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Invariance of decay rate with respect to boundary conditions in thermoelastic Timoshenko systems

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Abstract. This paper is mainly concerned with the polynomial stability of a thermoelastic Timoshenko system recently introduced by Almeida Júnior et al. (Z Angew Math Phys 65(6):1233–1249, 2014) that proved, in the general case when equal wave speeds are not assumed, different polynomial decay rates depending on the boundary conditions, namely, optimal rate $t^{-1/2}$ for mixed Dirichlet–Neumann boundary condition and rate $t^{-1/4}$ for full Dirichlet boundary condition. Here, our main achievement is to prove the same polynomial decay rate $t^{-1/2}$ (corresponding to the optimal one) independently of the boundary conditions, which improves the existing literature on the subject. As a complementary result, we also prove that the system is exponentially stable under equal wave speeds assumption. The technique employed here can probably be applied to other kind of thermoelastic systems.

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1. Introduction

Thermoelastic model and main goal. This paper addresses results on stability to the following thermoelastic Timoshenko system

$$\rho_1\varphi_{tt} - k(\varphi_x+\psi)_x + m\,\theta_x = 0 \quad \text{in} \quad (0,l)\times\mathbb{R}^+, \tag{1.1}$$

$$\rho_2\psi_{tt} - b\,\psi_{xx} + k(\varphi_x+\psi) - m\,\theta = 0 \quad \text{in} \quad (0,l)\times\mathbb{R}^+, \tag{1.2}$$

$$\rho_3\theta_t - c\,\theta_{xx} + m(\varphi_{xt}+\psi_t) = 0 \quad \text{in} \quad (0,l)\times\mathbb{R}^+, \tag{1.3}$$

with initial conditions

$$\varphi(\cdot,0)=\varphi_0(\cdot),\quad \varphi_t(\cdot,0)=\varphi_1(\cdot),\quad \psi(\cdot,0)=\psi_0(\cdot),\quad \psi_t(\cdot,0)=\psi_1(\cdot),\quad \theta(\cdot,0)=\theta_0(\cdot), \tag{1.4}$$

and either the full Dirichlet or mixed Dirichlet–Neumann boundary conditions

$$\begin{cases} (a) & \varphi(0,t)=\varphi(l,t)=\psi(0,t)=\psi(l,t)=\theta(0,t)=\theta(l,t)=0,\\ (b) & \varphi_x(0,t)=\varphi_x(l,t)=\psi(0,t)=\psi(l,t)=\theta(0,t)=\theta(l,t)=0,\\ (c) & \varphi(0,t)=\varphi(l,t)=\psi_x(0,t)=\psi_x(l,t)=\theta(0,t)=\theta(l,t)=0,\\ (d) & \varphi(0,t)=\varphi(l,t)=\psi(0,t)=\psi(l,t)=\theta_x(0,t)=\theta_x(l,t)=0,\\ (e) & \varphi(0,t)=\varphi(l,t)=\psi_x(0,t)=\psi_x(l,t)=\theta_x(0,t)=\theta_x(l,t)=0, \end{cases} \tag{1.5}$$

where the functions φ,ψ and θ stand for the transversal displacement, the rotation angle and the difference of temperature of a beam with length $l>0$, respectively, and $\rho_1,\rho_2,\rho_3,k,b,c,\,m>0$. The model (1.1)–(1.3) was recently proposed by Almeida Júnior et al. [1] by considering the classical Timoshenko model (see Timoshenko [27,28])

$$\rho_1\varphi_{tt} = S_x \quad \text{and} \quad \rho_2\psi_{tt} = M_x - S, \tag{1.6}$$

with constitutive thermal law acting on the shear stress

$$S = k(\varphi_x + \psi) - m\,\theta \qquad \text{and} \qquad M = b\,\psi_x. \tag{1.7}$$

More precisely, in [1] the authors only studied the system (1.1)–(1.4) with boundary conditions $(1.5)_a$ or $(1.5)_e$. They proved that the stability of the system depends of the boundary conditions and the difference of wave speeds

$$\chi := \frac{\rho_1}{k} - \frac{\rho_2}{b}. \tag{1.8}$$

Indeed, following [1, Theorems 3.2 and 4.4] one can see that problem (1.1)–$(1.5)_e$ is exponentially stable if and only if $\chi = 0$. In the case $\chi \neq 0$, it is reached in [1, Theorem 5.1] that the system decays polynomially with decay rate depending on the boundary conditions, namely, with optimal rate $t^{-1/2}$ for $(1.5)_e$ and rate $t^{-1/4}$ for $(1.5)_a$. It is worth mentioning that different polynomial decay rates depending on the boundary conditions also appear in other models such as thermoelastic Bresse systems, see e.g., [9,14]. The key point in these three later works in achieving different decay rates for different boundary conditions appears when it is necessary to estimate boundary point-wise terms with respect to the thermal component. However, as remarked in [14] (see Remark 4.1 therein), there is no physical explanation why such polynomial rates are different depending on their respective boundary conditions.

Therefore, motivated by the above works and the "problem" of different polynomial decay rates, the main goal in this paper is to show that, in general, the system (1.1)–(1.5) decays polynomially with the same rate independent of the boundary conditions. Concerning to thermoelastic Timoshenko systems, this seems to be the first work which unifies the same polynomial decay rate no matter what boundary condition is taken into account. In addition, the exponential stability (also independent of the boundary conditions) is achieved by requiring the equal wave speeds. The main contribution of this work is twofold:

1. In general, when equal wave speeds are not assumed, it is proved that the system (1.1)–(1.5) is polynomially stable with the same decay rate (corresponding to the optimal case) independent of boundary conditions, which improves the result on polynomial decay provided by [1]. This is presented in Theorem 4.1.
2. Under equal wave speeds it is proved that the system (1.1)–(1.5) is exponentially stable, which complements the result on exponential stability given by authors in [1]. This is stated in Theorem 4.3.

A short literature overview. In what follows we consider a short literature on Timoshenko systems and main results on the subject. In the isothermal case, say taking $m = 0$ in (1.7), then from relations (1.6)–(1.7) we obtain the conservative Timoshenko system

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \tag{1.9}$$

$$\rho_2\psi_{tt} - b\,\psi_{xx} + k(\varphi_x + \psi) = 0. \tag{1.10}$$

In Soufyane [24] it is shown that (1.9)–(1.10) with a damping $\beta\psi_t$, $\beta > 0$, added in the equation (1.10) is exponentially stable if and only if $\chi = 0$ in (1.8). Ever since, the assumption $\chi = 0$ has been widely used in the stabilization of elastic, viscoelastic and thermoelastic Timoshenko systems. See e.g. [2–4,7,12,13,16,17,19,25,26] and references therein. In the non-isothermal case, following [11,18], we can also consider the thermal effect acting only on the bending moment

$$S = k(\varphi_x + \psi) \qquad \text{and} \qquad M = b\,\psi_x - m\,\vartheta, \tag{1.11}$$

instead of (1.7). Then, from (1.6) and (1.11) we obtain the following thermoelastic Timoshenko system

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \tag{1.12}$$

$$\rho_2\psi_{tt} - b\,\psi_{xx} + k(\varphi_x + \psi) + \sigma\,\vartheta_x = 0, \tag{1.13}$$

$$\rho_3\vartheta_t - c\,\vartheta_{xx} + \sigma\psi_{xt} = 0, \tag{1.14}$$

where (1.14) is now the coupling heat equation with temperature ϑ and coefficient $\sigma > 0$. The system (1.12)–(1.14) with mixed Dirichlet–Neumann boundary conditions (see condition $(1.5)_{b,c,d}$) was first studied by Muñoz Rivera and Racke [18] that proved the system is exponentially stable when $\chi = 0$ is regarded. Moreover, the authors proved that the system (1.12)–(1.14) with boundary condition $(1.5)_c$ is exponentially stable if and only if $\chi = 0$. The same result is also obtained by Fernández Sare and Racke [11] with $\varphi_x = \psi = \theta_x = 0$ on $x = 0, l$. As observed at the end of this paper, see Remark 4.5, our main results on stability (Theorems 4.1 and 4.3) can be extended to problem (1.12)–(1.14) under a lot of modifications on the local estimates provided in Sect. 3. For systems which take into account other thermal laws acting on the bending moment we refer to [8,10,11,22,23] and references therein. We finally note that assumption $\chi = 0$ is only given from mathematical viewpoint since it is not satisfied from elasticity theory, see e.g., [14,20]. Therefore, our main result concerns the case where $\chi \neq 0$.

The remaining paper is organized as follows. In Sect. 2 we consider the well-posedness result with respect to problem (1.1)–(1.5). In Sect. 3 we provide some a priori estimates which consist the main tool used in the proof of the results on stability. Finally, in Sect. 4 we state and prove our main results concerning polynomial and exponential decay rates.

2. Well-posedness

Let us start this section by defining the phase space of solutions

$$\mathcal{H} = \begin{cases} H_0^1(0,l) \times L^2(0,l) \times H_0^1(0,l) \times L^2(0,l) \times L^2(0,l) & \text{for} \quad (1.5)_a, \\ H_*^1(0,l) \times L_*^2(0,l) \times H_0^1(0,l) \times L^2(0,l) \times L^2(0,l) & \text{for} \quad (1.5)_b, \\ H_0^1(0,l) \times L^2(0,l) \times H_*^1(0,l) \times L_*^2(0,l) \times L^2(0,l) & \text{for} \quad (1.5)_c, \\ H_0^1(0,l) \times L^2(0,l) \times H_0^1(0,l) \times L^2(0,l) \times L_*^2(0,l) & \text{for} \quad (1.5)_d, \\ H_0^1(0,l) \times L^2(0,l) \times H_*^1(0,l) \times L_*^2(0,l) \times L_*^2(0,l) & \text{for} \quad (1.5)_e, \end{cases}$$

where $H_*^1(0,l) = H^1(0,l) \cap L_*^2(0,l)$ and $L_*^2(0,l) = \left\{ u \in L^2(0,l);\ \frac{1}{l}\int_0^l u(x)\,\mathrm{d}x = 0 \right\}$. It is well-known that $\mathcal{H}$ is a Hilbert space with respect to the norm

$$\|U\|_{\mathcal{H}}^2 = \int_0^l \left[\rho_1|\Phi|^2 + \rho_2|\Psi|^2 + b|\psi_x|^2 + k|\varphi_x + \psi|^2 + \rho_3|\theta|^2 \right] \mathrm{d}x,$$

for $U = (\varphi, \Phi, \psi, \Psi, \theta)^T \in \mathcal{H}$, associated with the inner-product $(\cdot,\cdot)_{\mathcal{H}}$ induced by system on $\mathcal{H}$. Under above notations the system (1.1)–(1.5) can be rewritten as

$$\begin{cases} U_t = \mathcal{A}U, \quad t > 0, \\ U(0) := U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0)^T, \end{cases} \tag{2.1}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$\mathcal{A}U = \begin{pmatrix} \Phi \\ \dfrac{k}{\rho_1}(\varphi_x + \psi)_x - \dfrac{m}{\rho_1}\theta_x \\ \Psi \\ \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{k}{\rho_2}(\varphi_x + \psi) + \dfrac{m}{\rho_2}\theta \\ \dfrac{c}{\rho_3}\theta_{xx} - \dfrac{m}{\rho_3}(\Phi_x + \Psi) \end{pmatrix}, \tag{2.2}$$

for any $U=(\varphi,\Phi,\psi,\Psi,\theta)^T\in D(\mathcal{A})$, with domain

$$D(\mathcal{A})=\big\{U\in\mathcal{H}\mid \varphi,\psi,\theta\in H^2(0,l)\text{ and }(D)\text{ is satified}\big\},$$

where

$$(D)\ \begin{cases}\Phi,\Psi,\theta\in H_0^1(0,l) & \text{for}\quad (1.5)_a,\\ \Psi,\varphi_x,\theta\in H_0^1(0,l),\ \ \Phi\in H_*^1(0,l) & \text{for}\quad (1.5)_b,\\ \Phi,\psi_x,\theta\in H_0^1(0,l),\ \ \Psi\in H_*^1(0,l) & \text{for}\quad (1.5)_c,\\ \Phi,\Psi,\theta_x\in H_0^1(0,l) & \text{for}\quad (1.5)_d,\\ \Phi,\psi_x,\theta_x\in H_0^1(0,l),\Psi\in H_*^1(0,l) & \text{for}\quad (1.5)_e.\end{cases}\tag{2.3}$$

The result on existence and uniqueness of solution to the abstract Cauchy problem (2.1), and therefore to the equivalent system (1.1)–(1.5), is stated as follows.

Theorem 2.1. *Under above notations we have:*

(i) *If $U_0\in\mathcal{H}$, then problem (2.1) has a unique mild solution $U\in C^0([0,\infty),\mathcal{H})$.*

(ii) *If $U_0\in D(\mathcal{A})$, then the above weak solution is strong one with*

$$U\in C^0\left([0,\infty),D(\mathcal{A})\right)\cap C^1\left([0,\infty),\mathcal{H}\right).$$

(iii) *If $U_0\in D(\mathcal{A}^n)$, $n\geq 2$ integer, then the above strong solution is more regular*

$$U\in\bigcap_{j=0}^{n}C^{n-j}\left([0,+\infty),D(\mathcal{A}^j)\right).$$

Proof. The proof of existence is analogous to the one given by authors in [1]. Indeed, it is not so difficult to check that $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions $T(t)=e^{\mathcal{A}t}$ on $\mathcal{H}$, namely, as shown in [1, Theorem 2.1] we have that $I_d-\mathcal{A}:D(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$ is onto and $\mathcal{A}$ is dissipative on $\mathcal{H}$ with

$$\mathrm{Re}\ (\mathcal{A}U,U)_{\mathcal{H}}=-c\int_0^l|\theta_x(x)|^2\,\mathrm{d}x\leq 0,\quad U\in D(\mathcal{A}),\tag{2.4}$$

for all boundary conditions taken in (1.5). Hence, the proof follows from the standard theory of linear semigroups, see Pazy [21].

3. A priori estimates

In this section we consider some results which provide local estimates independently of the boundary conditions. Our starting point is to consider the resolvent equation

$$i\beta U-\mathcal{A}U=F,\tag{3.1}$$

with $U=(\varphi,\Phi,\psi,\Psi,\theta)^T$, $F=(f_1,f_2,f_3,f_4,f_5)^T$ and $\mathcal{A}$ defined in (2.2), which can be rewritten in terms of its components

$$i\beta\varphi-\Phi=f_1,\tag{3.2}$$

$$i\beta\rho_1\Phi-k(\varphi_x+\psi)_x+m\,\theta_x=\rho_1f_2,\tag{3.3}$$

$$i\beta\psi-\Psi=f_3,\tag{3.4}$$

$$i\beta\rho_2\Psi-b\,\psi_{xx}+k(\varphi_x+\psi)-m\,\theta=\rho_2f_4,\tag{3.5}$$

$$i\beta\rho_3\theta-c\,\theta_{xx}+m(\Phi_x+\Psi)=\rho_3f_5.\tag{3.6}$$

Hereafter, we shall denote by $C>0$ different constants and by $\|\cdot\|_{L^p}$ the norm in $L^p(0,l)$. Besides, the well-known Hölder and Poincaré inequalities, and $|\beta|>1$ large enough, shall be used in most estimates without being mentioned.

Lemma 3.1. *Under above notations there exists a constant $C > 0$ such that*

$$\|\theta_x\|_{L^2}^2 \le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{3.7}$$

Proof. From (2.4) and (3.1) we obtain

$$c\int_0^l |\theta_x|^2 \,\mathrm{d}x = \mathrm{Re}\,(U,F)_{\mathcal{H}},$$

from which (3.7) follows.

Lemma 3.2. *Under above notations we have $i\mathbb{R} \subseteq \rho(\mathcal{A})$, where $\rho(\mathcal{A})$ is resolvent set of $\mathcal{A}$.*

Proof. Let us suppose that there exists $i\beta \in \sigma(\mathcal{A})$, $\beta \neq 0$, with corresponding eigenvector $U = (\varphi,\Phi,\psi,\Psi,\theta)^T \neq 0$. From (3.7) with $F = 0$, we obtain $\theta \equiv 0$. Returning to equations (3.2)–(3.6) with $F = 0$, we get $\Phi,\varphi \equiv 0$ and then $\Psi,\psi \equiv 0$. This implies that $U \equiv 0$ which is a contradiction. Thus, there are no purely imaginary eigenvalues in the spectrum $\sigma(\mathcal{A}) = \mathbb{C}\backslash\rho(\mathcal{A})$, which in turn is made by eigenvalues only. Therefore, $i\mathbb{R} \subseteq \rho(\mathcal{A})$.

In what follows we shall obtain local estimates by using auxiliary cut-off functions. The main aim is to avoid different estimates provided by boundary point-wise terms. In this way, let us first consider $l_0 \in (0,l)$ and $\delta > 0$ arbitrary numbers such that $(l_0-\delta, l_0+\delta) \subset (0,l)$, and a function $s \in C^2(0,l)$ satisfying

$$\mathrm{supp}\ s \subset (l_0-\delta, l_0+\delta), \quad 0 \le s(x) \le 1, \quad x \in (0,l), \tag{3.8}$$

and

$$s(x) = 1 \quad \text{for} \quad x \in [l_0-\delta/2, l_0+\delta/2]\,. \tag{3.9}$$

Proposition 3.3. *Under above notations there exists a constant $C > 0$ such that*

$$\begin{aligned}\int_{l_0-\delta/2}^{l_0+\delta/2} \left(|\varphi_x+\psi|^2 + |\Phi|^2\right) \mathrm{d}x \le{}& \frac{C}{|\beta|^{3/2}}\left(\|\theta_x\|_{L^2}^{1/2}\|U\|_{\mathcal{H}}^{1/2} + \|U\|_{\mathcal{H}}^{1/2}\|F\|_{\mathcal{H}}^{1/2}\right)\|U\|_{\mathcal{H}} \\ &+ \frac{C}{|\beta|}\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}} + \frac{C}{|\beta|^{4/3}}\|\theta_x\|_{L^2}^{2/3}\|U\|_{\mathcal{H}}^{4/3} \\ &+ C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|_{\mathcal{H}}^2.\end{aligned} \tag{3.10}$$

In particular, given $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that

$$\int_{l_0-\delta/2}^{l_0+\delta/2} \left(|\varphi_x+\psi|^2 + |\Phi|^2\right) \mathrm{d}x \le \epsilon\|U\|_{\mathcal{H}}^2 + C_\epsilon\|F\|_{\mathcal{H}}^2. \tag{3.11}$$

Proof. On the one hand, from expressions (3.2), (3.4) and (3.6), we obtain

$$i\beta\rho_3\theta - c\,\theta_{xx} + i\beta m(\varphi_x+\psi) = \rho_3 f_5 + m(f_{1,x}+f_3). \tag{3.12}$$

Taking the multiplier $sk\overline{[\varphi_x+\psi]}$ in (3.12) and performing integration by parts, we have

$$\begin{aligned} i\beta km\int_0^l s|\varphi_x+\psi|^2\,\mathrm{d}x ={}& \underbrace{-c\int_0^l s\,\theta_x\overline{[k(\varphi_x+\psi)_x]}\,\mathrm{d}x}_{:=I_1} + \underbrace{k\rho_3\int_0^l s\theta\overline{[i\beta(\varphi_x+\psi)]}\,\mathrm{d}x}_{:=I_2} \\ &- kc\int_0^l s'\,\theta_x\overline{[\varphi_x+\psi]}\,\mathrm{d}x + k\int_0^l s\left[\rho_3 f_5 + m(f_{1,x}+f_3)\right]\overline{[\varphi_x+\psi]}\,\mathrm{d}x. \end{aligned} \tag{3.13}$$

Using (3.3) one has

$$I_1 = i\beta c\rho_1 \int_0^l s\,\theta_x \overline{\Phi}\,\mathrm{d}x - cm \int_0^l s\,|\theta_x|^2\,\mathrm{d}x + c\rho_1 \int_0^l s\theta_x \overline{f_2}\,\mathrm{d}x.$$

In addition, applying (3.2), (3.4), and integration by parts, then

$$I_2 = -k\rho_3 \int_0^l [s\,\theta]_x \overline{\Phi}\,\mathrm{d}x + k\rho_3 \int_0^l s\,\theta\overline{\Psi}\,\mathrm{d}x + k\rho_3 \int_0^l s\,\theta\overline{[f_{1,x} + f_3]}\,\mathrm{d}x.$$

Replacing these two last identities in (3.13) we deduce

$$i\beta km \int_0^l s|\varphi_x + \psi|^2\,\mathrm{d}x \;=\; i\beta c\rho_1 \int_0^l s\theta_x \overline{\Phi}\,\mathrm{d}x + I_3, \tag{3.14}$$

where

$$\begin{aligned} I_3 = &- cm \int_0^l s\,|\theta_x|^2\,\mathrm{d}x - kc \int_0^l s'\,\theta_x \overline{[\varphi_x + \psi]}\,\mathrm{d}x - k\rho_3 \int_0^l [s\,\theta]_x \overline{\Phi}\,\mathrm{d}x + k\rho_3 \int_0^l s\,\theta\overline{\Psi}\,\mathrm{d}x \\ &+ k\rho_3 \int_0^l s\,\theta\overline{[f_{1,x} + f_3]}\,\mathrm{d}x + c\rho_1 \int_0^l s\theta_x \overline{f_2}\,\mathrm{d}x + k \int_0^l s\,[\rho_3 f_5 + m(f_{1,x} + f_3)]\,\overline{[\varphi_x + \psi]}\,\mathrm{d}x. \end{aligned}$$

From estimate (3.7) and keeping in mind the definition of $\mathcal{H}$-norm, we infer

$$|I_3| \le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}} + C\|\theta_x\|_{L^2}\|F\|_{\mathcal{H}},$$

for some constant $C > 0$. Going back to (3.14) and using condition (3.8) on s, we conclude

$$\begin{aligned} |\beta| \int_{l_0-\delta}^{l_0+\delta} s|\varphi_x + \psi|^2\,\mathrm{d}x \;\le\; & C|\beta|\|\theta_x\|_{L^2} \left(\int_{l_0-\delta}^{l_0+\delta} s|\Phi|^2\,\mathrm{d}x \right)^{1/2} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} \\ &+ C\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}} + C\|\theta_x\|_{L^2}\|F\|_{\mathcal{H}}. \end{aligned}$$

Moreover, applying Young inequality and estimate (3.7), we obtain

$$\begin{aligned} \int_{l_0-\delta}^{l_0+\delta} s|\varphi_x + \psi|^2\,\mathrm{d}x \le & C\,\|\theta_x\|_{L^2} \left(\int_{l_0-\delta}^{l_0+\delta} s|\Phi|^2\,\mathrm{d}x \right)^{1/2} + \frac{C}{|\beta|}\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}} \\ &+ \frac{C}{|\beta|}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{C}{|\beta|}\|F\|^2_{\mathcal{H}}. \end{aligned} \tag{3.15}$$

On the other hand, taking the multiplier $-s\overline{\varphi}$ in (3.3), performing integration by parts and applying (3.2), we get

$$\rho_1 \int_0^l s|\Phi|^2\,\mathrm{d}x = k \int_0^l s|\varphi_x + \psi|^2\,\mathrm{d}x - k \int_0^l s(\varphi_x + \psi)\overline{\psi}\,\mathrm{d}x + I_4 + I_5, \tag{3.16}$$

where

$$I_4 = \frac{i}{\beta} m \int_0^l s\,\theta_x \overline{[\Phi + f_1]}\,\mathrm{d}x - \rho_1 \int_0^l s[\Phi\overline{f_1} + f_2\overline{\varphi}]\,\mathrm{d}x \quad \text{and} \quad I_5 = k \int_0^l s'(\varphi_x + \psi)\overline{\varphi}\,\mathrm{d}x.$$

From (3.7) it is easy to see that

$$|I_4| \le \frac{C}{|\beta|}\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}} + \frac{C}{|\beta|}\|\theta_x\|_{L^2}\|F\|_{\mathcal{H}} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}},$$

for some constant $C > 0$. In addition, from equations (3.2) and (3.4), it follows that

$$|\text{Re } I_5| \le \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}} + \frac{C}{|\beta|^2}\|F\|^2_{\mathcal{H}},$$

for some constant $C > 0$. Thus, taking the real part in (3.16) and observing that supp $s \subset (l_0 - \delta, l_0 + \delta)$, we have

$$\begin{aligned}\int_{l_0-\delta}^{l_0+\delta} s|\Phi|^2 \,\mathrm{d}x &\le C\int_{l_0-\delta}^{l_0+\delta} s|\varphi_x + \psi|^2 \,\mathrm{d}x + C\int_{l_0-\delta}^{l_0+\delta} s|\varphi_x+\psi||\psi| \,\mathrm{d}x + \frac{C}{|\beta|}\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}} \\ &\quad + \frac{C}{|\beta|}\|\theta_x\|_{L^2}\|F\|_{\mathcal{H}} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}} + \frac{C}{|\beta|^2}\|F\|^2_{\mathcal{H}} \\ &\le C\int_{l_0-\delta}^{l_0+\delta} s|\varphi_x + \psi|^2 \,\mathrm{d}x + C\left(\int_{l_0-\delta}^{l_0+\delta} s|\varphi_x+\psi|^2 \,\mathrm{d}x\right)^{1/2}\|\psi\|_{L^2} + \frac{C}{|\beta|}\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}} \\ &\quad + \frac{C}{|\beta|}\|\theta_x\|_{L^2}\|F\|_{\mathcal{H}} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}} + \frac{C}{|\beta|^2}\|F\|^2_{\mathcal{H}}.\end{aligned}$$

From estimate (3.15), Young inequality and (3.7), results

$$\begin{aligned}\int_{l_0-\delta}^{l_0+\delta} s|\Phi|^2 \,\mathrm{d}x &\le C\,\|\theta_x\|_{L^2}\left(\int_{l_0-\delta}^{l_0+\delta} s|\Phi|^2 \,\mathrm{d}x\right)^{1/2} + C\,\|\theta_x\|^{1/2}_{L^2}\left(\int_{l_0-\delta}^{l_0+\delta} s|\Phi|^2 \,\mathrm{d}x\right)^{1/4}\|\psi\|_{L^2} \\ &\quad + \frac{C}{|\beta|^{1/2}}\left(\|\theta_x\|^{1/2}_{L^2}\|U\|^{1/2}_{\mathcal{H}} + \|U\|^{1/2}_{\mathcal{H}}\|F\|^{1/2}_{\mathcal{H}} + \|F\|_{\mathcal{H}}\right)\|\psi\|_{L^2} \\ &\quad + \frac{C}{|\beta|}\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|^2_{\mathcal{H}} + \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}}.\end{aligned}$$

Using again Young inequality and (3.7), and then equation (3.4), it follows that

$$\begin{aligned}\int_{l_0-\delta}^{l_0+\delta} s|\Phi|^2 \,\mathrm{d}x &\le \frac{C}{|\beta|^{3/2}}\left(\|\theta_x\|^{1/2}_{L^2}\|U\|^{1/2}_{\mathcal{H}} + \|U\|^{1/2}_{\mathcal{H}}\|F\|^{1/2}_{\mathcal{H}} + \|F\|_{\mathcal{H}}\right)\left(\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}\right) \\ &\quad + \frac{C}{|\beta|^{4/3}}\|\theta_x\|^{2/3}_{L^2}\left(\|U\|^{4/3}_{\mathcal{H}} + \|F\|^{4/3}_{\mathcal{H}}\right) + \frac{C}{|\beta|}\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}} \\ &\quad + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|^2_{\mathcal{H}} + \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}}.\end{aligned}$$

Applying once more Young inequality and estimate (3.7), we conclude

$$\begin{aligned}\int_{l_0-\delta}^{l_0+\delta} s|\Phi|^2 \,\mathrm{d}x &\le \frac{C}{|\beta|^{3/2}}\left(\|\theta_x\|^{1/2}_{L^2}\|U\|^{1/2}_{\mathcal{H}} + \|U\|^{1/2}_{\mathcal{H}}\|F\|^{1/2}_{\mathcal{H}}\right)\|U\|_{\mathcal{H}} + \frac{C}{|\beta|^{4/3}}\|\theta_x\|^{2/3}_{L^2}\|U\|^{4/3}_{\mathcal{H}} \\ &\quad + \frac{C}{|\beta|}\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|^2_{\mathcal{H}} + \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}}. \end{aligned} \tag{3.17}$$

Finally, combining (3.15) and (3.17) we obtain

$$\int_{l_0-\delta}^{l_0+\delta} s\left(|\varphi_x+\psi|^2+|\Phi|^2\right) \mathrm{d}x \leq \frac{C}{|\beta|^{3/2}}\left(\|\theta_x\|_{L^2}^{1/2}\|U\|_{\mathcal{H}}^{1/2}+\|U\|_{\mathcal{H}}^{1/2}\|F\|_{\mathcal{H}}^{1/2}\right)\|U\|_{\mathcal{H}} + \frac{C}{|\beta|}\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}}+\frac{C}{|\beta|^{4/3}}\|\theta_x\|_{L^2}^{2/3}\|U\|_{\mathcal{H}}^{4/3} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+C\|F\|_{\mathcal{H}}^2,$$

where we use Young inequality and estimate (3.7) over again, and from definition of s in (3.8)–(3.9), the estimate (3.10) holds true. In particular, using Young inequality and (3.7) repeatedly, and observing conditions (3.8)–(3.9) on s, the estimate (3.11) is also achieved.

Now we consider another auxiliary cut-off function $s_1 \in C^2(0,l)$ such that

$$\operatorname{supp} s_1 \subset (l_0-\delta/2, l_0+\delta/2), \quad 0 \leq s_1(x) \leq 1, \quad x \in (0,l), \tag{3.18}$$

and

$$s_1(x)=1 \quad \text{for} \quad x \in [l_0-\delta/3, l_0+\delta/3]. \tag{3.19}$$

Proposition 3.4. *Under above notations there exists a constant $C>0$ such that*

$$\int_{l_0-\delta/2}^{l_0+\delta/2} s_1|\psi_x|^2 \mathrm{d}x \leq C|\beta|\,|b\rho_1-k\rho_2| \int_{l_0-\delta/2}^{l_0+\delta/2} s_1|\varphi_x+\psi||\Psi| \mathrm{d}x + C\int_{l_0-\delta/2}^{l_0+\delta/2} |\varphi_x+\psi|^2 \mathrm{d}x + C\|U\|_{\mathcal{H}}\left(\int_{l_0-\delta/2}^{l_0+\delta/2} |\varphi_x+\psi|^2 \mathrm{d}x\right)^{1/2} + C\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}}+C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+C\|F\|_{\mathcal{H}}^2+\frac{C}{|\beta|^2}\|U\|_{\mathcal{H}}^2. \tag{3.20}$$

Proof. Multiplying (3.5) by $s_1\frac{b}{\rho_2}\overline{\psi}$ and performing integration by parts on $(0,l)$ we get

$$\frac{b^2}{\rho_2}\int_0^l s_1|\psi_x|^2 \mathrm{d}x = -i\beta b\int_0^l s_1\Psi\overline{\psi}\,\mathrm{d}x-\frac{bk}{\rho_2}\int_0^l s_1(\varphi_x+\psi)\overline{\psi}\,\mathrm{d}x + \frac{bm}{\rho_2}\int_0^l s_1\theta\overline{\psi}\,\mathrm{d}x+b\int_0^l s_1 f_4\overline{\psi}\,\mathrm{d}x-\frac{b^2}{\rho_2}\int_0^l s_1'\,\psi_x\overline{\psi}\,\mathrm{d}x. \tag{3.21}$$

On the other hand, deriving (3.3), taking the multiplier $s_1 b\,\overline{\psi}$ in the resulting expression and then integrating by parts, we infer

$$i\beta b\int_0^l s_1\,\Phi_x\overline{\psi}\,\mathrm{d}x+\underbrace{\frac{b}{\rho_1}\int_0^l s_1[k(\varphi_x+\psi)_x]\overline{\psi}_x\,\mathrm{d}x}_{:=I_6}+\underbrace{\frac{b}{\rho_1}\int_0^l s_1'[k(\varphi_x+\psi)_x]\overline{\psi}\,\mathrm{d}x}_{:=I_7} - \frac{mb}{\rho_1}\int_0^l \theta_x[s_1\overline{\psi}]_x\,\mathrm{d}x=-b\int_0^l f_2[s_1\overline{\psi}]_x\,\mathrm{d}x. \tag{3.22}$$

Using integration by parts and equation (3.5) we rewrite I_6 and I_7 as follows

$$
\begin{aligned}
I_6 &= i\beta\frac{k\rho_2}{\rho_1}\int_0^l s_1(\varphi_x+\psi)\overline{\Psi}\,\mathrm{d}x - \frac{k^2}{\rho_1}\int_0^l s_1|\varphi_x+\psi|^2\,\mathrm{d}x \\
&+ \frac{km}{\rho_1}\int_0^l s_1(\varphi_x+\psi)\overline{\theta}\,\mathrm{d}x + \frac{k\rho_2}{\rho_1}\int_0^l s_1(\varphi_x+\psi)\overline{f_4}\,\mathrm{d}x - \frac{bk}{\rho_1}\int_0^l s_1'(\varphi_x+\psi)\overline{\psi}_x\,\mathrm{d}x,
\end{aligned}
$$

and

$$
I_7 = -\frac{bk}{\rho_1}\int_0^l s_1''(\varphi_x+\psi)\overline{\psi}\,\mathrm{d}x - \frac{bk}{\rho_1}\int_0^l s_1'(\varphi_x+\psi)\overline{\psi}_x\,\mathrm{d}x.
$$

Replacing these two last identities in (3.22), adding the resulting expression with (3.21), using again equations (3.2) and (3.4), and integration by parts, results

$$
\begin{aligned}
\frac{b^2}{\rho_2}\int_0^l s_1|\psi_x|^2\,\mathrm{d}x &= i\beta\left[\frac{b\rho_1-k\rho_2}{\rho_1}\right]\int_0^l s_1(\varphi_x+\psi)\overline{\Psi}\,\mathrm{d}x + \frac{k^2}{\rho_1}\int_0^l s_1|\varphi_x+\psi|^2\,\mathrm{d}x \\
&+ k\int_0^l\left\{\frac{b}{\rho_1}s_1''-\frac{b}{\rho_2}s_1\right\}(\varphi_x+\psi)\overline{\psi}\,\mathrm{d}x \\
&+ 2\frac{bk}{\rho_1}\int_0^l s_1'(\varphi_x+\psi)\overline{\psi}_x\,\mathrm{d}x + I_8 + I_9,
\end{aligned}
\tag{3.23}
$$

where

$$
\begin{aligned}
I_8 &= b\int_0^l s_1\Psi\overline{f_3}\,\mathrm{d}x - b\int_0^l s_1(f_{1,x}+f_3)\overline{\Psi}\,\mathrm{d}x - b\int_0^l \Phi[s_1\overline{f_3}]_x\,\mathrm{d}x + b\int_0^l s_1 f_4\overline{\psi}\,\mathrm{d}x \\
&- \frac{k\rho_2}{\rho_1}\int_0^l s_1(\varphi_x+\psi)\overline{f_4}\,\mathrm{d}x + \frac{bm}{\rho_2}\int_0^l s_1\theta\overline{\psi}\,\mathrm{d}x - \frac{km}{\rho_1}\int_0^l s_1(\varphi_x+\psi)\overline{\theta}\,\mathrm{d}x \\
&+ b\int_0^l\left(\frac{m}{\rho_1}\theta_x - f_2\right)[s_1\,\overline{\psi}]_x\,\mathrm{d}x, \\
I_9 &= -\frac{b^2}{\rho_2}\int_0^l s_1'\,\psi_x\overline{\psi}\,\mathrm{d}x.
\end{aligned}
$$

Applying (3.7) it is easy to check that

$$
|I_8| \le C\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|^2_{\mathcal{H}},
$$

for some constant $C>0$. Further, using integrating by parts and equation (3.4), we infer

$$
|\mathrm{Re}\ I_9| \le \frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}} + \frac{C}{|\beta|^2}\|F\|^2_{\mathcal{H}},
$$

for some constant $C>0$. Going back to (3.23) and taking its real part, and noting that

$$
\mathrm{supp}\ s_1'' \subset \mathrm{supp}\ s_1' \subset \mathrm{supp}\ s_1 \subset (l_0-\delta/2, l_0+\delta/2),
$$

we obtain

$$\int_{l_0-\delta/2}^{l_0+\delta/2} s_1|\psi_x|^2\,\mathrm{d}x \le C|\beta|\,|b\rho_1-k\rho_2|\int_{l_0-\delta/2}^{l_0+\delta/2} s_1|\varphi_x+\psi||\Psi|\,\mathrm{d}x$$
$$+C\int_{l_0-\delta/2}^{l_0+\delta/2}|\varphi_x+\psi|^2\,\mathrm{d}x+C\|U\|_{\mathcal{H}}\left(\int_{l_0-\delta/2}^{l_0+\delta/2}|\varphi_x+\psi|^2\,\mathrm{d}x\right)^{1/2}$$
$$+C\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}}+C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+C\|F\|^2_{\mathcal{H}}+\frac{C}{|\beta|^2}\|U\|^2_{\mathcal{H}}.$$

This completes the proof of (3.20).

Proposition 3.5. *Under above notations there exists a constant $C>0$ such that*

$$\int_{l_0-\delta/2}^{l_0+\delta/2} s_1|\Psi|^2\,\mathrm{d}x \le C|\beta|\,|b\rho_1-k\rho_2|\int_{l_0-\delta/2}^{l_0+\delta/2} s_1|\varphi_x+\psi||\Psi|\,\mathrm{d}x$$
$$+C\int_{l_0-\delta/2}^{l_0+\delta/2}|\varphi_x+\psi|^2\,\mathrm{d}x+C\|U\|_{\mathcal{H}}\left(\int_{l_0-\delta/2}^{l_0+\delta/2}|\varphi_x+\psi|^2\,\mathrm{d}x\right)^{1/2} \tag{3.24}$$
$$+C\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}}+C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+C\|F\|^2_{\mathcal{H}}+\frac{C}{|\beta|}\|U\|^2_{\mathcal{H}}.$$

Proof. Taking the multiplier $-s_1\overline{\psi}$ in (3.5), performing integration by parts and using (3.4), we obtain

$$\rho_2\int_0^l s_1\,|\Psi|^2\,\mathrm{d}x = b\int_0^l s_1\,|\psi_x|^2\,\mathrm{d}x + I_{10}+I_{11}, \tag{3.25}$$

where

$$I_{10} = -\,\rho_2\int_0^l s_1\,\Psi\overline{f_3}\,\mathrm{d}x-\rho_2\int_0^l s_1 f_4\overline{\psi}\,\mathrm{d}x - m\int_0^l s_1\theta\overline{\psi}\,\mathrm{d}x,$$
$$I_{11} = k\int_0^l s_1(\varphi_x+\psi)\overline{\psi}\,\mathrm{d}x + b\int_0^l s_1'\psi_x\overline{\psi}\,\mathrm{d}x.$$

From equation (3.4) it is easy to see that there exists a constant $C>0$ such that

$$|I_{10}|\ \le\ C\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}}+C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}\quad\text{and}\quad |I_{11}|\ \le\ \frac{C}{|\beta|}\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+\frac{C}{|\beta|}\|U\|^2_{\mathcal{H}}.$$

Therefore, inserting these last two estimates in (3.25), using (3.20) and since supp $s_1\subset(l_0-\delta/2,l_0+\delta/2)$, we conclude (3.24) as desired.

Corollary 3.6. *Under above notations, let us also consider $\epsilon>0$.*

(i) *If $\chi\neq 0$ in (1.8), then there exists a constant $C_\epsilon>0$ such that*

$$\int_{l_0-\delta/3}^{l_0+\delta/3}\left(|\psi_x|^2+|\Psi|^2\right)\,\mathrm{d}x\ \le\ \epsilon\|U\|^2_{\mathcal{H}}+C_\epsilon|\beta|^4\|F\|^2_{\mathcal{H}}. \tag{3.26}$$

(ii) *If $\chi = 0$ in (1.8), then there exists a constant $C_\epsilon > 0$ such that*

$$\int_{l_0-\delta/3}^{l_0+\delta/3} \left(|\psi_x|^2 + |\Psi|^2\right) \mathrm{d}x \,\le\, \epsilon\|U\|_{\mathcal{H}}^2 + C_\epsilon\|F\|_{\mathcal{H}}^2. \tag{3.27}$$

Proof. Adding the estimates (3.20) and (3.24) provided by Propositions 3.4 and 3.5, respectively, using Young inequality and conditions (3.18)–(3.19) on s_1, we get

$$\begin{aligned}\int_{l_0-\delta/3}^{l_0+\delta/3} \left(|\psi_x|^2 + |\Psi|^2\right) \mathrm{d}x \le{}& C|\beta|^2\,|b\rho_1 - k\rho_2|^2 \int_{l_0-\delta/2}^{l_0+\delta/2} |\varphi_x + \psi|^2\,\mathrm{d}x + C\|\theta_x\|_{L^2}\|U\|_{\mathcal{H}} \\ &+ C\int_{l_0-\delta/2}^{l_0+\delta/2} |\varphi_x + \psi|^2\,\mathrm{d}x + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|_{\mathcal{H}}^2 \\ &+ C\left(\int_{l_0-\delta/2}^{l_0+\delta/2} |\varphi_x + \psi|^2\,\mathrm{d}x\right)^{1/2} \|U\|_{\mathcal{H}} + \frac{C}{|\beta|}\|U\|_{\mathcal{H}}^2,\end{aligned} \tag{3.28}$$

for some constant $C > 0$. Thus, we have:

(i) Since $\chi \neq 0$, then $b\rho_1 - k\rho_2 \neq 0$. From (3.28) with $|\beta| > 1$ large enough, Young inequality with $\epsilon > 0$ and (3.7), we have

$$\int_{l_0-\delta/3}^{l_0+\delta/3} \left(|\psi_x|^2 + |\Psi|^2\right) \mathrm{d}x \,\le\, \epsilon\|U\|_{\mathcal{H}}^2 + C_\epsilon\|F\|_{\mathcal{H}}^2 + |\beta|^2 \left(C_\epsilon \int_{l_0-\delta/2}^{l_0+\delta/2} |\varphi_x + \psi|^2\,\mathrm{d}x\right),$$

for some constant $C_\epsilon > 0$. In addition, using the estimate (3.10) of Proposition 3.3, proper Young inequalities with $\epsilon > 0$ and estimate (3.7), we conclude

$$C_\epsilon \int_{l_0-\delta/2}^{l_0+\delta/2} |\varphi_x + \psi|^2\,\mathrm{d}x \le \frac{\epsilon}{|\beta|^2}\|U\|_{\mathcal{H}}^2 + C_\epsilon|\beta|^2\|F\|_{\mathcal{H}}^2.$$

Hence, the conclusion (3.26) follows.

(ii) In this case, $\chi = 0$ implies that $b\rho_1 - k\rho_2 = 0$. Therefore, the desired estimate (3.27) follows from (3.28) with $|\beta| > 1$ large enough, estimates (3.7) and (3.11) of Proposition 3.3 and Young inequality with $\epsilon > 0$.

In what follows we consider an observability inequality for the homogeneous problem related to Timoshenko systems. Such result was first proved in Muñoz Rivera and Ávila [17] for systems with variable coefficients. See also Alves et al. [5].

Let us consider the system

$$i\beta u - v = g_1 \quad \text{in} \quad (0,l), \tag{3.29}$$
$$i\beta\rho_1 v - k(u_x + w)_x = g_2 \quad \text{in} \quad (0,l), \tag{3.30}$$
$$i\beta w - z = g_3 \quad \text{in} \quad (0,l), \tag{3.31}$$
$$i\beta\rho_2 z - b\,w_{xx} + k(u_x + w) = g_4 \quad \text{in} \quad (0,l), \tag{3.32}$$

where $g_1, g_3 \in H_0^1(0,l)$ (or $H_*^1(0,l)$), $g_2, g_4 \in L^2(0,l)$. We denote by V and G the vector-valued functions $V = (u,v,w,z)^T$ and $G = (g_1,g_2,g_3,g_4)^T$, respectively. Besides, given any $a_1, a_2 \in \mathbb{R}$ with $0 \le a_1 < a_2 \le l$, the notations $\|\cdot\|_{a_1,a_2}$ and $\mathcal{I}(\cdot)$ stand for

$$\|V\|^2_{a_1,\,a_2} := \int_{a_1}^{a_2} \left(|u_x+w|^2+|v|^2+|w_x|^2+|z|^2\right)\,\mathrm{d}x,$$

$$\mathcal{I}(a_j) := |u_x(a_j)+w(a_j)|^2+|v(a_j)|^2+|w_x(a_j)|^2+|z(a_j)|^2, \quad j=1,2.$$

Proposition 3.7. *Under above notations, let us consider a strong solution $V=(u,v,w,z)^T$ of (3.29)-(3.32) and any $0\le a_1<a_2\le l$. Then there exist constants $C_0,\,C_1>0$ such that*

$$\mathcal{I}(a_j) \le C_0\|V\|^2_{a_1,\,a_2} + C_0\|G\|^2_{0,\,l}, \quad j=1,2, \tag{3.33}$$

$$\|V\|^2_{a_1,\,a_2} \le C_1\,\mathcal{I}(a_j) + C_1\|G\|^2_{0,\,l}, \quad j=1,2. \tag{3.34}$$

Proof. The proof is analogous to [17, Lemma 3.2] and can be done without requiring extra informations on the boundary conditions. See also [5, Proposition 3.13].

Corollary 3.8. *Let $V=(u,v,w,z)^T$ be a strong solution of the system (3.29)–(3.32). If for any sub-interval $(a_1,a_2)\subset(0,l)$ we have*

$$\|V\|^2_{a_1,\,a_2} \le \Lambda, \tag{3.35}$$

then there exists a constant $C>0$ such that

$$\|V\|^2_{0,\,l} \le C\Lambda + C\|G\|^2_{0,\,l}. \tag{3.36}$$

Proof. Indeed, from (3.33) and (3.35), we have

$$\mathcal{I}(a_j) \le C_0\Lambda + C_0\|G\|^2_{0,\,l}, \quad j=1,2. \tag{3.37}$$

Using (3.34) with $a_1 := 0$ and (3.37) with $j=2$, we obtain

$$\int_0^{a_2} \left(|u_x+w|^2+|v|^2+|w_x|^2+|z|^2\right)\mathrm{d}x \le C_2\,\Lambda + C_2\|G\|^2_{0,\,l},$$

where $C_2 = C_1C_0 + C_1 > 0$. Analogously, using (3.34) with $a_1 := a_2$, $a_2 := l$, and (3.37) with $j=1$, we also obtain

$$\int_{a_2}^{l} \left(|u_x+w|^2+|v|^2+|w_x|^2+|z|^2\right)\mathrm{d}x \le C_2\,\Lambda + C_2\|G\|^2_{0,\,l}.$$

Hence, adding the above last two estimates, we conclude (3.36).

4. Main results

The first main result on stability asserts that, in general, the problem (1.1)–(1.5) is polynomially stable with rates depending on the regularity of the initial data, but independent of the boundary conditions considered in (1.5).

Theorem 4.1. *Under above notations, let us assume that $\chi\ne 0$ in (1.8). Then, there exists a constant $C_n>0$ independent of $U_0\in D(\mathcal{A}^n)$, $n\ge 1$ integer, such that the semigroup solution $U(t)=e^{\mathcal{A}t}U_0$ decays as*

$$\|U(t)\|_{\mathcal{H}} \le \frac{C_n}{t^{n/2}}\|U_0\|_{D(\mathcal{A}^n)}, \quad t\to+\infty. \tag{4.1}$$

Proof. Let $\epsilon > 0$ be given. From estimates (3.11) of Proposition 3.3 and (3.26) of Corollary 3.6, there exists a constant $C_\epsilon > 0$ such that

$$\int_{l_0-\delta/3}^{l_0+\delta/3} \left(|\varphi_x + \psi|^2 + |\Phi|^2 + |\psi_x|^2 + |\Psi|^2\right) \mathrm{d}x \,\le\, \epsilon \|U\|_{\mathcal{H}}^2 + C_\epsilon |\beta|^4 \|F\|_{\mathcal{H}}^2 := \Lambda,$$

for some constant $C_\epsilon > 0$. Since $V := (\varphi, \Phi, \psi, \Psi)^T$ is a solution of (3.29)-(3.32) with

$$g_1 := f_1, \quad g_2 := \rho_1 f_2 - m\,\theta_x, \quad g_3 = f_3, \quad g_4 = \rho_2 f_4 + m\,\theta,$$

and (3.35) is verified with $a_1 = l_0 - \delta/3$ and $a_2 = l_0 + \delta/3$, then Corollary 3.8, Lemma 3.1 and Young inequality imply

$$\int_0^l \left(|\varphi_x + \psi|^2 + |\Phi|^2 + |\psi_x|^2 + |\Psi|^2\right) \mathrm{d}x \,\le\, \epsilon C \|U\|_{\mathcal{H}}^2 + C_\epsilon |\beta|^4 \|F\|_{\mathcal{H}}^2, \tag{4.2}$$

for some constants C, $C_\epsilon > 0$. In addition, from estimates (3.7) and (4.2), we arrive at

$$\|U\|_{\mathcal{H}}^2 \le \epsilon C \|U\|_{\mathcal{H}}^2 + C_\epsilon |\beta|^4 \|F\|_{\mathcal{H}}^2.$$

Choosing $\epsilon > 0$ small enough and observing the resolvent equation (3.1), then

$$\|(i\beta I_d - \mathcal{A})^{-1} F\|_{\mathcal{H}} \le C\,|\beta|^2 \|F\|_{\mathcal{H}}, \quad |\beta| \to \infty, \tag{4.3}$$

for some constant $C > 0$. Finally, applying Lemma 3.2 and estimate (4.3), we conclude from [6, Theorem 2.4] that

$$\|U(t)\|_{\mathcal{H}} \,\le\, \frac{C}{t^{1/2}} \|U_0\|_{D(\mathcal{A})}, \quad t \to \infty,$$

for $U_0 \in D(\mathcal{A})$, which proves (4.1) for $N = 1$. The remaining decay rates in (4.1) follows by using induction over $N \ge 2$.

Remark 4.2. It is worth pointing out two issues with respect to estimate (4.1):

(i) the polynomial decay rate $t^{-1/2}$ achieved in (4.1), for $U_0 \in D(\mathcal{A})$, is independent of the boundary conditions in (1.5). Therefore, in the particular case of full Dirichlet condition $(1.5)_a$, this achievement improves the decay $t^{-1/4}$ obtained in [1, Theorem 5.1];

(ii) the optimality of the decay rate $t^{-1/2}$ is only ensured for boundary conditions $(1.5)_b$ and $(1.5)_e$. This is proved in [1, Theorem 5.1] for the case $(1.5)_e$. We can also extend such result for $(1.5)_b$ with minor adjustments on the arguments. In particular, the homogeneous thermoelastic Timoshenko system with boundary condition $(1.5)_b$ or $(1.5)_e$ is exponential stable if and only if $\chi = 0$, see e.g. [1, Theorem 3.2]. The same result must be expected for the other boundary conditions $(1.5)_{a,c,d}$ because they are of conservative nature, although there is no concrete proof of this fact so far.

The second main result states that the system (1.1)–(1.5) is exponentially stable under equal wave speeds. It reads as follows.

Theorem 4.3. *Under above notations, let us assume that $\chi = 0$ in (1.8). Then, there exist constants $C, \gamma > 0$ independent of $U_0 \in \mathcal{H}$ such that the semigroup solution $U(t) = e^{\mathcal{A}t} U_0$ decays as*

$$\|U(t)\|_{\mathcal{H}} \le C e^{-\gamma\, t} \|U_0\|_{\mathcal{H}}, \quad t > 0. \tag{4.4}$$

Proof. Let $\epsilon > 0$ be given. Since $\chi = 0$ we can apply Corollary 3.6-(ii). Thus, from (3.11) and (3.27), we obtain

$$\int_{l_0-\delta/3}^{l_0+\delta/3} \left(|\varphi_x + \psi|^2 + |\Phi|^2 + |\psi_x|^2 + |\Psi|^2\right) \mathrm{d}x \,\le\, \epsilon \|U\|_{\mathcal{H}}^2 + C_\epsilon \|F\|_{\mathcal{H}}^2 := \Lambda,$$

for some constant $C_\epsilon > 0$. As above, applying Corollary 3.8, Lemma 3.1 and Young inequality, we deduce

$$\int_0^l \left(|\varphi_x + \psi|^2 + |\Phi|^2 + |\psi_x|^2 + |\Psi|^2\right) \mathrm{d}x \;\leq\; \epsilon C\|U\|_{\mathcal{H}}^2 + C_\epsilon \|F\|_{\mathcal{H}}^2, \tag{4.5}$$

for some constants C, $C_\epsilon > 0$. Combining (3.7) and (4.5), we get

$$\|U\|_{\mathcal{H}}^2 \leq \epsilon C\|U\|_{\mathcal{H}}^2 + C_\epsilon \|F\|_{\mathcal{H}}^2.$$

Taking $\epsilon > 0$ small enough and regarding the resolvent equation (3.1), we conclude

$$\|(i\beta I_d - \mathcal{A})^{-1}F\|_{\mathcal{H}} \leq C\|F\|_{\mathcal{H}}, \quad |\beta| \to \infty. \tag{4.6}$$

The estimate (4.6) and Lemma 3.2 are enough to conclude (4.4) from the classical result on exponential stability given e.g. in [15, Theorem 1.3.2].

Remark 4.4. We observe that all local estimates given in Sect. 3 rely on the use of auxiliary cut-off multipliers which do not require further information on boundary point-wise terms. Therefore, the same results on stability (Theorems 4.1 and 4.3) can be considered with other boundary conditions. Among others, we quote:

$$\begin{aligned}
&S(0,t) = S(l,t) = \psi_x(0,t) = \psi_x(l,t) = \theta(0,t) = \theta(l,t) = 0,\\
&S(0,t) = S(l,t) = \psi(0,t) = \psi(l,t) = \theta_x(0,t) = \theta_x(l,t) = 0,\\
&S(0,t) = S(l,t) = \psi_x(0,t) = \psi_x(l,t) = \theta_x(0,t) = \theta_x(l,t) = 0,
\end{aligned}$$

where S is the shear stress given in (1.7). In addition, mixed boundary condition such as Dirichlet (or Neumann) on $x = 0$ and Neumann (or Dirichlet) on $x = l$ are also allowed provided the dissipation property (2.4) is satisfied.

Remark 4.5. We finally observe that both results on stability can be proved with respect to thermoelastic system (1.12)–(1.14) with initial-boundary conditions (1.4)–(1.5), excepting for $(1.5)_e$ which is replaced by $\varphi_x = \psi = \theta_x = 0$ on $x = 0, l$. This is not so trivial and requires a substantial number of local estimates which must be given in a reverse way from the ones provided in Sect. 3.

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