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Non-periodic discrete Schrödinger equations: ground state solutions

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Abstract. In this paper, we study a class of non-periodic discrete Schrödinger equations with superlinear non-linearities at infinity. Under conditions weaker than those previously assumed, we obtain the existence of ground state solutions, i.e., non-trivial solutions with least possible energy. In addition, an example is given to illustrate our results.

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1. Introduction and main results

Discrete nonlinear Schrödinger equations (DNLS) play an important role in describing many phenomena, ranging from solid-state and condensed matter physics to biology, including nonlinear optics [1], biomolecular chains [5], Bose–Einstein condensates [8]. Some authors have successfully applied the DNLS equations to the modeling of localized pulse propagation in optical fibers and wave guides, to the study of energy relaxation in solids and to the modeling of self-trapping of vibrational energy in proteins; see [6,7,17].

In this paper, we study the following discrete nonlinear equation

$$\begin{cases} -\Delta u_n + V_n - \omega u_n = \sigma g_n(u_n), & n \in \mathbb{Z}, \\ \lim_{|n|\to\infty} u_n = 0, \end{cases} \tag{1.1}$$

where $\Delta u_n := u_{n+1} + u_{n-1} - 2u_n$ is the discrete Laplacian in one spatial dimension. The discrete potential $V = (V_n)_{n\in\mathbb{Z}}$ is a sequence of real numbers, $\omega \in \mathbb{R}$, $\sigma = \pm 1$ and $(g_n)_{n\in\mathbb{Z}}$ is a function sequence. The problem (1.1) appears when we look for *standing waves* of the *discrete nonlinear Schrödinger* (DNLS) equation

$$i\dot{\psi}_n = -\Delta\psi_n + V_n\psi_n - \sigma|\psi_n|^2\psi_n, \quad n \in \mathbb{Z}. \tag{1.2}$$

By the definition of standing waves, we want $\psi_n = u_n e^{-i\omega t}$ and $\lim_{|n|\to\infty} \psi_n = 0$, where $\{u_n\}$ is a real-valued sequence and $\omega \in \mathbb{R}$ is the temporal frequency. Then (1.2) becomes

$$\begin{cases} -\Delta u_n + V_n - \omega u_n = \sigma |u_n|^2 u_n, & n \in \mathbb{Z}, \\ \lim_{|n|\to\infty} u_n = 0. \end{cases} \tag{1.3}$$

Thus, the problem of the existence of standing waves of (1.2) has been reduced to that of the existence of solutions of (1.3) in the space l^2 of two-sided infinite sequences. Note that every element of l^2 automatically satisfies $\lim_{|n|\to\infty} u_n = 0$. Clearly, (1.3) is a special case of (1.1) with $g_n(u_n) = |u_n|^2 u_n$.

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In contrast to the *periodic* case of (1.1) which was studied by many authors (see [2–4,9,10,14,15,18,21–23] and their references), we are interested in the *non-periodic* case of (1.1) (i.e., V_n and g_n are all non-periodic). As we know, periodic assumptions are very important in the study of DNLS equations since periodicity is used to control the lack of compactness due to the fact that DNLS equations are set on all $\mathbb{Z}$. But non-periodic equations are quite different from the ones described in periodic cases. Recently, some authors [11,19,20] studied *non-periodic* DNLS equations. The authors of [11] obtained the existence and multiplicity of non-trivial solutions for (1.1) with $g_n(s)$ being *asymptotically linear* as $|s| \to \infty$ by using the critical point theory for smooth functionals. The authors of [19] obtained the existence of non-trivial solutions for a special *superlinear* case of (1.1) with $g_n(s) = |s|^{p-2}s$ $(p > 2)$ by using the Nehari manifold approach. We will give some comparisons between the results of [20] and our results below (our Theorem 1.2).

In this paper, we are mainly interested in the case where $\sigma = 1$. The other case when $\sigma = -1$ can also be discussed if we replace $\triangle$ by $-\triangle$, V by $-V$ and ω by $-\omega$, respectively. But the corresponding results will be slightly different from the case $\sigma = 1$, see Theorem 2.3 in [20]. Next, we need the standard real sequence spaces l^q, $q \in [1, \infty)$, endowed with the norm

$$\|u\|_{l^q} := \left(\sum_{n=-\infty}^{+\infty} |u_n|^q\right)^{1/q}, \qquad \|u\|_{l^\infty} := \max_{n\in\mathbb{Z}} |u_n|.$$

We have the well-known embedding between such spaces:

$$l^q \subset l^p, \quad \|u\|_{l^p} \le \|u\|_{l^q}, \quad 1 \le q \le p \le \infty.$$

Note that the domain $\mathbb{Z}$ is unbounded. Thus, to overcome the loss of compactness caused by the unboundedness of the domain $\mathbb{Z}$, we will use the following assumption:

$(\mathbf{V_1})$ $V = (V_n)_{n\in\mathbb{Z}}$ is bounded from below and satisfies $\lim_{|n|\to\infty} V_n = +\infty$.

Then condition (V_1) implies that the spectrum $\sigma(-\triangle + V)$ is discrete and consists of simple eigenvalues accumulating to $+\infty$ (see [19]). Now we can assume that

$$\gamma_1 < \gamma_2 < \cdots < \gamma_k < \cdots \to +\infty$$

are all eigenvalues of $L := -\triangle + V$, which is defined by $Lu_n = -\triangle u_n + V_n u_n$ for $u = (u_n)_{n\in\mathbb{Z}} \in l^2$. Obviously, the operator L is an unbounded self-adjoint operator in l^2. Let E be the forms domain of L, i.e., the domain of $L^{1/2}$. Since the operator $-\triangle$ is bounded in l^2, it is easy to see that

$$E = \{u \in l^2 : \ V^{1/2}u \in l^2\},$$

which is a Hilbert space. Here, $V^{1/2}u$ is defined by $(V^{1/2}u)_n = V_n^{1/2}u_n$. The corresponding action functional of (1.1) is

$$\Phi(u) := \frac{1}{2}((L-\omega)u, u)_{l^2} - \sum_{n=-\infty}^{+\infty} G_n(u_n), \quad u \in E, \tag{1.4}$$

where $(\cdot,\cdot)_{l^2}$ is the inner product of l^2 and the corresponding norm of l^2 is denoted by $\|\cdot\|_{l^2}$. In this paper, we focus on the following cases:

$(\mathbf{L_1})$ $\gamma_{k_0} - \omega := a < 0 < b := \gamma_{k_0+1} - \omega$ for some $k_0 \ge 1$ (the indefinite case).

$(\mathbf{L_2})$ $\omega < \gamma_1$ (the positive definite case).

$(\mathbf{L_3})$ $\omega := \gamma_{k_0'}$ for some $k_0' \ge 1$ (ω is an eigenvalue of L).

We should mention that Schechter [13] obtained the existence of ground state solutions for a *periodic* Schrödinger equation by using the variant weak linking theorem in [12]. In this paper, we shall adopt the variant weak linking theorem to study the *non-periodic* discrete Schrödinger equation (1.1). To the best of our knowledge, this technique has not been used for discrete equations. In addition, our results improve and generalize the related results. For non-linearities g_n, we assume that

($\mathbf{G_1}$) $g_n \in C(\mathbb{R},\mathbb{R})$, $|g_n(s)| \le c(1+|s|^{p-1})$ for some $c>0$ and $p>2$, $\quad n\in\mathbb{Z},\ s\in\mathbb{R}$.

($\mathbf{G_2}$) $G_n(s) := \int_0^s g_n(t)\mathrm{d}t \ge \frac{1}{2}as^2$, here the constant a is defined in (L_1), $\quad n\in\mathbb{Z},\ s\in\mathbb{R}$.

($\mathbf{G_3}$) $|g_n(s)| \le \gamma|s|$ if $|s|<\delta$ for some $0<\gamma<b$ and $\delta>0$, $\quad n\in\mathbb{Z},\ s\in\mathbb{R}$.

($\mathbf{G_4}$) $\lim_{|s|\to\infty}\frac{G_n(s)}{s^2} = +\infty$ and $G_n(s)\ge -W_n$ for some $W=(W_n)_{n\in\mathbb{Z}}\in l^1$, $\quad n\in\mathbb{Z},\ s\in\mathbb{R}$.

($\mathbf{G_5}$) $G_n(s+l)-G_n(s)-rg_n(s)l+\frac{(r-1)^2}{2}g_n(s)s \ge -W_n$, $\quad r\in[0,1],\ n\in\mathbb{Z},\ s\in\mathbb{R}$.

Our results read as follows:

Theorem 1.1. *If $\sigma=1$, (V_1), (G_1)–(G_5) and (L_1) (or (L_2), or (L_3)) hold, then (1.1) has at least one non-trivial solution u. Moreover, if*

$$g_n(s)=o(s) \quad \text{as } s\to 0, \quad n\in\mathbb{Z},\ s\in\mathbb{R}, \tag{1.5}$$

then u decays exponentially at infinity, i.e., there are two positive constants $\tau,\nu>0$ such that

$$|u_n| \le \tau \mathrm{e}^{-\nu|n|}, \quad n\in\mathbb{Z}. \tag{1.6}$$

Theorem 1.2. *Let $\sigma=1$ and $\mathcal{M}$ be the collection of solutions of (1.1). Then there is a solution that minimizes Φ in (1.4) over $\mathcal{M}$. In addition, if (1.5) holds, then (1.1) has a ground state solution, i.e., non-trivial solution with least possible energy of (1.1).*

We mention that the authors of [20] also considered the cases ((L_1)-(L_3)) and obtained the existence and multiplicity (if g_n is odd) of non-trivial solutions for (1.1) with $g_n(s)$ being *superlinear* as $|s|\to\infty$ by using a linking theorem. But it is worth pointing out that the results in [20] are based on the following assumptions

$$g_n\in C^1(\mathbb{R},\mathbb{R}), \quad |g_n(s)|\le c(1+|s|^{p-1}) \text{ for some } c>0 \text{ and } p>2, \tag{1.7}$$

$$\lim_{s\to 0}\frac{g_n(s)}{|s|}=0 \quad \text{(i.e., } g_n \text{ is superliner near 0)}, \tag{1.8}$$

and the Ambrosetti–Rabinowitz condition

$$\exists\nu>2, \quad \text{s.t.} \quad 0<\nu G_n(s)\le g_n(s)s, \quad \forall s\in\mathbb{R}\backslash\{0\}. \tag{1.9}$$

However, we obtain the existence of *ground state* solutions of (1.1) by using the variant weak linking theorem in [12] under weaker conditions (G_1)–(G_4) than the above conditions (1.7)–(1.9). The condition (G_5) is the discrete counterpart of the Schechter condition [13], which is not comparable with the Ambrosetti–Rabinowitz condition (1.9).

Example 1.1. Here we give an example to illustrate our Theorems 1.1 and 1.2. Let

$$g_n(s) := a_n|s|^{p-2}s, \quad p>2, \qquad s\in\mathbb{R},\ n\in\mathbb{Z},$$

where $\{a_n\}$ is non-periodic and $0<a_n<C$ for some $C>0$, $\forall n\in\mathbb{Z}$. Clearly, the function g_n satisfies conditions (G_1)-(G_4) with $W_n=0$ for all $n\in\mathbb{Z}$. Note that

$$\frac{g_n(s)}{|s|}=\frac{a_n|s|^{p-2}s}{|s|}=\begin{cases}-a_n|s|^{p-2}, & s\in(-\infty,0),\\ a_n|s|^{p-2}, & s\in(0,+\infty),\end{cases}$$

so it is strictly increasing on $(-\infty,0)$ and $(0,+\infty)$. Therefore, g_n satisfies the condition (G_5) by the argument in [13].

The remainder of this paper is organized as follows. In Sect. 2, we give some preliminary lemmas, which are useful in the proofs of Theorems 1.1–1.2. In Sect. 3, we give the detailed proofs of our main results.

2. Variational frameworks and preliminary lemmas

In the following, we will denote generic constants by C, and we always assume that $\sigma = 1$, (V_1) and (G_1)–(G_5) are satisfied. As before, we let $L := -\triangle + V$.

If (L_1) holds, then we have $l^2 = (l^2)^- \oplus (l^2)^+$, where $(l^2)^+$ and $(l^2)^-$ are the positive and negative spectral subspaces of $L - \omega$ in l^2, respectively. It is easy to see that the Hilbert space

$$E = \{u \in l^2 : \ V^{1/2}u \in l^2\} = E^- \oplus E^+, \tag{2.1}$$

where $E^\pm := E \cap (l^2)^\pm$.

For any $u = u^+ + u^-, v = v^+ + v^- \in E = E^+ \oplus E^-$, we can define an inner product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$ on E by

$$(u,v) = ((L-\omega)u^+, v^+)_{l^2} - ((L-\omega)u^-, v^-)_{l^2} \quad \text{and} \quad \|u\| = (u,u)^{\frac{1}{2}}, \tag{2.2}$$

respectively. Hence, by (L_1), we have

$$-\|u^-\|^2 = ((L-\omega)u^-, u^-)_{l^2} \le a\|u^-\|^2_{l^2}, \quad \forall u^- \in E^- \tag{2.3}$$

and

$$\|u^+\|^2 = ((L-\omega)u^+, u^+)_{l^2} \ge b\|u^+\|^2_{l^2}, \quad \forall u^+ \in E^+, \tag{2.4}$$

where a and b are defined in (L_1). Thus, Φ defined in (1.4) can be rewritten as

$$\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \sum_{n=-\infty}^{+\infty} G_n(u_n), \quad u \in E.$$

Let $\Psi(u) := \sum_{n=-\infty}^{+\infty} G_n(u_n)$. Then $\Phi, \Psi \in C^1(E,\mathbb{R})$, and the derivative is given by

$$\langle \Psi'(u), v\rangle = \sum_{n=-\infty}^{+\infty} g_n(u_n)v_n, \quad \langle \Phi'(u), v\rangle = (u^+,v^+) - (u^-,v^-) - \langle \Psi'(u), v\rangle, \quad \forall u,v \in E,$$

which imply that (1.1) is the corresponding Euler–Lagrange equation for Φ. Therefore, we have reduced the problem of finding a non-trivial solution of (1.1) to that of seeking a nonzero critical point of the functional Φ on E.

Remark 2.1. The following facts show that the proofs of (L_2)–(L_3) will be similar to the proof of (L_1), so we will only give the detailed proof in the indefinite case (L_1).
(1) If (L_2) holds (i.e., the positive definite case), we will let $E^- := \{0\}$. Then $E = E^- \oplus E^+$, where $E^+ = E$. Thus, by (L_2), we have

$$(u,v) = ((L-\omega)u, v)_{l^2} \quad \text{and} \quad \|u\|^2 = (u,u) \ge (\gamma_1 - \omega)\|u\|^2_{l^2}.$$

Thus,

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \sum_{n=-\infty}^{+\infty} G_n(u_n), \quad u \in E.$$

(2) If (L_3) holds (i.e., ω is an eigenvalue of L), we let

$$W^- := \text{span}\{e_1, ..., e_{k_0'-1}\} \cap E, \ W^0 := \text{span}\{e_{k_0'}\} \cap E, \ E^+ := \overline{\text{span}\{e_{k_0'+1}, ..., \}} \cap E,$$

where $W^- = \{0\}$ if $k_0' = 1$ and $\{e_k\}$ are the associated normalized eigenfunctions with γ_k, that is, $Le_k = \gamma_k e_k$, $\|e_k\|_{l^2} = 1$. Then $E = W^- \oplus W^0 \oplus E^+ = E^- \oplus E^+$, where $E^- = W^- \oplus W^0$. Obviously, the quadratic part of Φ, $((L-\omega)u, v)_{l^2}$ is positive on E^+ and negative on W^- and zero on W^0. We introduce, respectively, on E the following new inner product and norm:

$$(u,v) := (u^0, v^0)_{l^2} + (|L|^{1/2}u, |L|^{1/2}v)_{l^2}, \quad \|u\| = (u,u)^{1/2},$$

where $u, v \in E = W^- \oplus W^0 \oplus E^+$ with $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+$. Clearly, the decomposition $E = W^- \oplus W^0 \oplus E^+$ is orthogonal with respect to both inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_{l^2}$. Thus, by (L_3), we have

$$-\|u^-\|^2 = ((L-\omega)u^-, u^-)_{l^2} \le (\gamma_{k_0'-1} - \omega)\|u^-\|_{l^2}^2, \quad \forall u^- \in E^-$$

and

$$\|u^+\|^2 = ((L-\omega)u^+, u^+)_{l^2} \ge (\gamma_{k_0'+1} - \omega)\|u^+\|_{l^2}^2, \quad \forall u^+ \in E^+.$$

Thus,

$$\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \sum_{n=-\infty}^{+\infty} G_n(u_n), \quad u \in E,$$

where $\|u^-\| = 0$ if $k_0' = 1$ in (L_3). For the cases (L_2) and (L_3), if $\|u^-\| = 0$, we will replace the condition $G_n(s) \ge \frac{1}{2}as^2$ in (G_2) by $G_n(s) \ge 0$.

The following abstract critical point theorem plays an important role in proving our main results (cf. [12]). Let E be a Hilbert space with norm $\|\cdot\|$ and have an orthogonal decomposition $E = N \oplus N^\perp$, where $N \subset E$ is a closed and separable subspace. There exists a norm $|v|_\omega$ satisfying $|v|_\omega \le \|v\|$ for all $v \in N$ which induces a topology equivalent to the weak topology of N on bounded subset of N. For $u = v + w \in E = N \oplus N^\perp$ with $v \in N,\ w \in N^\perp$, we define $|u|_\omega^2 = |v|_\omega^2 + \|w\|^2$. Particularly, if $(u_n = v_n + w_n)$ is $\|\cdot\|$-bounded and $u_n \overset{|\cdot|_\omega}{\to} u$, then $v_n \rightharpoonup v$ weakly in N, $w_n \to w$ strongly in $N^\perp$, $u_n \rightharpoonup v + w$ weakly in E.

Let $E = E^- \oplus E^+,\ z_0 \in E^+$ with $\|z_0\| = 1$. Let $N := E^- \oplus \mathbb{R}z_0$ and $E_0^+ := N^\perp = (E^- \oplus \mathbb{R}z_0)^\perp$. For $R > 0$, let

$$Q := \{u := u^- + sz_0 :\ s \in \mathbb{R}^+,\ u^- \in E^-,\ \|u\| < R\}$$

with $p_0 = s_0 z_0 \in Q,\ s_0 > 0$. We define

$$B := \{u := sz_0 + w^+ :\ s \in \mathbb{R},\ w^+ \in E_0^+,\ \|sz_0 + w^+\| = s_0\}.$$

For $\Phi \in C^1(E,\mathbb{R})$, define $\Gamma := \{h | h : [0,1] \times \bar{Q} \mapsto E$ is $|\cdot|_\omega$-continuous $, h(0,u) = u,\ \Phi(h(s,u)) \le \Phi(u),\ \forall u \in \bar{Q}$. For any $(s_0,u_0) \in [0,1] \times \bar{Q}$, there is a $|\cdot|_\omega$-neighborhood $U_{(s_0,u_0)}$ such that $\{u - h(t,u) : (t,u) \in U_{(s_0,u_0)} \cap ([0,1] \times \bar{Q})\} \subset E_{fin}.\}$, where E_{fin} denotes various finite-dimensional subspaces of E, $\Gamma \ne 0$ since $id \in \Gamma$.

The variant weak linking theorem is:

Theorem A. ([12]) *The family of C^1-functional $\{\Phi_\lambda\}$ has the form*

$$\Phi_\lambda(u) := \lambda I(u) - J(u), \quad \forall \lambda \in [1,\lambda_0].$$

where $\lambda_0 > 1$. Assume
(a) $I(u) \ge 0,\ \forall u \in E,\ \Phi_1 = \Phi$.
(b) $I(u) + |J(u)| \to +\infty$ as $\|u\| \to \infty$.
(c) Φ_λ is $|\cdot|_\omega$-upper semicontinuous, Φ_λ' is weakly sequentially continuous on E. Moreover, Φ_λ maps bounded sets to bounded sets.
(d) $\sup_{\partial Q} \Phi_\lambda < \inf_B \Phi_\lambda,\ \forall \lambda \in [1,\lambda_0]$.
Then for almost all $\lambda \in [1,\lambda_0]$, there exists a sequence $\{u_n\}$ such that

$$\sup_n \|u_n\| < \infty, \quad \Phi_\lambda'(u_n) \to 0, \quad \Phi_\lambda(u_n) \to c_\lambda,$$

where $c_\lambda := \inf_{h\in\Gamma} \sup_{u\in\bar{Q}} \Phi_\lambda(h(t,u)) \in [\inf_B \Phi_\lambda,\ \sup_{\bar{Q}} \Phi_\lambda]$.

In order to apply Theorem A, we shall prove a few lemmas. We pick λ_0 such that $1 < \lambda_0 < \min[2, b/\gamma]$. For $1 \leq \lambda \leq \lambda_0$, we consider

$$\Phi_\lambda(u) := \frac{\lambda}{2}\|u^+\|^2 - \left(\frac{1}{2}\|u^-\|^2 + \sum_{n=-\infty}^{+\infty} G_n(u_n)\right) := \lambda I(u) - J(u).$$

It is easy to see that Φ_λ satisfies conditions (a) in Theorem A. Conditions (b) and (d) will be proved later. To see (c), if $u^j \overset{|\cdot|_\omega}{\to} u$ and $\Phi_\lambda(u^j) \geq c$, then there is a renamed subsequence satisfying $(u^j)^+ \to u^+$ and $(u^j)^- \rightharpoonup u^-$ in E, $u_n^j \to u_n$ for all $n \in \mathbb{Z}$. It follows from the weak lower semicontinuity of the norm, Fatou's lemma and the fact $G_n(s) \geq -W_n$ for all $n \in \mathbb{Z}$ and $s \in \mathbb{R}$ in (G_4) that

$$\begin{aligned}
c &\leq \limsup_{j\to\infty} \Phi_\lambda(u^j) \\
&= \limsup_{j\to\infty} \left[\tfrac{\lambda}{2}\|(u^j)^+\|^2 - \left(\tfrac{1}{2}\|(u^j)^-\|^2 + \sum_{n=-\infty}^{+\infty}\left(G_n(u_n^j) + W_n\right)\right) + \sum_{n=-\infty}^{+\infty} W_n\right] \\
&\leq \tfrac{\lambda}{2}\|u^+\|^2 - \liminf_{j\to\infty}\left[\tfrac{1}{2}\|(u^j)^-\|^2 + \sum_{n=-\infty}^{+\infty}\left(G_n(u_n^j) + W_n\right)\right] + \sum_{n=-\infty}^{+\infty} W_n \\
&\leq \tfrac{\lambda}{2}\|u^+\|^2 - \left(\tfrac{1}{2}\|u^-\|^2 + \sum_{n=-\infty}^{+\infty} G_n(u_n)\right) = \Phi_\lambda(u).
\end{aligned}$$

Hence, $\Phi_\lambda(u) \geq c$, which means that Φ_λ is $|\cdot|_\omega$-upper semicontinuous. Next, we prove Φ'_λ is weakly sequentially continuous. If $u^j \rightharpoonup u$ in E, then there is renamed subsequence satisfying $(u^j)^+ \rightharpoonup u^+$, $(u^j)^- \rightharpoonup u^-$ in E and $u_n^j \to u_n$ for all $n \in \mathbb{Z}$. Thus,

$$((u^j)^+, \varphi) \to ((u)^+, \varphi), \quad ((u^j)^-, \varphi) \to ((u)^-, \varphi), \quad \forall \varphi \in E.$$

Note that

$$\langle \Phi'_\lambda(u^j), \varphi\rangle = \lambda((u^j)^+, \varphi) - ((u^j)^-, \varphi) - \sum_{n=-\infty}^{+\infty} g_n(u_n^j)\varphi_n, \quad \forall \varphi \in E.$$

Hence, we only need to prove

$$\sum_{n=-\infty}^{+\infty} \left(g_n(u_n^j) - g_n(u_n)\right)\varphi_n \to 0 \quad \text{as } j \to \infty.$$

If this is so, then

$$\langle \Phi'_\lambda(u^j), \varphi\rangle \to \lambda((u)^+, \varphi) - ((u)^-, \varphi) - \sum_{n=-\infty}^{+\infty} g_n(u_n)\varphi_n = \langle \Phi'_\lambda(u), \varphi\rangle \quad \text{as } j \to \infty,$$

that is, Φ'_λ is weakly sequentially continuous. Note that if $\varphi \in l^2 \subset l^p$ for $p \geq 2$, then for any $\varepsilon > 0$ there exists a positive constant $N_0 \in \mathbb{Z}$ such that

$$\left(\sum_{\{n\in\mathbb{Z}:\ |n|>N_0\}} \varphi_n^2\right)^{1/2} \leq \varepsilon, \quad \left(\sum_{\{n\in\mathbb{Z}:\ |n|>N_0\}} |\varphi_n|^p\right)^{1/p} \leq \varepsilon.$$

It follows from (G_1), (G_3), Hölder's inequality, $\|u^j\|_{l^p} \le \|u^j\|_{l^2} \le C'\|u^j\| \le C$, $\|u\|_{l^p} \le \|u\|_{l^2} \le C'\|u\| \le C$ (which is due to $E \subset l^2 \subset l^p$ and $u^j \rightharpoonup u$ in E), that

$$\begin{aligned}&\sum_{\{n\in\mathbb{Z}:\ |n|>N_0\}} \left|\left(g_n(u_n^j)-g_n(u_n)\right)\varphi_n\right|\\ &\le C\left[\sum_{\{n\in\mathbb{Z}:\ |n|>N_0\}}\left(|u_n^j|+|u_n^j|^{p-1}\right)|\varphi_n| + \sum_{\{n\in\mathbb{Z}:\ |n|>N_0\}}\left(|u_n|+|u_n|^{p-1}\right)|\varphi_n|\right]\\ &\le C\varepsilon,\end{aligned}$$

where C', C are generic constants. It follows from $u_n^j \to u_n$ for all $n \in \mathbb{Z}$ that

$$\begin{aligned}\left|\sum_{n=-\infty}^{+\infty}\left(g_n(u_n^j)-g_n(u_n)\right)\varphi_n\right| &\le \sum_{\{n\in\mathbb{Z}:\ |n|\le N_0\}}\left|\left(g_n(u_n^j)-g_n(u_n)\right)\varphi_n\right|\\ &\quad+\sum_{\{n\in\mathbb{Z}:\ |n|>N_0\}}\left|\left(g_n(u_n^j)-g_n(u_n)\right)\varphi_n\right|\\ &\to 0 \quad \text{as } j\to\infty,\end{aligned}$$

that is, $\sum_{n=-\infty}^{+\infty}\left(g_n(u_n^j)-g_n(u_n)\right)\varphi_n \to 0$ as $j \to \infty$. It remains to verify conditions (b) and (d). We do this by means of the following three lemmas:

Lemma 2.1. $J(u) + I(u) \to +\infty$ *as* $\|u\| \to \infty$.

Proof. By the definition of $\Phi(u)$ and (G_4), we have

$$\begin{aligned}J(u)+I(u) &= \tfrac{1}{2}\|u^+\|^2 + \tfrac{1}{2}\|u^-\|^2 + \sum_{n=-\infty}^{+\infty} G_n(u_n)\\ &\ge \tfrac{1}{2}\|u\|^2 - \sum_{n=-\infty}^{+\infty}|W_n| \to +\infty \quad \text{as } \|u\|\to+\infty,\end{aligned}$$

which is due to $\sum_{n=-\infty}^{+\infty}|W_n| < +\infty$. □

Therefore, Lemma 2.1 implies that condition (b) holds. We proceed to verify condition (d) by means of the following two lemmas:

Lemma 2.2. *There are two positive constants* $\epsilon, s_0 > 0$ *such that*

$$\Phi_\lambda(u) \ge \epsilon, \quad u \in E^+, \quad \|u\| = s_0, \quad \lambda \in [1, \lambda_0].$$

Proof. For $u \in E^+$, by (G_1), (G_3), (2.4) and the fact that $\|u\|_{l^p} \le \|u\|_{l^2} \le \|u\|$, we have

$$\begin{aligned}\Phi_\lambda(u) &\ge \tfrac{1}{2}\|u\|^2 - \sum_{n=-\infty}^{+\infty} G_n(u_n)\\ &= \tfrac{1}{2}\|u\|^2 - \sum_{\{n\in\mathbb{Z}:\ |u_n|<\delta\}} G_n(u_n) - \sum_{\{n\in\mathbb{Z}:\ |u_n|\ge\delta\}} G_n(u_n)\\ &\ge \tfrac{1}{2}\|u\|^2 - \tfrac{1}{2}\gamma\sum_{\{n\in\mathbb{Z}:\ |u_n|<\delta\}}|u_n|^2 - \underline{c\sum_{\{n\in\mathbb{Z}:\ |u_n|\ge\delta\}}\left(|u_n|^p+|u_n|\right)}\\ &\ge \tfrac{1}{2}\|u\|^2 - \tfrac{\gamma}{b}\tfrac{1}{2}\|u\|^2 - \underline{C\|u\|^p}\\ &= \tfrac{1}{2}\|u\|^2\left(1-\tfrac{\gamma}{b}-2C\|u\|^{p-2}\right), \quad 0<\gamma<b,\end{aligned}$$

where the inequality marked by underlining is due to the following facts. Note that $|u_n| \geq \delta$. if we pick a constant C_1 such that $C_1 \geq \frac{1}{\delta^{p-1}}$, then $|u_n| \leq C_1|u_n|^p$, which together with the Sobolev imbedding theorem implies

$$c \sum_{\{n\in\mathbb{Z}:\ |u_n|\geq\delta\}} (|u_n|^p + |u_n|) \leq c(1+C_1) \sum_{\{n\in\mathbb{Z}:\ |u_n|\geq\delta\}} |u_n|^p \leq C\|u\|^p.$$

This implies the conclusion if we take $\|u\|$ sufficiently small. □

It is not hard to check that Lemma 2.2 implies that $\inf_B \Phi_\lambda > 0$. We shall prove that $\sup_{\partial Q} \Phi_\lambda \leq 0$, that is, the following Lemma:

Lemma 2.3. *There is an $R > 0$ such that*

$$\Phi_\lambda(u) \leq 0, \quad u \in \partial Q_R, \quad \lambda \in [1, \lambda_0],$$

where $Q_R := \{u := v + sz_0 : \ s \geq 0,\ v \in E^-,\ z_0 \in E^+$ with $\|z_0\| = 1,\ \|u\| \leq R\}$.

Proof. If not, then there exist $R_j \to +\infty$, $\lambda_j \in [1, \lambda_0]$ and $u^j = v^j + s_j z_0 \in \partial Q_{R_j}$ ($v^j \in E^-$) such that $\Phi_{\lambda_j}(u^j) > 0$. If $s_j = 0$, then by (G_2) and (2.3), we have

$$\Phi_{\lambda_j}(u^j) = \Phi_{\lambda_j}(v^j) = -\frac{1}{2}\|v^j\|^2 - \sum_{n=-\infty}^{+\infty} G_n(v_n^j) \leq -\frac{1}{2}\|v^j\|^2 - \frac{1}{2}a\|v^j\|_{l^2}^2 \leq 0.$$

This produces a contradiction. Therefore, $s_j \neq 0$ and $\|u^j\|^2 = \|v^j\|^2 + s_j^2 = R_j^2$. Let $\widetilde{u}^j = \frac{u^j}{\|u^j\|} = \widetilde{s}_j z_0 + \widetilde{v}^j$, then

$$\|\widetilde{u}^j\|^2 = \|\widetilde{v}^j\|^2 + \widetilde{s}_j^2 = 1.$$

It follows from $\Phi_{\lambda_j}(u^j) > 0$ and the definition of Φ_{λ_j} that

$$\begin{aligned} 0 < \frac{\Phi_{\lambda_j}(u^j)}{\|u^j\|^2} &= \frac{1}{2}\left(\lambda_j \widetilde{s}_j^2 - \|\widetilde{v}^j\|^2\right) - \sum_{n=-\infty}^{+\infty} \frac{G_n(u_n^j)}{|u_n^j|^2}|\widetilde{u}_n^j|^2 \\ &= \frac{1}{2}\left[(\lambda_j + 1)\widetilde{s}_j^2 - 1\right] - \sum_{n=-\infty}^{+\infty} \frac{G_n(u_n^j)}{|u_n^j|^2}|\widetilde{u}_n^j|^2. \end{aligned} \tag{2.5}$$

Obviously, $\{\widetilde{s}_j\}$ and $\{\lambda_j\}$ are bounded. Thus, there are renamed subsequences such that $\widetilde{s}_j \to \widetilde{s}$ and $\lambda_j \to \lambda$ as $j \to \infty$, and there is a renamed subsequence such that $\widetilde{u}^j = \frac{u^j}{\|u^j\|} = \widetilde{s}_j z_0 + \widetilde{v}^j \rightharpoonup \widetilde{u} = \widetilde{s} z_0 + \widetilde{v}$ in E and $\widetilde{u}_n^j \to \widetilde{u}_n$ for all $n \in \mathbb{Z}$ as $j \to \infty$.

Case 1 If $\widetilde{u} \neq 0$. Without loss of generality, we let Ω_0 be the subset of $\mathbb{Z}$ where $\widetilde{u}_n \neq 0$. Then for all $n \in \Omega_0$, we have $|u_n^j| = |\widetilde{u}_n^j| \cdot \|u^j\| \to +\infty$ since $\|u^j\| = R_j \to +\infty$ as $j \to \infty$. It follows from (G_4) and the facts $\sum_{n=-\infty}^{+\infty} |W_n| < +\infty$ and $\|u^j\| = R_j \to +\infty$ as $j \to \infty$ that

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} \frac{G_n(u_n^j)}{|u_n^j|^2}|\widetilde{u}_n^j|^2 &= \sum_{n\in\Omega_0} \frac{G_n(u_n^j)}{|u_n^j|^2}|\widetilde{u}_n^j|^2 + \sum_{n\in\mathbb{Z}\setminus\Omega_0} \frac{G_n(u_n^j)}{|u_n^j|^2}|\widetilde{u}_n^j|^2 \\ &\geq \sum_{n\in\Omega_0} \frac{G_n(u_n^j)}{|u_n^j|^2}|\widetilde{u}_n^j|^2 - \frac{1}{\|u^j\|^2} \sum_{n\in\mathbb{Z}\setminus\Omega_0} |W_n| \to +\infty \quad \text{as } j \to \infty, \end{aligned}$$

which contradicts (2.5).

Case 2 If $\widetilde{u} = 0$. By (G_4) and the facts $\sum_{n=-\infty}^{+\infty} |W_n| < +\infty$ and $\|u^j\| = R_j \to +\infty$ as $j \to \infty$,

$$\sum_{n=-\infty}^{+\infty} \frac{G_n(u_n^j)}{|u_n^j|^2}|\widetilde{u}_n^j|^2 = \frac{1}{\|u^j\|^2} \sum_{n=-\infty}^{+\infty} G_n(u_n^j) \geq -\frac{1}{\|u^j\|^2} \sum_{n=-\infty}^{+\infty} |W_n| \to 0 \quad \text{as } j \to \infty.$$

Thus,

$$\liminf_{n\to\infty}\sum_{n=-\infty}^{+\infty}\frac{G_n(u_n^j)}{|u_n^j|^2}|\widetilde{u}_n^j|^2\geq 0. \tag{2.6}$$

Therefore, by (2.5) and (2.6), we get

$$(\lambda+1)\widetilde{s}^2-1\geq 0,$$

that is, $\widetilde{s}^2\geq\frac{1}{1+\lambda}\geq\frac{1}{1+\lambda_0}>0$. Thus, $\widetilde{u}=\widetilde{s}z_0+\widetilde{v}\neq 0$. We also get a contradiction.

Therefore, the proof is finished by Cases 1 and 2. □

Therefore, Lemmas 2.2 and 2.3 imply condition (d) of Theorem A holds. Applying Theorem A, we soon obtain the following fact:

Lemma 2.4. *For almost all $\lambda\in[1,\lambda_0]$, there exists a sequence $\{u^j\}$ such that*

$$\sup_j\|u^j\|<\infty,\quad \Phi_\lambda'(u^j)\to 0\quad\text{and}\quad \Phi_\lambda(u^j)\to c_\lambda\quad\text{as } j\to\infty,$$

where the definition of c_λ is given in Theorem A.

Lemma 2.5. ([19]) *If (V_1) holds, then the embedding map from E into l^p is compact for all $p\in[2,\infty]$.*

Lemma 2.6. *For almost all $\lambda\in[1,\lambda_0]$, there exists a $u_\lambda\in E$ such that*

$$\Phi_\lambda'(u_\lambda)=0,\quad \Phi_\lambda(u_\lambda)=c_\lambda.$$

Proof. Let $\{u^j\}$ be the sequence obtained in Lemma 2.4. Since $\{u^j\}$ is bounded, we can assume $u^j\rightharpoonup u_\lambda$ in E and $u_n^j\to u_{\lambda,n}$ for all $n\in\mathbb{Z}$. By Lemma 2.4 and the fact that Φ_λ' is weakly sequentially continuous, we have

$$\langle\Phi_\lambda'(u_\lambda),\varphi\rangle=\lim_{j\to\infty}\langle\Phi_\lambda'(u^j),\varphi\rangle=0,\quad\forall\varphi\in E.$$

That is, $\Phi_\lambda'(u_\lambda)=0$.

Note that (G_1) and (G_3) imply that there exists a constant C such that

$$|g_n(s)s|\leq C(|s|^2+|s|^p),\quad |G_n(s)|\leq C(|s|^2+|s|^p),\quad n\in\mathbb{Z},\ s\in\mathbb{R}.$$

It follows from $u^j\rightharpoonup u_\lambda$ in E, $u_n^j\to u_{\lambda,n}$ and $g_n(u_n^j)u_n^j\to g_n(u_{\lambda,n})u_{\lambda,n}$ for all $n\in\mathbb{Z}$, Lemma 2.5 and the Lebesgue's dominated convergence theorem that

$$\sum_{n=-\infty}^{+\infty}\frac{1}{2}g_n(u_n^j)u_n^j\to\sum_{n=-\infty}^{+\infty}\frac{1}{2}g_n(u_{\lambda,n})u_{\lambda,n} \tag{2.7}$$

and

$$\sum_{n=-\infty}^{+\infty}G_n(u_n^j)\to\sum_{n=-\infty}^{+\infty}G_n(u_{\lambda,n}) \tag{2.8}$$

as $j\to\infty$. By Lemma 2.4, we have

$$\Phi_\lambda(u^j)-\frac{1}{2}\langle\Phi_\lambda'(u^j),u^j\rangle=\sum_{n=-\infty}^{+\infty}\left(\frac{1}{2}g_n(u_n^j)u_n^j-G_n(u_n^j)\right)\to c_\lambda,\quad j\to\infty.$$

It follows from $\Phi_\lambda'(u_\lambda)=0$ and (2.7)–(2.8) that

$$\Phi_\lambda(u_\lambda)=\Phi_\lambda(u_\lambda)-\frac{1}{2}\langle\Phi_\lambda'(u_\lambda),u_\lambda\rangle=\sum_{n=-\infty}^{+\infty}\left(\frac{1}{2}g_n(u_{\lambda,n})u_{\lambda,n}-G_n(u_{\lambda,n})\right)=c_\lambda.$$

This completes the proof. □

Applying Lemma 2.6, we obtain the following fact:

Lemma 2.7. *For every $\lambda \in [1, \lambda_0]$, there are sequences $\{u^j\} \subset E$ and $\{\lambda_j\} \subset [1, \lambda_0]$ with $\lambda_j \to \lambda$ such that*

$$\Phi'_{\lambda_j}(u^j) = 0, \quad \Phi_{\lambda_j}(u^j) = c_{\lambda_j}.$$

Lemma 2.8. *Let $u = (u_n)_{n\in\mathbb{Z}} \in E$, $w = (w_n)_{n\in\mathbb{Z}} \in E^+$ and $0 \le r \le 1$, then*

$$\sum_{n=-\infty}^{+\infty} \left(G_n(u_n) - G_n(rw_n) + r^2 g_n(u_n)w_n - \frac{1+r^2}{2} g_n(u_n)u_n \right) \le C,$$

where the constant $C := \sum_{n=-\infty}^{+\infty} |W_n|$ does not depend on u, w and r.

Proof. This follows from (G_5) if we take $s = u_n$ and $l = rw_n - u_n$. □

Lemma 2.9. *The sequence $\{u^j\}$ given in Lemma 2.7 is bounded.*

Proof. Suppose by contradiction that

$$\|u^j\| \to \infty \quad \text{as } j \to \infty.$$

We write $u^j = (u^j)^+ + (u^j)^-$, $(u^j)^\pm \in E^\pm$. Let $v^j := \frac{u^j}{\|u^j\|}$, then $(v^j)^+ = \frac{(u^j)^+}{\|u^j\|}$, $(v^j)^- = \frac{(u^j)^-}{\|u^j\|}$ and $\|v^j\|^2 = \|(v^j)^+\|^2 + \|(v^j)^-\|^2 = 1$. Thus, we can assume that $(v^j)^\pm \rightharpoonup v^\pm$ in E and $(v^j)^\pm_n \to v^\pm_n$ for all $n \in \mathbb{Z}$, after passing to a subsequence.

Case 1 If $v^+ \ne 0$, then $v \ne 0$. Let Ω_1 be the subset of $\mathbb{Z}$ where $v_n \ne 0$, then we have $|u^j_n| = |v^j_n| \cdot \|u^j\| \to +\infty$ on Ω_1. It follows from (G_4) and $\sum_{n=-\infty}^{+\infty} |W_n| < +\infty$ that

$$\sum_{n=-\infty}^{+\infty} \frac{G_n(u^j_n)}{|u^j_n|^2} |v^j_n|^2 \ge \sum_{n\in\Omega_1} \frac{G_n(u^j_n)}{|u^j_n|^2} |v^j_n|^2 - \sum_{n\in\mathbb{Z}\setminus\Omega_1} \frac{|W_n|}{\|u^j\|^2} \to +\infty \quad \text{as } j \to \infty,$$

which together with the fact that the $\|(v^j)^\pm\|$ are bounded and Lemmas 2.2 and 2.7 hold implies that

$$0 \le \frac{c_{\lambda_j}}{\|u^j\|^2} = \frac{\Phi_{\lambda_j}(u^j)}{\|u^j\|^2} = \frac{\lambda_j}{2}\|(v^j)^+\|^2 - \frac{1}{2}\|(v^j)^-\|^2 - \sum_{n=-\infty}^{+\infty} \frac{G_n(u^j_n)}{|u^j_n|^2}|v^j_n|^2 \to -\infty \quad \text{as } j \to \infty.$$

This creates a contradiction.

Case 2 If $v^+ = 0$. We claim that there is a constant $C > 0$ independent of u^j and λ_j such that

$$\Phi_{\lambda_j}(r(u^j)^+) - \Phi_{\lambda_j}(u^j) \le C, \quad \forall r \in [0, 1]. \tag{2.9}$$

Note that Lemma 2.7 and the definition of Φ_{λ_j} imply that

$$\frac{1}{2}\langle \Phi'_{\lambda_j}(u^j), \varphi\rangle = \underline{\frac{1}{2}\lambda_j((u^j)^+, \varphi^+) - \frac{1}{2}((u^j)^-, \varphi^-) - \frac{1}{2}\sum_{n=-\infty}^{+\infty} g_n(u^j_n)\varphi_n} = 0, \quad \forall \varphi \in E.$$

It follows from the definition of Φ_{λ_j} that

$$\begin{aligned}
&\Phi_{\lambda_j}(r(u^j)^+) - \Phi_{\lambda_j}(u^j) \\
&= \tfrac{1}{2}\lambda_j(r^2-1)\|(u^j)^+\|^2 + \tfrac{1}{2}\|(u^j)^-\|^2 + \sum_{n=-\infty}^{+\infty}\left[G_n(u^j_n) - G_n(r(u^j)^+_n)\right] \\
&\quad \underline{+\tfrac{1}{2}\lambda_j((u^j)^+, \varphi^+) - \tfrac{1}{2}((u^j)^-, \varphi^-) - \tfrac{1}{2}\sum_{n=-\infty}^{+\infty} g_n(u^j_n)\varphi_n}.
\end{aligned} \tag{2.10}$$

Take

$$\varphi = (r^2+1)(u^j)^- - (r^2-1)(u^j)^+ = (r^2+1)u^j - 2r^2(u^j)^+,$$

which together with Lemma 2.8 and (2.10) implies that

$$\begin{aligned}
&\Phi_{\lambda_j}(r(u^j)^+) - \Phi_{\lambda_j}(u^j)\\
&= -\tfrac{r^2}{2}\|(u^j)^-\|^2 + \sum_{n=-\infty}^{+\infty}\Big[G_n(u_n^j) - G_n(r(u^j)_n^+) + r^2 g_n(u_n^j)(u^j)_n^+ - \tfrac{1+r^2}{2} g_n(u_n^j)u_n^j\Big]\\
&\le \sum_{n=-\infty}^{+\infty}\Big[G_n(u_n^j) - G_n(r(u^j)_n^+) + r^2 g_n(u_n^j)(u^j)_n^+ - \tfrac{1+r^2}{2} g_n(u_n^j)u_n^j\Big]\\
&\le C.
\end{aligned}$$

Thus, (2.9) holds.

Let $C_0 > 0$ be fixed constant. Then $\|u^j\| \to \infty$ as $j \to \infty$ implies that

$$r_j := \frac{C_0}{\|u^j\|} \to 0 \quad \text{as } j \to \infty.$$

Therefore, (2.9) implies that

$$\Phi_{\lambda_j}(r_j(u^j)^+) - \Phi_{\lambda_j}(u^j) \le C$$

for all sufficiently large j. It follows from $(v^j)^+ = \frac{(u^j)^+}{\|u^j\|}$ and Lemma 2.7 that for all sufficiently large j we have

$$\Phi_{\lambda_j}(C_0(v^j)^+) \le C' \tag{2.11}$$

for some constant C'. Note that Lemmas 2.2 and 2.7 and (G_4) imply that

$$\begin{aligned}
0 \le \frac{c_{\lambda_j}}{\|u^j\|^2} = \frac{\Phi_{\lambda_j}(u^j)}{\|u^j\|^2} &= \frac{\lambda_j}{2}\|(v^j)^+\|^2 - \frac{1}{2}\|(v^j)^-\|^2 - \frac{\sum_{n=-\infty}^{+\infty} G_n(u_n^j)}{\|u^j\|^2}\\
&\le \frac{\lambda_0}{2}\|(v^j)^+\|^2 - \frac{1}{2}\|(v^j)^-\|^2 + \frac{\sum_{n=-\infty}^{+\infty} |W_n|}{\|u^j\|^2}.
\end{aligned}$$

It follows from $\frac{\sum_{n=-\infty}^{+\infty} |W_n|}{\|u^j\|^2} \to 0$ as $j \to \infty$ (since $W \in l^1$ and $\|u^j\| \to \infty$) that

$$\frac{\lambda_0}{2}\|(v^j)^+\|^2 - \frac{1}{2}\|(v^j)^-\|^2 + \varepsilon \ge 0, \quad \forall \varepsilon > 0 \tag{2.12}$$

for all sufficiently large j. We take $\varepsilon = \frac{1}{4}$, by (2.12) and $\|v^j\|^2 = \|(v^j)^+\|^2 + \|(v^j)^-\|^2 = 1$, we have

$$\|(v^j)^+\|^2 \ge \frac{1}{2(1+\lambda_0)} \tag{2.13}$$

for all sufficiently large j. By (G_1) and (G_3), we have

$$\begin{aligned}
&\sum_{n=-\infty}^{+\infty} G_n(C_0(v^j)_n^+)\\
&\le \tfrac{1}{2}\gamma C_0^2 \sum_{\{n\in\mathbb{Z}:\ |C_0(v^j)_n^+|<\delta\}} |(v^j)_n^+|^2 + \tfrac{1}{2}c \underline{\sum_{\{n\in\mathbb{Z}:\ |C_0(v^j)_n^+|\ge\delta\}} \big(C_0|(v^j)_n^+| + C_0^p|(v^j)_n^+|^p\big)}\\
&\le \tfrac{1}{2}\gamma C_0^2 \sum_{\{n\in\mathbb{Z}:\ |C_0(v^j)_n^+|<\delta\}} |(v^j)_n^+|^2 + C_1 \underline{\sum_{\{n\in\mathbb{Z}:\ |C_0(v^j)_n^+|\ge\delta\}} |(v^j)_n^+|^p}\\
&\le \tfrac{1}{2}\gamma C_0^2\|(v^j)^+\|_{l^2}^2 + C_1\|(v^j)^+\|_{l^p}^p
\end{aligned} \tag{2.14}$$

for some constant C_1, where the inequality marked by underlining is similar to the expression in Lemma 2.2. For all sufficiently large j, (2.13) and (2.14) follow from Lemma 2.5 and the fact that $v^+ = 0$ and

$$\begin{aligned}\Phi_{\lambda_j}(C_0(v^j)^+) &= \tfrac{1}{2}\lambda_j C_0^2\|(v^j)^+\|^2 - \sum_{n=-\infty}^{+\infty} G_n(C_0(v^j)_n^+)\\ &\geq \tfrac{1}{2}\lambda_j C_0^2 \tfrac{1}{2(1+\lambda_0)} - \tfrac{1}{2}\gamma C_0^2\|(v^j)^+\|_{l^2}^2 - C_1\|(v^j)^+\|_{l^p}^p\\ &\to \tfrac{\lambda C_0^2}{4(1+\lambda_0)} \qquad \text{as } j\to\infty.\end{aligned}$$

This implies that $\Phi_{\lambda_j}(C_0(v^j)^+) \to +\infty$ as $C_0 \to +\infty$, contrary to (2.11).

Therefore, the sequence $\{u^j\}$ is bounded. The proof is finished. □

3. Proofs of main results

Proof of Theorem 1.1. From Lemma 2.7, there are sequences $1 \leq \lambda_j \to 1$ and $\{u^j\} \subset E$ such that $\Phi'_{\lambda_j}(u^j) = 0$ and $\Phi_{\lambda_j}(u^j) = c_{\lambda_j}$. Lemma 2.9 implies $\{u^j\}$ is bounded; thus, we can assume $u^j \rightharpoonup u$ in E, $u_n^j \to u_n$ for all $n \in \mathbb{Z}$. By the fact Φ'_{λ_j} is weakly sequentially continuous on E (it is similar to the fact Φ'_λ is weakly sequentially continuous on E, which is below Theorem A) and $\Phi'_{\lambda_j}(u^j) = 0$, we have

$$0 = \lim_{j\to\infty}\langle \Phi'_{\lambda_j}(u^j), \varphi\rangle = \langle \Phi'(u), \varphi\rangle, \quad \forall \varphi \in E.$$

Therefore, $\Phi'(u) = 0$.

The facts $\Phi'_{\lambda_j}(u^j) = 0$ and $\Phi_{\lambda_j}(u^j) = c_{\lambda_j}$ imply that

$$\Phi_{\lambda_j}(u^j) - \frac{1}{2}\langle \Phi'_{\lambda_j}(u^j), u^j\rangle = \sum_{n=-\infty}^{+\infty}\left(\frac{1}{2}g_n(u_n^j)u_n^j - G_n(u_n^j)\right) = c_{\lambda_j} \geq c_1. \tag{3.1}$$

Similar to (2.7) and (2.8), we know

$$\sum_{n=-\infty}^{+\infty}\left(\frac{1}{2}g_n(u_n^j)u_n^j - G_n(u_n^j)\right) \to \sum_{n=-\infty}^{+\infty}\left(\frac{1}{2}g_n(u_n)u_n - G_n(u_n)\right) \quad \text{as } j\to\infty.$$

It follows from (3.1), Lemma 2.2 and $\Phi'(u) = 0$ that

$$\begin{aligned}\Phi(u) = \Phi(u) - \tfrac{1}{2}\langle \Phi'(u), u\rangle &= \sum_{n=-\infty}^{+\infty}\left(\tfrac{1}{2}g_n(u_n)u_n - G_n(u_n)\right)\\ &= \lim_{j\to\infty}\sum_{n=-\infty}^{+\infty}\left(\tfrac{1}{2}g_n(u_n^j)u_n^j - G_n(u_n^j)\right) \geq c_1 \geq \epsilon > 0.\end{aligned}$$

It implies that $u \neq 0$. Therefore, (1.1) has a non-trivial solution u.

By using the argument borrowed from [20], we will show that u satisfies (1.6). In fact, let

$$z_n = -\sigma\frac{g_n(u_n)}{u_n} \text{ if } u_n \neq 0 \quad \text{and} \quad z_n = 0 \text{ if } u_n = 0, \quad n \in \mathbb{Z},$$

then

$$\tilde{L}u_n = \omega u_n, \tag{3.2}$$

where $\tilde{L}u_n = Lu_n + z_n u_n$. Note that

$$g_n(s) = o(s) \quad \text{as } s \to 0 \ \text{ for all } n \in \mathbb{Z}, \quad s \in \mathbb{R}$$

and $\lim_{|n|\to\infty} u_n = 0$ imply that $\lim_{|n|\to\infty} z_n = 0$. Thus, the multiplication by z_n is a compact operator in l^2, which implies that $\sigma_e(\tilde{L}) = \sigma_e(L)$, where σ_e stands for the essential spectrum. Equation (3.2) means that $u = \{u_n\}$ is an eigenfunction that corresponds to the eigenvalue of finite multiplicity $\omega \notin \sigma_e(\tilde{L})$ of the operator $\tilde{L}$. Equation (1.6) follows from the standard theorem on exponential decay for such eigenfunctions [16]. □

Proof of Theorem 1.2. By Theorem 1.1, $\mathcal{M} \neq \emptyset$, where $\mathcal{M}$ is the collection of solutions of (1.1). Let

$$\alpha := \inf_{u\in\mathcal{M}} \Phi(u).$$

If u is a solution of (1.1), then by Lemma 2.8 (take $r = 0$),

$$\Phi(u) = \Phi(u) - \frac{1}{2}\langle \Phi'(u), u\rangle = \sum_{n=-\infty}^{+\infty} \left(\frac{1}{2} g_n(u_n)_n - G_n(u_n)\right) \geq -C = -\sum_{n=-\infty}^{+\infty} |W_n|.$$

Thus, $\alpha > -\infty$. Let $\{u^j\}$ be a sequence in $\mathcal{M}$ such that

$$\Phi(u^j) \to \alpha. \tag{3.3}$$

Similar to the proof of Lemma 2.9, we conclude that the sequence $\{u^j\}$ is bounded in E. Thus, $u^j \rightharpoonup u$ in E and $u_n^j \to u_n$ for all $n \in \mathbb{Z}$, after passing to a subsequence. Therefore, by the facts that Φ' is weakly sequentially continuous on E and $\Phi'(u^j) = 0$, we have

$$\langle \Phi'(u), \varphi\rangle = \lim_{j\to\infty} \langle \Phi'(u^j), \varphi\rangle = 0, \quad \forall \varphi \in E.$$

That is, $\Phi'(u) = 0$.

Note that $\Phi'(u^j) = 0$, similar to (2.7) and (2.8), we have

$$\begin{aligned} \Phi(u^j) &= \Phi(u^j) - \tfrac{1}{2}\langle \Phi'(u^j), u^j\rangle \\ &= \sum_{n=-\infty}^{+\infty} \left(\tfrac{1}{2} g_n(u_n^j)u_n^j - G_n(u_n^j)\right) \to \sum_{n=-\infty}^{+\infty} \left(\tfrac{1}{2} g_n(u_n)u_n - G_n(u_n)\right) \quad \text{as } j \to \infty. \end{aligned}$$

It follows from $\Phi'(u) = 0$ and (3.3) that

$$\begin{aligned} \Phi(u) = \Phi(u) - \tfrac{1}{2}\langle \Phi'(u), u\rangle &= \sum_{n=-\infty}^{+\infty} \left[\tfrac{1}{2} g_n(u_n)u_n - G_n(u_n)\right] \\ &= \lim_{j\to\infty} \Phi(u^j) = \alpha. \end{aligned}$$

Now, we suppose that

$$g_n(s) = o(s) \quad \text{as } s \to 0 \;\; \text{for all } n \in \mathbb{Z}, \quad s \in \mathbb{R}.$$

It follows form (G_1) that for any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$|g_n(s)| \leq \varepsilon|s| + C_\varepsilon|s|^{p-1} \;\; \text{for all } n \in \mathbb{Z} \text{ and } s \in \mathbb{R}. \tag{3.4}$$

Let

$$\beta := \inf_{u\in M'} \Phi(u),$$

where $M' := \mathcal{M}\backslash\{0\}$. Let $\{u^j\}$ be a sequence in $\mathcal{M}\backslash\{0\}$ such that

$$\Phi(u^j) \to \beta. \tag{3.5}$$

Similar to the proof of Lemma 2.9, we conclude that the sequence $\{u^j\}$ is bounded in E. Thus, $u^j \rightharpoonup u$ in E, after passing to a subsequence. Note that

$$0 = \langle \Phi'(u^j), (u^j)^+ \rangle = \|(u^j)^+\|^2 - \sum_{n=-\infty}^{+\infty} g_n(u_n^j)(u^j)_n^+,$$

which together with (3.4), the Hölder's inequality and the fact $\|u\|_{l^p} \le \|u\|_{l^2} \le C\|u\|$ for $u \in E$ implies

$$\begin{aligned} \|(u^j)^+\|^2 &= \sum_{n=-\infty}^{+\infty} g_n(u_n^j)(u^j)_n^+ \\ &\le \varepsilon \sum_{n=-\infty}^{+\infty} |u_n^j| \cdot |(u^j)_n^+| + C_\varepsilon \sum_{n=-\infty}^{+\infty} |u_n^j|^{p-1}|(u^j)_n^+| \\ &\le \varepsilon C\|u^j\| \cdot \|(u^j)^+\| + C'_\varepsilon \|u^j\|_{l^p}^{p-1}\|(u^j)^+\| \\ &\le \varepsilon C\|u^j\| \cdot \|(u^j)^+\| + C''_\varepsilon \|u^j\|_{l^p}^{p-2}\|u^j\| \cdot \|(u^j)^+\| \\ &\le \varepsilon C\|u^j\|^2 + C''_\varepsilon \|u^j\|_{l^p}^{p-2}\|u^j\|^2. \end{aligned} \tag{3.6}$$

Similarly, we have

$$\|(u^j)^-\|^2 \le \varepsilon C\|u^j\|^2 + C''_\varepsilon \|u^j\|_{l^p}^{p-2}\|u^j\|^2. \tag{3.7}$$

From (3.6) and (3.7), we get

$$\|u^j\|^2 = \|(u^j)^+\|^2 + \|(u^j)^-\|^2 \le 2\varepsilon C\|u^j\|^2 + 2C''_\varepsilon \|u^j\|_{l^p}^{p-2}\|u^j\|^2,$$

which means $\|u^j\|_{l^p} \ge C$ for some constant $C > 0$. Since Lemma 2.5 implies $u^j \to u$ in l^p, we know $u \ne 0$. As before, we can easily get $\Phi(u^j) \to \Phi(u) = \beta$ as $j \to \infty$. Therefore, (1.1) has a ground state solution. □

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