3 a zoom on the right extremum is shown); this appears to be another form of the paradox related to the presence of the nonholonomic constraint imposing constant horizontal component of the velocity of the mass, which in case of Fig. 2 (left) leads to a jump discontinuity. (ii) The qualitative behavior in the second half of the graph, since the Galerkin simulation shows bumps that are not visible in the simulation based on the analytical iteration, showing the inadequacy of the numerical method for this case.

In Fig. 4, left panel, the simulation shown in Fig. 2 (right) has been performed by means of the same Galerkin method as in the previous one. In this case, even larger differences are observable in the last part of the graph. In Fig. 4, right panel, a zoom helps noticing that Galerkin simulation in this case is again not reliable.

On the whole, it appears that Galerkin simulations are useful to grasp the general behavior of the solution and the maximum amplitude, but one cannot rely on them to get an accurate pointwise estimate of the solution, especially in the final part.

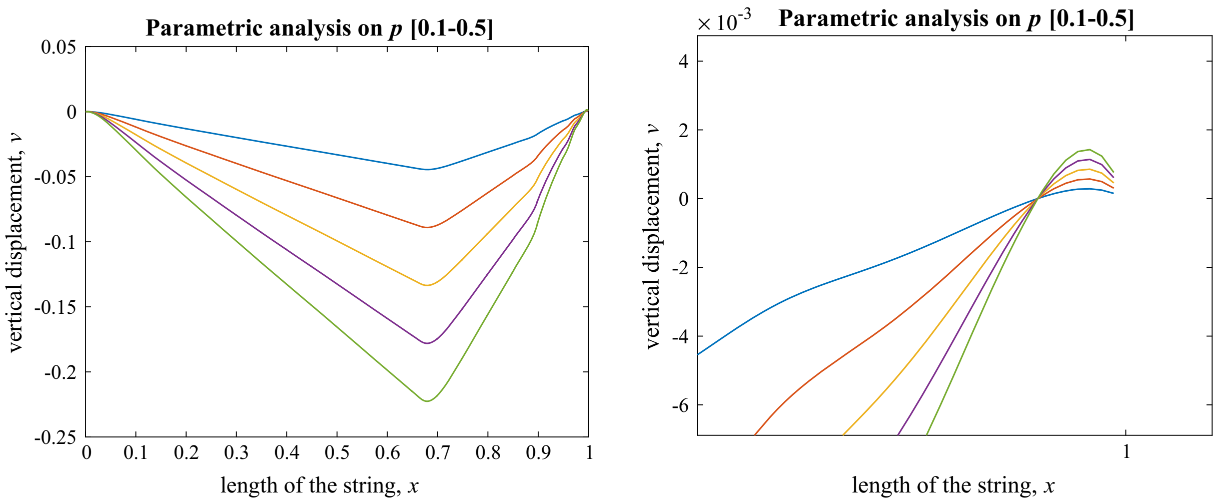

Fig. 3. *On the left* standard Galerkin method applied to the simulation shown in Fig. 2 (*left*); on the *right* the corresponding zoom on the *right* extremum

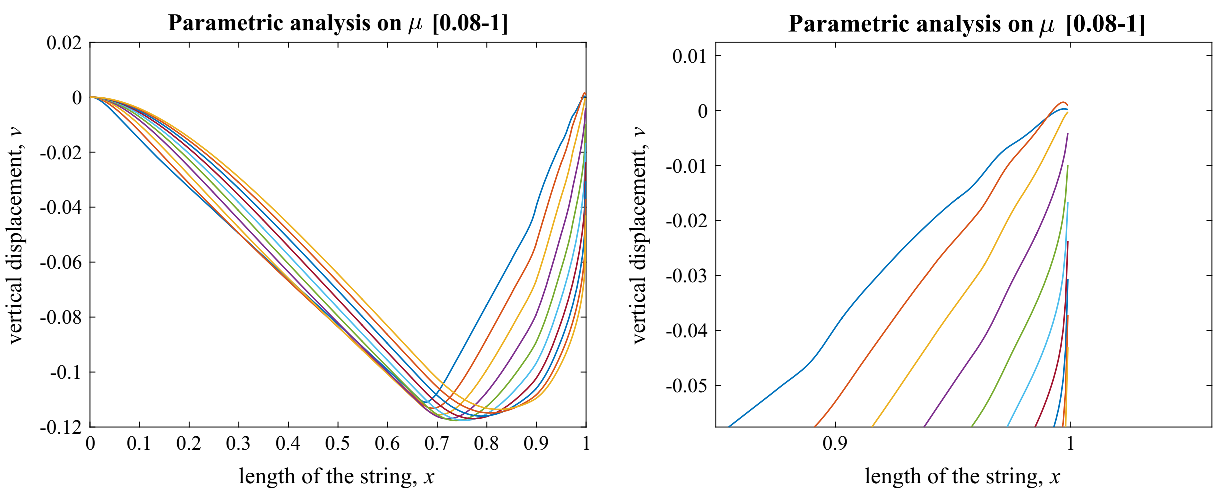

Fig. 4. *On the left* standard Galerkin method applied to the simulation shown in Fig. 2 (*right*); *on the right* the corresponding zoom on the right extremum

5. Conclusion and discussion

In this paper we determined the explicit solution of the dynamical equilibrium of a taut string of finite length carrying a traveling mass, in the subsonic case, building the solution in terms of a power series in time, for every segment into which the line $x = Ut$ is divided by the characteristic lines. This approach allows us to confirm the paradox highlighted in [30,34], due to a lack of wellposedness of the problem.

This paradoxical situation will be subject of further investigation. Let us however remark that, from a physical point of view, setting a constant velocity U implies that, in order to allow the mass to reach the end point, we should have to apply an infinite force. This was already observed in previous investigations on traveling masses. In particular, in [38], dealing with the problem in large deformations and including extensibility, the author points out that, even with finite speed of the moving mass, an infinitely large longitudinal resistance force can be exerted on the load. Aim of our future research will be to approach the problem where an external force is introduced, without imposing a kinematic condition on the traveling mass.

The technique here described could be in the future applied to determine the explicit solution of other problems, like, for example, the study of the dynamics of strings traveled by train of forces, as in [49,53]. The more complex problem involving a beam instead of a string is already known in the literature ([1,13,56]). In this respect, impact effects between the mass and the beam, which can be of significant practical relevance since the constraint is of course never perfectly verified by real-world objects, could be investigated with the theoretical, numerical and experimental methods developed in [4,5,17,20,48]. Also, the onset of buckling due to the traveling load (see e.g., [3,58,64,65]) may concern this more complex case of study in case of compressive prestress. It is also natural to consider the possibility of further generalizing the method to 2D elastic bodies, and in this case, the nature of the problem, entailing the presence of concentrated loads, suggests the consideration of higher gradient energy models which can sustain loads of this type (for related theoretical results see [7,23,26,28,36,54,55,70], while for numerical investigations, see [19,27,40,62,66–68]; a formulation of elasticity which is suitably general for this kind of continua is provided in [31]). The interaction of the traveling mass with a microstructure (as for instance the generalized continua studied e.g., in [2,22,24,32,39,41,42,59–61,63]) can also be of great interest both theoretically and for applications of the results to metamaterials ([10,21,25]).

Future numerical studies of this problem can employ suitably refined forms of finite element methods, such as isogeometric analysis (see e.g., [6,9,11,12,16,43]). Specifically, it will be useful to employ a nesting form-finding procedure, typical for cable structures (as in [44,45]) in isogeometric formulation with singularities, as developed in [46,47], reducing the formulation to a standard shape optimization problem on the lengths of two portion of the cable. As the problem is of dynamical nature, the tools developed in [14,15] may also be of help. Finally, the generalization to multiple masses may entail the onset of instabilities, for the investigation of which the numerical techniques employed in [18,33,50–52] can be useful.

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References

1. Akin, J.E., Mofid, M.: Numerical solution for response of beams with moving mass. J. Struct. Eng. **115**(1), 120–131 (1989)
2. Altenbach, H., Eremeyev, V.A.: On the linear theory of micropolar plates. ZAMM J. Appl. Math. Mech. **89**(4), 242–256 (2009)
3. Aminpour, H., Rizzi, N.: A one-dimensional continuum with microstructure for single-wall carbon nanotubes bifurcation analysis. Math. Mech. Solids **21**(2), 168–181 (2016)
4. Andreaus, Ugo, Baragatti, Paolo, Placidi, L.: Experimental and numerical investigations of the responses of a cantilever beam possibly contacting a deformable and dissipative obstacle under harmonic excitation. Int. J. Non-Linear Mech. **80**, 96–106 (2016)
5. Andreaus, Ugo, Chiaia, Bernardino, Placidi, L.: Soft-impact dynamics of deformable bodies. Contin. Mech. Thermodyn. **25**, 375–398 (2013)
6. Aristodemo, M., Turco, E.: Boundary element discretization of plane elasticity and plate bending problems. Int. J. Numer. Methods Eng. **37**(6), 965–987 (1994)

7. Auffray, N., dell'Isola, F., Eremeyev, V.A., Madeo, A., Rosi, G.: Analytical continuum mechanics à la Hamilton-Piola least action principle for second gradient continua and capillary fluids. Math. Mech. Solids **20**(4), 375–417 (2015)
8. Bilello, C., Bergman, L.A., Kuchma, D.: Experimental investigation of a small-scale bridge model under a moving mass. J. Struct. Eng. **130**(5), 799–804 (2004)
9. Bilotta, A., Formica, G., Turco, E.: Performance of a high-continuity finite element in three-dimensional elasticity. Int. J. Numer. Methods Biomed. Eng. **26**(9), 1155–1175 (2010)
10. Carcaterra, A., dell'Isola, F., Esposito, R., Pulvirenti, M.: Macroscopic description of microscopically strongly inhomogenous systems: A mathematical basis for the synthesis of higher gradients metamaterials. Arch. Ration. Mech. Anal. **218**(3), 1239–1262 (2015)
11. Cazzani, A., Stochino, F., Turco, E.: An analytical assessment of finite element and isogeometric analyses of the whole spectrum of Timoshenko beams. ZAMM J. Appl. Math. Mech. (2016). doi:10.1002/zamm.201500280
12. Cazzani, A., Malagù, M., Turco, E.: Isogeometric analysis of plane-curved beams. Math. Mech. Solids. **21**(5), 562–577 (2016). doi:10.1177/1081286514531265
13. Cazzani, A., Wagner, N., Ruge, P., Stochino, F.: Continuous transition between traveling mass and traveling oscillator using mixed variables. Int. J. Non-Linear Mech. **80**, 82–95 (2016)
14. Cazzani, A., Ruge, P.: Symmetric matrix-valued transmitting boundary formulation in the time-domain for soil-structure interaction problems. Soil Dyn. Earthq. Eng. **57**, 104–120 (2014)
15. Cazzani, A., Ruge, P.: Stabilization by deflation for sparse dynamical systems without loss of sparsity. Mech. Syst. Signal Process. **70**(71), 664–681 (2016)
16. Cazzani, A., Malagù, M., Turco, E., Stochino, F.: Constitutive models for strongly curved beams in the frame of isogeometric analysis. Math. Mech. Solids **21**(2), 182–209 (2016)
17. Chiaia, B., Kumpyak, O., Placidi, L., Maksimov, V.: Experimental analysis and modelling of two-way reinforced concrete slabs over different kinds of yielding supports under short-term dynamic loading. Eng. Struct. **96**, 88–99. ISSN: 0141-0296. doi:10.1016/j.engstruct.2015.03.054 (2015)
18. Contento, A., Luongo, A.: Static and dynamic consistent perturbation analysis for nonlinear inextensible planar frames. Comput. Struct. **123**, 79–92 (2013)
19. Cuomo, M., dell'Isola, F., Greco, L.: Simplified analysis of a generalized bias-test for fabrics with two families of inextensible fibres. Zeitschrift für angewandte Mathematik und Physik. (2016). doi:10.1007/s00033-016-0653-z
20. Cuomo, M., Ventura, G.: Complementary energy approach to contact problems based on consistent augmented Lagrangian formulation. Math. Comput. Model. **28**(4-8), 185–204 (1998)
21. Del Vescovo, D., Giorgio, I.: Dynamic problems for metamaterials: review of existing models and ideas for further research. Int. J. Eng. Sci. **80**, 153–172 (2014)
22. dell'Isola, F., Guarascio, M., Hutter, K.: A variational approach for the deformation of a saturated porous solid. A second-gradient theory extending Terzaghi's effective stress principle. Arch. Appl. Mech. **70**(5), 323–337 (2000)
23. dell'Isola, F., Seppecher, P., Madeo, A.: How contact interactions may depend on the shape of Cauchy cuts in Nth gradient continua: approach "à la D'Alembert". Zeitschrift für Angewandte Mathematik Und Physik **63**(6), 1119–1141 (2012)
24. dell'Isola, F., Batra, R.C.: Saint-Venant's problem for porous linear elastic materials. J. Elast. **47**(1), 73–81 (1997)
25. dell'Isola, F., Bucci, S., Battista, A.: Against the fragmentation of knowledge: the power of multidisciplinary research for the design of metamaterials. In: Advanced Methods of Continuum Mechanics for Materials and Structures, vol. 60 of the series, pp. 523–545. Advanced Structured Materials (2016)
26. dell'Isola, F., Madeo, A., Seppecher, P.: Cauchy tetrahedron argument applied to higher contact interactions. Arch. Rational Mech. Anal. **219**(3), 1305–1341 (2016)
27. dell'Isola, F., d'Agostino, M.V., Madeo, A., Boisse, P., Steigmann, D.: Minimization of shear energy in two dimensional continua with two orthogonal families of inextensible fibers: The case of standard bias extension test. J. Elast. **122**(2), 131–155 (2016)
28. dell'Isola, F., Steigmann, D.: A two-dimensional gradient-elasticity theory for woven fabrics. J. Elast. **118**(1), 113–125 (2015)
29. Derendyayev, N.V., Soldatov, I.N.: The motion of a point mass along a vibrating string. J. Appl. Math. Mech. **61**, 681–684 (1997)
30. Dyniewicz, B., Bajer, C.I.: Paradox of a particle's trajectory moving on a string. Arch. Appl. Mech. **79**, 213–223 (2009)
31. Eremeyev, V.E., Lebedev, L.P.: Existence of weak solutions in elasticity. Math. Mech. Solids **18**(2), 204–217 (2013)
32. Eremeyev, V.A., Morozov, N.F.: The effective stiffness of a nanoporous rod. In: Doklady Physics, vol. 55, No. 6, pp. 279–282. MAIK Nauka/Interperiodica. (2010, June)
33. Favino, M., Grillo, A., Krause, R.: A stability condition for the numerical simulation of poroelastic systems. In: Poromechanics V: Proceedings of the Fifth Biot Conference on Poromechanics, pp. 919–928. ASCE (2013)
34. Ferretti, M., Piccardo, G.: Dynamic modeling of taut string carrying a traveling mass. Contin. Mech. Thermodyn. **25**(2-4), 469–488 (2013)
35. Fryba, L.: Vibrations of Solids and Structures Under Moving Loads. Academia, Prague (1972)

36. Gabriele, S., Rizzi, N., Varano, V.: A 1D nonlinear TWB model accounting for in plane cross-section deformation. Int. J. Solids Struct. (2015). doi:10.1016/j.ijsolstr.2016.04.017
37. Gao, Q., Zhang, J., Zhang, H.W., Zhong, W.X.: The exact solutions for a point mass moving along a stretched string on a Winkler foundation. Shock Vib. **2014**, 1–11 (2014)
38. Gavrilov, S.: Nonlinear investigation of the possibility to exceed the critical speed by a load on a string. Acta Mech. **154**(1–4), 47–60 (2002)
39. Giorgio, I., Andreaus, U., Scerrato, D., Braidotti, P.: Modeling of a non-local stimulus for bone remodeling process under cyclic load: application to a dental implant using a bioresorbable porous material. Math. Mech. Solids (2016) doi:10.1177/1081286516644867
40. Giorgio, I., Grygoruk, R., dell'Isola, F., Steigmann, D.J.: Pattern formation in the three-dimensional deformations of fibered sheets. Mech. Res. Commun. **69**, 164–171 (2015)
41. Giorgio, I., Scerrato, D.: Multi-scale concrete model with rate dependent internal friction. Eur. J. Environ. Civil Eng. (2016). (doi:10.1080/19648189.2016.1144539)
42. Giorgio, I., Andreaus, U., Scerrato, D., dell'Isola, F.: A visco-poroelastic model of functional adaptation in bones reconstructed with bio-resorbable materials. Biomech. Model. Mechanobiol. (2016). (doi:10.1007/s10237-016-0765-6)
43. Greco, L., Cuomo, M.: B-Spline interpolation of Kirchhoff–Love space rods. Comput. Methods Appl. Mech. Eng. **256**, 251–269 (2013)
44. Greco, L., Cuomo, M.: On the force density method for slack cable nets. Int. J. Solids Struct. **49**(13), 1526–1540 (2012)
45. Greco, L., Impollonia, N., Cuomo, M.: A procedure for the static analysis of cable structures following elastic catenary theory. Int. J. Solids Struct. **51**(7), 1521–1533 (2014)
46. Greco, L., Cuomo, M.: An implicit G1 multi patch B-spline interpolation for Kirchhoff-Love space rod. Comput. Methods Appl. Mech. Eng. **269**, 173–197 (2014)
47. Greco, L., Cuomo, M.: An isogeometric implicit G1 mixed finite element for Kirchhoff space rods. Comput. Methods Appl. Mech. Eng. **298**, 325–349 (2016)
48. Kezmane, A., Chiaia, B., Kumpyak, O., Maksimov, V., Placidi, L.: 3D Modeling of reinforced concrete slab with yielding supports subject to impact load. Eur. J. Environ. Civil Eng. (2016). ISSN: 1964-8189. doi:10.1080/19648189.2016.1194330
49. Luongo, A., Piccardo, G.: Perturbation and numerical analysis of suspended cables traveled by a single mass. In: Proceedings of XIX AIMETA, Ancona, IT. (2009)
50. Luongo, A., Paolone, A.: Multiple scale analysis for divergence-Hopf bifurcation of imperfect symmetric systems. J. Sound Vib. **218**(3), 527–539 (1998)
51. Luongo, A., Di Egidio, A., Paolone, A.: On the proper form of the amplitude modulation equations for resonant systems. Nonlinear Dyn. **27**(3), 237–254 (2002)
52. Luongo, A., Paolone, A., Di Egidio, A.: Multiple timescales analysis for 1: 2 and 1: 3 resonant Hopf bifurcations. Nonlinear Dyn. **34**(3-4), 269–291 (2003)
53. Luongo, A., Piccardo, G.: Dynamics of taut strings traveled by train of forces. Contin. Mech. Thermodyn. **28**(1–2), 603–616 (2016)
54. Madeo, A., Placidi, L., Rosi, G.: Towards the design of metamaterials with enhanced damage sensitivity: second gradient porous materials. Res Nondestr. Eval. **25**(2), 99–124 (2014)
55. Madeo, A., dell'Isola, F., Darve, F.: A continuum model for deformable, second gradient porous media partially saturated with compressible fluids. J. Mech. Phys. Solids **61**(11), 2196–2211 (2013)
56. Mofid, M., Akin, J.E.: Discrete element response of beams with traveling mass. Adv. Eng. Softw. **25**(2), 321–331 (1996)
57. Ouyang, H.: Moving-load dynamic problems: A tutorial (with a brief overview). Mech. Syst. Signal Process. **25**(6), 2039–2060 (2011)
58. Pignataro, M., Ruta, G., Rizzi, N., Varano, V.: Effects of warping constraints and lateral restraint on the buckling of thin-walled frames. In: ASME 2009 International Mechanical Engineering Congress and Exposition, pp. 803–810. American Society of Mechanical Engineers. (2009, January)
59. Placidi, L., Faria, S.H., Hutter, K.: On the role of grain growth, recrystallization and polygonization in a continuum theory for anisotropic ice sheets. Annal. Glaciol. **39**(1), 49–52 (2004)
60. Placidi, L., Hutter, K., Faria, S.H.: A critical review of the mechanics of polycrystalline polar ice. GAMM-Mitteilungen **29**(1), 80–117 (2006)
61. Placidi, L., Rosi, G., Giorgio, I., Madeo, A.: Reflection and transmission of plane waves at surfaces carrying material properties and embedded in second-gradient materials. Math. Mech. Solids **19**(5), 555–578 (2014)
62. Placidi, L., Andreaus, U., Giorgio, I.: Identification of two-dimensional pantographic structure via a linear D4 orthotropic second gradient elastic model. J. Eng. Math. (2016). doi:10.1007/s10665-016-9856-8
63. Rinaldi, A., Placidi, L.: A microscale second gradient approximation of the damage parameter of quasi-brittle heterogeneous lattices. ZAMM J. Appl. Math. Mech. **94**(10), 862–877 (2014)
64. Rizzi, N., Varano, V.: On the postbuckling analysis of thin-walled frames. In: Thirteenth International Conference on Civil, Structural and Environmental Engineering Computing. Civil-Comp Press. (2011)

65. Gabriele, S., Rizzi, N., Varano, V.: On the imperfection sensitivity of thin-walled frames. In: Topping, B.H.V. (ed.) Proceedings of the Eleventh International Conference on Computational Structures Technology. Civil-Comp Press, Stirlingshire (2012). doi:10.4203/ccp.99.15
66. Scerrato, D., Zhurba Eremeeva, I.A., Lekszycki, T., Rizzi, N.L.: On the effect of shear stiffness on the plane deformation of linear second gradient pantographic sheets. ZAMM J. Appl. Math. Mech. doi:10.1002/zamm.201600066 (2016)
67. Scerrato, D., Giorgio, I., Rizzi, N.L.: Three-dimensional instabilities of pantographic sheets with parabolic lattices: numerical investigations. Zeitschrift für Angewandte Mathematik Und Physik, **67**(3), 1–19 (2016)
68. dell'Isola, F., Giorgio, I., Pawlikowski, M., Rizzi, N.L.: Large deformations of planar extensible beams and pantographic lattices: heuristic homogenization, experimental and numerical examples of equilibrium. Proc. R. Soc. A **472**(2185), 20150790 (2016)
69. Smith, C.E.: Motion of a stretched string carrying a moving mass particle. J. Appl. Mech. **31**, 29–37 (1964)
70. Steigmann, D.J., dell'Isola, F.: Mechanical response of fabric sheets to three-dimensional bending, twisting, and stretching. Acta Mech. Sin. **31**(3), 373–382 (2015)

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