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Ground state sign-changing solutions for a class of Schrödinger–Poisson type problems in $\mathbb{R}^3$

Sitong Chen and Xianhua Tang

Abstract. This paper is dedicated to studying the following Schrödinger–Poisson system

$$\begin{cases} -\triangle u + V(x)u + \lambda\phi u = K(x)f(u), & x \in \mathbb{R}^3, \\ -\triangle \phi = u^2, \quad x \in \mathbb{R}^3, \end{cases}$$

where V, K are positive continuous potentials, f is a continuous function and λ is a positive parameter. We develop a direct approach to establish the existence of one ground state sign-changing solution u_λ with precisely two nodal domains, by introducing a weaker condition that there exists $\theta_0 \in (0, 1)$ such that

$$K(x)\left[\frac{f(\tau)}{\tau^3} - \frac{f(t\tau)}{(t\tau)^3}\right]\operatorname{sign}(1-t) + \theta_0 V(x)\frac{|1-t^2|}{(t\tau)^2} \geq 0, \quad \forall\, x \in \mathbb{R}^3,\ t > 0,\ \tau \neq 0$$

than the usual increasing condition on $f(t)/|t|^3$. Under the above condition, we also prove that the energy of any sign-changing solution is strictly larger than two times the least energy, and give a convergence property of u_λ as $\lambda \searrow 0$.

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1. Introduction

In this paper, we are concerned with the existence of sign-changing solutions for the following Schrödinger–Poisson system

$$\begin{cases} -\triangle u + V(x)u + \lambda\phi u = K(x)f(u), & x \in \mathbb{R}^3, \\ -\triangle \phi = u^2, \quad x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $V, K : \mathbb{R}^3 \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ and λ are a positive parameter. Such a system, also known as the nonlinear Schrödinger–Maxwell system, arises in many mathematical physics contexts. For instance, in Abelian Gauge theories, (1.1) provides a model to describe the interaction of a nonlinear Schrödinger field with an electromagnetic field (see [12,13,25]). In quantum electrodynamics, (1.1) describes the interaction between a charge particle interacting with the electromagnetic field. Moreover, (1.1) also appears in semiconductor theory, nonlinear optics and plasma physics. For more details in the physical aspects, we refer the readers to [5,6,38].

In recent years, there has been increasing attention to systems like (1.1) on the existence of positive solutions, ground state solutions, multiple solutions and semiclassical states; see for examples [8,15–17,19,20,24,31,41] and the references therein. To the authors' knowledge, there are very few results on the existence of sign-changing solutions for Problem (1.1).

As in [39], to avoid involving too much details when checking the compactness, we assume V satisfies

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(V) $V \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$, $V(x) > 0$ for all $x \in \mathbb{R}^3$ and $H \subset H^1(\mathbb{R}^3)$, such that, for $2 < q < 6$, the embedding

$$H \hookrightarrow L^q(\mathbb{R}^3)$$

is compact, where

$$H := \begin{cases} H_r^1(\mathbb{R}^3) = \left\{u \in H^1(\mathbb{R}^3) : u(x) = u(|x|)\right\}, & \text{if } V(x) \text{ is a constant,} \\ \left\{u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 \mathrm{d}x < \infty\right\}, & \text{if } V(x) \text{ is not a constant} \end{cases} \tag{1.2}$$

with the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x \right)^{1/2}.$$

It is well known that for $u \in H$, if ϕ_u be the unique solution of $-\triangle \phi_u = u^2$ in $D^{1,2}(\mathbb{R}^3)$, then

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} \mathrm{d}y. \tag{1.3}$$

Define the energy functional $\Phi_\lambda : H \to \mathbb{R}$ by

$$\Phi_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} K(x)F(u)\mathrm{d}x, \tag{1.4}$$

where $F(u) = \int_0^u f(s)\mathrm{d}s$. The functional Φ_λ is well defined for every $u \in H$ and $\Phi_\lambda \in \mathcal{C}^1(H, \mathbb{R})$. Moreover, for any $u, \varphi \in H$, we have

$$\langle \Phi_\lambda'(u), \varphi \rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla \varphi + V(x)u\varphi) \mathrm{d}x + \lambda \int_{\mathbb{R}^3} \phi_u u \varphi \mathrm{d}x - \int_{\mathbb{R}^3} K(x)f(u)\varphi \mathrm{d}x. \tag{1.5}$$

Clearly, critical points are the weak solutions of (1.1). Furthermore, if $u \in H$ is a solution of (1.1) and $u^\pm \neq 0$, then u is a sign-changing solution of (1.1), where

$$u^+(x) := \max\{u(x), 0\} \quad \text{and} \quad u^-(x) := \min\{u(x), 0\}.$$

When $\lambda = 0$, Eq. (1.1) reduces to the following semilinear Schrödinger equation

$$-\triangle u + V(x)u = K(x)f(u), \quad x \in \mathbb{R}^3. \tag{1.6}$$

In the last two decades, this equation has been studied extensively under various hypotheses on the potential and the nonlinearities; see for example [9–11, 14, 27, 28, 32–36, 42] and the references therein.

As far as we know, there are different ways to get the sign-changing solutions of Eq. (1.6), for example by constructing invariant sets and descending flow (see Bartsch et al. [3]), using the Ekeland's variational principle and the implicit function theorem (see Noussair and Wei [26]), and applying variational method together with the Brouwer degree theory (see Bartsh and Weth [4]). For more discussions on the existence of sign-changing solutions of (1.6), we refer the reader to the book [43] and the references therein. However, these methods of finding sign-changing solutions for (1.6) heavily rely on the following decomposition, $u \in H$,

$$\Phi_0(u) = \Phi_0(u^+) + \Phi_0(u^-), \tag{1.7}$$

$$\langle \Phi_0'(u), u^+ \rangle = \langle \Phi_0'(u^+), u^+ \rangle, \quad \langle \Phi_0'(u), u^- \rangle = \langle \Phi_0'(u^-), u^- \rangle, \tag{1.8}$$

where $\Phi_0 : H \to \mathbb{R}$ is the energy functional of (1.6) given by

$$\Phi_0(u) = \frac{1}{2}\int\limits_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x - \int\limits_{\mathbb{R}^3} K(x)F(u)\mathrm{d}x \tag{1.9}$$

and

$$\langle \Phi_0'(u), \varphi\rangle = \int\limits_{\mathbb{R}^3} \left(\nabla u \cdot \nabla\varphi + V(x)u\varphi\right) \mathrm{d}x - \int\limits_{\mathbb{R}^3} K(x)f(u)\varphi\mathrm{d}x, \quad \forall\ u, \varphi \in H. \tag{1.10}$$

When $\lambda > 0$, the nonlocal term $\phi_u(x)$ is involved in the energy functional Φ_λ, it is easy to see that

$$\phi_u(x) = \phi_{u^+}(x) + \phi_{u^-}(x), \tag{1.11}$$

but for the functional Φ_λ, we have

$$\Phi_\lambda(u) = \Phi_\lambda(u^+) + \Phi_\lambda(u^-) + \frac{\lambda}{4}\int\limits_{\mathbb{R}^3} \left[\phi_{u^+}(u^-)^2 + \phi_{u^-}(u^+)^2\right] \mathrm{d}x, \tag{1.12}$$

$$\langle \Phi_\lambda'(u), u^+\rangle = \langle \Phi_\lambda'(u^+), u^+\rangle + \lambda\int\limits_{\mathbb{R}^3} \phi_{u^-}(u^+)^2\mathrm{d}x, \tag{1.13}$$

$$\langle \Phi_\lambda'(u), u^-\rangle = \langle \Phi_\lambda'(u^-), u^-\rangle + \lambda\int\limits_{\mathbb{R}^3} \phi_{u^+}(u^-)^2\mathrm{d}x. \tag{1.14}$$

Clearly, the functional Φ_λ does no longer satisfy the decompositions (1.7) and (1.8). Hence, the methods of getting sign-changing solutions of (1.6) seem not applicable to Problem (1.1), and it is much more difficult to find sign-changing solutions of (1.1).

Define

$$\mathcal{M}_\lambda := \left\{u \in H : u^\pm \neq 0, \langle \Phi_\lambda'(u), u^+\rangle = \langle \Phi_\lambda'(u), u^-\rangle = 0\right\}, \tag{1.15}$$

$$\mathcal{M}_0 := \left\{u \in H : u^\pm \neq 0, \langle \Phi_0'(u), u^+\rangle = \langle \Phi_0'(u), u^-\rangle = 0\right\}, \tag{1.16}$$

$$\mathcal{N}_\lambda := \left\{u \in H : u \neq 0, \langle \Phi_\lambda'(u), u\rangle = 0\right\}, \tag{1.17}$$

$$\mathcal{N}_0 := \left\{u \in H : u \neq 0, \langle \Phi_0'(u), u\rangle = 0\right\}, \tag{1.18}$$

$$m_\lambda := \inf_{u\in\mathcal{M}_\lambda} \Phi_\lambda(u), \quad c_\lambda := \inf_{u\in\mathcal{N}_\lambda} \Phi_\lambda(u), \tag{1.19}$$

$$m_0 := \inf_{u\in\mathcal{M}_0} \Phi_0(u), \quad c_0 := \inf_{u\in\mathcal{N}_0} \Phi_0(u). \tag{1.20}$$

When (V) holds, $K \equiv 1$, $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ and verifies the following hypotheses:

(F1) $f(t) = o(|t|)$ as $t \to 0$;
(F2) f has a "quasicritical" growth, namely $\lim_{|t|\to\infty} f(t)/t^5 = 0$;
(F3) $\lim_{|t|\to\infty} f(t)/t^3 = \infty$;
(F4′) $\frac{f(t)}{|t|^3}$ is nondecreasing on $(-\infty, 0) \cup (0, \infty)$,

Shuai and Wang [30] investigated the existence and asymptotic behavior of sign-changing solutions to (1.1). More precisely, by the parametric method and implicit function theorem, they showed that for each $u \in H$ with $u^\pm \neq 0$ there is a unique pair $(s, t) \in (\mathbb{R}^+ \times \mathbb{R}^+)$ such that $su^+ + tu^- \in \mathcal{M}_\lambda$; then, via the quantitative deformation lemma and degree theory, they obtained the minimizer of the energy functional Φ_λ over the constraint $\mathcal{M}_\lambda$ is a critical point. Note that $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ plays a crucial role in using implicit function theorem and (F4′) guarantees the uniqueness of (s, t) mentioned above.

We also mention that when $f(u) = |u|^{p-2}u$ with $p \in (4, 6)$, Wang and Zhou [39] proved $\mathcal{M}_\lambda \neq \emptyset$ by Brouwer degree and obtained the existence of sign-changing solutions to (1.1). Ianni [18] employed a dynamical (not variational) approach and showed the existence of sign-changing radial solutions to (1.1) with $\lambda = 1$ and $V = K \equiv 1$, having a prescribed number of nodal domains.

Recently, when V satisfies $\lim_{|x|\to\infty} V(x) = V_\infty := \sup_{x\in\mathbb{R}^3} V(x) \geq \inf_{x\in\mathbb{R}^3} V(x) > 0$ and $|V(x) - V_\infty|$ does a limited growth condition, $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ and satisfies (F1)–(F3) and (F4′), Alves et al. [2] proved the existence of sign-changing solutions to (1.1) with $\lambda = 1$. In this paper, they introduced some new ideas and techniques association with the deformation lemma and Miranda's theorem. They started by establishing some estimates involving functions that change sign and then successfully overcame the difficulty lack of compactness about embedding $H^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ with $s \in [2, 6)$. Finally, they found a sign-changing solution as an existence result by minimization in a closed subset containing all the sign-changing solutions of the equation. However, we must point out that both $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ and (F4′) are needed in [2].

In the nonautonomous case, Liang et al. [21], via the constraint variational method and quantitative deformation lemma, proved the existence of sign-changing solutions to (1.1) with $\lambda = 1$ when $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ satisfies (F1)–(F3) and (F4′), and V, K satisfy:

(V0) $V(x), K(x) > 0$ for all $x \in \mathbb{R}^3$, $V \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$ and $K \in \mathcal{C}(\mathbb{R}^3, \mathbb{R}) \cap L^\infty(\mathbb{R}^3, \mathbb{R})$;

(V1) if $\{A_n\} \subset \mathbb{R}^3$ is a sequence of Borel sets such that the Lebesgue measure of A_n is less than R, for all n and some $R > 0$, then

$$\lim_{r\to\infty} \int\limits_{A_n \cap B_r^c(0)} K(x)\mathrm{d}x = 0, \quad \text{uniformly in } n \in \mathbb{N};$$

(V2) $K/V \in L^\infty(\mathbb{R}^3)$; or

(V3) there exists $p \in (2, 6)$ such that

$$\frac{K(x)}{[V(x)]^{(6-p)/4}} \to 0, \quad |x| \to \infty.$$

This kind of conditions was first introduced by Alves and Souto [1] to get a positive ground state solution of (1.6), which can be used to certify that the space E given by

$$E = \left\{ u \in D^{1,2}(\mathbb{R}^3) : \int\limits_{\mathbb{R}^3} V(x)u^2\mathrm{d}x < \infty \right\}$$

with the same norm as in H is compactly embedded into the weighted Lebesgue space

$$L^q_K(\mathbb{R}^3) = \left\{ u : u \text{ is measurable on } \mathbb{R}^3 \text{ and } \int\limits_{\mathbb{R}^3} K(x)|u|^q\mathrm{d}x < \infty \right\}$$

for some $q \in (2, 6)$; see [1, Proposition 2.1]. Hereafter, we say that $(V, K) \in \mathcal{K}$ if these conditions hold. Note that they used the Brouwer fixed point theorem to show that $\mathcal{M}_\lambda \neq \emptyset$.

To the authors' knowledge, in the autonomous (see [2,18,30,39]) or nonautonomous case (see [21]), without $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ and condition (F4′), there seem no results on the existence of least energy sign-changing solutions and the convergence property as $\lambda \searrow 0$ for Problem (1.1) in the literature.

To state our results, we introduce the following assumptions:

(K) $K \in \mathcal{C}(\mathbb{R}^3, \mathbb{R}) \cap L^\infty(\mathbb{R}^3, \mathbb{R})$ and $K(x) > 0$ for all $x \in \mathbb{R}^3$;

(F4) there exists a $\theta_0 \in (0, 1)$ such that for any $x \in \mathbb{R}^3$, $t > 0$ and $\tau \neq 0$

$$K(x)\left[\frac{f(\tau)}{\tau^3} - \frac{f(t\tau)}{(t\tau)^3}\right] \mathrm{sign}(1 - t) + \theta_0 V(x)\frac{|1 - t^2|}{(t\tau)^2} \geq 0. \tag{1.21}$$

Obviously, (F4′) implies (F4). In fact, either (V) or $(V, K) \in \mathcal{K}$ holds, and there exist many functions satisfying assumptions (F1)–(F4), but not (F4′). For example, let $V \equiv 1$ and $0 < K(x) \leq M$ for all $x \in \mathbb{R}^3$, or $(V, K) \in \mathcal{K}$ with $|K/V|_\infty := M > 0$, and

$$f(t)=\begin{cases}|t|^3t, & |t|\le\varrho,\\ \alpha|t|^3t+\frac{1}{3M}t, & |t|>\varrho\end{cases}$$

with $\alpha,\varrho>0$ and $3(1-\alpha)\varrho^3M=1$. Then, f satisfies (F1)–(F4) with $\theta_0=1/2$ but not satisfy (F4′).

Motivated by the works [30,37], we have developed a more direct approach to show that $\mathcal{M}_\lambda\neq\emptyset$ and the minimizer is a sign-changing solution **without** $f\in\mathcal{C}^1(\mathbb{R},\mathbb{R})$ **and (F4′)**. Roughly speaking, we first establish an inequality between $\Phi_\lambda(u)$ and $\Phi_\lambda(su^++tu^-)$ (see Lemma 2.1 below) under a weaker condition (F4) and then prove that if $u\in H$ with $u^\pm\neq0$, there is a unique pair $(s,t)\in(\mathbb{R}^+\times\mathbb{R}^+)$ such that $su^++tu^-\in\mathcal{M}_\lambda$ (see Lemma 2.4), which is entirely different from the method presented in [21,30,39]. Taking advantage of the above-mentioned inequality, we can show the minimizer of the constrained problem is a sign-changing solution, which has precisely two nodal domains. We can also give a convergence property of u_λ as $\lambda\searrow0$, which reflects some relationship between $\lambda>0$ and $\lambda=0$ in Problem (1.1).

Our main results can be stated as follows.

Theorem 1.1. *Assume that (V), (K) and* (F1)–(F4) *hold. Then, Problem* (1.1) *has a sign-changing solution* $u_\lambda\in\mathcal{M}_\lambda$ *such that* $\Phi_\lambda(u_\lambda)=\inf_{\mathcal{M}_\lambda}\Phi_\lambda>0$*, which has precisely two nodal domains.*

Theorem 1.2. *Assume that (V), (K) and* (F1)–(F4) *hold. Then, Problem* (1.1) *has a solution* $\bar u\in\mathcal{N}_\lambda$ *such that* $\Phi_\lambda(\bar u)=\inf_{\mathcal{N}_\lambda}\Phi_\lambda$*; moreover,* $m_\lambda>2c_\lambda$ *for all* $\lambda>0$.

Theorem 1.3. *Assume that (V), (K) and* (F1)–(F4) *hold. Then, Problem* (1.6) *has a sign-changing solution* $v_0\in\mathcal{M}_0$ *such that* $\Phi_0(v_0)=\inf_{\mathcal{M}_0}\Phi_0>0$*, which has precisely two nodal domains. Furthermore, for any sequence* $\{\lambda_n\}$ *with* $\lambda_n\searrow0$ *as* $n\to\infty$*, there exists a subsequence which we label in the same way such that* $u_{\lambda_n}\to u_0$ *in* H*, where* $u_0\in\mathcal{M}_0$ *is a sign-changing solution of* (1.6) *with* $\Phi_0(u_0)=\inf_{\mathcal{M}_0}\Phi_0>0$.

We must point out that if $(V,K)\in\mathcal{K}$, our method is still valid after a slight modification. In this sense, our results not only unify but also generalize the previous results.

Corollary 1.4. *Assume that* $(V,K)\in\mathcal{K}$ *and* (F1)–(F4) *hold. Then, all conclusions in Theorems* 1.1–1.3 *hold in* E.

The paper is organized as follows. In Sect. 2, we provide several lemmas, which are crucial in proving our main results. In Sect. 3, we show the existence of a ground sign-changing solution with precisely two nodal domains. In Sect. 4, we first investigate the ground state solutions of Nehari type and then prove Theorem 1.2. We complete the proofs of Theorem 1.3 and Corollary 1.4 in Sects. 5 and 6, respectively.

Throughout this paper, we denote the norm of $L^s(\mathbb{R}^3)$ by $\|u\|_s=\left(\int_{\mathbb{R}^3}|u|^s\mathrm{d}x\right)^{1/s}$ for $s\ge2$, $B_r(x)=\{y\in\mathbb{R}^3:|y-x|<r\}$.

2. Some preliminary lemmas

In this section, we first show the following lemmas and corollaries which will play crucial roles in this paper.

Lemma 2.1. *Assume that (V), (K) and* (F1)–(F4) *hold. Then,*

$$\begin{aligned}\Phi_\lambda(u)\ge{}&\Phi_\lambda(su^++tu^-)+\frac{1-s^4}{4}\langle\Phi_\lambda'(u),u^+\rangle+\frac{1-t^4}{4}\langle\Phi_\lambda'(u),u^-\rangle\\&+\frac{(1-\theta_0)(1-s^2)^2}{4}\|u^+\|^2+\frac{(1-\theta_0)(1-t^2)^2}{4}\|u^-\|^2\\&+\frac{\lambda(s^2-t^2)^2}{4}\int_{\mathbb{R}^3}\phi_{u^+}(u^-)^2\mathrm{d}x,\quad\forall\ u=u^++u^-\in H,\quad s,t\ge0.\end{aligned}\tag{2.1}$$

Proof. For any $x \in \mathbb{R}^3$, $t \geq 0$, $\tau \in \mathbb{R}$, (F4) yields

$$\begin{aligned} & K(x)\left[\frac{1-t^4}{4}\tau f(\tau) + F(t\tau) - F(\tau)\right] + \frac{\theta_0 V(x)}{4}(1-t^2)^2\tau^2 \\ & = \int_t^1 \left\{K(x)\left[\frac{f(\tau)}{\tau^3} - \frac{f(s\tau)}{(s\tau)^3}\right] + \theta_0 V(x)\frac{(1-s^2)}{(s\tau)^2}\right\} s^3\tau^4 \mathrm{d}s \geq 0. \end{aligned} \tag{2.2}$$

By (1.3) and Fubini theorem, we see that

$$\int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 \mathrm{d}x = \int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 \mathrm{d}x. \tag{2.3}$$

Hence, it follows from (1.4), (1.5), (2.2) and (2.3) that

$$\begin{aligned} & \Phi_\lambda(u) - \Phi_\lambda(su^+ + tu^-) \\ & = \frac{1}{2}\left(\|u\|^2 - \|su^+ + tu^-\|^2\right) + \frac{\lambda}{4}\int_{\mathbb{R}^3}\left[\phi_u u^2 - \phi_{su^+ + tu^-}(su^+ + tu^-)^2\right]\mathrm{d}x \\ & \quad + \int_{\mathbb{R}^3} K(x)[F(su^+ + tu^-) - F(u)]\mathrm{d}x \\ & = \frac{1-s^4}{4}\left(\|u^+\|^2 + \lambda\int_{\mathbb{R}^3}\phi_{u^+}(u^+)^2\mathrm{d}x\right) + \frac{1-t^4}{4}\left(\|u^-\|^2 + \lambda\int_{\mathbb{R}^3}\phi_{u^-}(u^-)^2\mathrm{d}x\right) \\ & \quad + \frac{(1-s^2)^2}{4}\|u^+\|^2 + \frac{(1-t^2)^2}{4}\|u^-\|^2 + \frac{\lambda(1-s^2t^2)}{2}\int_{\mathbb{R}^3}\phi_{u^+}(u^-)^2\mathrm{d}x \\ & \quad + \int_{\mathbb{R}^3} K(x)[F(su^+) + F(tu^-) - F(u^+) - F(u^-)]\mathrm{d}x \\ & = \frac{1-s^4}{4}\langle\Phi_\lambda'(u), u^+\rangle + \frac{1-t^4}{4}\langle\Phi_\lambda'(u), u^-\rangle \\ & \quad + \frac{(1-s^2)^2}{4}\|u^+\|^2 + \frac{(1-t^2)^2}{4}\|u^-\|^2 + \frac{\lambda(s^2-t^2)^2}{4}\int_{\mathbb{R}^3}\phi_{u^+}(u^-)^2\mathrm{d}x \\ & \quad + \int_{\mathbb{R}^3} K(x)\left[\frac{1-s^4}{4}f(u^+)u^+ + F(su^+) - F(u^+)\right]\mathrm{d}x \\ & \quad + \int_{\mathbb{R}^3} K(x)\left[\frac{1-t^4}{4}f(u^-)u^- + F(tu^-) - F(u^-)\right]\mathrm{d}x \\ & \geq \frac{1-s^4}{4}\langle\Phi_\lambda'(u), u^+\rangle + \frac{1-t^4}{4}\langle\Phi_\lambda'(u), u^-\rangle + \frac{(1-\theta_0)(1-s^2)^2}{4}\|u^+\|^2 \\ & \quad + \frac{(1-\theta_0)(1-t^2)^2}{4}\|u^-\|^2 + \frac{\lambda(s^2-t^2)^2}{4}\int_{\mathbb{R}^3}\phi_{u^+}(u^-)^2\mathrm{d}x \\ & \quad + \int_{\mathbb{R}^3}\left\{K(x)\left[\frac{1-s^4}{4}f(u^+)u^+ + F(su^+) - F(u^+)\right] + \frac{\theta_0 V(x)}{4}(1-s^2)^2|u^+|^2\right\}\mathrm{d}x \end{aligned}$$

$$+\int_{\mathbb{R}^3}\left\{K(x)\left[\frac{1-t^4}{4}f(u^-)u^-+F(tu^-)-F(u^-)\right]+\frac{\theta_0 V(x)}{4}(1-t^2)^2|u^-|^2\right\}\mathrm{d}x$$
$$\geq \frac{1-s^4}{4}\langle\Phi_\lambda'(u),u^+\rangle+\frac{1-t^4}{4}\langle\Phi_\lambda'(u),u^-\rangle+\frac{(1-\theta_0)(1-s^2)^2}{4}\|u^+\|^2$$
$$+\frac{(1-\theta_0)(1-t^2)^2}{4}\|u^-\|^2+\frac{\lambda(s^2-t^2)^2}{4}\int_{\mathbb{R}^3}\phi_{u^+}(u^-)^2\mathrm{d}x,\quad \forall\ s,t\geq 0.$$

This shows that (2.1) holds. □

Corollary 2.2. *Assume that (V), (K) and* (F1)–(F4) *hold. Then, for* $u=u^++u^-\in\mathcal{M}_\lambda$

$$\Phi_\lambda(u)\geq\Phi_\lambda(su^++tu^-)+\frac{(1-\theta_0)(1-s^2)^2}{4}\|u^+\|^2+\frac{(1-\theta_0)(1-t^2)^2}{4}\|u^-\|^2$$
$$+\frac{\lambda(s^2-t^2)^2}{4}\int_{\mathbb{R}^3}\phi_{u^+}(u^-)^2\mathrm{d}x,\quad\forall\ s,t\geq 0. \tag{2.4}$$

Corollary 2.3. *Assume that (V), (K) and* (F1)–(F4) *hold. Then, for* $u=u^++u^-\in\mathcal{M}_\lambda$

$$\Phi_\lambda(u^++u^-)=\max_{s,t\geq 0}\Phi_\lambda(su^++tu^-). \tag{2.5}$$

Secondly, we check that $\mathcal{M}_\lambda\neq\emptyset$ and $m_\lambda>0$ can be achieved, which are the key points in this paper.

Lemma 2.4. *Assume that (V), (K) and* (F1)–(F4) *hold. If* $u\in H$ *with* $u^\pm\neq 0$*, then there exists a unique pair* (s_u,t_u) *of positive numbers such that* $s_uu^++t_uu^-\in\mathcal{M}_\lambda$.

Proof. Let

$$g_1(s,t)=s^2\|u^+\|^2+\lambda s^4\int_{\mathbb{R}^3}\phi_{u^+}(u^+)^2\mathrm{d}x+\lambda s^2t^2\int_{\mathbb{R}^3}\phi_{u^-}(u^+)^2\mathrm{d}x$$
$$-\int_{\mathbb{R}^3}K(x)f(su^+)su^+\mathrm{d}x \tag{2.6}$$

and

$$g_2(s,t)=t^2\|u^-\|^2+\lambda t^4\int_{\mathbb{R}^3}\phi_{u^-}(u^-)^2\mathrm{d}x+\lambda s^2t^2\int_{\mathbb{R}^3}\phi_{u^+}(u^-)^2\mathrm{d}x$$
$$-\int_{\mathbb{R}^3}K(x)f(tu^-)tu^-\mathrm{d}x. \tag{2.7}$$

For any fixed $t\geq 0$, it is easy to verify using (F1) and (F3) that $g_1(0,t)=0$, $g_1(s,t)>0$ for $s>0$ small and $g_1(s,t)<0$ for s large. From continuity of $g_1(s,t)$ on s, there is a $s_t>0$ such that $g_1(s_t,t)=0$ for $t\geq 0$. We claim that $s_t>0$ is unique for any $t\geq 0$. In fact, for any given $t_0\geq 0$, let $\hat{s}_1,\hat{s}_2>0$ such that

$$g_1(\hat{s}_1,t_0)=g_1(\hat{s}_2,t_0)=0. \tag{2.8}$$

Then, it follows from (1.5), (2.6) and (2.8) that

$$\langle\Phi_\lambda'(\hat{s}_1u^++t_0u^-),u^+\rangle=\langle\Phi_\lambda'(\hat{s}_2u^++t_0u^-),u^+\rangle=0. \tag{2.9}$$

From (2.1) and (2.9), we have

$$\begin{aligned}\Phi_\lambda(\hat{s}_1u^+ + t_0u^-) &\geq \Phi_\lambda(\hat{s}_2u^+ + t_0u^-) + \frac{\hat{s}_1^4 - \hat{s}_2^4}{4\hat{s}_1^4}\langle \Phi'_\lambda(\hat{s}_1u^+ + t_0u^-), \hat{s}_1u^+\rangle \\ &\quad + \frac{(1-\theta_0)(\hat{s}_1^2 - \hat{s}_2^2)^2}{4\hat{s}_1^2}\|u^+\|^2 \\ &= \Phi_\lambda(\hat{s}_2u^+ + t_0u^-) + \frac{(1-\theta_0)(\hat{s}_1^2 - \hat{s}_2^2)^2}{4\hat{s}_1^2}\|u^+\|^2 \end{aligned} \tag{2.10}$$

and

$$\begin{aligned}\Phi_\lambda(\hat{s}_2u^+ + t_0u^-) &\geq \Phi_\lambda(\hat{s}_1u^+ + t_0u^-) + \frac{\hat{s}_2^4 - \hat{s}_1^4}{4\hat{s}_2^4}\langle \Phi'_\lambda(\hat{s}_2u^+ + t_0u^-), \hat{s}_2u^+\rangle \\ &\quad + \frac{(1-\theta_0)(\hat{s}_2^2 - \hat{s}_1^2)^2}{4\hat{s}_2^2}\|u^+\|^2 \\ &= \Phi_\lambda(\hat{s}_1u^+ + t_0u^-) + \frac{(1-\theta_0)(\hat{s}_2^2 - \hat{s}_1^2)^2}{4\hat{s}_2^2}\|u^+\|^2. \end{aligned} \tag{2.11}$$

(2.10) and (2.11) imply $\hat{s}_1 = \hat{s}_2$. Therefore, $s_t := \tilde{s}(t) > 0$ is unique for $t \geq 0$, i.e., $g_1(s,t) = 0$ defines an implicit function $s = \tilde{s}(t)$ for $t \geq 0$. Since $s_0 = \tilde{s}(0)$, and for every $t \geq 0$, $g_1(s,s) > 0$ for small $s > 0$ and $g_1(t,t) < 0$ for large $s > 0$, then one has

$$g_1(s_t, t) = 0, \quad \forall\, t \geq 0; \quad s_t > t \text{ for small } t \geq 0, \quad s_t < t \text{ for large } t \geq 0. \tag{2.12}$$

It is easy to verify that $\tilde{s}(t)$ is continuous on $[0, \infty)$.

Analogously, $g_2(s,t) = 0$ also defines an implicit function $t = t_s := \tilde{t}(s) > 0$ for $s \geq 0$ such that

$$g_2(s, t_s) = 0, \quad \forall\, s \geq 0; \quad t_s > s \text{ for small } s \geq 0, \quad t_s < s \text{ for large } s \geq 0. \tag{2.13}$$

(2.12) and (2.13) imply that the planar curves $s = \tilde{s}(t)$ and $t = \tilde{t}(s)$ intersect at some point (s_u, t_u) with $s_u, t_u > 0$, which implies that $g_1(s_u, t_u) = g_2(s_u, t_u) = 0$. Therefore, $s_uu^+ + t_uu^- \in \mathcal{M}_\lambda$.

Next, we prove the uniqueness. Let (s_1, t_1) and (s_2, t_2) such that $s_iu^+ + t_iu^- \in \mathcal{M}_\lambda, i = 1, 2$. In view of Corollary 2.2, one has

$$\begin{aligned}\Phi_\lambda(s_1u^+ + t_1u^-) &\geq \Phi_\lambda(s_2u^+ + t_2u^-) + \frac{(1-\theta_0)(s_1^2 - s_2^2)^2}{4s_1^2}\|u^+\|^2 \\ &\quad + \frac{(1-\theta_0)(t_1^2 - t_2^2)^2}{4t_1^2}\|u^-\|^2 \end{aligned} \tag{2.14}$$

and

$$\begin{aligned}\Phi_\lambda(s_2u^+ + t_2u^-) &\geq \Phi_\lambda(s_1u^+ + t_1u^-) + \frac{(1-\theta_0)(s_1^2 - s_2^2)^2}{4s_2^2}\|u^+\|^2 \\ &\quad + \frac{(1-\theta_0)(t_1^2 - t_2^2)^2}{4t_2^2}\|u^-\|^2. \end{aligned} \tag{2.15}$$

Both (2.14) and (2.15) imply $(s_1, t_1) = (s_2, t_2)$. □

Lemma 2.5. *Assume that (V), (K) and* (F1)–(F4) *hold. Then,*

$$\inf_{u \in \mathcal{M}_\lambda} \Phi_\lambda(u) = m_\lambda = \inf_{u \in H, u^\pm \neq 0} \max_{s,t \geq 0} \Phi_\lambda(su^+ + tu^-).$$

Proof. Both Corollary 2.3 and Lemma 2.4 imply the above lemma. □

Lemma 2.6. *Assume that (V), (K) and* (F1)–(F4) *hold. Then, $m_\lambda > 0$ is achieved.*

Proof. Similar as [30, (2.14)], we can derive from (F1)–(F3) that there exists a constant $\alpha > 0$ such that $\|u^{\pm}\|^2 > \alpha$ for all $u \in \mathcal{M}_\lambda$. Let $\{u_n\} \subset \mathcal{M}_\lambda$ be such that $\Phi_\lambda(u_n) \to m_\lambda$. Observe that (2.2) with $t = 0$ yields

$$K(x)\left[\frac{1}{4}f(\tau)\tau - F(\tau)\right] + \frac{\theta_0 V(x)}{4}\tau^2 \geq 0, \quad \forall\, x \in \mathbb{R}^3,\ \tau \in \mathbb{R}. \tag{2.16}$$

Thus, from (1.4), (1.5) and (2.16), one has for large $n \in \mathbb{N}$

$$\begin{aligned} m_\lambda + 1 &\geq \Phi_\lambda(u_n) - \frac{1}{4}\langle \Phi_\lambda'(u_n), u_n\rangle \\ &\geq \frac{(1-\theta_0)}{4}\|u_n\|^2 + \int_{\mathbb{R}^3}\left\{K(x)\left[\frac{1}{4}f(u_n)u_n - F(u_n)\right] + \frac{\theta_0 V(x)}{4}|u_n|^2\right\}\mathrm{d}x \\ &\geq \frac{(1-\theta_0)}{4}\|u_n\|^2. \end{aligned} \tag{2.17}$$

This shows that $\{u_n\}$ is bounded in H, and then, there exists a $u_\lambda \in H$ such that $u_n^{\pm} \rightharpoonup u_\lambda^{\pm}$ in H. By (V), (K), (F1)–(F3), (1.5) and [40, A.1], we can get

$$0 < \alpha \leq \|u_n^{\pm}\|^2 + \int_{\mathbb{R}^3}\phi_{u_n}(u_n^{\pm})^2\mathrm{d}x = \int_{\mathbb{R}^3}K(x)f(u_n^{\pm})u_n^{\pm}\mathrm{d}x = \int_{\mathbb{R}^3}K(x)f(u_\lambda^{\pm})u_\lambda^{\pm}\mathrm{d}x + o(1), \tag{2.18}$$

which yields $u_\lambda^{\pm} \neq 0$. Moreover, by (1.3) and Hardy–Littlewood–Sobolev inequality (see [22] or [23, page 98]), one has

$$\liminf_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n}(u_n^{\pm})^2\mathrm{d}x = \int_{\mathbb{R}^3}\phi_{u_\lambda}(u_\lambda^{\pm})^2\mathrm{d}x.$$

From this, (2.18), the weak semicontinuity of norm and Fatou's Lemma, we have

$$\|u_\lambda^{\pm}\|^2 + \int_{\mathbb{R}^3}\phi_{u_\lambda}(u_\lambda^{\pm})^2\mathrm{d}x \leq \liminf_{n\to\infty}\left[\|u_n^{\pm}\|^2 + \int_{\mathbb{R}^3}\phi_{u_n}(u_n^{\pm})^2\mathrm{d}x\right] = \int_{\mathbb{R}^3}K(x)f(u_\lambda^{\pm})u_\lambda^{\pm}\mathrm{d}x, \tag{2.19}$$

which implies

$$\langle \Phi_\lambda'(u_\lambda), u_\lambda^{\pm}\rangle \leq 0. \tag{2.20}$$

Thus, from (1.4), (1.5), (2.1), (2.16), (2.20), the weak semicontinuity of norm, Fatou's Lemma and Lemma 2.5, we have

$$\begin{aligned} m_\lambda &= \lim_{n\to\infty}\left[\Phi_\lambda(u_n) - \frac{1}{4}\langle \Phi_\lambda'(u_n), u_n\rangle\right] \\ &= \lim_{n\to\infty}\left\{\frac{1}{4}\|u_n\|^2 + \int_{\mathbb{R}^3}K(x)\left[\frac{1}{4}f(u_n)u_n - F(u_n)\right]\mathrm{d}x\right\} \\ &\geq \frac{1}{4}\liminf_{n\to\infty}\left[\|\nabla u_n\|_2^2 + (1-\theta_0)\int_{\mathbb{R}^3}V(x)|u_n|^2\mathrm{d}x\right] \\ &\quad + \liminf_{n\to\infty}\int_{\mathbb{R}^3}\left\{K(x)\left[\frac{1}{4}f(u_n)u_n - F(u_n)\right] + \frac{\theta_0}{4}V(x)|u_n|^2\right\}\mathrm{d}x \\ &\geq \frac{1}{4}\left[\|\nabla u_\lambda\|_2^2 + (1-\theta_0)\int_{\mathbb{R}^3}V(x)|u_\lambda|^2\mathrm{d}x\right] \end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^3}\left\{K(x)\left[\frac{1}{4}f(u_\lambda)u_\lambda-F(u_\lambda)\right]+\frac{\theta_0}{4}V(x)|u_\lambda|^2\right\}\mathrm{d}x\\
&=\frac{1}{4}\|u_\lambda\|^2+\int_{\mathbb{R}^3}K(x)\left[\frac{1}{4}f(u_\lambda)u_\lambda-F(u_\lambda)\right]\mathrm{d}x\\
&=\Phi_\lambda(u_\lambda)-\frac{1}{4}\langle\Phi_\lambda'(u_\lambda),u_\lambda\rangle\\
&\geq\sup_{s,t\geq 0}\left[\Phi_\lambda(su_\lambda^++tu_\lambda^-)+\frac{1-s^4}{4}\langle\Phi_\lambda'(u_\lambda),u_\lambda^+\rangle+\frac{1-t^4}{4}\langle\Phi_\lambda'(u_\lambda),u_\lambda^-\rangle\right]-\frac{1}{4}\langle\Phi_\lambda'(u_\lambda),u_\lambda\rangle\\
&\geq\sup_{s,t\geq 0}\Phi_\lambda(su_\lambda^++tu_\lambda^-)\geq m_\lambda,
\end{aligned}
$$

which implies

$$
\lim_{n\to\infty}\|\nabla u_n\|_2^2=\|\nabla u_\lambda\|_2^2,\quad \lim_{n\to\infty}\int_{\mathbb{R}^3}V(x)|u_n|^2\mathrm{d}x=\int_{\mathbb{R}^3}V(x)|u_\lambda|^2\mathrm{d}x. \tag{2.21}
$$

Therefore, $u_n\to u_\lambda$ in H, then $\Phi_\lambda(u_\lambda)=m_\lambda$ and $u_\lambda\in\mathcal{M}_\lambda$. □

Finally, with the help of the above conclusions, we prove the minimizer of the constrained problem is a critical point.

Lemma 2.7. *Assume that (V), (K) and* (F1)–(F4) *hold. If* $u_0\in\mathcal{M}_\lambda$ *and* $\Phi_\lambda(u_0)=m_\lambda$*, then* u_0 *is a critical point of* Φ_λ*.*

Proof. Assume that $u_0=u_0^++u_0^-\in\mathcal{M}_\lambda$, $\Phi_\lambda(u_0)=m_\lambda$ and $\Phi_\lambda'(u_0)\neq 0$. Then, there exist $\delta>0$ and $\varrho>0$ such that

$$
u\in H,\ \|u-u_0\|\leq 3\delta\Rightarrow\|\Phi_\lambda'(u)\|\geq\varrho. \tag{2.22}
$$

In view of Corollary 2.2, one has for all $s,t\geq 0$,

$$
\begin{aligned}
\Phi_\lambda(su_0^++tu_0^-)&\leq\Phi_\lambda(u_0)-\frac{(1-\theta_0)(1-s^2)^2}{4}\|u_0^+\|^2-\frac{(1-\theta_0)(1-t^2)^2}{4}\|u_0^-\|^2\\
&=m_\lambda-\frac{(1-\theta_0)(1-s^2)^2}{4}\|u_0^+\|^2-\frac{(1-\theta_0)(1-t^2)^2}{4}\|u_0^-\|^2.
\end{aligned} \tag{2.23}
$$

Let $D=(0.5,1.5)\times(0.5,1.5)$. Then,

$$
\kappa:=\max_{(s,t)\in\partial D}\Phi_\lambda(su_0^++tu_0^-)<m_\lambda. \tag{2.24}
$$

For $\varepsilon:=\min\{(m_\lambda-\kappa)/3,1,\varrho\delta/8\}$, $S:=B(u_0,\delta)$, [40, Lemma 2.3] yields a deformation $\eta\in\mathcal{C}([0,1]\times H,H)$ such that

i) $\eta(1,u)=u$ if $\Phi_\lambda(u)<m_\lambda-2\varepsilon$ or $\Phi_\lambda(u)>m_\lambda+2\varepsilon$;
ii) $\eta\left(1,\Phi_\lambda^{m_\lambda+\varepsilon}\cap B(u_0,\delta)\right)\subset\Phi_\lambda^{m_\lambda-\varepsilon}$;
iii) $\Phi_\lambda(\eta(1,u))\leq\Phi_\lambda(u)$, $\forall\ u\in H$.

By Corollary 2.3, $\Phi_\lambda(su_0^++tu_0^-)\leq\Phi_\lambda(u_0)=m_\lambda$ for $s,t\geq 0$; then, it follows from ii) that

$$
\Phi_\lambda(\eta(1,su_0^++tu_0^-))\leq m_\lambda-\varepsilon,\quad\forall\ s,t\geq 0,\quad |s-1|^2+|t-1|^2<\delta^2/\|u_0\|^2. \tag{2.25}
$$

On the other hand, by iii) and (2.23), one has

$$\begin{aligned}\Phi_\lambda(\eta(1, su_0^+ + tu_0^-)) &\le \Phi_\lambda(su_0^+ + tu_0^-)\\ &\le m_\lambda - \frac{(1-\theta_0)(1-s^2)^2}{4}\|u_0^+\|^2 - \frac{(1-\theta_0)(1-t^2)^2}{4}\|u_0^-\|^2\\ &\le m_\lambda - \frac{(1-\theta_0)\delta^2}{8\|u_0\|^2}\min\{\|u_0^+\|^2, \|u_0^-\|^2\},\\ &\qquad \forall\ s,t \ge 0,\quad |s-1|^2 + |t-1|^2 \ge \delta^2/\|u_0\|^2. \end{aligned} \tag{2.26}$$

Combining (2.25) with (2.26), we have

$$\max_{(s,t)\in\bar{D}} \Phi_\lambda(\eta(1, su_0^+ + tu_0^-)) < m_\lambda. \tag{2.27}$$

Define $g(s,t) := su_0^+ + tu_0^-$. By an argument similar as [29, Page 1268] or [30, Page 3277], we get $\eta(1, g(D)) \cap \mathcal{M}_\lambda \ne \emptyset$, which contradicts to the definition of m_λ. □

3. Sign-changing solutions

Proof of Theorem 1.1. In view of Lemmas 2.6 and 2.7, there exists a $u_\lambda \in \mathcal{M}_\lambda$ such that $\Phi_\lambda(u_\lambda) = m_\lambda$ and $\Phi'_\lambda(u_\lambda) = 0$. Thus, u_λ is a sign-changing solution of (1.1).

Now, we show that u_λ has exactly two nodal domains. Let $u_\lambda = u_1 + u_2 + u_3$, where

$$u_1 \ge 0,\quad u_2 \le 0,\quad \Omega_1 \cap \Omega_2 = \emptyset,\quad u_1|_{\Omega_2\cup\Omega_3} = u_2|_{\Omega_1\cup\Omega_3} = u_3|_{\Omega_1\cup\Omega_2} = 0, \tag{3.1}$$

$$\Omega_1 := \{x \in \mathbb{R}^3 : u_1(x) > 0\},\quad \Omega_2 := \{x \in \mathbb{R}^3 : u_2(x) < 0\},\quad \Omega_3 := \mathbb{R}^3 \setminus (\Omega_1 \cup \Omega_2), \tag{3.2}$$

and Ω_1, Ω_2 are connected open subsets of $\mathbb{R}^3$.

Setting $v = u_1 + u_2$, we see that $v^+ = u_1$ and $v^- = u_2$, i.e., $v^\pm \ne 0$. From (1.4), (1.5), (2.1), (2.16) and (3.1), one has

$$\begin{aligned} m_\lambda = \Phi_\lambda(u_\lambda) &= \Phi_\lambda(u_\lambda) - \frac{1}{4}\langle \Phi'_\lambda(u_\lambda), u_\lambda\rangle\\ &= \Phi_\lambda(v) + \Phi_\lambda(u_3) + \frac{\lambda}{4}\int_{\mathbb{R}^3} \left(\phi_{u_3}v^2 + \phi_v u_3^2\right)\mathrm{d}x\\ &\quad -\frac{1}{4}\left[\langle \Phi'_\lambda(v), v\rangle + \langle \Phi'_\lambda(u_3), u_3\rangle + \lambda\int_{\mathbb{R}^3}\left(\phi_{u_3}v^2 + \phi_v u_3^2\right)\mathrm{d}x\right]\\ &\ge \sup_{s,t\ge 0}\left[\Phi_\lambda(sv^+ + tv^-) + \frac{1-s^4}{4}\langle \Phi'_\lambda(v), v^+\rangle + \frac{1-t^4}{4}\langle \Phi'_\lambda(v), v^-\rangle\right]\\ &\quad -\frac{1}{4}\langle \Phi'_\lambda(v), v\rangle + \Phi_\lambda(u_3) - \frac{1}{4}\langle \Phi'_\lambda(u_3), u_3\rangle\\ &\ge \sup_{s,t\ge 0}\left[\Phi_\lambda(sv^+ + tv^-) + \frac{\lambda s^4}{4}\int_{\mathbb{R}^3}\phi_{u_3}(v^+)^2\mathrm{d}x + \frac{\lambda t^4}{4}\int_{\mathbb{R}^3}\phi_{u_3}(v^-)^2\mathrm{d}x\right]\\ &\quad +\frac{1}{4}\|u_3\|^2 + \int_{\mathbb{R}^3} K(x)\left[\frac{1}{4}f(u_3)u_3 - F(u_3)\right]\mathrm{d}x \end{aligned}$$

$$\geq \sup_{s,t\geq 0} \Phi_\lambda(sv^+ + tv^-) + \frac{(1-\theta_0)}{4}\|u_3\|^2$$
$$\geq m_\lambda + \frac{(1-\theta_0)}{4}\|u_3\|^2,$$

which shows $u_3 = 0$. Therefore, u_λ has exactly two nodal domains. □

4. Ground state solutions and sign-changing solutions

As Sect. 2, we can prove the following lemmas and corollaries.

Lemma 4.1. *Assume that (V), (K) and* (F1)–(F4) *hold. Then,*

$$\Phi_\lambda(u) \geq \Phi_\lambda(tu) + \frac{1-t^4}{4}\langle \Phi'_\lambda(u), u\rangle + \frac{(1-\theta_0)(1-t^2)^2}{4}\|u\|^2, \quad \forall\, u \in H, \quad t \geq 0. \tag{4.1}$$

Corollary 4.2. *Assume that (V), (K) and* (F1)–(F4) *hold. Then, for $u \in \mathcal{N}_\lambda$*

$$\Phi_\lambda(u) \geq \Phi_\lambda(tu) + \frac{(1-\theta_0)(1-t^2)^2}{4}\|u\|^2, \quad \forall\, t \geq 0. \tag{4.2}$$

Corollary 4.3. *Assume that (V), (K) and* (F1)–(F4) *hold. Then, for $u \in \mathcal{N}_\lambda$*

$$\Phi_\lambda(u) = \max_{t\geq 0} \Phi_\lambda(tu). \tag{4.3}$$

Lemma 4.4. *Assume that (V), (K) and* (F1)–(F4) *hold. If $u \in H \setminus \{0\}$, then there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_\lambda$.*

Lemma 4.5. *Assume that (V), (K) and* (F1)–(F4) *hold. Then,*

$$\inf_{u\in\mathcal{N}_\lambda} \Phi_\lambda(u) = c_\lambda = \inf_{u\in H, u\neq 0} \max_{t\geq 0} \Phi_\lambda(tu).$$

Lemma 4.6. *Assume that (V), (K) and* (F1)–(F4) *hold. Then, there exist a constant $c_* \in (0, c_\lambda]$ and a sequence $\{u_n\} \subset H$ satisfying*

$$\Phi_\lambda(u_n) \to c_*, \quad \|\Phi'_\lambda(u_n)\|(1+\|u_n\|) \to 0. \tag{4.4}$$

Proof. It follows from (F1), (F2) and (1.4) that there exist $\delta_0 > 0$ and $\rho_0 > 0$ such that

$$\Phi_\lambda(u) \geq \rho_0, \quad \|u\| = \delta_0. \tag{4.5}$$

Choose $v_k \in \mathcal{N}_\lambda$ such that

$$c_\lambda \leq \Phi_\lambda(v_k) < c_\lambda + \frac{1}{k}, \quad k \in \mathbb{N}. \tag{4.6}$$

Since $\Phi_\lambda(0) = 0$ and $\Phi_\lambda(tv_k) < 0$ for large $t > 0$, then according to [7], there exists a sequence $\{u_{k,n}\}_{n\in\mathbb{N}} \subset H$ satisfying

$$\Phi_\lambda(u_{k,n}) \to c^k, \quad \|\Phi'_\lambda(u_{k,n})\|(1+\|u_{k,n}\|) \to 0, \quad k \in \mathbb{N}, \tag{4.7}$$

where $c^k \in [\rho_0, \sup_{t\geq 0} \Phi_\lambda(tv_k)]$. By virtue of Corollary 4.2, one has

$$\Phi_\lambda(v_k) \geq \Phi_\lambda(tv_k), \quad \forall\, t \geq 0,$$

which implies $\Phi_\lambda(v_k) = \sup_{t\geq 0} \Phi_\lambda(tv_k)$. Hence, by (4.5) and (4.7), one has

$$\Phi_\lambda(u_{k,n}) \to c^k \in \left[\rho_0, c_\lambda + \frac{1}{k}\right), \quad \|\Phi'_\lambda(u_{k,n})\|(1+\|u_{k,n}\|) \to 0, \quad k \in \mathbb{N}. \tag{4.8}$$

Now, we can choose a sequence $\{n_k\} \subset \mathbb{N}$ such that

$$\Phi_\lambda(u_{k,n_k}) \in \left[\rho_0, c_\lambda + \frac{1}{k}\right), \quad \|\Phi'_\lambda(u_{k,n_k})\|(1 + \|u_{k,n_k}\|) < \frac{1}{k}, \quad k \in \mathbb{N}. \tag{4.9}$$

Let $u_k = u_{k,n_k}, k \in \mathbb{N}$. Then, going if necessary to a subsequence, we have

$$\Phi_\lambda(u_n) \to c_* \in [\rho_0, c_\lambda], \quad \|\Phi'_\lambda(u_n)\|(1 + \|u_n\|) \to 0.$$

□

Proof of Theorem 1.2. Lemma 4.6 implies the existence of a sequence $\{u_n\} \subset H$ satisfying (4.4), which implies that

$$\Phi_\lambda(u_n) \to c_*, \quad \langle \Phi'_\lambda(u_n), u_n\rangle \to 0. \tag{4.10}$$

From (1.4), (1.5), (2.16) and (4.10), one has for large $n \in \mathbb{N}$

$$c_* + 1 \geq \Phi_\lambda(u_n) - \frac{1}{4}\langle \Phi'_\lambda(u_n), u_n\rangle \geq \frac{(1-\theta_0)}{4}\|u_n\|^2.$$

This shows that $\{u_n\}$ is bounded in H. By a standard argument, we can prove that there exists a $u_0 \in H \setminus \{0\}$ such that $\Phi'_\lambda(u_0) = 0$. This shows that $u_0 \in \mathcal{N}_\lambda$ is a nontrivial of (1.1) and $\Phi_\lambda(u_0) \geq c_\lambda$. On the other hand, by using (1.4), (1.5), (2.16), the weak semicontinuity of norm and Fatou's Lemma, we have

$$\begin{aligned}
c_\lambda \geq c_* &= \lim_{n\to\infty}\left[\Phi_\lambda(u_n) - \frac{1}{4}\langle \Phi'_\lambda(u_n), u_n\rangle\right] \\
&= \lim_{n\to\infty}\left\{\frac{1}{4}\|u_n\|^2 + \int_{\mathbb{R}^3} K(x)\left[\frac{1}{4}f(u_n)u_n - F(u_n)\right]\mathrm{d}x\right\} \\
&\geq \frac{1}{4}\liminf_{n\to\infty}\left(\|\nabla u_n\|_2^2 + (1-\theta_0)\int_{\mathbb{R}^3} V(x)|u_n|^2\mathrm{d}x\right) \\
&\quad + \liminf_{n\to\infty}\int_{\mathbb{R}^3}\left\{K(x)\left[\frac{1}{4}f(u_n)u_n - F(u_n)\right] + \frac{\theta_0 V(x)}{4}|u_n|^2\right\}\mathrm{d}x \\
&\geq \frac{1}{4}\left(\|\nabla u_0\|_2^2 + (1-\theta_0)\int_{\mathbb{R}^3} V(x)|u_0|^2\mathrm{d}x\right) \\
&\quad + \int_{\mathbb{R}^3}\left\{K(x)\left[\frac{1}{4}f(u_0)u_0 - F(u_0)\right] + \frac{\theta_0 V(x)}{4}|u_0|^2\right\}\mathrm{d}x \\
&= \frac{1}{4}\|u_0\|^2 + \int_{\mathbb{R}^3} K(x)\left[\frac{1}{4}f(u_0)u_0 - F(u_0)\right]\mathrm{d}x \\
&= \Phi_\lambda(u_0) - \frac{1}{4}\langle \Phi'_\lambda(u_0), u_0\rangle = \Phi_\lambda(u_0).
\end{aligned}$$

This shows that $\Phi_\lambda(u_0) \leq c_*$ and so $\Phi_\lambda(u_0) = c_\lambda = \inf_{\mathcal{N}_\lambda}\Phi_\lambda > 0$.

In view of Theorem 1.1, there exists a $u_\lambda \in \mathcal{M}_\lambda$ such that $\Phi_\lambda(u_\lambda) = m_\lambda$. Thus, from (1.4), Lemma 2.1, Corollary 2.3 and Lemma 4.5, we have

$$
\begin{aligned}
m_\lambda = \Phi_\lambda(u_\lambda) &= \sup_{s,t\geq 0} \Phi_\lambda(su_\lambda^+ + tu_\lambda^-) \\
&= \sup_{s,t\geq 0} \left\{ \Phi_\lambda(su_\lambda^+) + \Phi_\lambda(tu_\lambda^-) + \frac{\lambda s^2 t^2}{4} \int_{\mathbb{R}^3} \left[\phi_{u_\lambda^+}(u_\lambda^-)^2 + \phi_{u_\lambda^-}(u_\lambda^+)^2 \right] \mathrm{d}x \right\} \\
&> \sup_{s\geq 0} \Phi_\lambda(su_\lambda^+) + \sup_{t\geq 0} \Phi_\lambda(tu_\lambda^-) \geq 2c_\lambda.
\end{aligned} \tag{4.11}
$$

□

5. The convergence property

In this section, we give the proof of Theorem 1.3.

Proof of Theorem 1.3. In the arguments of Sect. 2, $\lambda = 0$ is allowed. Therefore, under the assumptions of Theorem 1.3, there exists a $v_0 \in \mathcal{M}_0$ such that $\Phi_0'(v_0) = 0$ and $\Phi_0(v_0) = m_0$, i.e., (1.6) has the least energy sign-changing solution, which changes sign only once.

For any $\lambda > 0$, let $u_\lambda \in \mathcal{M}_\lambda$ be a sign-changing solution of (1.1) obtained in Theorem 1.1, which changes sign only once and satisfies $\Phi_\lambda(u_\lambda) = m_\lambda$.

Choose $w_0 \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ such that $w_0^\pm \neq 0$. By (K) and (F1)–(F3), there exist $\beta_1 > 0$ and $\beta_2 \geq \frac{1}{2}\int_{\mathbb{R}^3} \phi_{w_0}(w_0)^2 \mathrm{d}x$ such that

$$
\int_{\mathbb{R}^3} K(x)F(sw_0^+)\mathrm{d}x \geq \beta_2|s|^4 - \beta_1, \quad \int_{\mathbb{R}^3} K(x)F(tw_0^-)\mathrm{d}x \geq \beta_2|t|^4 - \beta_1, \quad \forall\, s,t \in \mathbb{R}. \tag{5.1}
$$

For any $\lambda \in [0,1]$, by (1.4), (1.19), (2.3), (5.1) and Lemma 2.4 we have

$$
\begin{aligned}
\Phi_\lambda(u_\lambda) &= m_\lambda \leq \max_{s,t\geq 0} \Phi_\lambda(sw_0^+ + tw_0^-) \\
&= \max_{s,t\geq 0} \left\{ \frac{s^2}{2}\|w_0^+\|^2 + \frac{\lambda s^4}{4}\int_{\mathbb{R}^3} \phi_{w_0^+}(w_0^+)^2 \mathrm{d}x - \int_{\mathbb{R}^3} K(x)F(sw_0^+)\mathrm{d}x \right. \\
&\quad + \frac{t^2}{2}\|w_0^-\|^2 + \frac{\lambda t^4}{4}\int_{\mathbb{R}^3} \phi_{w_0^-}(w_0^-)^2 \mathrm{d}x - \int_{\mathbb{R}^3} K(x)F(tw_0^-)\mathrm{d}x \\
&\quad \left. + \frac{\lambda s^2 t^2}{4}\int_{\mathbb{R}^3} \left[\phi_{w_0^+}(w_0^-)^2 + \phi_{w_0^-}(w_0^+)^2 \right] \mathrm{d}x \right\} \\
&\leq \max_{s,t\geq 0} \left[\frac{s^2}{2}\|w_0^+\|^2 + \frac{\lambda s^4}{4}\int_{\mathbb{R}^3} \phi_{w_0^+}(w_0^+)^2 \mathrm{d}x + 2\beta_1 - \beta_2 s^4 \right. \\
&\quad + \frac{t^2}{2}\|w_0^-\|^2 + \frac{\lambda t^4}{4}\int_{\mathbb{R}^3} \phi_{w_0^-}(w_0^-)^2 \mathrm{d}x - \beta_2 t^4 \\
&\quad \left. + \frac{\lambda s^4}{4}\int_{\mathbb{R}^3} \phi_{w_0^-}(w_0^+)^2 \mathrm{d}x + \frac{\lambda t^4}{4}\int_{\mathbb{R}^3} \phi_{w_0^+}(w_0^-)^2 \mathrm{d}x \right] \\
&= \max_{s,t\geq 0} \left[\frac{s^2}{2}\|w_0^+\|^2 + \frac{\lambda s^4}{4}\int_{\mathbb{R}^3} \phi_{w_0}(w_0^+)^2 \mathrm{d}x + 2\beta_1 - \beta_2 s^4 \right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad+\frac{t^2}{2}\|w_0^-\|^2+\frac{\lambda t^4}{4}\int\limits_{\mathbb{R}^3}\phi_{w_0}(w_0^-)^2\mathrm{d}x-\beta_2 t^4\Bigg]\\
&\le \max_{s\ge 0}\left[\frac{s^2}{2}\|w_0^+\|^2-\frac{s^4}{4}\int\limits_{\mathbb{R}^3}\phi_{w_0}(w_0)^2\mathrm{d}x\right]\\
&\quad+\max_{t\ge 0}\left[\frac{t^2}{2}\|w_0^-\|^2-\frac{t^4}{4}\int\limits_{\mathbb{R}^3}\phi_{w_0}(w_0)^2\mathrm{d}x\right]+2\beta_1\\
&:=\Lambda_0>0.
\end{aligned}
$$

From this, (1.4), (1.5) and (2.16), we get

$$
\Lambda_0+1\ge\Phi_{\lambda_n}(u_{\lambda_n})-\frac{1}{4}\langle\Phi'_{\lambda_n}(u_{\lambda_n}),u_{\lambda_n}\rangle\ge\frac{(1-\theta_0)}{4}\|u_{\lambda_n}\|^2.
$$

This shows that $\{u_{\lambda_n}\}$ is bounded in H. Hence, there exists a subsequence of $\{\lambda_n\}$, still denoted by $\{\lambda_n\}$ and $u_0\in H$ such that $u_{\lambda_n}\rightharpoonup u_0$ in H. Proceeding as in Lemma 2.6, we conclude that $u_{\lambda_n}^{\pm}\to u_0^{\pm}\neq 0$ in H. Moreover,

$$
\begin{aligned}
\langle\Phi'_0(u_0),\varphi\rangle&=\int\limits_{\mathbb{R}^3}\left(\nabla u_0\cdot\nabla\varphi+V(x)u_0\varphi\right)\mathrm{d}x-\int\limits_{\mathbb{R}^3}K(x)f(u_0)\varphi\mathrm{d}x\\
&=\lim_{n\to\infty}\Bigg[\int\limits_{\mathbb{R}^3}\left(\nabla u_{\lambda_n}\cdot\nabla\varphi+V(x)u_{\lambda_n}\varphi\right)\mathrm{d}x+\frac{\lambda_n}{4}\int\limits_{\mathbb{R}^3}\phi_{u_{\lambda_n}}u_{\lambda_n}\varphi\mathrm{d}x\\
&\quad-\int\limits_{\mathbb{R}^3}K(x)f(u_{\lambda_n})\varphi\mathrm{d}x\Bigg]\\
&=\lim_{n\to\infty}\langle\Phi'_{\lambda_n}(u_{\lambda_n}),\varphi\rangle=0.
\end{aligned}
$$

This shows $\Phi'_0(u_0)=0$, and so, $u_0\in\mathcal{M}_0$ and $\Phi_0(u_0)\ge m_0$. Next, we prove that $\Phi_0(u_0)=m_0$. Let $\lambda_n\in[0,1]$. Then, it follows from (K) and (F3) that there exists a $K_0>0$ such that for all $s\ge K_0$ or $t\ge K_0$,

$$
\begin{aligned}
\Phi_{\lambda_n}(sv_0^++tv_0^-)&=\frac{s^2}{2}\|v_0^+\|^2+\frac{\lambda_n s^4}{4}\int\limits_{\mathbb{R}^3}\phi_{v_0^+}(v_0^+)^2\mathrm{d}x-\int\limits_{\mathbb{R}^3}K(x)F(sv_0^+)\mathrm{d}x\\
&\quad+\frac{t^2}{2}\|v_0^-\|^2+\frac{\lambda_n t^4}{4}\int\limits_{\mathbb{R}^3}\phi_{v_0^-}(v_0^-)^2\mathrm{d}x-\int\limits_{\mathbb{R}^3}K(x)F(tv_0^-)\mathrm{d}x\\
&\quad+\frac{\lambda_n s^2t^2}{4}\int\limits_{\mathbb{R}^3}\left[\phi_{v_0^-}(v_0^+)^2+\phi_{v_0^+}(v_0^-)^2\right]\mathrm{d}x\\
&\le\frac{s^2}{2}\|v_0^+\|^2+\frac{s^4}{4}\int\limits_{\mathbb{R}^3}\phi_{v_0}(v_0^+)^2\mathrm{d}x-\int\limits_{\mathbb{R}^3}K(x)F(sv_0^+)\mathrm{d}x\\
&\quad+\frac{t^2}{2}\|v_0^-\|^2+\frac{t^4}{4}\int\limits_{\mathbb{R}^3}\phi_{v_0}(v_0^-)^2\mathrm{d}x-\int\limits_{\mathbb{R}^3}K(x)F(tv_0^-)\mathrm{d}x<0.
\end{aligned}\tag{5.2}
$$

In view of Lemma 2.4, there exists (s_n, t_n) such that $s_n v_0^+ + t_n v_0^- \in \mathcal{M}_{\lambda_n}$, which, together with (5.2), implies $0 < s_n, t_n < K_0$. Hence, from (1.4), (1.5), (1.9) and (2.1), we have

$$\begin{aligned} m_0 &= \Phi_0(v_0) \\ &= \Phi_{\lambda_n}(v_0) - \frac{\lambda_n}{4} \int_{\mathbb{R}^3} \phi_{v_0}(v_0)^2 \mathrm{d}x \\ &\geq \Phi_{\lambda_n}(s_n v_0^+ + t_n v_0^-) + \frac{1 - s_n^4}{4} \langle \Phi'_{\lambda_n}(v_0), v_0^+ \rangle + \frac{1 - t_n^4}{4} \langle \Phi'_{\lambda_n}(v_0), v_0^- \rangle - \frac{\lambda_n}{4} \int_{\mathbb{R}^3} \phi_{v_0}(v_0)^2 \mathrm{d}x \\ &\geq m_{\lambda_n} + \frac{1 - s_n^4}{4} \lambda_n \int_{\mathbb{R}^3} \phi_{v_0}(v_0^+)^2 \mathrm{d}x + \frac{1 - t_n^4}{4} \lambda_n \int_{\mathbb{R}^3} \phi_{v_0}(v_0^-)^2 \mathrm{d}x - \frac{\lambda_n}{4} \int_{\mathbb{R}^3} \phi_{v_0}(v_0)^2 \mathrm{d}x \\ &\geq m_{\lambda_n} - \frac{K_0^4 \lambda_n}{4} \int_{\mathbb{R}^3} \phi_{v_0}(v_0)^2 \mathrm{d}x, \end{aligned}$$

which yields

$$\limsup_{n \to \infty} m_{\lambda_n} \leq m_0. \tag{5.3}$$

From (1.4), (1.9) and (5.3), we have

$$m_0 \leq \Phi_0(u_0) = \limsup_{n \to \infty} \Phi_{\lambda_n}(u_{\lambda_n}) = \limsup_{n \to \infty} m_{\lambda_n} \leq m_0.$$

This shows that $\Phi_0(u_0) = m_0$. □

6. Proof of Corollary 1.4

When $(V, K) \in \mathcal{K}$, for readers' convenience, we first state the following important consequence to recover compactness.

Proposition 6.1. ([1, Proposition 2.1]) *Assume that $(V, K) \in \mathcal{K}$. If (V2) holds, then E is compactly embedded in $L_K^q(\mathbb{R}^3)$ for every $q \in (2, 6)$; if (V3) holds, then E is compactly embedded in $L_K^p(\mathbb{R}^3)$.*

Proof of Corollary 1.4. Due to Proposition 6.1 and [21, Lemmas 2.4–2.6], under the assumptions of Corollary 1.4, it is easy to verify that Φ_λ satisfies the similar geometry structure as the case where (V) and (K) hold. Thus, Corollary 1.4 follows by slightly modifying the arguments of Sects. 2–5. □

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Sitong Chen and Xianhua Tang
School of Mathematics and Statistics
Central South University
Changsha, 410083, Hunan
People's Republic of China
e-mail: tangxh@mail.csu.edu.cn

Sitong Chen
e-mail: mathsitongchen@163.com

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