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The application of trigonal curve to the Mikhailov–Shabat–Sokolov flows

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Abstract. Resorting to the characteristic polynomial of Lax matrix for the Mikhailov–Shabat–Sokolov hierarchy associated with a 3×3 matrix spectral problem, we introduce a trigonal curve, from which we deduce the associated Baker–Akhiezer function, meromorphic functions and Dubrovin-type equations. The straightening out of the Mikhailov–Shabat–Sokolov flows is exactly given through the Abel map. On the basis of these results and the theory of trigonal curve, we obtain the explicit theta function representations of the Baker–Akhiezer function, the meromorphic functions, and in particular, that of solutions for the entire Mikhailov–Shabat–Sokolov hierarchy.

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1. Introduction

Soliton theory brings together many branches of mathematics, some of which touch on deep ideas. The theory has many practical applications to the physical sciences such as optical fiber communication, superconductivity, nonlinear waves, plasma physics and magnetic fluid. Soliton equations are nonlinear partial differential equations having specific solutions with nice mathematical and physical properties. However, it is very difficult to solve them due to their high nonlinearity. An important application of the inverse scattering transformation is to obtain N-soliton solutions of nonlinear equations, which exhibit the particle behavior in the interaction of solitary waves [1,2]. There also exist various approaches to obtain algebro-geometric solutions of soliton equations, which show the quasi-periodic behavior or characteristic for Liouville integrability [2–11]. Using these approaches, algebro-geometric solutions for a lot of soliton equations associated with 2×2 matrix spectral problems have been obtained, such as the KdV, nonlinear Schrödinger, mKdV, sine-Gordon, and Toda lattice, and Camassa–Holm equations. As compared with the 2×2 case, the study of algebro-geometric solutions of soliton equations associated with 3×3 matrix spectral problems has received relatively less attention. In Refs. [3,12–18], certain algebro-geometric solutions of the Boussinesq equation related to a third-order differential operator were found as special solutions of the Kadomtsev–Peiviashvili equation or by the reduction theory of Riemann theta functions. In Refs. [19], a unified framework was proposed which yields all algebro-geometric solutions of the entire Boussinesq hierarchy. Recently, based on the characteristic polynomial of Lax matrix associated with a 3×3 matrix spectral problem, we introduce the trigonal curve [20–22] and obtain the explicit theta function representations of the entire three-wave resonant interaction hierarchy and the Kaup–Kupershmidt hierarchy [23–25].

The main aim of the present paper is to study algebro-geometric constructions of the Mikhailov–Shabat–Sokolov (MSS) flows associated with a 3×3 matrix spectral problem [26], from which we deduce the explicit theta function representations of the Baker–Akhiezer function, the meromorphic functions, and that of solutions for the entire MSS hierarchy. The first typical member in the hierarchy reads

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$$\begin{aligned} u_t &= u_{xx} + 2uu_x - 2(uv)_x - 2v_{xx}, \\ v_t &= -v_{xx} + 2vv_x - 2(uv)_x + 2u_{xx}, \end{aligned} \tag{1.1}$$

which is reduced to the MSS equation [27]

$$\begin{aligned} p_t &= p_{xx} - 2qq_x, \\ q_t &= -q_{xx} + 2pp_x \end{aligned} \tag{1.2}$$

by a change of variables

$$u = p + q, \quad v = \frac{1}{2}(1 - i\sqrt{3})p + \frac{1}{2}(1 + i\sqrt{3})q \tag{1.3}$$

and then by rescaling $t \to \frac{t}{i\sqrt{3}}$. Therefore, the hierarchy of nonlinear evolution equations associated with the 3×3 matrix spectral problem is called the MSS hierarchy. It is easy to see that the transformation (1.3) between Eq. (1.1) and the MSS Eq. (1.2) is linear, so the sets of solutions of the two equations are isomorphism in the complex plane. Then, using the inverse transformation of (1.3) we can directly obtain the algebro-geometric solutions to the MSS Eq. (1.2) from ones of Eq. (1.1).

The structure of this paper is as follows. In Sect. 2, we construct the hierarchy of the MSS equations associated with a 3×3 matrix spectral problem. In Sect. 3, we introduce the Baker–Akhiezer function, from which we deduce the associated meromorphic functions. With the help of the characteristic polynomial of Lax matrix for the MSS equations, a trigonal curve $\mathcal{K}_{m-1}$ of arithmetic genus $m-1$ is defined. Then the MSS equations are decomposed into the system of Dubrovin-type ordinary differential equations. In Sect. 4, we introduce the holomorphic differentials on $\mathcal{K}_{m-1}$ and the Abel map, through which the straightening out of the MSS flows is exactly given. At last, we derive the explicit theta function representations of the Baker–Akhiezer function, the meromorphic functions, and in particular, that of the potentials u and v for the entire MSS hierarchy.

2. The Mikhailov–Shabat–Sokolov hierarchy

In this section, we briefly review the construction of the MSS hierarchy associated with the 3×3 matrix spectral problem [26]

$$\psi_x = U\psi, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad U = \begin{pmatrix} u & 1 & 0 \\ 0 & v & \lambda \\ 1 & 0 & 0 \end{pmatrix}, \tag{2.1}$$

where u and v are two potentials, and λ is a constant spectral parameter. To this end, we introduce two sets of Lenard recursion equations

$$Kg_{j-1} = Jg_j, \quad g_j \,|_{(u,v)=0} = 0, \quad j \geq 0, \tag{2.2}$$

$$K\hat{g}_{j-1} = J\hat{g}_j, \quad \hat{g}_j \,|_{(u,v)=0} = 0, \quad j \geq 0 \tag{2.3}$$

with two starting points

$$g_{-1} = \begin{pmatrix} 1 \\ \frac{u+v}{3} \\ 0 \\ 1 \end{pmatrix}, \quad \hat{g}_{-1} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{2.4}$$

where the initial conditions mean to identify constants of integration as zero, and two operators are defined as

$$K = \begin{pmatrix} -(\partial^3 + \partial^2 v + \partial u \partial + \partial uv) & \partial^2 + \partial u & 0 & 0 \\ -(\partial^3 + \partial^2 v + \partial u \partial + \partial uv) & 2\partial^2 + \partial v & 0 & 0 \\ 2\partial + v - u & 0 & \partial^2 - u\partial & \partial + u - v \\ \partial(u+v) + (u+v)\partial & -3\partial & -(\partial^3 - \partial u \partial - v\partial^2 + uv\partial) & 0 \end{pmatrix},$$

$$J=\begin{pmatrix} -1 & 0 & -2\partial & 1\\ 1 & 0 & -\partial & -1\\ 2\partial+v-u & 0 & \partial^2-u\partial & \partial+u-v\\ \partial(u+v)+(u+v)\partial & -3\partial & -(\partial^3-\partial u\partial-v\partial^2+uv\partial) & 0\end{pmatrix}.$$

It is easy to see that

$$\ker J=\{\alpha_0 g_{-1}+\beta_0\hat{g}_{-1}+\gamma_0\check{g}_{-1}\quad|\quad\forall\alpha_0,\beta_0,\gamma_0\in\mathbb{R}\},$$

and $\check{g}_{-1}\in\ker K$, where $\check{g}_{-1}=(0,0,1,0)^T$. Hence, g_j and $\hat{g}_j$ are uniquely determined, for example, the first two members read as

$$g_0=\begin{pmatrix} \frac{1}{9}\left[(2u-v)_{xx}+(2v-u)v_x-3vu_x-\frac{4}{9}(u^3+v^3)+\frac{2}{3}(uv^2+vu^2)\right]\\ \frac{1}{27}\Big[3(u-v)_{xxx}+3(u-v)(u-v)_{xx}-(2u^2+2v^2+uv)(u-v)_x+u_x^2+v_x^2\\ -7u_xv_x+\frac{1}{3}u^2v^2+\frac{8}{9}uv(u^2+v^2)-\frac{7}{9}(u^4+v^4)\Big]\\ \frac{1}{9}(3v_x-3u_x-u^2-v^2+4uv)\\ \frac{1}{9}\left[-(u+v)_{xx}+(2u-v)u_x+(u-2v)v_x-\frac{4}{9}(u^3+v^3)+\frac{2}{3}(uv^2+vu^2)\right]\end{pmatrix},$$

$$\hat{g}_0=\begin{pmatrix} \frac{1}{9}(3v_x+uv-u^2-v^2)\\ \frac{1}{27}\left[3(u+v)_{xx}+3(u+v)(v-u)_x+u^2v+uv^2-\frac{5}{3}(u^3+v^3)\right]\\ -\frac{1}{3}(u+v)\\ \frac{1}{9}\left[3(u-v)_x+uv-u^2-v^2\right]\end{pmatrix}.$$

In order to generate a hierarchy of evolution equations associated with the spectral problem (2.1), we solve the stationary zero-curvature equation

$$V_x-[U,V]=0,\quad V=(V_{ij})_{3\times3}=\begin{pmatrix} V_{11} & V_{12} & \lambda V_{13}\\ \lambda V_{21} & V_{22} & \lambda V_{23}\\ V_{31} & V_{32} & V_{33}\end{pmatrix},\tag{2.5}$$

which is equivalent to

$$\begin{aligned} &V_{11,x}+\lambda V_{13}-\lambda V_{21}=0,\\ &V_{12,x}+(v-u)V_{12}+V_{11}-V_{22}=0,\\ &V_{13,x}+V_{12}-uV_{13}-V_{23}=0,\\ &V_{21,x}+(u-v)V_{21}+V_{23}-V_{31}=0,\\ &V_{22,x}+\lambda V_{21}-\lambda V_{32}=0,\\ &V_{23,x}-vV_{23}+V_{22}-V_{33}=0,\\ &V_{31,x}+uV_{31}+V_{33}-V_{11}=0,\\ &V_{32,x}+vV_{32}+V_{31}-V_{12}=0,\\ &V_{33,x}+\lambda V_{32}-\lambda V_{13}=0,\end{aligned}\tag{2.6}$$

where each entry $V_{ij}=V_{ij}(a,b,c,d)$ is a Laurent expansion in λ:

$$\begin{aligned} &V_{11}=(\partial+u)b-(\partial^2+\partial v+u\partial+uv)a+\lambda c,\\ &V_{12}=b,\quad V_{13}=a+\partial c,\quad V_{21}=d,\\ &V_{22}=(2\partial+v)b-(\partial^2+\partial v+u\partial+uv)a+\lambda c,\\ &V_{23}=(\partial-u)a+b+(\partial^2-u\partial)c,\\ &V_{31}=b-(\partial+v)a,\quad V_{32}=a,\quad V_{33}=\lambda c,\end{aligned}\tag{2.7}$$

$$a=\sum_{j\geq 0} a_{j-1}\lambda^{-j},\quad b=\sum_{j\geq 0} b_{j-1}\lambda^{-j},\quad c=\sum_{j\geq 0} c_{j-1}\lambda^{-j},\quad d=\sum_{j\geq 0} d_{j-1}\lambda^{-j}. \tag{2.8}$$

A direct calculation shows that (2.6) and (2.7) imply the Lenard equation

$$KG=\lambda JG,\quad G=(a,b,c,d)^T \tag{2.9}$$

with the conditions

$$\begin{aligned}&(2\partial+v-u)a+(\partial^2-u\partial)c+(\partial+u-v)d=0,\\ &(\partial(u+v)+(u+v)\partial)a-3\partial b-(\partial^3-\partial u\partial-v\partial^2+uv\partial)c=0.\end{aligned} \tag{2.10}$$

Substituting (2.8) into (2.9) and collecting terms with the same powers of λ, we arrive at the following recursion relation

$$KG_{j-1}=JG_j,\quad JG_{-1}=0,\quad j\geq 0, \tag{2.11}$$

where $G_j=(a_j,b_j,c_j,d_j)^T$. It is easy to see that g_j, $\hat{g}_j$ determined by (2.2)–(2.4) and G_j satisfy the same equation as (2.11). Since equation $JG_{-1}=0$ has a solution

$$G_{-1}=\alpha_0 g_{-1}+\beta_0\hat{g}_{-1}, \tag{2.12}$$

then G_j can be expressed as

$$G_j=\alpha_0 g_j+\beta_0\hat{g}_j+\cdots+\alpha_j g_0+\beta_j\hat{g}_0+\alpha_{j+1}g_{-1}+\beta_{j+1}\hat{g}_{-1},\quad j\geq 0, \tag{2.13}$$

where α_j and β_j are arbitrary constants.

Let ψ satisfy the spectral problem (2.1) and an auxiliary problem

$$\psi_{t_n}=V^{(n)}\psi,\quad V^{(n)}=(\lambda^n V)_+=(V_{ij}^{(n)})_{3\times 3}, \tag{2.14}$$

where each entry $V_{ij}^{(n)}=V_{ij}(a^{(n)},b^{(n)},c^{(n)},d^{(n)})$,

$$a^{(n)}=\sum_{j=0}^{n} a_{j-1}\lambda^{n-j},\quad b^{(n)}=\sum_{j=0}^{n} b_{j-1}\lambda^{n-j},\quad c^{(n)}=\sum_{j=0}^{n} c_{j-1}\lambda^{n-j},\quad d^{(n)}=\sum_{j=0}^{n} d_{j-1}\lambda^{n-j}. \tag{2.15}$$

Then the compatibility condition of (2.1) and (2.14) yields the zero-curvature equation, $U_{t_n}-V_x^{(n)}+[U,V^{(n)}]=0$, which is equivalent to the hierarchy of nonlinear evolution equations

$$(u_{t_n},v_{t_n})^T=X_n,\quad n\geq 0, \tag{2.16}$$

where the vector fields $X_j=X(u,v;\underline{\alpha}_j,\underline{\beta}_j)=P(KG_{j-1})=P(JG_j)$, P is the projective map $\chi=(\chi^{(1)},\chi^{(2)},\chi^{(3)},\chi^{(4)})^T\to(\chi^{(1)},\chi^{(2)})^T$, and $\underline{\alpha}_j=(\alpha_0,\ldots,\alpha_j),\underline{\beta}_j=(\beta_0,\ldots,\beta_j)$. The first nontrivial member in the hierarchy (2.16) is as follows

$$\begin{aligned}u_{t_0}&=\tfrac{\alpha_0}{3}[u_{xx}+2uu_x-2(uv)_x-2v_{xx}]+\beta_0 u_x,\\ v_{t_0}&=\tfrac{\alpha_0}{3}[-v_{xx}+2vv_x-2(uv)_x+2u_{xx}]+\beta_0 v_x,\end{aligned} \tag{2.17}$$

which is just (1.1) as $\alpha_0=3$ and $\beta_0=0$. And the second one is as follows

$$\begin{aligned}u_{t_1}=&\tfrac{1}{243}\alpha_0[-7u^5+20u^4v-15u^3v^2-10u^2v^3+5uv^4\\ &-5(2u^3+2v^3-3u^2v-3uv^2+9vu_x-9v_{xx})(u-2v)_x\\ &+45(u^2+v^2-uv+u_x+v_x)u_{xx}+45u(u_x^2-v_x^2)-27u_{xxxx}]_x,\\ v_{t_1}=&\tfrac{1}{243}\alpha_0[-7v^5+20v^4u-15v^3u^2-10v^2u^3+5vu^4\\ &-5(2u^3+2v^3-3u^2v-3uv^2-9uv_x-9u_{xx})(2u-v)_x\\ &+45(u^2+v^2-uv-u_x-v_x)v_{xx}-45v(u_x^2-v_x^2)-27v_{xxxx}]_x\end{aligned} \tag{2.18}$$

with taking $\alpha_1=\beta_0=\beta_1=0$ for the sake of simplicity. It is easy to see that (2.18) is reduced to a fifth-order nonlinear evolution equation [28,29], the modified Sawada–Kotera–Kaup–Kupershmidt (mSKKK) equation

$$v_t = v_{5x} - \left(5v_xv_{xx} + 5vv_x^2 + 5v^2v_{xx} - v^5\right)_x \tag{2.19}$$

as $u = 2v$ and $\alpha_0 = -9$.

Moreover, the authors in [28] found that there are two Miura transformations

$$u = v_x - v^2, \quad w = -v_x - \frac{1}{2}v^2 \tag{2.20}$$

between the mSKKK Eq. (2.19) and the classical Sawada–Kotera (SK) equation associated with the 3×3 matrix spectral problem

$$u_t = u_{xxxxx} + 5u_xu_{xxx} + 5u_xu_{xx} + 5u^2u_x$$

and the Kaup–Kupershmidt (KK) equation associated with the 3×3 matrix spectral problem

$$w_t = w_{xxxxx} + 10w_xw_{xxx} + 25w_xw_{xx} + 20w^2w_x.$$

So from solutions to (2.19) and using (2.20), we can easily derive solutions to the SK and KK equations, respectively. But by utilizing solutions to the SK and KK equations and the Miura transformation (2.20), it is difficult to obtain solutions to the mSKKK Eq. (2.19).

3. The Baker–Akhiezer function

In this section, we first introduce the Baker–Akhiezer function and the associated meromorphic functions. The trigonal curve $\mathcal{K}_{m-1}$ is given with the help of the characteristic polynomial of Lax matrix for the MSS equations. Then the nonlinear evolution equations are decomposed into the system of solvable ordinary differential equations. For the sake of convenience, we employ the notations $\widetilde{G}_j, \tilde{a}_j, \tilde{b}_j, \tilde{c}_j, \tilde{d}_j, \widetilde{V}, \widetilde{V}_{ij}^{(r)}$ to stand for the quantities $G_j, a_j, b_j, c_j, d_j, V, V_{ij}^{(n)}$ by replacing $\{\alpha_l, \beta_l\}_{l=0,\ldots,n}$ with $\{\tilde{\alpha}_l, \tilde{\beta}_l\}_{l=0,\ldots,r}$, where the integration constants $\{\alpha_l, \beta_l\}_{l=0,\ldots,n} \subset \mathbb{R}$ and $\{\tilde{\alpha}_l, \tilde{\beta}_l\}_{l=0,\ldots,r} \subset \mathbb{R}$ are independent of each other. In addition, we indicate the individual rth MSS flow by a separate time variable $t_r \in \mathbb{C}$. Here we introduce the Baker–Akhiezer function $\psi(P, x, x_0, t_r, t_{0,r})$ by

$$\begin{aligned}
&\psi_x(P, x, x_0, t_r, t_{0,r}) = U(u(x,t_r), v(x,t_r); \lambda(P))\psi(P, x, x_0, t_r, t_{0,r}),\\
&\psi_{t_r}(P, x, x_0, t_r, t_{0,r}) = \widetilde{V}^{(r)}(u(x,t_r), v(x,t_r); \lambda(P))\psi(P, x, x_0, t_r, t_{0,r}),\\
&V^{(n)}(u(x,t_r), v(x,t_r); \lambda(P))\psi(P, x, x_0, t_r, t_{0,r}) = y(P)\psi(P, x, x_0, t_r, t_{0,r}),\\
&\psi_1(P, x_0, x_0, t_{0,r}, t_{0,r}) = 1, \qquad \psi_3(P, x_0, x_0, t_{0,r}, t_{0,r}) = 1
\end{aligned} \tag{3.1}$$

with $\widetilde{V}^{(r)} = (\lambda^r\widetilde{V})_+ = (\widetilde{V}_{ij}^{(r)})_{3\times 3}$, and $\widetilde{V}_{ij}^{(r)} = \widetilde{V}_{ij}(\tilde{a}^{(r)}, \tilde{b}^{(r)}, \tilde{c}^{(r)}, \tilde{d}^{(r)})$,

$$\tilde{a}^{(r)} = \sum_{j=0}^{r} \tilde{a}_{j-1}\lambda^{r-j}, \quad \tilde{b}^{(r)} = \sum_{j=0}^{r} \tilde{b}_{j-1}\lambda^{r-j}, \quad \tilde{c}^{(r)} = \sum_{j=0}^{r} \tilde{c}_{j-1}\lambda^{r-j}, \quad \tilde{d}^{(r)} = \sum_{j=0}^{r} \tilde{d}_{j-1}\lambda^{r-j},$$

where $\tilde{a}_j, \tilde{b}_j, \tilde{c}_j, \tilde{d}_j$ are determined by $\widetilde{G}_j = (\tilde{a}_j, \tilde{b}_j, \tilde{c}_j, \tilde{d}_j)^T$,

$$\widetilde{G}_j = \tilde{\alpha}_0 g_j + \tilde{\beta}_0 \hat{g}_j + \cdots + \tilde{\alpha}_j g_0 + \tilde{\beta}_j \hat{g}_0 + \tilde{\alpha}_{j+1} g_{-1} + \tilde{\beta}_{j+1}\hat{g}_{-1}, \quad j \geq 0.$$

Closely related to $\psi(P, x, x_0, t_r, t_{0,r})$ are the succeeding meromorphic functions $\phi_1(P, x, t_r)$ and $\phi_3(P, x, t_r)$ defined by

$$\phi_1(P, x, t_r) = \frac{\partial_x \psi_1(P, x, x_0, t_r, t_{0,r})}{\psi_1(P, x, x_0, t_r, t_{0,r})} - u(x, t_r) \tag{3.2}$$

and

$$\phi_3(P, x, t_r) = \frac{\partial_x \psi_3(P, x, x_0, t_r, t_{0,r})}{\psi_3(P, x, x_0, t_r, t_{0,r})}, \tag{3.3}$$

which can be, respectively, written as

$$\begin{aligned}\phi_1(P,x,t_r) &= \frac{\varepsilon(m)\lambda\mathcal{E}_{m-1}(\lambda,x,t_r)}{y^2V_{23}^{(n)}(\lambda,x,t_r)-y\mathcal{C}_m(\lambda,x,t_r)+\mathcal{H}_m(\lambda,x,t_r)}\\ &= \frac{y^2V_{13}^{(n)}(\lambda,x,t_r)-y\mathcal{A}_m(\lambda,x,t_r)+\mathcal{B}_m(\lambda,x,t_r)}{-\varepsilon(m)F_{m-1}(\lambda,x,t_r)}\\ &= \frac{yV_{23}^{(n)}(\lambda,x,t_r)+\mathcal{C}_m(\lambda,x,t_r)}{yV_{13}^{(n)}(\lambda,x,t_r)+\mathcal{A}_m(\lambda,x,t_r)}\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\phi_3(P,x,t_r) &= \frac{\varepsilon(m)\lambda F_{m-1}(\lambda,x,t_r)}{y^2V_{12}^{(n)}(\lambda,x,t_r)-yC_m(\lambda,x,t_r)+D_m(\lambda,x,t_r)}\\ &= \frac{y^2V_{32}^{(n)}(\lambda,x,t_r)-yA_m(\lambda,x,t_r)+B_m(\lambda,x,t_r)}{-\varepsilon(m)E_{m-1}(\lambda,x,t_r)}\\ &= \frac{yV_{12}^{(n)}(\lambda,x,t_r)+C_m(\lambda,x,t_r)}{yV_{32}^{(n)}(\lambda,x,t_r)+A_m(\lambda,x,t_r)},\end{aligned} \tag{3.5}$$

where $P=(\lambda,y)\in\mathcal{K}_{m-1},(x,t_r)\in\mathbb{C}^2$,

$$\begin{aligned}A_m &= V_{12}^{(n)}V_{31}^{(n)}-V_{32}^{(n)}V_{11}^{(n)},\\ B_m &= V_{32}^{(n)}(V_{22}^{(n)}V_{33}^{(n)}-\lambda V_{23}^{(n)}V_{32}^{(n)})+V_{31}^{(n)}(V_{12}^{(n)}V_{33}^{(n)}-\lambda V_{13}^{(n)}V_{32}^{(n)}),\\ C_m &= \lambda V_{13}^{(n)}V_{32}^{(n)}-V_{12}^{(n)}V_{33}^{(n)},\\ D_m &= V_{12}^{(n)}(V_{11}^{(n)}V_{22}^{(n)}-\lambda V_{12}^{(n)}V_{21}^{(n)})+\lambda V_{13}^{(n)}(V_{11}^{(n)}V_{32}^{(n)}-V_{12}^{(n)}V_{31}^{(n)}),\\ \mathcal{A}_m &= V_{12}^{(n)}V_{23}^{(n)}-V_{13}^{(n)}V_{22}^{(n)},\\ \mathcal{B}_m &= V_{13}^{(n)}(V_{11}^{(n)}V_{33}^{(n)}-\lambda V_{13}^{(n)}V_{31}^{(n)})+V_{12}^{(n)}(V_{11}^{(n)}V_{23}^{(n)}-\lambda V_{13}^{(n)}V_{21}^{(n)}),\\ \mathcal{C}_m &= \lambda V_{13}^{(n)}V_{21}^{(n)}-V_{11}^{(n)}V_{23}^{(n)},\\ \mathcal{H}_m &= V_{23}^{(n)}(V_{22}^{(n)}V_{33}^{(n)}-\lambda V_{23}^{(n)}V_{32}^{(n)})+\lambda V_{21}^{(n)}(V_{13}^{(n)}V_{22}^{(n)}-V_{12}^{(n)}V_{23}^{(n)}),\end{aligned} \tag{3.6}$$

$$\begin{aligned}E_{m-1} &= -\varepsilon(m)[V_{32}^{(n)}(\lambda V_{21}^{(n)}V_{32}^{(n)}-V_{22}^{(n)}V_{31}^{(n)})+V_{31}^{(n)}(V_{11}^{(n)}V_{32}^{(n)}-V_{12}^{(n)}V_{31}^{(n)})],\\ F_{m-1} &= \varepsilon(m)[V_{12}^{(n)}(V_{12}^{(n)}V_{23}^{(n)}-V_{13}^{(n)}V_{22}^{(n)})+V_{13}^{(n)}(V_{12}^{(n)}V_{33}^{(n)}-\lambda V_{13}^{(n)}V_{32}^{(n)})],\\ \mathcal{E}_{m-1} &= \varepsilon(m)[V_{23}^{(n)}(V_{23}^{(n)}V_{31}^{(n)}-V_{21}^{(n)}V_{33}^{(n)})+V_{21}^{(n)}(V_{11}^{(n)}V_{23}^{(n)}-\lambda V_{13}^{(n)}V_{21}^{(n)})]\end{aligned} \tag{3.7}$$

with

$$\varepsilon(m)=\begin{cases}-1, & \text{if } m=3n+2,\\ 1, & \text{if } m=3n+1,\end{cases} \tag{3.8}$$

which is introduced to ensure that E_{m-1}, F_{m-1} and $\mathcal{E}_{m-1}$ are monic.

The compatibility conditions of the first three expressions in (3.1) yield that

$$U_{t_r}-\widetilde{V}_x^{(r)}+[U,\widetilde{V}^{(r)}]=0, \tag{3.9}$$

$$-V_x^{(n)}+[U,V^{(n)}]=0, \tag{3.10}$$

$$-V_{t_r}^{(n)}+[\widetilde{V}^{(r)},V^{(n)}]=0. \tag{3.11}$$

Through a direct calculation we can show that $V_{11}^{(n)}+V_{22}^{(n)}+V_{33}^{(n)}=0$ and $yI-V^{(n)}$ satisfy Eqs. (3.10) and (3.11). Then the characteristic polynomial of the Lax matrix $V^{(n)}$, $\mathcal{F}_m(\lambda,y)=\det(yI-V^{(n)})$, is an independent constant of the variables x and t_r with the expansion

$$\det(yI-V^{(n)})=y^3+yS_m(\lambda)-T_m(\lambda),$$

where $S_m(\lambda)$ and $T_m(\lambda)$ are polynomials with constant coefficients of λ

$$S_m(\lambda)=\begin{vmatrix} V_{11}^{(n)} & V_{12}^{(n)} \\ \lambda V_{21}^{(n)} & V_{22}^{(n)} \end{vmatrix} + \begin{vmatrix} V_{11}^{(n)} & \lambda V_{13}^{(n)} \\ V_{31}^{(n)} & V_{33}^{(n)} \end{vmatrix} + \begin{vmatrix} V_{22}^{(n)} & \lambda V_{23}^{(n)} \\ V_{32}^{(n)} & V_{33}^{(n)} \end{vmatrix}, \quad T_m(\lambda) = \begin{vmatrix} V_{11}^{(n)} & V_{12}^{(n)} & \lambda V_{13}^{(n)} \\ \lambda V_{21}^{(n)} & V_{22}^{(n)} & \lambda V_{23}^{(n)} \\ V_{31}^{(n)} & V_{32}^{(n)} & V_{33}^{(n)} \end{vmatrix}. \tag{3.12}$$

Note that $T_m(\lambda)$ is a $(3n+2)$ or $(3n+1)$ order polynomial of λ. Then $\mathcal{F}_m(\lambda, y) = 0$ naturally leads to a trigonal curve

$$\mathcal{K}_{m-1}: \quad \mathcal{F}_m(\lambda, y) = y^3 + yS_m(\lambda) - T_m(\lambda) = 0, \tag{3.13}$$

where $m = 3n+2$ as $\alpha_0 = 1$ or $m = 3n+1$ as $\alpha_0 = 0$ and $\beta_0 = 1$. For the convenience, we also denote the compactification of the curve $\mathcal{K}_{m-1}$ by $\mathcal{K}_{m-1}$. Thus, $\mathcal{K}_{m-1}$ becomes a three-sheeted Riemann surface of arithmetic genus $m-1$. Points P on $\mathcal{K}_{m-1}$ are represented as pairs $P = (\lambda, y(P))$ satisfying (3.13) together with $P_\infty = (\infty, \infty)$, the point at infinity. The complex structure on $\mathcal{K}_{m-1}$ is defined by introducing local coordinates $\zeta_{P_0} : P \to (\lambda - \lambda_0)$ near points $P_0 \in \mathcal{K}_{m-1}$ which are neither branch nor singular points of $\mathcal{K}_{m-1}$, $\zeta_{P_\infty} : P \to 1/\lambda^{\frac{1}{3}}$ near the branch point $P_\infty \in \mathcal{K}_{m-1}$ and similar at branch points of $\mathcal{K}_{m-1}$.

By the definitions of $\phi_1(P, x, t_r)$ and $\phi_3(P, x, t_r)$, we can see that there are various interrelationships among the polynomials $A_m, B_m, C_m, D_m, E_{m-1}, \mathcal{A}_m, \mathcal{B}_m, \mathcal{C}_m, \mathcal{H}_m, \mathcal{E}_{m-1}, F_{m-1}$ and S_m, T_m:

$$\begin{aligned} &\varepsilon(m)\lambda V_{32}^{(n)} F_{m-1} = V_{12}^{(n)} D_m - S_m (V_{12}^{(n)})^2 - C_m^2, \\ &\varepsilon(m)\lambda A_m F_{m-1} = T_m (V_{12}^{(n)})^2 + C_m D_m, \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\varepsilon(m) V_{12}^{(n)} E_{m-1} = S_m (V_{32}^{(n)})^2 - V_{32}^{(n)} B_m + A_m^2, \\ &-\varepsilon(m) C_m E_{m-1} = T_m (V_{32}^{(n)})^2 + A_m B_m, \end{aligned} \tag{3.15}$$

$$\begin{aligned} &V_{12}^{(n)} B_m + V_{32}^{(n)} D_m - V_{12}^{(n)} V_{32}^{(n)} S_m + A_m C_m = 0, \\ &V_{12}^{(n)} V_{32}^{(n)} T_m + V_{12}^{(n)} A_m S_m + V_{32}^{(n)} C_m S_m - B_m C_m - A_m D_m = 0, \\ &V_{12}^{(n)} A_m T_m + V_{32}^{(n)} C_m T_m - \lambda E_{m-1} F_{m-1} - B_m D_m = 0, \end{aligned} \tag{3.16}$$

$$\begin{aligned} &\varepsilon(m)\lambda V_{13}^{(n)} \mathcal{E}_{m-1} = V_{23}^{(n)} \mathcal{H}_m - S_m (V_{23}^{(n)})^2 - \mathcal{C}_m^2, \\ &\varepsilon(m)\lambda \mathcal{A}_m \mathcal{E}_{m-1} = T_m (V_{23}^{(n)})^2 + \mathcal{C}_m \mathcal{H}_m, \end{aligned} \tag{3.17}$$

$$\begin{aligned} &\varepsilon(m) V_{23}^{(n)} F_{m-1} = S_m (V_{13}^{(n)})^2 - V_{13}^{(n)} \mathcal{B}_m + \mathcal{A}_m^2, \\ &-\varepsilon(m) \mathcal{C}_m F_{m-1} = T_m (V_{13}^{(n)})^2 + \mathcal{A}_m \mathcal{B}_m, \end{aligned} \tag{3.18}$$

$$\begin{aligned} &V_{23}^{(n)} \mathcal{B}_m + V_{13}^{(n)} \mathcal{H}_m - V_{13}^{(n)} V_{23}^{(n)} S_m + \mathcal{A}_m \mathcal{C}_m = 0, \\ &V_{13}^{(n)} V_{23}^{(n)} T_m + V_{23}^{(n)} \mathcal{A}_m S_m + V_{13}^{(n)} \mathcal{C}_m S_m - \mathcal{B}_m \mathcal{C}_m - \mathcal{A}_m \mathcal{H}_m = 0, \\ &V_{23}^{(n)} \mathcal{A}_m T_m + V_{13}^{(n)} \mathcal{C}_m T_m - \lambda \mathcal{E}_{m-1} F_{m-1} - \mathcal{B}_m \mathcal{H}_m = 0, \end{aligned} \tag{3.19}$$

$$\begin{aligned} &\varepsilon(m) E_{m-1,x} = 2S_m V_{32}^{(n)} - 3B_m - \varepsilon(m)(u+v) E_{m-1}, \\ &V_{12}^{(n)} F_{m-1,x} = -(3V_{11}^{(n)} + (v-2u) V_{12}^{(n)}) F_{m-1} + \varepsilon(m) V_{13}^{(n)} (2V_{12}^{(n)} S_m - 3D_m), \end{aligned} \tag{3.20}$$

$$\begin{aligned} &\varepsilon(m) F_{m-1,x} = 2S_m V_{13}^{(n)} - 3\mathcal{B}_m + \varepsilon(m)(2u-v) F_{m-1}, \\ &V_{23}^{(n)} \mathcal{E}_{m-1,x} = -(3V_{22}^{(n)} + (u-2v) V_{23}^{(n)}) \mathcal{E}_{m-1} + \varepsilon(m) V_{21}^{(n)} (2V_{23}^{(n)} S_m - 3\mathcal{H}_m). \end{aligned} \tag{3.21}$$

By investigating (2.15) and (3.7), we can see that E_{m-1}, F_{m-1} and $\mathcal{E}_{m-1}$ are monic polynomials with respect to λ of degree $m-1$. Hence, we may write them as three finite products

$$E_{m-1}(\lambda, x, t_r) = \prod_{j=1}^{m-1} (\lambda - \mu_j(x, t_r)), \tag{3.22}$$

$$F_{m-1}(\lambda,x,t_r)=\prod_{j=1}^{m-1}(\lambda-\nu_j(x,t_r)), \tag{3.23}$$

$$\mathcal{E}_{m-1}(\lambda,x,t_r)=\prod_{j=1}^{m-1}(\lambda-\xi_j(x,t_r)). \tag{3.24}$$

Defining

$$\hat{\mu}_j(x,t_r)=\Big(\mu_j(x,t_r),y(\hat{\mu}_j(x,t_r))\Big)=\Big(\mu_j(x,t_r),-\frac{A_m(\mu_j(x,t_r),x,t_r)}{V_{32}^{(n)}(\mu_j(x,t_r),x,t_r)}\Big)\in\mathcal{K}_{m-1}, \tag{3.25}$$

$$\begin{aligned}\hat{\nu}_j(x,t_r)&=\Big(\nu_j(x,t_r),y(\hat{\nu}_j(x,t_r))\Big)\\&=\Big(\nu_j(x,t_r),-\frac{C_m(\nu_j(x,t_r),x,t_r)}{V_{12}^{(n)}(\nu_j(x,t_r),x,t_r)}\Big)\\&=\Big(\nu_j(x,t_r),-\frac{\mathcal{A}_m(\nu_j(x,t_r),x,t_r)}{V_{13}^{(n)}(\nu_j(x,t_r),x,t_r)}\Big)\in\mathcal{K}_{m-1},\end{aligned} \tag{3.26}$$

$$\begin{aligned}\hat{\xi}_j(x,t_r)&=\Big(\xi_j(x,t_r),y(\hat{\xi}_j(x,t_r))\Big)\\&=\Big(\xi_j(x,t_r),-\frac{\mathcal{C}_m(\xi_j(x,t_r),x,t_r)}{V_{23}^{(n)}(\xi_j(x,t_r),x,t_r)}\Big)\in\mathcal{K}_{m-1},\\&\qquad 1\le j\le m-1,(x,t_r)\in\mathbb{C}^2,\end{aligned} \tag{3.27}$$

and $P_0=(0,0)$, $\mathcal{P}_0=(0,V_{11}^{(n)}|_{\lambda=0})$ with $V_{11}^{(n)}|_{\lambda=0}=(\partial+u)b_{n-1}-(\partial^2+\partial v+u\partial+uv)a_{n-1}$, and the positive divisors on $\mathcal{K}_{m-1}$ of degree $m-1$

$$\mathcal{D}_{P_1,\ldots,P_{m-1}}:\begin{cases}\mathcal{K}_{m-1}\to\mathbb{N}_0,\\ P\to\mathcal{D}_{P_1,\ldots,P_{m-1}}(P)=\begin{cases}k, \text{ if } P \text{ occurs } k \text{ times in } \{P_1,\ldots,P_{m-1}\},\\ 0, \text{ if } P\notin\{P_1,\ldots,P_{m-1}\}\end{cases}\end{cases}$$

with $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. One infers from (3.4) and (3.5) that the divisors $(\phi_1(P,x,t_r))$ of $\phi_1(P,x,t_r)$ and $(\phi_3(P,x,t_r))$ of $\phi_3(P,x,t_r)$ are given by

$$(\phi_1(P,x,t_r))=\mathcal{D}_{\mathcal{P}_0,\hat{\xi}_1(x,t_r),\ldots,\hat{\xi}_{m-1}(x,t_r)}(P)-\mathcal{D}_{P_\infty,\hat{\nu}_1(x,t_r),\ldots,\hat{\nu}_{m-1}(x,t_r)}(P), \tag{3.28}$$

$$(\phi_3(P,x,t_r))=\mathcal{D}_{P_0,\hat{\nu}_1(x,t_r),\ldots,\hat{\nu}_{m-1}(x,t_r)}(P)-\mathcal{D}_{P_\infty,\hat{\mu}_1(x,t_r),\ldots,\hat{\mu}_{m-1}(x,t_r)}(P), \tag{3.29}$$

which means that $\mathcal{P}_0,\hat{\xi}_1(x,t_r),\ldots,\hat{\xi}_{m-1}(x,t_r)$ and $P_0,\hat{\nu}_1(x,t_r),\ldots,\hat{\nu}_{m-1}(x,t_r)$ are m zeros of $\phi_1(P,x,t_r)$ and $\phi_3(P,x,t_r)$, respectively, and $P_\infty,\hat{\nu}_1(x,t_r),\ldots,\hat{\nu}_{m-1}(x,t_r)$ and $P_\infty,\hat{\mu}_1(x,t_r),\ldots,\hat{\mu}_{m-1}(x,t_r)$ their m poles, respectively.

Next, we introduce the holomorphic map $*$, changing sheets, which is defined by

$$\begin{aligned}&*:\begin{cases}\mathcal{K}_{m-1}\to\mathcal{K}_{m-1}\\ P=(\lambda,y_i(\lambda))\to P^*=(\lambda,y_{i+1(mod3)}(\lambda)),\quad i=0,1,2\end{cases},\\&P^{**}:=(P^*)^*,\quad etc.,\end{aligned} \tag{3.30}$$

where $y_i(\lambda),\ \ i=0,1,2$, denote the three branches of $y(P)$ satisfying $\mathcal{F}_m(\lambda,y)=0$, that is, $(y-y_0(\lambda))(y-y_1(\lambda))(y-y_2(\lambda))=y^3+yS_m(\lambda)-T_m(\lambda)=0$, which is used to exactly get

$$\begin{aligned}&y_0+y_1+y_2=0,\\&y_0y_1+y_0y_2+y_1y_2=S_m(\lambda),\\&y_0y_1y_2=T_m(\lambda),\\&y_0^2+y_1^2+y_2^2=-2S_m(\lambda),\\&y_0^3+y_1^3+y_2^3=3T_m(\lambda),\\&y_0^2y_1^2+y_0^2y_2^2+y_1^2y_2^2=S_m^2(\lambda).\end{aligned} \tag{3.31}$$

Then through a direct calculation, by using (3.1)–(3.5), (3.14)–(3.21), (3.30) and (3.31), we will obtain further properties of $\phi_1(P,x,t_r)$ and $\phi_3(P,x,t_r)$.

Lemma 3.1. *Assume* (3.1) *and* (3.3), $P=(\lambda,y(P))\in\mathcal{K}_{m-1}\setminus\{P_\infty\}$, *and let* $(\lambda,x,x_0,t_r,t_{0,r})\in\mathbb{C}^5$. *Then*

$$\phi_{1,xx}(P,x,t_r)+3\phi_1(P,x,t_r)\phi_{1,x}(P,x,t_r)+\phi_1^3(P,x,t_r)+[u_x(x,t_r)-v_x(x,t_r)+u^2(x,t_r) -u(x,t_r)v(x,t_r)]\phi_1(P,x,t_r)+[2u(x,t_r)-v(x,t_r)][\phi_{1,x}(P,x,t_r)+\phi_1^2(P,x,t_r)]=\lambda, \tag{3.32}$$

$$\phi_{3,xx}(P,x,t_r)+3\phi_3(P,x,t_r)\phi_{3,x}(P,x,t_r)+\phi_3^3(P,x,t_r)+[u(x,t_r)v(x,t_r) -u_x(x,t_r)]\phi_3(P,x,t_r)-[u(x,t_r)+v(x,t_r)][\phi_{3,x}(P,x,t_r)+\phi_3^2(P,x,t_r)]=\lambda, \tag{3.33}$$

$$[\phi_1(P,x,t_r)+u(x,t_r)]_{t_r}=\partial_x\{\widetilde{V}_{13}^{(r)}(\lambda,x,t_r)[\phi_{1,x}(P,x,t_r)+\phi_1^2(P,x,t_r)+(u(x,t_r) -v(x,t_r))\phi_1(P,x,t_r)]+\widetilde{V}_{12}^{(r)}(\lambda,x,t_r)\phi_1(P,x,t_r)+\widetilde{V}_{11}^{(r)}(\lambda,x,t_r)\}, \tag{3.34}$$

$$\phi_{3,t_r}(P,x,t_r)=\partial_x[\widetilde{V}_{32}^{(r)}(\lambda,x,t_r)(\phi_{3,x}(P,x,t_r)+\phi_3^2(P,x,t_r))+(\widetilde{V}_{31}^{(r)}(\lambda,x,t_r) -u(x,t_r)\widetilde{V}_{32}^{(r)}(\lambda,x,t_r))\phi_3(P,x,t_r)+\widetilde{V}_{33}^{(r)}(\lambda,x,t_r)], \tag{3.35}$$

$$\phi_1(P,x,t_r)\phi_1(P^*,x,t_r)\phi_1(P^{**},x,t_r)=\frac{\lambda\mathcal{E}_{m-1}(\lambda,x,t_r)}{F_{m-1}(\lambda,x,t_r)}, \tag{3.36}$$

$$\phi_3(P,x,t_r)\phi_3(P^*,x,t_r)\phi_3(P^{**},x,t_r)=\frac{\lambda F_{m-1}(\lambda,x,t_r)}{E_{m-1}(\lambda,x,t_r)}, \tag{3.37}$$

$$\phi_1(P,x,t_r)+\phi_1(P^*,x,t_r)+\phi_1(P^{**},x,t_r)=\frac{F_{m-1,x}(\lambda,x,t_r)}{F_{m-1}(\lambda,x,t_r)}+v(x,t_r)-2u(x,t_r), \tag{3.38}$$

$$\phi_3(P,x,t_r)+\phi_3(P^*,x,t_r)+\phi_3(P^{**},x,t_r)=\frac{E_{m-1,x}(\lambda,x,t_r)}{E_{m-1}(\lambda,x,t_r)}+u(x,t_r)+v(x,t_r), \tag{3.39}$$

$$\frac{1}{\phi_1(P,x,t_r)}+\frac{1}{\phi_1(P^*,x,t_r)}+\frac{1}{\phi_1(P^{**},x,t_r)}=-\frac{V_{23}^{(n)}(\lambda,x,t_r)}{\lambda V_{21}^{(n)}(\lambda,x,t_r)}\frac{\mathcal{E}_{m-1,x}(\lambda,x,t_r)}{\mathcal{E}_{m-1}(\lambda,x,t_r)} -\frac{3V_{22}^{(n)}(\lambda,x,t_r)+(u(x,t_r)-2v(x,t_r))V_{23}^{(n)}(\lambda,x,t_r)}{\lambda V_{21}^{(n)}(\lambda,x,t_r)}, \tag{3.40}$$

$$\frac{1}{\phi_3(P,x,t_r)}+\frac{1}{\phi_3(P^*,x,t_r)}+\frac{1}{\phi_3(P^{**},x,t_r)}=-\frac{V_{12}^{(n)}(\lambda,x,t_r)}{\lambda V_{13}^{(n)}(\lambda,x,t_r)}\frac{F_{m-1,x}(\lambda,x,t_r)}{F_{m-1}(\lambda,x,t_r)} -\frac{3V_{11}^{(n)}(\lambda,x,t_r)+(v(x,t_r)-2u(x,t_r))V_{12}^{(n)}(\lambda,x,t_r)}{\lambda V_{13}^{(n)}(\lambda,x,t_r)}, \tag{3.41}$$

$$y(P)\phi_1(P,x,t_r)+y(P^*)\phi_1(P^*,x,t_r)+y(P^{**})\phi_1(P^{**},x,t_r) =\frac{3T_m(\lambda)V_{13}^{(n)}(\lambda,x,t_r)+2S_m(\lambda)\mathcal{A}_m(\lambda,x,t_r)}{-\varepsilon(m)F_{m-1}(\lambda,x,t_r)}, \tag{3.42}$$

$$y(P)\phi_3(P,x,t_r)+y(P^*)\phi_3(P^*,x,t_r)+y(P^{**})\phi_3(P^{**},x,t_r) =\frac{3T_m(\lambda)V_{32}^{(n)}(\lambda,x,t_r)+2S_m(\lambda)A_m(\lambda,x,t_r)}{-\varepsilon(m)E_{m-1}(\lambda,x,t_r)}. \tag{3.43}$$

Lemma 3.2. *Assume* (3.1), (3.7) *and let* $(\lambda,x,t_r)\in\mathbb{C}^3$. *Then*

$$E_{m-1,t_r}(\lambda,x,t_r)=E_{m-1,x}(\lambda,x,t_r)\left(\widetilde{V}_{31}^{(r)}-\frac{\widetilde{V}_{32}^{(r)}}{V_{32}^{(n)}}V_{31}^{(n)}\right)+E_{m-1}(\lambda,x,t_r)\left[3\left(\widetilde{V}_{33}^{(r)}-\frac{\widetilde{V}_{32}^{(r)}}{V_{32}^{(n)}}V_{33}^{(n)}\right) +\left(\widetilde{V}_{31}^{(r)}-\frac{\widetilde{V}_{32}^{(r)}}{V_{32}^{(n)}}V_{31}^{(n)}\right)(u+v)+3\tilde{c}_r\right],$$

$$F_{m-1,t_r}(\lambda,x,t_r)=F_{m-1,x}(\lambda,x,t_r)\left(\widetilde{V}_{12}^{(r)}-\frac{\widetilde{V}_{13}^{(r)}}{V_{13}^{(n)}}V_{12}^{(n)}\right)+F_{m-1}(\lambda,x,t_r)\left[3\left(\widetilde{V}_{11}^{(r)}-\frac{\widetilde{V}_{13}^{(r)}}{V_{13}^{(n)}}V_{11}^{(n)}\right)\right.$$
$$\left.+\left(\widetilde{V}_{12}^{(r)}-\frac{\widetilde{V}_{13}^{(r)}}{V_{13}^{(n)}}V_{12}^{(n)}\right)(v-2u)+3\tilde{c}_r\right],$$
$$\mathcal{E}_{m-1,t_r}(\lambda,x,t_r)=\mathcal{E}_{m-1,x}(\lambda,x,t_r)\left(\widetilde{V}_{23}^{(r)}-\frac{\widetilde{V}_{21}^{(r)}}{V_{21}^{(n)}}V_{23}^{(n)}\right)+\mathcal{E}_{m-1}(\lambda,x,t_r)\left[3\left(\widetilde{V}_{22}^{(r)}-\frac{\widetilde{V}_{21}^{(r)}}{V_{21}^{(n)}}V_{22}^{(n)}\right)\right.$$
$$\left.+\left(\widetilde{V}_{23}^{(r)}-\frac{\widetilde{V}_{21}^{(r)}}{V_{21}^{(n)}}V_{23}^{(n)}\right)(u-2v)+3\tilde{c}_r\right]. \tag{3.44}$$

Proof. See "Appendix A." □

An explicit computation gives immediately the properties of $\psi_3(P,x,x_0,t_r,t_{0,r})$ as we will see below.

$$\psi_{3,t_r}(P,x,x_0,t_r,t_{0,r})=[\widetilde{V}_{32}^{(r)}(\lambda,x,t_r)(\phi_{3,x}(P,x,t_r)+\phi_3^2(P,x,t_r))$$
$$+(\widetilde{V}_{31}^{(r)}(\lambda,x,t_r)-u(x,t_r)\widetilde{V}_{32}^{(r)}(\lambda,x,t_r))\phi_3(P,x,t_r)+\widetilde{V}_{33}^{(r)}(\lambda,x,t_r)]\psi_3(P,x,x_0,t_r,t_{0,r}), \tag{3.45}$$

$$\psi_3(P,x,x_0,t_r,t_{0,r})$$
$$=\exp\Big(\int_{x_0}^{x}\phi_3(P,x',t_r)\mathrm{d}x'+\int_{t_{0,r}}^{t_r}\left[\widetilde{V}_{32}^{(r)}(\lambda,x_0,t')\left(\frac{y(P)-V_{33}^{(n)}(\lambda,x_0,t')}{V_{32}^{(n)}(\lambda,x_0,t')}\right.\right.$$
$$\left.\left.-\frac{V_{31}^{(n)}(\lambda,x_0,t')}{V_{32}^{(n)}(\lambda,x_0,t')}\phi_3(P,x_0,t')\right)+\widetilde{V}_{31}^{(r)}(\lambda,x_0,t')\phi_3(P,x_0,t')+\widetilde{V}_{33}^{(r)}(\lambda,x_0,t')\right]\mathrm{d}t'\Big), \tag{3.46}$$

$$\psi_3(P,x,x_0,t_r,t_{0,r})\psi_3(P^*,x,x_0,t_r,t_{0,r})\psi_3(P^{**},x,x_0,t_r,t_{0,r})$$
$$=\frac{E_{m-1}(\lambda,x,t_r)}{E_{m-1}(\lambda,x_0,t_{0,r})}\exp\Big(\int_{x_0}^{x}[u(x',t_r)+v(x',t_r)]\mathrm{d}x'-3\int_{t_{0,r}}^{t_r}\tilde{c}_r(x_0,t')\mathrm{d}t'\Big), \tag{3.47}$$

$$\psi_{3,x}(P,x,x_0,t_r,t_{0,r})\psi_{3,x}(P^*,x,x_0,t_r,t_{0,r})\psi_{3,x}(P^{**},x,x_0,t_r,t_{0,r})$$
$$=\frac{\lambda F_{m-1}(\lambda,x,t_r)}{E_{m-1}(\lambda,x_0,t_{0,r})}\exp\Big(\int_{x_0}^{x}[u(x',t_r)+v(x',t_r)]\mathrm{d}x'-3\int_{t_{0,r}}^{t_r}\tilde{c}_r(x_0,t')\mathrm{d}t'\Big), \tag{3.48}$$

$$\psi_3(P,x,x_0,t_r,t_{0,r})=\left[\frac{E_{m-1}(\lambda,x,t_r)}{E_{m-1}(\lambda,x_0,t_{0,r})}\right]^{1/3}$$
$$\times\exp\Big\{\int_{x_0}^{x}\Big(\frac{y^2(P)V_{32}^{(n)}(\lambda,x',t_r)-y(P)A_m(\lambda,x',t_r)+\frac{2}{3}S_m(\lambda)V_{32}^{(n)}(\lambda,x',t_r)}{-\varepsilon(m)E_{m-1}(\lambda,x',t_r)}+\frac{1}{3}[u(x',t_r)$$
$$+v(x',t_r)]\Big)\mathrm{d}x'-\int_{t_{0,r}}^{t_r}\Big(\frac{y^2(P)V_{32}^{(n)}(\lambda,x_0,t')-y(P)A_m(\lambda,x_0,t')+\frac{2}{3}S_m(\lambda)V_{32}^{(n)}(\lambda,x_0,t')}{-\varepsilon(m)E_{m-1}(\lambda,x_0,t')}$$
$$\times\left[\widetilde{V}_{31}^{(r)}(\lambda,x_0,t')-\frac{\widetilde{V}_{32}^{(r)}(\lambda,x_0,t')}{V_{32}^{(n)}(\lambda,x_0,t')}V_{31}^{(n)}(\lambda,x_0,t')\right]+y(P)\frac{\widetilde{V}_{32}^{(r)}(\lambda,x_0,t')}{V_{32}^{(n)}(\lambda,x_0,t')}-\tilde{c}_r(x_0,t')\Big)\mathrm{d}t'\Big\}. \tag{3.49}$$

The dynamics of the zeros $\mu_j(x,t_r)$, $\nu_j(x,t_r)$ and $\xi_j(x,t_r)$ of $E_{m-1}(\lambda,x,t_r)$, $F_{m-1}(\lambda,x,t_r)$ and $\mathcal{E}_{m-1}(\lambda,x,t_r)$ are then described in terms of Dubrovin-type equations.

Lemma 3.3. *(i) Suppose the zeros $\{\mu_j(x,t_r)\}_{j=1,\ldots,m-1}$ of $E_{m-1}(\cdot,x,t_r)$ remain distinct for $(x,t_r)\in\Omega_\mu$, where $\Omega_\mu\subseteq\mathbb{C}^2$ is open and connected. Then $\{\mu_j(x,t_r)\}_{j=1,\ldots,m-1}$ satisfy the system of differential equations*

$$\mu_{j,x}(x,t_r)=\frac{\varepsilon(m)V_{32}^{(n)}(\mu_j(x,t_r),x,t_r)[3y^2(\hat{\mu}_j(x,t_r))+S_m(\mu_j(x,t_r))]}{\prod\limits_{\substack{k=1\\k\neq j}}^{m-1}(\mu_j(x,t_r)-\mu_k(x,t_r))},\quad 1\le j\le m-1, \tag{3.50}$$

$$\begin{aligned}\mu_{j,t_r}(x,t_r)=&[V_{32}^{(n)}(\mu_j(x,t_r),x,t_r)\widetilde{V}_{31}^{(r)}(\mu_j(x,t_r),x,t_r)-\widetilde{V}_{32}^{(r)}(\mu_j(x,t_r),x,t_r)V_{31}^{(n)}(\mu_j(x,t_r),x,t_r)]\\&\times\frac{\varepsilon(m)[3y^2(\hat{\mu}_j(x,t_r))+S_m(\mu_j(x,t_r))]}{\prod\limits_{\substack{k=1\\k\neq j}}^{m-1}(\mu_j(x,t_r)-\mu_k(x,t_r))},\quad 1\le j\le m-1.\end{aligned} \tag{3.51}$$

(ii) Suppose the zeros $\{\nu_j(x,t_r)\}_{j=1,\ldots,m-1}$ of $F_{m-1}(\cdot,x,t_r)$ remain distinct for $(x,t_r)\in\Omega_\nu$, where $\Omega_\nu\subseteq\mathbb{C}^2$ is open and connected. Then $\{\nu_j(x,t_r)\}_{j=1,\ldots,m-1}$ satisfy the system of differential equations

$$\nu_{j,x}(x,t_r)=\frac{\varepsilon(m)V_{13}^{(n)}(\nu_j(x,t_r),x,t_r)[3y^2(\hat{\nu}_j(x,t_r))+S_m(\nu_j(x,t_r))]}{\prod\limits_{\substack{k=1\\k\neq j}}^{m-1}(\nu_j(x,t_r)-\nu_k(x,t_r))},\quad 1\le j\le m-1, \tag{3.52}$$

$$\begin{aligned}\nu_{j,t_r}(x,t_r)=&[V_{13}^{(n)}(\nu_j(x,t_r),x,t_r)\widetilde{V}_{12}^{(r)}(\nu_j(x,t_r),x,t_r)-\widetilde{V}_{13}^{(r)}(\nu_j(x,t_r),x,t_r)V_{12}^{(n)}(\nu_j(x,t_r),x,t_r)]\\&\times\frac{\varepsilon(m)[3y^2(\hat{\nu}_j(x,t_r))+S_m(\nu_j(x,t_r))]}{\prod\limits_{\substack{k=1\\k\neq j}}^{m-1}(\nu_j(x,t_r)-\nu_k(x,t_r))},\quad 1\le j\le m-1.\end{aligned} \tag{3.53}$$

(iii) Suppose the zeros $\{\xi_j(x,t_r)\}_{j=1,\ldots,m-1}$ of $\mathcal{E}_{m-1}(\cdot,x,t_r)$ remain distinct for $(x,t_r)\in\Omega_\xi$, where $\Omega_\xi\subseteq\mathbb{C}^2$ is open and connected. Then $\{\xi_j(x,t_r)\}_{j=1,\ldots,m-1}$ satisfy the system of differential equations

$$\xi_{j,x}(x,t_r)=\frac{\varepsilon(m)V_{21}^{(n)}(\xi_j(x,t_r),x,t_r)[3y^2(\hat{\xi}_j(x,t_r))+S_m(\xi_j(x,t_r))]}{\prod\limits_{\substack{k=1\\k\neq j}}^{m-1}(\xi_j(x,t_r)-\xi_k(x,t_r))},\quad 1\le j\le m-1, \tag{3.54}$$

$$\begin{aligned}\xi_{j,t_r}(x,t_r)=&[V_{21}^{(n)}(\xi_j(x,t_r),x,t_r)\widetilde{V}_{23}^{(r)}(\xi_j(x,t_r),x,t_r)-\widetilde{V}_{21}^{(r)}(\xi_j(x,t_r),x,t_r)V_{23}^{(n)}(\xi_j(x,t_r),x,t_r)]\\&\times\frac{\varepsilon(m)[3y^2(\hat{\xi}_j(x,t_r))+S_m(\xi_j(x,t_r))]}{\prod\limits_{\substack{k=1\\k\neq j}}^{m-1}(\xi_j(x,t_r)-\xi_k(x,t_r))},\quad 1\le j\le m-1.\end{aligned} \tag{3.55}$$

Proof. Substituting $\lambda = \mu_j(x,t_r)$ or $\nu_j(x,t_r)$ or $\xi_j(x,t_r)$ into (3.14) or (3.15) or (3.17), we have

$$\begin{aligned} B_m(\mu_j(x,t_r),x,t_r) &= S_m(\mu_j(x,t_r))V_{32}^{(n)}(\mu_j(x,t_r),x,t_r) + \frac{A_m^2(\mu_j(x,t_r),x,t_r)}{V_{32}^{(n)}(\mu_j(x,t_r),x,t_r)} \\ &= [S_m(\mu_j(x,t_r)) + y^2(\hat{\mu}_j(x,t_r))]V_{32}^{(n)}(\mu_j(x,t_r),x,t_r), \end{aligned}$$

$$\begin{aligned} D_m(\nu_j(x,t_r),x,t_r) &= S_m(\nu_j(x,t_r))V_{12}^{(n)}(\nu_j(x,t_r),x,t_r) + \frac{C_m^2(\nu_j(x,t_r),x,t_r)}{V_{12}^{(n)}(\nu_j(x,t_r),x,t_r)} \\ &= [S_m(\nu_j(x,t_r)) + y^2(\hat{\nu}_j(x,t_r))]V_{12}^{(n)}(\nu_j(x,t_r),x,t_r), \\ \mathcal{H}_m(\xi_j(x,t_r),x,t_r) &= S_m(\xi_j(x,t_r))V_{23}^{(n)}(\xi_j(x,t_r),x,t_r) + \frac{\mathcal{C}_m^2(\xi_j(x,t_r),x,t_r)}{V_{23}^{(n)}(\xi_j(x,t_r),x,t_r)} \\ &= [S_m(\xi_j(x,t_r)) + y^2(\hat{\xi}_j(x,t_r))]V_{23}^{(n)}(\xi_j(x,t_r),x,t_r). \end{aligned} \tag{3.56}$$

After substituting (3.56) into (3.20) and (3.21), we get

$$\begin{aligned} \varepsilon(m)E_{m-1,x}(\mu_j(x,t_r),x,t_r) &= -V_{32}^{(n)}(\mu_j(x,t_r),x,t_r)[3y^2(\hat{\mu}_j(x,t_r)) + S_m(\mu_j(x,t_r))], \\ F_{m-1,x}(\nu_j(x,t_r),x,t_r) &= -\varepsilon(m)V_{13}^{(n)}(\nu_j(x,t_r),x,t_r)[3y^2(\hat{\nu}_j(x,t_r)) + S_m(\nu_j(x,t_r))], \\ \mathcal{E}_{m-1,x}(\xi_j(x,t_r),x,t_r) &= -\varepsilon(m)V_{21}^{(n)}(\xi_j(x,t_r),x,t_r)[3y^2(\hat{\xi}_j(x,t_r)) + S_m(\xi_j(x,t_r))]. \end{aligned} \tag{3.57}$$

On the other hand, differentiating (3.22)–(3.24) with respect to x gives rise to

$$E_{m-1,x}|_{\lambda=\mu_j(x,t_r)} = -\mu_{j,x}(x,t_r)\prod_{\substack{k=1\\k\neq j}}^{m-1}(\mu_j(x,t_r) - \mu_k(x,t_r)), \tag{3.58}$$

$$F_{m-1,x}|_{\lambda=\nu_j(x,t_r)} = -\nu_{j,x}(x,t_r)\prod_{\substack{k=1\\k\neq j}}^{m-1}(\nu_j(x,t_r) - \nu_k(x,t_r)), \tag{3.59}$$

$$\mathcal{E}_{m-1,x}|_{\lambda=\xi_j(x,t_r)} = -\xi_{j,x}(x,t_r)\prod_{\substack{k=1\\k\neq j}}^{m-1}(\xi_j(x,t_r) - \xi_k(x,t_r)), \tag{3.60}$$

which together with (3.57) indicates (3.50), (3.52) and (3.54). Analogously we can deduce the Dubrovin-type equations (3.51), (3.53) and (3.55). □

4. Algebro-geometric solutions to the MSS hierarchy

In the last section, we will obtain theta function representations for the Baker–Akhiezer function, the meromorphic functions and algebro-geometric solutions of the MSS hierarchy.

In order to view the properties of $\phi_1(P,x,t_r)$ and $\phi_3(P,x,t_r)$ near $P_\infty \in \mathcal{K}_{m-1}$, we take the local coordinate $\zeta = \lambda^{-\frac{1}{3}}$. After substituting the power series ansatz of $\phi_1(P,x,t_r)$ and $\phi_3(P,x,t_r)$ in ζ into (3.32) and (3.33) then yields the indicated Laurent series:

$$\phi_1(P,x,t_r) \underset{\zeta\to 0}{=} \frac{1}{\zeta}\sum_{j=0}^{\infty}\Lambda_j(x,t_r)\zeta^j, \quad \text{as} \quad P \to P_\infty \tag{4.1}$$

with

$$\begin{aligned}
&\Lambda_0 = 1, \quad \Lambda_1 = \frac{1}{3}(v-2u), \quad \Lambda_2 = \frac{1}{9}(3u_x + u^2 - uv + v^2),\\
&\Lambda_3 = -\frac{1}{9}\left[u_{xx} + v_{xx} + vv_x + uv_x + \frac{1}{3}u^2v + \frac{1}{3}uv^2 - \frac{2}{9}u^3 - \frac{2}{9}v^3\right],\\
&\Lambda_4 = \frac{1}{27}\left[3v_{xx} - 2uu_x + 4uv_x + vu_x + vv_x + u^2v + uv^2 - \frac{2}{3}u^3 - \frac{2}{3}v^3\right]_x,\\
&\Lambda_j = -\frac{1}{3}\left[\Lambda_{j-2,xx} + (2u-v)\Lambda_{j-2,x} + 3\sum_{i=0}^{j-1}\Lambda_{j-1-i}\Lambda_{i,x} + (2u-v)\sum_{i=0}^{j-1}\Lambda_{j-1-i}\Lambda_i\right.\\
&\qquad \left. + \sum_{i=1}^{j-1}\Lambda_i\Lambda_{j-i} + \sum_{i=1}^{j-1}\sum_{l=0}^{j-i}\Lambda_i\Lambda_l\Lambda_{j-i-l} + (u_x - v_x + u^2 - uv)\Lambda_{j-2}\right], \quad (j \geq 3),
\end{aligned} \tag{4.2}$$

and

$$\phi_3(P,x,t_r) \underset{\zeta\to 0}{=} \frac{1}{\zeta}\sum_{j=0}^{\infty}\kappa_j(x,t_r)\zeta^j, \quad \text{as} \quad P \to P_\infty, \tag{4.3}$$

where

$$\begin{aligned}
&\kappa_0 = 1, \quad \kappa_1 = \frac{1}{3}(u+v), \quad \kappa_2 = -\frac{1}{3}(uv + v_x) + \frac{1}{9}(u+v)^2,\\
&\kappa_3 = \frac{1}{9}\left[2u_xv + uv_x - uu_x - 2vv_x - u^2v - uv^2 - u_{xx} + 2v_{xx} + \frac{2}{9}(u+v)^3\right],\\
&\kappa_4 = -\frac{1}{9}\left[-\frac{1}{3}uu_x + \frac{2}{3}uv_x + \frac{5}{3}vu_x - \frac{4}{3}vv_x - u^2v - uv^2 - u_{xx} + v_{xx} + \frac{2}{9}(u+v)^3\right]_x,\\
&\kappa_j = -\frac{1}{3}\left[\kappa_{j-2,xx} - (u+v)\kappa_{j-2,x} + 3\sum_{i=0}^{j-1}\kappa_{j-1-i}\kappa_{i,x} - (u+v)\sum_{i=0}^{j-1}\kappa_{j-1-i}\kappa_i\right.\\
&\qquad \left. + \sum_{i=1}^{j-1}\kappa_i\kappa_{j-i} + \sum_{i=1}^{j-1}\sum_{l=0}^{j-i}\kappa_i\kappa_l\kappa_{j-i-l} + (uv - u_x)\kappa_{j-2}\right], \quad (j \geq 3).
\end{aligned} \tag{4.4}$$

Like the method used in [24], we can present some asymptotic properties of y and S_m near $P_\infty \in \mathcal{K}_{m-1}$ in the following

$$y(P) \underset{\zeta\to 0}{=} \begin{cases} \zeta^{-3n-2}[1 + \beta_0\zeta + \alpha_1\zeta^3 + \beta_1\zeta^4 + O(\zeta^6)], & \text{as} \ \ P \to P_\infty, \quad m = 3n+2,\\ \zeta^{-3n-1}[1 + \alpha_1\zeta^2 + \beta_1\zeta^3 + O(\zeta^5)], & \text{as} \ \ P \to P_\infty, \quad m = 3n+1, \end{cases} \tag{4.5}$$

$$S_m \underset{\zeta\to 0}{=} \begin{cases} -3\zeta^{-6n-3}[\beta_0 + (\beta_1 + \alpha_1\beta_0)\zeta^3 + O(\zeta^6)], & \text{as} \ \ P \to P_\infty, \quad m = 3n+2,\\ -3\zeta^{-6n}[\alpha_1 + O(\zeta^3)], & \text{as} \ \ P \to P_\infty, \quad m = 3n+1. \end{cases} \tag{4.6}$$

We now introduce the holomorphic differentials $\eta_l(P)$ on $\mathcal{K}_{m-1}$ defined by

$$\eta_l(P) = \frac{1}{3y(P)^2 + S_m}\begin{cases} \lambda^{l-1}d\lambda, & 1 \leq l \leq m-n-1,\\ y(P)\lambda^{l+n-m}d\lambda, & m-n \leq l \leq m-1, \end{cases} \tag{4.7}$$

and choose a homology basis $\{a_j, b_j\}_{j=1}^{m-1}$ on $\mathcal{K}_{m-1}$ in such a way that the intersection matrix of the cycles satisfies

$$a_j \circ b_k = \delta_{j,k}, \quad a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad j,k = 1,\ldots,m-1.$$

Define the invertible matrix $E \in GL(m-1,\mathbb{C})$ by

$$E = (E_{j,k})_{(m-1)\times(m-1)}, \qquad E_{j,k} = \int_{a_k}\eta_j, \tag{4.8}$$

and the normalized holomorphic differentials ω_j by

$$\omega_j = \sum_{l=1}^{m-1}e_j(l)\eta_l, \quad \int_{a_k}\omega_j = \delta_{j,k}, \quad \int_{b_k}\omega_j = \tau_{j,k}, \quad j,k = 1,\ldots,m-1, \tag{4.9}$$

where

$$\underline{e}(k) = (e_1(k), \ldots, e_{m-1}(k)), \qquad e_j(k) = (E^{-1})_{j,k}.$$

The matrix τ can be shown to be symmetric ($\tau_{j,k} = \tau_{k,j}$) and it has a positive-definite imaginary part [30,31]. Let $\omega^{(2)}_{P_\infty,2}(P)$ denote the normalized Abelian differential of the second kind defined by

$$\omega^{(2)}_{P_\infty,2}(P) = -\sum_{j=1}^{m-1} z_j \eta_j(P) - \frac{1}{3y(P)^2 + S_m} \begin{cases} y(P)\lambda^n d\lambda, & m = 3n+2, \\ \lambda^{2n} d\lambda, & m = 3n+1, \end{cases} \tag{4.10}$$

which is holomorphic on $\mathcal{K}_{m-1} \setminus \{P_\infty\}$ with a pole of order 2 at P_∞ and the constants $\{z_j\}_{j=1,\ldots,m-1}$ are determined by the normalization condition

$$\int_{a_j} \omega^{(2)}_{P_\infty,2}(P) = 0, \quad j = 1, \ldots, m-1. \tag{4.11}$$

The b-periods of the differential $\omega^{(2)}_{P_\infty,2}$ are denoted by

$$\underline{U}^{(2)}_2 = (U^{(2)}_{2,1}, \cdots, U^{(2)}_{2,m-1}), \quad U^{(2)}_{2,j} = \frac{1}{2\pi i} \int_{b_j} \omega^{(2)}_{P_\infty,2}(P), \quad j = 1, \ldots, m-1. \tag{4.12}$$

In particular, if we set $\omega_j = \sum_{l=0}^{\infty} \rho_{l,j} \zeta^l d\zeta$, then we can know $U^{(2)}_{2,j} = \rho_{0,j}$ [20].

On the other hand, $\omega^{(2)}_{P_\infty,3}(P)$ denotes the normalized Abelian differential of the second kind which is holomorphic on $\mathcal{K}_{m-1} \setminus \{P_\infty\}$ with a pole of order 3 at P_∞

$$\begin{aligned} &\underline{U}^{(2)}_3 = (U^{(2)}_{3,1}, \ldots, U^{(2)}_{3,m-1}), \quad U^{(2)}_{3,j} = \frac{1}{2\pi i} \int_{b_j} \omega^{(2)}_{P_\infty,3}(P), \quad j = 1, \ldots, m-1, \\ &\omega^{(2)}_{P_\infty,3}(P) \underset{\zeta \to 0}{=} (\zeta^{-3} + O(1)) \mathrm{d}\zeta, \quad as \quad P \to P_\infty. \end{aligned} \tag{4.13}$$

Furthermore, the normalized Abelian differential of the third kind $\omega^{(3)}_{P_\infty,P_0}(P)$ is holomorphic on $\mathcal{K}_{m-1} \setminus \{P_\infty, P_0\}$ with simple poles at P_∞ and P_0 with residues ± 1, respectively, that is,

$$\begin{aligned} \omega^{(3)}_{P_\infty,P_0}(P) &\underset{\zeta \to 0}{=} (\zeta^{-1} + O(1)) \mathrm{d}\zeta, \quad \text{as } P \to P_\infty, \\ \omega^{(3)}_{P_\infty,P_0}(P) &\underset{\zeta \to 0}{=} (-\zeta^{-1} + O(1)) \mathrm{d}\zeta, \quad \text{as } P \to P_0. \end{aligned} \tag{4.14}$$

Similarly, the other normalized Abelian differential of the third kind $\omega^{(3)}_{P_\infty,\mathcal{P}_0}(P)$ is also holomorphic on $\mathcal{K}_{m-1} \setminus \{P_\infty, \mathcal{P}_0\}$ with simple poles at P_∞ and $\mathcal{P}_0$ with residues ± 1, respectively, that is,

$$\begin{aligned} \omega^{(3)}_{P_\infty,\mathcal{P}_0}(P) &\underset{\zeta \to 0}{=} (\zeta^{-1} + O(1)) \mathrm{d}\zeta, \quad \text{as } P \to P_\infty, \\ \omega^{(3)}_{P_\infty,\mathcal{P}_0}(P) &\underset{\zeta \to 0}{=} (-\zeta^{-1} + O(1)) \mathrm{d}\zeta, \quad \text{as } P \to \mathcal{P}_0. \end{aligned}$$

Thus, we have

$$\begin{aligned} &\int_{Q_0}^{P} \omega^{(3)}_{P_\infty,\mathcal{P}_0}(P) = \ln \zeta + e^{(3)}(Q_0) + O(\zeta), \quad \text{as} \quad P \to P_\infty, \\ &\int_{Q_0}^{P} \omega^{(3)}_{P_\infty,\mathcal{P}_0}(P) = -\ln \zeta + e^{(3)}(Q_0) + O(\zeta), \quad \text{as} \quad P \to \mathcal{P}_0, \end{aligned} \tag{4.15}$$

with $e^{(3)}(Q_0)$ an integration constant.

A straightforward Laurent expansion of (4.7), (4.9) and (4.10) near P_∞ yields the subsequent results.

Lemma 4.1. *Assume the curve $\mathcal{K}_{m-1}$ to be nonsingular. Then near P_∞ in the local coordinate $\zeta=\lambda^{-\frac{1}{3}}$, the differentials $\underline{\omega}$ and $\omega^{(2)}_{P_\infty,2}$ have the Laurent series*

$$\underline{\omega}=(\omega_1,\ldots,\omega_{m-1}) \underset{\zeta\to 0}{=} (\underline{\rho}_0+\underline{\rho}_1\zeta+\underline{\rho}_2\zeta^3+O(\zeta^4))d\zeta \tag{4.16}$$

with

$$\underline{\rho}_0=\begin{cases}-\underline{e}(m-n-1), & m=3n+2,\\ -\underline{e}(m-1), & m=3n+1,\end{cases}$$
$$\underline{\rho}_1=\begin{cases}-\underline{e}(m-1)+\beta_0\underline{e}(m-n-1), & m=3n+2,\\ -\underline{e}(m-n-1), & m=3n+1,\end{cases}$$
$$\underline{\rho}_2=\begin{cases}(2\alpha_1-\beta_0^3)\underline{e}(m-n-1)+\beta_0^2\underline{e}(m-1)-\underline{e}(m-n-2), & m=3n+2,\\ \beta_1\underline{e}(m-1)+\alpha_1\underline{e}(m-n-1)-\underline{e}(m-2), & m=3n+1,\end{cases}$$

and

$$\omega^{(2)}_{P_\infty,2}(P) \underset{\zeta\to 0}{=} \begin{cases}(\zeta^{-2}+z_{m-n-1}-\beta_0^2+(-\alpha_1+\beta_0^3-\beta_0 z_{m-n-1}+z_{m-1})\zeta+O(\zeta^2))d\zeta, \\ \qquad m=3n+2,\\ (\zeta^{-2}+z_{m-1}-\alpha_1+(z_{m-n-1}-2\beta_1)\zeta+O(\zeta^2))d\zeta, \qquad m=3n+1.\end{cases} \tag{4.17}$$

Consequently, from (4.17) we infer

$$\int_{Q_0}^{P}\omega^{(2)}_{P_\infty,2}(P) \underset{\zeta\to 0}{=} -\zeta^{-1}+e_2^{(2)}(Q_0)-q_1\zeta+q_2\zeta^2+O(\zeta^3), \quad \text{as} \quad P\to P_\infty, \tag{4.18}$$

where $e_2^{(2)}(Q_0)$ is an appropriate constant, and

$$q_1=\begin{cases}-z_{m-n-1}+\beta_0^2, & m=3n+2,\\ -z_{m-1}+\alpha_1, & m=3n+1,\end{cases} \tag{4.19}$$

$$q_2=\begin{cases}\frac{1}{2}(-\alpha_1+\beta_0^3-\beta_0 z_{m-n-1}+z_{m-1}), & m=3n+2,\\ \frac{1}{2}z_{m-n-1}-\beta_1, & m=3n+1.\end{cases} \tag{4.20}$$

Let $\theta(\underline{\lambda})$ denote the Riemann theta function [30,31] associated with $\mathcal{K}_{m-1}$ and an appropriately fixed homology basis. We assume $\mathcal{K}_{m-1}$to be nonsingular for the remainder of this section. Next, we choose a convenient base point $Q_0\in\mathcal{K}_{m-1}\setminus\{P_\infty\}$. For brevity, define the function $\underline{\lambda}:\mathcal{K}_{m-1}\times\sigma^{m-1}\mathcal{K}_{m-1}\to\mathbb{C}$ by

$$\begin{aligned}\underline{\lambda}(P,\underline{Q})=\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P)+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}), \qquad & P\in\mathcal{K}_{m-1},\\ \underline{Q}=(Q_1,\ldots,Q_{m-1})\in\sigma^{m-1}\mathcal{K}_{m-1}, &\end{aligned} \tag{4.21}$$

where $\sigma^n\mathcal{K}_{m-1}=\{\{Q_1,\ldots,Q_n\}|Q_j\in\mathcal{K}_{m-1}, j=1,\ldots,n\}$ denotes the nth symmetric power of $\mathcal{K}_{m-1}$, $\underline{\Xi}_{Q_0}$ is the vector of Riemann constants, and the Abel maps $\underline{A}_{Q_0}(\cdot)$ and $\underline{\alpha}_{Q_0}(\cdot)$ are defined by (period lattice $L_{m-1}=\{\underline{z}\in\mathbb{C}^{m-1}|\underline{z}=\underline{N}+\tau\underline{M},\underline{N},\underline{M}\in\mathbb{Z}^{m-1}\}$)

$$\underline{A}_{Q_0}:\mathcal{K}_{m-1}\to\mathcal{J}(\mathcal{K}_{m-1})=\mathbb{C}^{m-1}/L_{m-1},$$
$$P\mapsto\underline{A}_{Q_0}(P)=(A_{Q_0,1}(P),\ldots,A_{Q_0,m-1}(P))=\left(\int_{Q_0}^{P}\omega_1,\ldots,\int_{Q_0}^{P}\omega_{m-1}\right)(\mathrm{mod}L_{m-1}),$$

and

$$\underline{\alpha}_{Q_0}:\mathrm{Div}(\mathcal{K}_{m-1})\to\mathcal{J}(\mathcal{K}_{m-1}),$$
$$\mathcal{D}\mapsto\underline{\alpha}_{Q_0}(\mathcal{D})=\sum_{P\in\mathcal{K}_{m-1}}\mathcal{D}(P)\underline{A}_{Q_0}(P).$$

Then the theta function representations of $\phi_1(P,x,t_r)$ and $\phi_3(P,x,t_r)$ read later.

Theorem 4.2. *Assume that the curve $\mathcal{K}_{m-1}$ is nonsingular. Furthermore, let $P=(\lambda,y)\in\mathcal{K}_{m-1}\setminus\{P_\infty\}$, and let $(x,t_r),(x_0,t_{0,r})\in\Omega_\mu$, where $\Omega_\mu\subseteq\mathbb{C}^2$ is open and connected. Suppose also that $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$, or equivalently, $\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}$, or $\mathcal{D}_{\underline{\hat{\xi}}(x,t_r)}$ is nonspecial for $(x,t_r)\in\Omega_\mu$. Then*

$$\phi_1(P,x,t_r)=\frac{\theta(\underline{\lambda}(P,\underline{\hat{\xi}}(x,t_r)))\theta(\underline{\lambda}(P_\infty,\underline{\hat{\nu}}(x,t_r)))}{\theta(\underline{\lambda}(P_\infty,\underline{\hat{\xi}}(x,t_r)))\theta(\underline{\lambda}(P,\underline{\hat{\nu}}(x,t_r)))}\exp\Big(e^{(3)}(Q_0)-\int_{Q_0}^P\omega^{(3)}_{P_\infty,\mathcal{P}_0}\Big),\tag{4.22}$$

and

$$\phi_3(P,x,t_r)=\frac{\theta(\underline{\lambda}(P,\underline{\hat{\nu}}(x,t_r)))\theta(\underline{\lambda}(P_\infty,\underline{\hat{\mu}}(x,t_r)))}{\theta(\underline{\lambda}(P_\infty,\underline{\hat{\nu}}(x,t_r)))\theta(\underline{\lambda}(P,\underline{\hat{\mu}}(x,t_r)))}\exp\Big(e^{(3)}(Q_0)-\int_{Q_0}^P\omega^{(3)}_{P_\infty,P_0}\Big).\tag{4.23}$$

Proof. Using (3.28) we immediately know that ϕ_1 has simple poles at $\underline{\hat{\nu}}(x,t_r)$ and P_∞, and simple zeros at $\underline{\hat{\xi}}(x,t_r)$ and $\mathcal{P}_0$. It is easily got from (4.15) that

$$\begin{aligned}&\exp\Big(e^{(3)}(Q_0)-\int_{Q_0}^P\omega^{(3)}_{P_\infty,\mathcal{P}_0}\Big)\underset{\zeta\to0}{=}\zeta^{-1}+O(1),\quad\text{as}\ \ P\to P_\infty,\\&\exp\Big(e^{(3)}(Q_0)-\int_{Q_0}^P\omega^{(3)}_{P_\infty,\mathcal{P}_0}\Big)\underset{\zeta\to0}{=}\zeta+O(1),\quad\text{as}\ \ P\to\mathcal{P}_0,\end{aligned}\tag{4.24}$$

Let Φ be defined by the right-hand side of (4.22). Then Φ has the same properties in view of (4.24) and the Riemann's vanishing theorem. Using the Riemann–Roch theorem, we conclude that the holomorphic function $\frac{\Phi}{\phi_1}=\gamma$, where γ is a constant. It is easy to see that (4.24) together with (4.1) implies

$$\frac{\Phi}{\phi_1}\underset{\zeta\to0}{=}\frac{(1+O(\zeta))(\zeta^{-1}+O(1))}{(\zeta^{-1}+O(1))}\underset{\zeta\to0}{=}1+O(\zeta),\quad\text{as } P\to P_\infty.\tag{4.25}$$

Then we derive $\gamma=1$. Homoplastically we get the theta function representation of $\phi_3(P,x,t_r)$ as (4.23). □

Let $\omega^{(2)}_{P_\infty,j}$, $j=3l+2$ or $3l+1$, $l\in\mathbb{N}_0$, be the normalized differentials of the second kind holomorphic on $\mathcal{K}_{m-1}\setminus\{P_\infty\}$ with a pole of order j at P_∞,

$$\omega^{(2)}_{P_\infty,j}(P)\underset{\zeta\to0}{=}(\zeta^{-j}+O(1))d\zeta,\quad\text{as } P\to P_\infty.\tag{4.26}$$

Furthermore, introduce the normalized differentials of the second kind

$$\widetilde{\Omega}^{(2)}_{P_\infty,s+1}=\sum_{l=0}^r\tilde{\alpha}_{r-l}(3l+2)\omega^{(2)}_{P_\infty,3l+3}+\sum_{l=0}^r\tilde{\beta}_{r-l}(3l+1)\omega^{(2)}_{P_\infty,3l+2},\tag{4.27}$$

where $s=3r+2$ or $3r+1, r\in\mathbb{N}_0$, and

$$(\tilde{\alpha}_0,\tilde{\beta}_0)=\begin{cases}(1,\tilde{\beta}_0),& s=3r+2,\\(0,1),& s=3r+1,\end{cases}\ \tilde{\beta}_0\in\mathbb{R}.$$

In addition, we define the vector of b-periods of the normalized differential of the second kind $\tilde{\Omega}^{(2)}_{P_\infty,s+1}$ by

$$\begin{aligned}&\underline{\widetilde{U}}^{(2)}_{s+1}=(\widetilde{U}^{(2)}_{s+1,1},\dots,\widetilde{U}^{(2)}_{s+1,m-1}),\quad \widetilde{U}^{(2)}_{s+1,j}=\frac{1}{2\pi i}\int_{b_j}\widetilde{\Omega}^{(2)}_{P_\infty,s+1},\\&j=1,\dots,m-1,\quad s=3r+2\ \text{or}\ 3r+1,\quad r\in\mathbb{N}_0.\end{aligned}\tag{4.28}$$

For the sake of splitting the two different choices of s, we introduce the notations

$$\widetilde{V}_{3j}^{(r,s)} = \begin{cases} \widetilde{V}_{3j}^{(r)}|_{\tilde{\alpha}_0=1,\tilde{\alpha}_1,\ldots,\tilde{\alpha}_r,\tilde{\beta}_0,\ldots,\tilde{\beta}_r\in\mathbb{R}} & s=3r+2, \\ \widetilde{V}_{3j}^{(r)}|_{\tilde{\alpha}_0=0,\tilde{\beta}_0=1,\tilde{\alpha}_1,\ldots,\tilde{\alpha}_r,\tilde{\beta}_1,\ldots,\tilde{\beta}_r\in\mathbb{R}} & s=3r+1, \end{cases} \quad j=1,2,3;$$

$$\tilde{c}_r^{(r,s)} = \begin{cases} \tilde{c}_r|_{\tilde{\alpha}_0=1,\tilde{\alpha}_1,\ldots,\tilde{\alpha}_r,\tilde{\beta}_0,\ldots,\tilde{\beta}_r\in\mathbb{R}} & s=3r+2, \\ \tilde{c}_r|_{\tilde{\alpha}_0=0,\tilde{\beta}_0=1,\tilde{\alpha}_1,\ldots,\tilde{\alpha}_r,\tilde{\beta}_1,\ldots,\tilde{\beta}_r\in\mathbb{R}} & s=3r+1, \end{cases};$$

and the corresponding homogeneous cases

$$\bar{\widetilde{V}}_{3j}^{(r,s)} = \begin{cases} \widetilde{V}_{3j}^{(r)}|_{\tilde{\alpha}_0=1,\tilde{\alpha}_1=\cdots=\tilde{\alpha}_r=\tilde{\beta}_0=\cdots=\tilde{\beta}_r=0} & s=3r+2, \\ \widetilde{V}_{3j}^{(r)}|_{\tilde{\alpha}_0=0,\tilde{\beta}_0=1,\tilde{\alpha}_1=\cdots=\tilde{\alpha}_r=\tilde{\beta}_1=\cdots=\tilde{\beta}_r=0} & s=3r+1, \end{cases} \quad j=1,2,3;$$

$$\bar{\tilde{c}}_r^{(r,s)} = \begin{cases} \tilde{c}_r|_{\tilde{\alpha}_0=1,\tilde{\alpha}_1=\cdots=\tilde{\alpha}_r=\tilde{\beta}_0=\cdots=\tilde{\beta}_r=0} & s=3r+2, \\ \tilde{c}_r|_{\tilde{\alpha}_0=0,\tilde{\beta}_0=1,\tilde{\alpha}_1=\cdots=\tilde{\alpha}_r=\tilde{\beta}_1=\cdots=\tilde{\beta}_r=0} & s=3r+1, \end{cases}.$$

Motivated by the second integration in (3.46) one defines the function $I_s(P,x,t_r)$, meromorphic on $\mathcal{K}_{m-1}\times\mathbb{C}^2$, by

$$\begin{aligned} I_s(P,x,t_r) &= [\widetilde{V}_{31}^{(r)}(\lambda,x,t_r) - u(x,t_r)\widetilde{V}_{32}^{(r)}(\lambda,x,t_r)]\phi_3(P,x,t_r) + \widetilde{V}_{33}^{(r)}(\lambda,x,t_r) \\ &\quad + \widetilde{V}_{32}^{(r)}(\lambda,x,t_r)(\phi_{3,x}(P,x,t_r) + \phi_3^2(P,x,t_r)). \end{aligned} \tag{4.29}$$

Denote by $\bar{I}_s(P,x,t_r)$ the associated homogeneous quantity replacing $\widetilde{V}_{31}^{(r)}, \widetilde{V}_{32}^{(r)}, \widetilde{V}_{33}^{(r)}$ by the corresponding homogeneous polynomials $\bar{\widetilde{V}}_{31}^{(r)}, \bar{\widetilde{V}}_{32}^{(r)}, \bar{\widetilde{V}}_{33}^{(r)}$ [24,25]. It is easy to conclude (the proof is in "Appendix B")

$$\bar{I}_s(P,x,t_r) \underset{\zeta\to 0}{=} \zeta^{-s} - \bar{\tilde{c}}_r^{(r,s)} + O(\zeta), \quad \text{as } P\to P_\infty. \tag{4.30}$$

By (3.1) one infers

$$I_s(P,x,t_r) = \sum_{l=0}^{r} \tilde{\alpha}_{r-l}\bar{I}_{3l+2}(P,x,t_r) + \sum_{l=0}^{r} \tilde{\beta}_{r-l}\bar{I}_{3l+1}(P,x,t_r) \tag{4.31}$$

with $s=3r+2$ or $3r+1$. Thus, (as $P\to P_\infty$)

$$\int_{t_{0,r}}^{t_r} I_s(P,x,t')\mathrm{d}t' \underset{\zeta\to 0}{=} (t_r-t_{0,r})\sum_{l=0}^{r}\left(\tilde{\alpha}_{r-l}\frac{1}{\zeta^{3l+2}} + \tilde{\beta}_{r-l}\frac{1}{\zeta^{3l+1}}\right) - \int_{t_{0,r}}^{t_r}\tilde{c}_r(x,t')\mathrm{d}t' + O(\zeta). \tag{4.32}$$

Furthermore, integrating (4.27) yields

$$\begin{aligned} \int_{P_0}^{P} \widetilde{\Omega}_{P_\infty,s+1}^{(2)} &= \sum_{l=0}^{r}\tilde{\alpha}_{r-l}(3l+2)\int_{\zeta_0}^{\zeta}\omega_{P_\infty,3l+3}^{(2)} + \sum_{l=0}^{r}\tilde{\beta}_{r-l}(3l+1)\int_{\zeta_0}^{\zeta}\omega_{P_\infty,3l+2}^{(2)} \\ &\underset{\zeta\to 0}{=} \sum_{l=0}^{r}\tilde{\alpha}_{r-l}(3l+2)\int_{\zeta_0}^{\zeta}\frac{1}{\zeta^{3l+3}}\mathrm{d}\zeta + \sum_{l=0}^{r}\tilde{\beta}_{r-l}(3l+1)\int_{\zeta_0}^{\zeta}\frac{1}{\zeta^{3l+2}}\mathrm{d}\zeta + O(\zeta) \\ &\underset{\zeta\to 0}{=} -\sum_{l=0}^{r}\tilde{\alpha}_{r-l}\frac{1}{\zeta^{3l+2}} - \sum_{l=0}^{r}\tilde{\beta}_{r-l}\frac{1}{\zeta^{3l+1}} + e_{s+1}^{(2)}(Q_0) + O(\zeta), \quad \text{as } P\to P_\infty, \end{aligned} \tag{4.33}$$

where $e_{s+1}^{(2)}(Q_0)$ is a constant. Combining (4.32) and (4.33) yields

$$\int_{t_{0,r}}^{t_r} I_s(P,x,t')\mathrm{d}t' \underset{\zeta\to 0}{=} (t_r-t_{0,r})(e_{s+1}^{(2)}(Q_0) - \int_{Q_0}^{P}\widetilde{\Omega}_{P_\infty,s+1}^{(2)}) - \int_{t_{0,r}}^{t_r}\tilde{c}_r(x,t')\mathrm{d}t' + O(\zeta), \text{ as } P\to P_\infty. \tag{4.34}$$

Next, we shall arrive at the theta function representation of $\psi_3(P,x,x_0,t_r,t_{0,r})$.

Theorem 4.3. *Assume that the curve $\mathcal{K}_{m-1}$ is nonsingular. Let $P=(\lambda,y)\in\mathcal{K}_{m-1}\setminus\{P_\infty,P_0\}$ and let $(x,t_r),(x_0,t_{0,r})\in\Omega_\mu$, where $\Omega_\mu\subseteq\mathbb{C}^2$ is open and connected. Suppose that $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$, or equivalently, $\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}$, or $\mathcal{D}_{\underline{\hat{\xi}}(x,t_r)}$ is nonspecial for $(x,t_r)\in\Omega_\mu$. Then*

$$\begin{aligned}\psi_3(P,x,x_0,t_r,t_{0,r})=&\frac{\theta(\underline{\lambda}(P,\underline{\hat{\mu}}(x,t_r)))\theta(\underline{\lambda}(P_\infty,\underline{\hat{\mu}}(x_0,t_{0,r})))}{\theta(\underline{\lambda}(P_\infty,\underline{\hat{\mu}}(x,t_r)))\theta(\underline{\lambda}(P,\underline{\hat{\mu}}(x_0,t_{0,r})))}\\&\times\exp\Big((x-x_0)(e_2^{(2)}(Q_0)-\int_{Q_0}^P\omega_{P_\infty,2}^{(2)})+(t_r-t_{0,r})\Big(e_{s+1}^{(2)}(Q_0)-\int_{Q_0}^P\tilde{\Omega}_{P_\infty,s+1}^{(2)}\Big)\\&+\frac{1}{3}\int_{x_0}^x[u(x',t_r)+v(x',t_r)]\,dx'-\int_{t_{0,r}}^{t_r}\tilde{c}_r(x_0,t')\,dt'\Big).\end{aligned}\tag{4.35}$$

Proof. Assume that $\mu_j(x,t_r)\neq\mu_k(x,t_r), j\neq k,(x,t_r)\in\widetilde{\Omega}_\mu\subseteq\Omega_\mu$, where $\widetilde{\Omega}_\mu$ is open and connected. Using (3.4), (3.20), (3.44), (3.50), (3.51), one can compute

$$\begin{aligned}\phi_3(P,x,t_r)&=\frac{y^2V_{32}^{(n)}-yA_m+B_m}{-\varepsilon(m)E_{m-1}}\\&=\frac{y^2V_{32}^{(n)}-yA_m+\frac{2}{3}V_{32}^{(n)}S_m-\frac{1}{3}\varepsilon(m)E_{m-1,x}-\frac{1}{3}\varepsilon(m)(u+v)E_{m-1}}{-\varepsilon(m)E_{m-1}}\\&=\frac{2}{3}V_{32}^{(n)}\frac{3y^2+S_m}{-\varepsilon(m)E_{m-1}}+\frac{1}{3}\partial_x\ln E_{m-1}+\frac{1}{3}(u+v)+\frac{V_{32}^{(n)}y^2+yA_m}{\varepsilon(m)E_{m-1}}\\&=-\frac{\mu_{j,x}}{\lambda-\mu_j}+O(1),\quad\text{as }\lambda\to\mu_j(x,t_r),\end{aligned}\tag{4.36}$$

and

$$\begin{aligned}I_s(P,x,t_r)&=\widetilde{V}_{31}^{(r)}\phi_3+\widetilde{V}_{32}^{(r)}(\phi_{3,x}+\phi_3^2-u\phi_3)+\widetilde{V}_{33}^{(r)}\\&=\Big(\widetilde{V}_{31}^{(r)}-\widetilde{V}_{32}^{(r)}\frac{V_{31}^{(n)}}{V_{32}^{(n)}}\Big)\phi_3+\widetilde{V}_{33}^{(r)}-\widetilde{V}_{32}^{(r)}\frac{V_{33}^{(n)}}{V_{32}^{(n)}}+y\frac{\widetilde{V}_{32}^{(r)}}{V_{32}^{(n)}}\\&=\Big(\widetilde{V}_{31}^{(r)}-\widetilde{V}_{32}^{(r)}\frac{V_{31}^{(n)}}{V_{32}^{(n)}}\Big)\Big(\frac{y^2V_{32}^{(n)}-yA_m+\frac{2}{3}S_mV_{32}^{(n)}-\frac{1}{3}\varepsilon(m)E_{m-1,x}}{-\varepsilon(m)E_{m-1}}+\frac{1}{3}(u+v)\Big)\\&\qquad+\widetilde{V}_{33}^{(r)}-\widetilde{V}_{32}^{(r)}\frac{V_{33}^{(n)}}{V_{32}^{(n)}}+y\frac{\widetilde{V}_{32}^{(r)}}{V_{32}^{(n)}}\\&=\frac{1}{3}\frac{E_{m-1,t_r}}{E_{m-1}}+\Big(\widetilde{V}_{31}^{(r)}-\widetilde{V}_{32}^{(r)}\frac{V_{31}^{(n)}}{V_{32}^{(n)}}\Big)\frac{y^2V_{32}^{(n)}-yA_m+\frac{2}{3}S_mV_{32}^{(n)}}{-\varepsilon(m)E_{m-1}}+y\frac{\widetilde{V}_{32}^{(r)}}{V_{32}^{(n)}}-\tilde{c}_r\\&=\frac{1}{3}\frac{E_{m-1,t_r}}{E_{m-1}}+\Big(\widetilde{V}_{31}^{(r)}V_{32}^{(n)}-\widetilde{V}_{32}^{(r)}V_{31}^{(n)}\Big)\left[\frac{2}{3}\frac{3y^2+S_m}{-\varepsilon(m)E_{m-1}}+\frac{y^2+y\frac{A_m}{V_{32}^{(n)}}}{\varepsilon(m)E_{m-1}}\right]+y\frac{\widetilde{V}_{32}^{(r)}}{V_{32}^{(n)}}-\tilde{c}_r\\&=-\frac{\mu_{j,t_r}}{\lambda-\mu_j}+O(1),\quad\text{as }\ P\to\hat{\mu}_j(x,t_r).\end{aligned}\tag{4.37}$$

More concisely,

$$\begin{aligned}\phi_3(P,x',t_r)&=\partial_{x'}\ln(\lambda-\mu_j(x',t_r))+O(1),\quad\text{for }P\text{ near }\ \hat{\mu}_j(x,t_r),\\I_s(P,x_0,t')&=\partial_{t'}\ln(\lambda-\mu_j(x_0,t'))+O(1),\quad\text{for }P\text{ near }\hat{\mu}_j(x_0,t'),\end{aligned}\tag{4.38}$$

which means that

$$
\begin{aligned}
&\exp\Big(\int_{x_0}^{x}[\partial_{x'}\ln(\lambda-\mu_j(x',t_r))+O(1)]\mathrm{d}x'+\int_{t_{0,r}}^{t_r}[\partial_{t'}\ln(\lambda-\mu_j(x_0,t'))+O(1)]\mathrm{d}t'\Big)\\
&\quad=\frac{\lambda-\mu_j(x,t_r)}{\lambda-\mu_j(x_0,t_r)}\times\frac{\lambda-\mu_j(x_0,t_r)}{\lambda-\mu_j(x_0,t_{0,r})}O(1)\\
&\quad=\begin{cases}(\lambda-\mu_j(x,t_r))O(1) & \text{for } P \text{ near } \hat{\mu}_j(x,t_r)\neq\hat{\mu}_j(x_0,t_{0,r}),\\ O(1) & \text{for } P \text{ near } \hat{\mu}_j(x,t_r)=\hat{\mu}_j(x_0,t_{0,r}),\\ (\lambda-\mu_j(x_0,t_{0,r}))^{-1}O(1) & \text{for } P \text{ near } \hat{\mu}_j(x_0,t_{0,r})\neq\hat{\mu}_j(x,t_r),\end{cases}
\end{aligned}
\tag{4.39}
$$

where $O(1)\neq 0$. Let $\Psi(P,x,x_0,t_r,t_{0,r})$ be denoted by the right-hand side of (4.35), we can immediately know that all zeros and poles of ψ_3 and Ψ on $\mathcal{K}_{m-1}\setminus\{P_\infty\}$ are simple and coincident. From (4.3), we have (as $P\to P_\infty$)

$$
\int_{x_0}^{x}\phi_3(P,x',t_r)\mathrm{d}x' \underset{\zeta\to 0}{=} (x-x_0)(\zeta^{-1}+O(\zeta))+\int_{x_0}^{x}\frac{1}{3}[u(x',t_r)+v(x',t_r)]\mathrm{d}x'. \tag{4.40}
$$

By (4.34) and (4.40), we see that the essential singularity of ψ_3 and Ψ at P_∞ coincide. The uniqueness result for Baker–Akhiezer functions completes the proof that $\psi_3=\Psi$ on $\widetilde{\Omega}_\mu$. The extension of the result from $(x,t_r)\in\widetilde{\Omega}_\mu$ to $(x,t_r)\in\Omega_\mu$ follows from the continuity of $\underline{\alpha}_{Q_0}$ and the hypothesis that $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$ is nonspecial for $(x,t_r)\in\Omega_\mu$. □

The MSS flows can be straightened out by the Abel map through the method used in [24,25].

Theorem 4.4. *Assume that the curve $\mathcal{K}_{m-1}$ is nonsingular and let $(x,t_r),(x_0,t_{0,r})\in\mathbb{C}^2$. Then*

$$
\begin{aligned}
\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)})&=\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})})+\underline{U}_2^{(2)}(x-x_0)+\underline{\widetilde{U}}_{s+1}^{(2)}(t_r-t_{0,r}),\\
\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)})&=\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x_0,t_{0,r})})+\underline{U}_2^{(2)}(x-x_0)+\underline{\widetilde{U}}_{s+1}^{(2)}(t_r-t_{0,r}),\\
\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\xi}}(x,t_r)})&=\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\xi}}(x_0,t_{0,r})})+\underline{U}_2^{(2)}(x-x_0)+\underline{\widetilde{U}}_{s+1}^{(2)}(t_r-t_{0,r}).
\end{aligned}
\tag{4.41}
$$

Theorem 4.5. *Assume that the curve $\mathcal{K}_{m-1}$ is nonsingular and let $(x,t_r)\in\Omega_\mu$, where $\Omega_\mu\subseteq\mathbb{C}^2$ is open and connected. Suppose also that $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$, or equivalently, $\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}$ or $\mathcal{D}_{\underline{\hat{\xi}}(x,t_r)}$ is nonspecial for $(x,t_r)\in\Omega_\mu$. Then*

$$
\begin{aligned}
u(x,t_r)&=\partial_x\ln\frac{\theta(\underline{\lambda}(P_\infty,\underline{\hat{\mu}}(x,t_r)))\theta(\underline{\lambda}(P_\infty,\underline{\hat{\xi}}(x,t_r)))}{\theta^2(\underline{\lambda}(P_\infty,\underline{\hat{\nu}}(x,t_r)))}+w_1-w_0,\\
v(x,t_r)&=-\partial_x\ln\frac{\theta(\underline{\lambda}(P_\infty,\underline{\hat{\nu}}(x,t_r)))\theta(\underline{\lambda}(P_\infty,\underline{\hat{\xi}}(x,t_r)))}{\theta^2(\underline{\lambda}(P_\infty,\underline{\hat{\mu}}(x,t_r)))}+w_0+2w_1,
\end{aligned}
\tag{4.42}
$$

where $w_0,\ w_1$ are constants.

Proof. The Taylor expansions about P_∞ of the ratios of the theta functions in (4.22) and (4.23) are

$$
\begin{aligned}
\frac{\theta(\underline{\lambda}(P,\underline{\hat{\mu}}(x,t_r)))}{\theta(\underline{\lambda}(P_\infty,\underline{\hat{\mu}}(x,t_r)))}&\underset{\zeta\to 0}{=}1-\partial_x\ln\theta_0\zeta+O(\zeta^2),\\
\frac{\theta(\underline{\lambda}(P,\underline{\hat{\nu}}(x,t_r)))}{\theta(\underline{\lambda}(P_\infty,\underline{\hat{\nu}}(x,t_r)))}&\underset{\zeta\to 0}{=}1-\partial_x\ln\theta_1\zeta+O(\zeta^2),\quad P\to P_\infty,\\
\frac{\theta(\underline{\lambda}(P,\underline{\hat{\xi}}(x,t_r)))}{\theta(\underline{\lambda}(P_\infty,\underline{\hat{\xi}}(x,t_r)))}&\underset{\zeta\to 0}{=}1-\partial_x\ln\theta_2\zeta+O(\zeta^2),
\end{aligned}
\tag{4.43}
$$

where $\theta_0 = \theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)})), \theta_1 = \theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}))$ and $\theta_2 = \theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\xi}}(x,t_r)}))$. Then we get the Taylor expansions about ϕ_1 and ϕ_3, as $P \to P_\infty$,

$$\begin{aligned}
\phi_1(P,x,t_r) &= \frac{\theta(\underline{\lambda}(P,\underline{\hat{\xi}}(x,t_r)))\theta(\underline{\lambda}(P_\infty,\underline{\hat{\nu}}(x,t_r)))}{\theta(\underline{\lambda}(P_\infty,\underline{\hat{\xi}}(x,t_r)))\theta(\underline{\lambda}(P,\underline{\hat{\nu}}(x,t_r)))} \exp\left(e^{(3)}(Q_0) - \int_{Q_0}^{P} \omega^{(3)}_{P_\infty,P_0}\right) \\
&\underset{\zeta\to 0}{=} (1 - \partial_x \ln \frac{\theta_2}{\theta_1}\zeta + O(\zeta^2))(\zeta^{-1} + w_0 + O(\zeta)) \\
&\underset{\zeta\to 0}{=} \zeta^{-1} - \partial_x \ln \frac{\theta_2}{\theta_1} + w_0 + O(\zeta), \\
\phi_3(P,x,t_r) &= \frac{\theta(\underline{\lambda}(P,\underline{\hat{\nu}}(x,t_r)))\theta(\underline{\lambda}(P_\infty,\underline{\hat{\mu}}(x,t_r)))}{\theta(\underline{\lambda}(P_\infty,\underline{\hat{\nu}}(x,t_r)))\theta(\underline{\lambda}(P,\underline{\hat{\mu}}(x,t_r)))} \exp\left(e^{(3)}(Q_0) - \int_{Q_0}^{P} \omega^{(3)}_{P_\infty,P_0}\right) \\
&\underset{\zeta\to 0}{=} (1 - \partial_x \ln \frac{\theta_1}{\theta_0}\zeta + O(\zeta^2))(\zeta^{-1} + w_1 + O(\zeta)) \\
&\underset{\zeta\to 0}{=} \zeta^{-1} - \partial_x \ln \frac{\theta_1}{\theta_0} + w_1 + O(\zeta),
\end{aligned} \tag{4.44}$$

where $w_0,\ w_1$ are constants. From (4.1) and (4.3), we also know

$$\begin{aligned}
\phi_1(P,x,t_r) &\underset{\zeta\to 0}{=} \zeta^{-1} + \tfrac{1}{3}[v(x,t_r) - 2u(x,t_r)] + O(\zeta), \\
\phi_3(P,x,t_r) &\underset{\zeta\to 0}{=} \zeta^{-1} + \tfrac{1}{3}[u(x,t_r) + v(x,t_r)] + O(\zeta).
\end{aligned} \tag{4.45}$$

Therefore, it is clear that

$$\begin{aligned}
\tfrac{1}{3}[v(x,t_r) - 2u(x,t_r)] &= -\partial_x \ln \frac{\theta_2}{\theta_1} + w_0, \\
\tfrac{1}{3}[u(x,t_r) + v(x,t_r)] &= -\partial_x \ln \frac{\theta_1}{\theta_0} + w_1,
\end{aligned} \tag{4.46}$$

which implies (4.42). □

5. Conclusions

In the present paper, we construct the solutions to the Mikhailov–Shabat–Sokolov hierarchy with the aid of the theory of the trigonal curve. By introducing a 3×3 matrix spectral problem, the Mikhailov–Shabat–Sokolov hierarchy are presented. Resorting to the characteristic polynomial of Lax matrix for it, we introduce a trigonal curve, from which we deduce the associated Baker–Akhiezer function, meromorphic functions and Dubrovin-type equations. In view of the approximation of the Baker function and meromorphic function near infinity and the properties of the differentials, we show the explicit theta function representation of the Baker function and meromorphic function.

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Appendix A: The proof of Lemma 3.2

From (3.1) and (3.2), we infer

$$\phi_{3,x} + \phi_3^2 = \frac{y - V_{33}^{(n)}}{V_{32}^{(n)}} + \left(u - \frac{V_{31}^{(n)}}{V_{32}^{(n)}}\right)\phi_3. \tag{A.1}$$

Thus,

$$
\begin{aligned}
&\phi_{3,x}(P,x,t_r)+\phi_{3,x}(P^*,x,t_r)+\phi_{3,x}(P^{**},x,t_r)+\phi_3^2(P,x,t_r)+\phi_3^2(P^*,x,t_r)+\phi_3^2(P^{**},x,t_r)\\
&\quad=\frac{y_0+y_1+y_2}{V_{32}^{(n)}}-3\frac{V_{33}^{(n)}}{V_{32}^{(n)}}+(u-\frac{V_{31}^{(n)}}{V_{32}^{(n)}})(\phi_3(P)+\phi_3(P^*)+\phi_3(P^{**}))\\
&\quad=-3\frac{V_{33}^{(n)}}{V_{32}^{(n)}}+\left(u-\frac{V_{31}^{(n)}}{V_{32}^{(n)}}\right)\left[\frac{E_{m-1,x}(\lambda,x,t_r)}{E_{m-1}(\lambda,x,t_r)}+u(x,t_r)+v(x,t_r)\right].
\end{aligned}
\tag{A.2}
$$

Differentiating (3.39) with respect to t_r indicates

$$
\begin{aligned}
\partial_{t_r}[\phi_3(P,x,t_r)+\phi_3(P^*,x,t_r)+\phi_3(P^{**},x,t_r)]&=\partial_{t_r}\left[\frac{E_{m-1,x}(\lambda,x,t_r)}{E_{m-1}(\lambda,x,t_r)}+u(x,t_r)+v(x,t_r)\right]\\
&=\partial_{t_r}\partial_x(\ln E_{m-1}(\lambda,x,t_r))-3\partial_x\tilde{c}_r(x,t_r).
\end{aligned}
\tag{A.3}
$$

On the other hand, noting (3.35), one deduces by choosing the constant of integration as zero for convenience

$$
\begin{aligned}
\partial_{t_r}(\ln E_{m-1}(\lambda,x,t_r))&=\widetilde{V}_{32}^{(r)}[\phi_{3,x}(P,x,t_r)+\phi_{3,x}(P^*,x,t_r)+\phi_{3,x}(P^{**},x,t_r)\\
&\quad+\phi_3^2(P,x,t_r)+\phi_3^2(P^*,x,t_r)+\phi_3^2(P^{**},x,t_r)]+3\widetilde{V}_{33}^{(r)}+3\tilde{c}_r\\
&\quad+[\widetilde{V}_{31}^{(r)}-u\widetilde{V}_{32}^{(r)}](\phi_3(P,x,t_r)+\phi_3(P^*,x,t_r)+\phi_3(P^{**},x,t_r))\\
&=\left(\widetilde{V}_{31}^{(r)}-\frac{\widetilde{V}_{32}^{(r)}}{V_{32}^{(n)}}V_{31}^{(n)}\right)\left[\frac{E_{m-1,x}(\lambda,x,t_r)}{E_{m-1}(\lambda,x,t_r)}+u(x,t_r)+v(x,t_r)\right]\\
&\quad-3\frac{\widetilde{V}_{32}^{(r)}}{V_{32}^{(n)}}V_{33}^{(n)}+3\widetilde{V}_{33}^{(r)}+3\tilde{c}_r(x,t_r),
\end{aligned}
\tag{A.4}
$$

which implies the first equality in (3.44). Resorting to (3.9), (3.32), (3.36) and (A.2), we arrive at

$$
\begin{aligned}
&\frac{F_{m-1,t_r}}{E_{m-1}}-\frac{F_{m-1}E_{m-1,t_r}}{E_{m-1}^2}\\
&\quad=\frac{F_{m-1}}{E_{m-1}}\Big[\left(\widetilde{V}_{12}^{(r)}-V_{12}^{(n)}\frac{\widetilde{V}_{13}^{(r)}}{V_{13}^{(n)}}\right)\frac{F_{m-1,x}}{F_{m-1}}-\left(\widetilde{V}_{31}^{(r)}-V_{31}^{(n)}\frac{\widetilde{V}_{32}^{(r)}}{V_{32}^{(n)}}\right)\left(\frac{E_{m-1,x}}{E_{m-1}}+u+v\right)\\
&\qquad+(v-2u)\left(\widetilde{V}_{12}^{(r)}-V_{12}^{(n)}\frac{\widetilde{V}_{13}^{(r)}}{V_{13}^{(n)}}\right)-3\left(\widetilde{V}_{33}^{(r)}-V_{33}^{(n)}\frac{\widetilde{V}_{32}^{(r)}}{V_{32}^{(n)}}-\widetilde{V}_{11}^{(r)}+V_{11}^{(n)}\frac{\widetilde{V}_{13}^{(r)}}{V_{13}^{(n)}}\right)\Big]
\end{aligned}
\tag{A.5}
$$

and in an alike way

$$
\begin{aligned}
&\frac{\mathcal{E}_{m-1,t_r}}{F_{m-1}}-\frac{\mathcal{E}_{m-1}F_{m-1,t_r}}{F_{m-1}^2}\\
&\quad=\frac{\mathcal{E}_{m-1}}{F_{m-1}}\Big[\left(\widetilde{V}_{23}^{(r)}-V_{23}^{(n)}\frac{\widetilde{V}_{21}^{(r)}}{V_{21}^{(n)}}\right)\frac{\mathcal{E}_{m-1,x}}{\mathcal{E}_{m-1}}-\left(\widetilde{V}_{12}^{(r)}-V_{12}^{(n)}\frac{\widetilde{V}_{13}^{(r)}}{V_{13}^{(n)}}\right)\left(\frac{F_{m-1,x}}{F_{m-1}}+v-2u\right)\\
&\qquad+(u-2v)\left(\widetilde{V}_{23}^{(r)}-V_{23}^{(n)}\frac{\widetilde{V}_{21}^{(r)}}{V_{21}^{(n)}}\right)-3\left(\widetilde{V}_{11}^{(r)}-V_{11}^{(n)}\frac{\widetilde{V}_{13}^{(r)}}{V_{13}^{(n)}}-\widetilde{V}_{22}^{(r)}+V_{22}^{(n)}\frac{\widetilde{V}_{21}^{(r)}}{V_{21}^{(n)}}\right)\Big].
\end{aligned}
\tag{A.6}
$$

By using (A.5), (A.6) and the first equality in (3.44), it is easy to see that the other two hold.

Appendix B: The proof of (4.30)

It is easy to get the first two members

$$\begin{aligned}\bar{I}_1 &= \phi_3(P,x,t_r) = \zeta^{-1} + \frac{1}{3}(u+v) + O(\zeta) = \zeta^{-1} - \bar{\bar{c}}_0^{(0,1)} + O(\zeta), \quad \text{as} \quad P \to P_\infty,\\ \bar{I}_2 &= \phi_{3,x}(P,x,t_r) + \phi_3^2(P,x,t_r) - \frac{2}{3}(u+v)\phi_3(P,x,t_r)\\ &= \zeta^{-2} + \frac{1}{9}(u^2+v^2-4uv+3u_x-3v_x) + O(\zeta)\\ &= \zeta^{-2} - \bar{\bar{c}}_0^{(0,2)} + O(\zeta), \quad \text{as} \quad P \to P_\infty\end{aligned}$$

from

$$\begin{aligned}\bar{I}_s(P,x,t_r) &= [\bar{\bar{V}}_{31}^{(r,s)}(\lambda,x,t_r) - u(x,t_r)\bar{\bar{V}}_{32}^{(r,s)}(\lambda,x,t_r)]\phi_3(P,x,t_r) + \bar{\bar{V}}_{33}^{(r,s)}(\lambda,x,t_r)\\ &\quad + \bar{\bar{V}}_{32}^{(r,s)}(\lambda,x,t_r)(\phi_{3,x}(P,x,t_r) + \phi_3^2(P,x,t_r))\\ &= (\bar{\bar{b}}^{(r,s)}(\lambda,x,t_r) - \bar{\bar{a}}_x^{(r,s)}(\lambda,x,t_r) - (u+v)\bar{\bar{a}}^{(r,s)}(\lambda,x,t_r))\phi_3(P,x,t_r)\\ &\quad + \bar{\bar{a}}^{(r,s)}(\lambda,x,t_r)(\phi_{3,x}(P,x,t_r) + \phi_3^2(P,x,t_r)) + \zeta^{-3}\bar{\bar{c}}^{(r,s)}(\lambda,x,t_r),\end{aligned} \tag{B.1}$$

which is obtained from (3.9) and (4.29). Then one may set $\bar{I}_s(P,x,t_r)$ as

$$\bar{I}_s(P,x,t_r) \underset{\zeta\to 0}{=} \zeta^{-s} + \sum_{j=0}^{\infty}\delta_j(x,t_r)\zeta^j, \quad \text{as} \quad P \to P_\infty \tag{B.2}$$

for some coefficients $\{\delta_j(x,t_r)\}_{j\in\mathbb{N}}$. From (3.35) and (4.29), we have $\partial_x \bar{I}_s(P,x,t_r) = \phi_{3,t_r}(P,x,t_r)$, that is

$$\partial_x\left(\zeta^{-s} + \sum_{j=0}^{\infty}\delta_j(x,t_r)\zeta^j\right) = \partial_{t_r}\left(\zeta^{-1} + \sum_{j=1}^{\infty}\kappa_j(x,t_r)\zeta^{j-1}\right) = \partial_{t_r}\left(\sum_{j=0}^{\infty}\kappa_{j+1}(x,t_r)\zeta^j\right). \tag{B.3}$$

Using (3.9), (4.4) and comparing terms with the same powers of ζ in (B.3) yield

$$\begin{aligned}\delta_{j,x}(x,t_r) &= \kappa_{j+1,t_r}(x,t_r), \quad j = 0,1,2,\ldots,\\ \delta_{0,x}(x,t_r) &= \kappa_{1,t_r}(x,t_r) = \frac{1}{3}(u+v)_{t_r}(x,t_r) = -\bar{\bar{c}}_{r,x}^{(r,s)}(x,t_r),\\ \delta_{1,x}(x,t_r) &= \kappa_{2,t_r}(x,t_r)\\ &= -\frac{1}{3}[u(x,t_r)v(x,t_r) + v_x(x,t_r)]_{t_r} + \frac{2}{9}[u(x,t_r)+v(x,t_r)][u(x,t_r)+v(x,t_r)]_{t_r}\\ &= -\bar{\bar{a}}_{r,x}^{(r,s)}(x,t_r),\\ \delta_{2,x}(x,t_r) &= \kappa_{3,t_r}(x,t_r)\\ &= \frac{1}{9}\left[2u_xv + uv_x - uu_x - 2vv_x - u^2v - uv^2 - u_{xx} + 2v_{xx} + \frac{2}{9}(u+v)^3\right]_{t_r},\\ &= -\bar{\bar{b}}_{r,x}^{(r,s)}(x,t_r) + \frac{1}{3}[(u(x,t_r)+v(x,t_r))\bar{\bar{a}}_r^{(r,s)}(x,t_r)]_x + \bar{\bar{a}}_{r,xx}^{(r,s)}(x,t_r),\\ \delta_{3,x}(x,t_r) &= \kappa_{4,t_r}(x,t_r)\\ &= -\frac{1}{9}\left[-\frac{1}{3}uu_x + \frac{2}{3}uv_x + \frac{5}{3}vu_x - \frac{4}{3}vv_x - u^2v - uv^2 - u_{xx} + v_{xx} + \frac{2}{9}(u+v)^3\right]_{x,t_r}\\ &= -\frac{1}{3}[(u(x,t_r)+v(x,t_r))\bar{\bar{a}}_r^{(r,s)}(x,t_r) + 2\bar{\bar{a}}_{r,x}^{(r,s)}(x,t_r) - 3\bar{\bar{b}}_r^{(r,s)}(x,t_r)]_{xx},\end{aligned} \tag{B.4}$$

which implies

$$\begin{aligned}
&\delta_0(x,t_r)=\epsilon_0(t_r)-\bar{\bar{c}}_r^{(r,s)}(x,t_r),\\
&\delta_1(x,t_r)=\epsilon_1(t_r)-\bar{\bar{a}}_r^{(r,s)}(x,t_r),\\
&\delta_2(x,t_r)=\epsilon_2(t_r)-\bar{\bar{b}}_r^{(r,s)}(x,t_r)+\frac{1}{3}(u(x,t_r)+v(x,t_r))\bar{\bar{a}}_r^{(r,s)}(x,t_r)+\bar{\bar{a}}_{r,x}^{(r,s)}(x,t_r),\\
&\delta_3(x,t_r)=\epsilon_3(t_r)-\frac{1}{3}[(u(x,t_r)+v(x,t_r))\bar{\bar{a}}_r^{(r,s)}(x,t_r)+2\bar{\bar{a}}_{r,x}^{(r,s)}(x,t_r)-3\bar{\bar{b}}_r^{(r,s)}(x,t_r)]_x,
\end{aligned} \tag{B.5}$$

where $\epsilon_0(t_r),\epsilon_1(t_r),\epsilon_2(t_r),\epsilon_3(t_r)$ are integration constants. Noting that there are no arbitrary integration constants either in the coefficients of the power series for $\phi_3(P,x,t_r)$ in the coordinate ζ near P_∞ or in the coefficients of the homogeneous polynomials $\bar{\bar{a}}^{(r,s)}(\zeta,x,t_r),\ldots,\bar{\bar{d}}^{(r,s)}(\zeta,x,t_r)$, it follows that $\bar{I}_s$ can also have no arbitrary integration constants. All these demonstrate that $\epsilon_0(t_r)=\epsilon_1(t_r)=\epsilon_2(t_r)=\epsilon_3(t_r)=0$. Hence,

$$\begin{aligned}
\bar{I}_s(P,x,t_r)=\zeta^{-s}-\bar{\bar{c}}_r^{(r,s)}-\bar{\bar{a}}_r^{(r,s)}\zeta+\left[-\bar{\bar{b}}_r^{(r,s)}+\frac{1}{3}(u+v)\bar{\bar{a}}_r^{(r,s)}+\bar{\bar{a}}_{r,x}^{(r,s)}\right]\zeta^2\\
-\frac{1}{3}[((u+v)\bar{\bar{a}}_r^{(r,s)})_x+2\bar{\bar{a}}_{r,xx}^{(r,s)}-3\bar{\bar{b}}_{r,x}^{(r,s)}]\zeta^3+O(\zeta^4),\quad \text{as}\quad P\to P_\infty.
\end{aligned} \tag{B.6}$$

On the other hand, we have

$$\begin{aligned}
\bar{I}_{s+3}(P,x,t_r)&=(\bar{\bar{b}}^{(r+1,s+3)}-\bar{\bar{a}}_x^{(r+1,s+3)}-(u+v)\bar{\bar{a}}^{(r+1,s+3)})\phi_3\\
&\quad+\bar{\bar{a}}^{(r+1,s+3)}(\phi_{3,x}+\phi_3^2)+\zeta^{-3}\bar{\bar{c}}^{(r+1,s+3)}\\
&=\zeta^{-3}\bar{I}_s+(\bar{\bar{b}}_r^{(r+1,s+3)}-\bar{\bar{a}}_{r,x}^{(r+1,s+3)}-(u+v)\bar{\bar{a}}_r^{(r+1,s+3)})\phi_3\\
&\quad+\bar{\bar{a}}_r^{(r+1,s+3)}(\phi_{3,x}+\phi_3^2)+\zeta^{-3}\bar{\bar{c}}_r^{(r+1,s+3)}\\
&=\zeta^{-s-3}-\bar{\bar{c}}_{r+1}^{(r+1,s+3)}+O(\zeta),\quad \text{as}\ P\to P_\infty.
\end{aligned} \tag{B.7}$$

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