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On the effect of damping on the stabilization of mechanical systems via parametric excitation

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Abstract. The effect of damping on the re-stabilization of statically unstable linear Hamiltonian systems, performed via parametric excitation, is studied. A general multi-degree-of-freedom mechanical system is considered, close to a divergence point, at which a mode is incipiently stable and $n-1$ modes are (marginally) stable. The asymptotic dynamics of system is studied via the Multiple Scale Method, which supplies amplitude modulation equations ruling the slow flow. Several resonances between the excitation and the natural frequencies, of direct 1:1, 1:2, 2:1, or sum and difference combination types, are studied. The algorithm calls for using integer or fractional asymptotic power expansions and performing nonstandard steps. It is found that a slight damping is able to increase the performances of the control system, but only far from resonance. Results relevant to a sample system are compared with numerical findings based on the Floquet theory.

Mathematics Subject Classification. 70E50 Stability problems · 70E55 Dynamics of multibody systems.

Keywords. Re-stabilization · Parametric excitation · Damped system · Multiple Scale Method · Triple pendulum.

1. Introduction

When a Hamiltonian system is subject to conservative loads, it can undergo a static bifurcation (divergence, or buckling) at a critical value of a load multiplier. Consequently, its equilibrium path (which can always be considered trivial by a proper change of variables) becomes unstable at loads higher than the critical one. The challenge to re-stabilize the equilibrium via a proper control (e.g., by internal moving masses, or by actuators for retroaction control laws) has attracted the interest of many researchers [1–5]. A simple form of control is offered by parametric excitation, in which a mass, stiffness, or length is periodically varied in time, changing the system from autonomous to non-autonomous and, possibly, making it stable [6–20]. A celebrated example of re-stabilization via parametric excitation is the Indian magic rope trick problem [13,14,19].

The beneficial effect of the parametric excitation on otherwise statically unstable systems has been explained in the literature via the concept of effective mechanical stiffness [16,19,21], which emerges when the problem is approached by the method of direct separation of motion [12]. A clear explanation of this phenomenon is given, in the opinion of the authors, by the Multiple Scale Method [22], that they applied in [23] to study a specific two-dof system, and in [24] to analyze a general multi-dof system. In both papers, it was proved that a static (zero frequency) forcing term, able to stabilize the system, appears in the right- hand member of higher-order perturbation equations, as generated by a combination of parametric and natural frequencies. The implementation of the perturbation scheme was not trivial, since it required the use of integer or fractional powers expansions, according to the resonance involved, proper ordering of parameters, higher-order expansions, addition of the homogeneous solutions, and refined technique of reconstitution.

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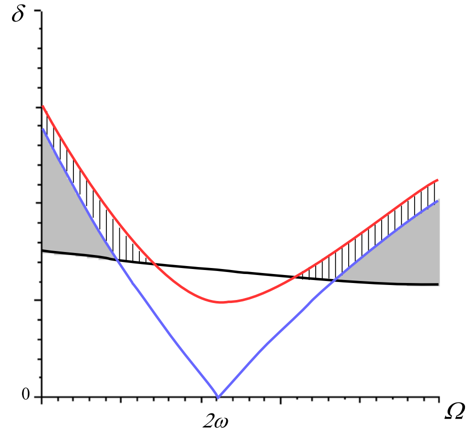

Fig. 1. Stability region (*shaded area*) in the frequency–amplitude plane, close to the principal parametric resonance; *black line* static bifurcation, *blue line* undamped system; *red line* damped system

However, it was stressed in [17,23] that although the parametric excitation is able to re-stabilize the statically unstable mode, on the other side, it has a detrimental effect on the otherwise stable modes, which can destabilize via the classical mechanism described by the Floquet theory. The result is sketched in Fig. 1, where the stability region close to the principal parametric resonance is plotted in the (Mathieu) excitation frequency (Ω)-amplitude (δ) plane. The bottom (black) line denotes the "lower stability boundary", at which the system undergoes a static bifurcation. It means that when a sufficiently high-amplitude excitation is provided, the unstable system regains stability. However, close to the parametric resonance, re-stabilization cannot occurs; moreover, the (blue) curves delimiting the unstable region constitute "upper stability boundaries" for non-resonant excitation.

The previous results concern undamped mechanical systems. It is therefore worth investigating if damping can improve, and at which extent, the performances of the parametric excitation. It is hoped, indeed, that similar to what happens for parametrically excited *stable* systems, where damping rises the stability boundary close to the resonance, the same occurs for the *unstable* systems here dealt with (red curve in Fig. 1), where a coupling between the static and dynamic instability forms is, in principle, possible. To this end, the whole analysis carried out in [24] is redeveloped in this paper, with damping now accounted for.

The paper is organized as follows. In Sect. 2, the equations in the problem are formulated and arranged in view of the perturbation analysis to be carried out ahead. In Sect. 3, the amplitude modulation equations governing the slow flow of the dynamical system are derived for non-resonant and several resonant cases, all handleable by standard integer series expansions. In Sect. 4, the more difficult 1:1 resonant case is treated by nonstandard fractional power series expansions (see, e.g., [25,26]). In Sect. 5, numerical results concerning a triple pendulum, taken as sample system, are displayed and commented. Section 6 is devoted to Conclusions. Since damping, as expected, renders the asymptotic treatment cumbersome, to make the reading easier, many details have been shifted to two Appendixes that close the paper.

2. Problem position

We consider a damped n-dof linear system, parametrically excited, whose motion is governed by the following equations:

$$\mathbf{M}\ddot{\mathbf{q}} + \gamma\mathbf{D}\dot{\mathbf{q}} + \left(\mathbf{C}(p) + \delta\Omega^2\cos\Omega t\mathbf{B}\right)\mathbf{q} = 0 \tag{1}$$

Here, $\mathbf{M} = \mathbf{M}^T$ is the mass matrix; $\mathbf{D} = \mathbf{D}^T$ is a positive-definite damping matrix, and γ is a damping multiplier; $\mathbf{C} = \mathbf{C}^T$ is the stiffness matrix, depending on a load parameter p; $\mathbf{B}$ is the parametric excitation matrix; δ and Ω are amplitude and frequency of the parametric excitation, respectively; $\mathbf{q}$ are Lagrangian coordinates, and a dot denotes differentiation with respect the time t.

We assume that the underlying Hamiltonian system ($\gamma = \delta = 0$) undergoes a static bifurcation (divergence) at the critical load $p = p_0$, such that the trivial equilibrium $\mathbf{q} = \mathbf{0}$ is stable at $p < p_0$ and unstable at $p > p_0$. Goal of the analysis is to find how to re-stabilize the system in the postcritical range via a suitable parametric excitation.

To this end, we put $p = p_0 + \Delta p$, where $\Delta p > 0$ is the incremental load, small with respect to p_0; moreover, we expand the stiffness matrix as $\mathbf{C}(p) = \mathbf{C}_0 + \Delta p\mathbf{C}_1 + O(\Delta p^2)$, where $\mathbf{C}_0 = \mathbf{C}(p_0)$. In view of a perturbation analysis, we introduce the rescaling $\Delta p \to \varepsilon^2 \Delta p$, $\gamma \to \varepsilon\gamma$, $\delta \to \varepsilon\delta$, where $0 < \varepsilon \ll 1$ is a perturbation parameter. Consequently, Eq. (1) becomes:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}_0\mathbf{q} + \varepsilon(\gamma\mathbf{D}\dot{\mathbf{q}} + \delta\Omega^2\cos\Omega t\mathbf{B}\mathbf{q}) + \varepsilon^2\Delta p\mathbf{C}_1\mathbf{q} = 0 \tag{2}$$

When $\varepsilon \to 0$, the system tends to the Hamiltonian system at the critical state, whose associate eigenvalue problem reads:

$$\left(\mathbf{C}_0 - \omega_k^2\mathbf{M}\right)\mathbf{u} = 0, \tag{3}$$

This latter admits the eigenfrequencies $(0, \omega_2, \ldots, \omega_n)$, supposed distinct, and the related real eigenvectors $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$.

We will deal Eq. (2) by the Multiple Scale Method (MSM, [22]). As it is well known [27], this method works as a *reduction method*, by furnishing few equations governing the so-called slow flow, i.e., the slow time-evolution of the amplitudes and phases of the harmonic components (including those of zero frequency, i.e., of static nature), which enter the essential part of the linear solution. Moreover, these equations appear to be directly in their *normal form* [28], i.e., deprived of any unessential terms. However, as discussed in [27], some 'difficult problems' are encountered in applying MSM; re-stabilization by parametric excitation is just one of them, as explained in "Appendixes 1, 2".

3. Multiple Scale analysis by integer power expansions

According to the MSM, we first introduce independent timescales $t_j = \varepsilon^j t$, $j = 0, 1, \ldots$, so that:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \sum_{k=0}^{\infty} \varepsilon^k \mathrm{d}_k, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \varepsilon^{k+j}\mathrm{d}_k\mathrm{d}_j \tag{4}$$

where $\mathrm{d}_k = \partial/\partial t_k$. Then, we expand the Lagrangian coordinates in series of integer powers of ε:

$$\mathbf{q} = \mathbf{q}_0 + \varepsilon\mathbf{q}_1 + \varepsilon^2\mathbf{q}_2 + \cdots \tag{5}$$

and get the following perturbation equations:

$$\begin{aligned}
\varepsilon^0: &\quad \mathbf{M}\mathrm{d}_0^2\mathbf{q}_0 + \mathbf{C}_0\mathbf{q}_0 = 0,\\
\varepsilon^1: &\quad \mathbf{M}\mathrm{d}_0^2\mathbf{q}_1 + \mathbf{C}_0\mathbf{q}_1 = -2\mathbf{M}\mathrm{d}_0\mathrm{d}_1\mathbf{q}_0 - \frac{\delta\Omega^2}{2}(\mathrm{e}^{\mathrm{i}\Omega t_0} + \mathrm{e}^{-\mathrm{i}\Omega t_0})\mathbf{B}\mathbf{q}_0 - \gamma\mathbf{D}\mathrm{d}_0\mathbf{q}_0,\\
\varepsilon^2: &\quad \mathbf{M}\mathrm{d}_2^2\mathbf{q}_2 + \mathbf{C}_0\mathbf{q}_2 = -2\mathbf{M}\mathrm{d}_0\mathrm{d}_1\mathbf{q}_1 - 2\mathbf{M}\mathrm{d}_0\mathrm{d}_2\mathbf{q}_0 - \mathbf{M}\mathrm{d}_1^2\mathbf{q}_0 - \Delta p\mathbf{C}_1\mathbf{q}_0\\
&\qquad -\frac{\delta\Omega^2}{2}(\mathrm{e}^{\mathrm{i}\Omega t_0} + \mathrm{e}^{-\mathrm{i}\Omega t_0})\mathbf{B}\mathbf{q}_1 - \gamma\mathbf{D}(\mathrm{d}_1\mathbf{q}_0 + \mathrm{d}_0\mathbf{q}_1),\\
&\qquad \ldots
\end{aligned} \tag{6}$$

The solution of the first of Eq. (6) (generating solution) reads:

$$\mathbf{q}_0 = \sum_{k=1}^{n} A_k(t_1, t_2, \ldots)\mathbf{u}_k \mathrm{e}^{\mathrm{i}\omega_k t_0} + \text{c.c.}, \tag{7}$$

where i is the imaginary unit and c.c. denotes the complex conjugate of the preceding terms. Here, A_k's are complex amplitudes, depending on slow timescales; since $\omega_1 = 0$, A_1 is real. With Eq. (7), the ε^1-order perturbation equations (6) reads:

$$\begin{aligned}\mathbf{M}\mathrm{d}_0^2\mathbf{q}_1 + \mathbf{C}_0\mathbf{q}_1 =& -2\mathrm{i}\sum_{k=2}^{n}\omega_k \mathrm{d}_1 A_k \mathbf{M}\mathbf{u}_k \mathrm{e}^{\mathrm{i}\omega_k t_0} - \frac{\delta\Omega^2}{2}\sum_{k=1}^{n} A_k(\mathrm{e}^{\mathrm{i}(\omega_k+\Omega)t_0} + \mathrm{e}^{\mathrm{i}(\omega_k-\Omega)t_0})\mathbf{B}\mathbf{u}_k \\ & -\mathrm{i}\gamma\sum_{k=2}^{n}\omega_k A_k \mathbf{D}\mathbf{u}_k \mathrm{e}^{\mathrm{i}\omega_k t_0} + \text{c.c.}\end{aligned} \tag{8}$$

Five different cases will be considered in this Section:

1. *non-resonant* case $\Omega \neq \omega_j, 2\omega_j, \omega_j/2, \omega_j \pm \omega_i, \quad i, j = 2, \ldots$;
2. *first-order resonant* case $\Omega = 2\omega_j$;
3. *second-order resonant* case $\Omega = \omega_j/2$;
4. *sum combination resonance* $\Omega = \omega_j + \omega_i$;
5. *difference combination resonance* $\Omega = \omega_j - \omega_i$.

A further case $\Omega = \omega_j$, requiring a different treatment, will be addressed later.

The non-resonant case. Any combination resonances are excluded, so that $\Omega \neq \omega_j \pm \omega_i, \quad \forall j, i$. By following the standard steps of the MSM, detailed in Appendix 1 "The non-resonant case", the following amplitude modulation equations (AME) are derived for the amplitudes:

$$\begin{aligned}\ddot{A}_1 &= \alpha_1\dot{A}_1 + \alpha_2 A_1 \\ \dot{A}_k &= \alpha_{3k}A_k, \qquad k = 2, 3, \ldots n,\end{aligned} \tag{9}$$

where the coefficients α's depend on the parameters and are defined in Eq. (47) in the Appendix 1 "The non-resonant case". In particular, damping enters the coefficients $\alpha_{1,}, \alpha_{3k}$ but not α_2.

Since $\Re(\alpha_{3k}) < 0$, the latter of Eq. (9) states that *all the dynamic components of motion decay.* Therefore, the essential asymptotic dynamics is governed by the first of Eq. (9) in the static component A_1; therefore, the n-dof system reduces to a single-dof system. Since $\alpha_1 < 0$, *stability can be lost only via a static bifurcation*, occurring at $\alpha_2 = 0$. This is exactly the stability boundary for the undamped system (named "lower" in [23,24]). In conclusion, when resonance is absent, damping is unable to increase the stability performances of the system.

The resonant case $\boldsymbol{\Omega = 2\omega_j}$. To express the closeness of Ω to $2\omega_j$ $(2\omega_j \neq \omega_k \forall k)$, we introduce a small detuning $\varepsilon\sigma$ and let $\Omega = 2\omega_j + \varepsilon\sigma$. Since amplitudes not involved in resonances are inessential in determining stability (via a mechanism similar to that discussed for the non-resonant case), we limit the generating solution to:

$$\mathbf{q}_0 = A_1(t_1, t_2, \ldots)\mathbf{u}_1 + A_j(t_1, t_2, \ldots)\mathbf{u}_j \mathrm{e}^{\mathrm{i}\omega_j t_0} + \text{c.c.}. \tag{10}$$

By applying the standard MSM algorithm (see the Appendix 1 "The resonant case $\Omega = 2\omega_j$"), we get the following AME:

$$\begin{aligned}\ddot{A}_1 &= \alpha_1\dot{A}_1 + \alpha_2 A_1, \\ \dot{A}_j &= \alpha_3 A_j + \mathrm{i}\alpha_4 A_j + \alpha_5\overline{A}_j\mathrm{e}^{\mathrm{i}\sigma t} + \mathrm{i}\alpha_6\overline{A}_j\mathrm{e}^{\mathrm{i}\sigma t},\end{aligned} \tag{11}$$

where the coefficients α's are defined in Eq. (57) of Appendix 1 "The resonant case $\Omega = 2\omega_j$". Since A_1 is real and A_j is complex, the asymptotic dynamics occur on a four-dimensional manifold. The first of Eq. (11) is identical to the first of Eq. (9), governing the static loss of stability and, moreover, is uncoupled from the second one. Therefore is this latter that rules the (dynamic) loss of stability. To transform it in an autonomous form, we let $A_j = R\,\mathrm{e}^{\mathrm{i}\sigma t/2}$ and, to pass to real quantities, $R = u + \mathrm{i}v$. Then, we write the system of equations for the new variables in the form:

$$\begin{aligned} \dot{u} &= (\alpha_3 + \alpha_5)u + (\sigma/2 - \alpha_4 + \alpha_6)v, \\ \dot{v} &= (-\sigma/2 + \alpha_4 + \alpha_6)u + (\alpha_3 - \alpha_5)v. \end{aligned} \tag{12}$$

The condition $\alpha_3^2 - \alpha_5^2 = \alpha_6^2 - (\sigma/2 - \alpha_4)^2$ determines the upper boundary of stability.

The resonant case $\Omega = \omega_j/2$. In this case, we have $\Omega = \omega_j/2 + \varepsilon\sigma$. As it was in previous case, we take a two-term generating solution, namely Eq. (10). By going to the ε^2-order, where resonance manifests itself for the first time, we obtain the following AME (see Appendix 1 "The resonant case $\Omega = \omega_j/2$"):

$$\begin{aligned} \ddot{A}_1 &= \alpha_1\dot{A}_1 + \alpha_2 A_1 + \alpha_3(\overline{A}_j\mathrm{e}^{2\mathrm{i}\sigma t} + A_j\mathrm{e}^{-2\mathrm{i}\sigma t}), \\ \dot{A}_j &= \mathrm{i}\alpha_4 A_1\mathrm{e}^{2\mathrm{i}\sigma t}A_j + \alpha_5 A_j + \mathrm{i}\alpha_6 A_j, \end{aligned} \tag{13}$$

where the α' s coefficients are defined in the Eq. (59) of Appendix 1 "The resonant case $\Omega = \omega_j/2$". The asymptotic dynamics still occurs on a four-dimensional manifold. However, since the equations are now coupled, *static and dynamic components of motion interact* in determining stability.

By letting $A_j = R\,\mathrm{e}^{2\mathrm{i}\sigma t}$, with $R = u + \mathrm{i}v$, we can rewrite the system (13) in the matrix form $\dot{\mathbf{x}} = \mathbf{H}\mathbf{x}$, where:

$$\mathbf{x} = \begin{pmatrix} A \\ \dot{A} \\ u \\ v \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha_2 & \alpha_1 & 2\alpha_3 & 0 \\ 0 & 0 & \alpha_5 & 2\sigma - \alpha_6 \\ \alpha_4 & 0 & \alpha_6 - 2\sigma & \alpha_5 \end{pmatrix}. \tag{14}$$

The characteristic equation for the characteristic number λ of the matrix $\mathbf{H}$ is:

$$\lambda^4 + k_1\lambda^3 + k_2\lambda^2 + k_3\lambda + k_4 = 0, \tag{15}$$

where:

$$\begin{aligned} &k_1 = -(\alpha_1 + 2\alpha_5) > 0, \qquad k_2 = 2\alpha_1\alpha_5 - \alpha_2 + \alpha_6^2 + 4\sigma^2 + \alpha_5^2 - 4\alpha_6\sigma, \\ &k_3 = 2\alpha_2\alpha_5 - 4\sigma^2\alpha_1 - \alpha_1\alpha_5^2 + 4\alpha_1\alpha_6\sigma - \alpha_1\alpha_6^2, \\ &k_4 = -4\alpha_2\sigma^2 - \alpha_2\alpha_5^2 + 4\alpha_2\alpha_6\sigma - \alpha_2\alpha_6^2 + 2\alpha_3\alpha_4\alpha_6 - 4\alpha_3\alpha_4\sigma. \end{aligned} \tag{16}$$

The stability conditions are drawn by the Hurwitz's theorem, which supplies the following stability conditions:

$$k_2 > 0, \quad k_3 > 0, \quad k_4 > 0, \quad k_1k_2k_3 - k_3^2 - k_1^2k_4 > 0. \tag{17}$$

The sum combination resonance $\Omega = \omega_j + \omega_i$. In this case we have $\Omega = \omega_j + \omega_i + \varepsilon\sigma$. The generating solution involves three components:

$$\mathbf{q}_0 = A_1(t_1, t_2, \ldots)\mathbf{u}_1 + A_i(t_1, t_2, \ldots)\mathbf{u}_i\mathrm{e}^{\mathrm{i}\omega_i t_0} + A_j(t_1, t_2, \ldots)\mathbf{u}_j\mathrm{e}^{\mathrm{i}\omega_j t_0} + \text{c.c.}. \tag{18}$$

By performing calculations detailed in Appendix 1 "The sum combination resonance $\Omega = \omega_j + \omega_i$", we get the following AME for the amplitudes:

$$\begin{aligned} \ddot{A}_1 &= \alpha_1\dot{A}_1 + \alpha_2 A_1 \\ \dot{A}_i &= \beta_{1i}A_i + \mathrm{i}\beta_{2i}A_i + \beta_{3i}\overline{A}_j\mathrm{e}^{\mathrm{i}\sigma t} + \mathrm{i}\beta_{4i}\overline{A}_j\mathrm{e}^{\mathrm{i}\sigma t}, \\ \dot{A}_j &= \beta_{1j}A_j + \mathrm{i}\beta_{2j}A_j + \beta_{3j}\overline{A}_i\mathrm{e}^{\mathrm{i}\sigma t} + \mathrm{i}\beta_{4j}\overline{A}_i\mathrm{e}^{\mathrm{i}\sigma t}, \end{aligned} \tag{19}$$

where the β's coefficients are defined in Eq. (67) of the Appendix 1 "The sum combination resonance $\Omega = \omega_j + \omega_i$". Motion occurs on a six-dimensional manifold. However, the equation for A_1 is uncoupled from the latter and coincides with the first of Eq. (9), holding in the non-resonant regime, so that it does not add any information on stability.

By letting $A_i = R_i e^{i\sigma t/2}$ and $A_j = R_j e^{i\sigma t/2}$, with $R_i = u_i + iv_i$ and $R_j = u_j + iv_j$, we can rewrite the last two Eq. (19) in the matrix form $\dot{\mathbf{x}} = \mathbf{H}\mathbf{x}$, where:

$$\mathbf{x} = \begin{pmatrix} u_i \\ v_i \\ u_j \\ v_j \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} \beta_{1i} & \sigma/2 - \beta_{2i} & \beta_{3i} & \beta_{4i} \\ \beta_{2i} - \sigma/2 & \beta_{1i} & \beta_{4i} & -\beta_{3i} \\ \beta_{3j} & \beta_{4j} & \beta_{1j} & \sigma/2 - \beta_{2j} \\ \beta_{4j} & -\beta_{3j} & \beta_{2j} - \sigma/2 & \beta_{1j} \end{pmatrix}. \tag{20}$$

The characteristic equation has the form (15) (with α's replaced by β's), and stability conditions the form of inequalities (17).

The difference combination resonance $\boldsymbol{\Omega = \omega_j - \omega_i}$. Here the closeness of Ω to the difference of the natural frequencies is defined by $\Omega = \omega_j - \omega_i + \varepsilon\sigma$ $(j > i)$. By following the procedure of Appendix 1 "The difference combination resonance $\Omega = \omega_j - \omega_i$", the AME are found to assume the form:

$$\begin{aligned} \ddot{A}_1 &= \alpha_1 \dot{A}_1 + \alpha_2 A_1 \\ \dot{A}_i &= \beta_{1i} A_i + i\beta_{2i} A_i + \beta_{3i} A_j e^{-i\sigma t} + i\beta_{4i} A_j e^{-i\sigma t}, \\ \dot{A}_j &= \beta_{1j} A_j + i\beta_{2j} A_j + \beta_{3j} A_i e^{i\sigma t} + i\beta_{4j} A_i e^{i\sigma t}, \end{aligned} \tag{21}$$

where the β's coefficients are defined in the Eq. (71) of the Appendix 1 "The difference combination resonance $\Omega = \omega_j - \omega_i$". As for the sum combination, the dynamics of the static component A_1 is uncoupled from those of the harmonic components A_i, A_j.

By letting $A_i = R_i e^{-i\sigma t/2}$ and $A_j = R_j e^{i\sigma t/2}$, with $R_i = u_i + iv_i$ and $R_j = u_j + iv_j$, the latter two equations are rewritten in matrix form, with:

$$\mathbf{H} = \begin{pmatrix} \beta_{1i} & -\sigma/2 - \beta_{2i} & \beta_{3i} & -\beta_{4i} \\ \beta_{2i} + \sigma/2 & \beta_{1i} & \beta_{4i} & \beta_{3i} \\ \beta_{3j} & -\beta_{4j} & \beta_{1j} & \sigma/2 - \beta_{2j} \\ \beta_{4j} & \beta_{3j} & \beta_{2j} - \sigma/2 & \beta_{1j} \end{pmatrix} \tag{22}$$

so that the stability conditions have the form of inequalities (17).

4. Multiple Scale analysis by fractional power expansions

The resonant case $\Omega = \omega_j$ is now addressed. As discussed in [24], this case calls for using fractional power expansions of $\varepsilon^{1/2}$. Since the procedure is not standard, we will describe it with a major detail.

First, we introduce a fractional expansion for the Lagrangian coordinates:

$$\mathbf{q}(t,\varepsilon) = \mathbf{q}_0(t_0,t_1,t_2,\ldots) + \varepsilon^{1/2}\mathbf{q}_1(t_0,t_1,t_2,\ldots) + \varepsilon\mathbf{q}_2(t_0,t_1,t_2,\ldots) + \varepsilon^{3/2}\mathbf{q}_3(t_0,t_1,t_2,\ldots) + \ldots \tag{23}$$

where:

$$t_0 = t, \quad t_1 = \varepsilon^{1/2} t, \quad t_2 = \varepsilon t, \quad \ldots \tag{24}$$

are fractional times, too. The chain rule now reads:

$$\frac{d}{dt} = \sum_{k=0}^{\infty} \varepsilon^{k/2} d_k, \quad \frac{d^2}{dt^2} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \varepsilon^{(k+j)/2} d_k d_j \tag{25}$$

so that the perturbation equations are:

$$
\begin{aligned}
\varepsilon^0: \quad & \mathbf{M}\mathrm{d}_0^2\mathbf{q}_0 + \mathbf{C}_0\mathbf{q}_0 = 0, \\
\varepsilon^{1/2}: \quad & \mathbf{M}\mathrm{d}_0^2\mathbf{q}_1 + \mathbf{C}_0\mathbf{q}_1 = -2\mathbf{M}\mathrm{d}_0\mathrm{d}_1\mathbf{q}_0, \\
\varepsilon^1: \quad & \mathbf{M}\mathrm{d}_0^2\mathbf{q}_2 + \mathbf{C}_0\mathbf{q}_2 = -2\mathbf{M}\mathrm{d}_0\mathrm{d}_2\mathbf{q}_0 - \mathbf{M}\mathrm{d}_1^2\mathbf{q}_0 - 2\mathbf{M}\mathrm{d}_0\mathrm{d}_1\mathbf{q}_1 - \frac{\delta\Omega^2}{2}\mathbf{B}\mathbf{q}_0(\mathrm{e}^{\mathrm{i}\Omega t_0} + \mathrm{e}^{-\mathrm{i}\Omega t_0}) \\
& \qquad -\gamma\mathbf{D}\mathrm{d}_0\mathbf{q}_0, \\
\varepsilon^{3/2}: \quad & \mathbf{M}\mathrm{d}_0^2\mathbf{q}_3 + \mathbf{C}_0\mathbf{q}_3 = -2\mathbf{M}\mathrm{d}_1\mathrm{d}_2\mathbf{q}_0 - 2\mathbf{M}\mathrm{d}_0\mathrm{d}_3\mathbf{q}_0 - 2\mathbf{M}\mathrm{d}_0\mathrm{d}_2\mathbf{q}_1 - \mathbf{M}\mathrm{d}_1^2\mathbf{q}_1 \\
& \qquad -2\mathbf{M}\mathrm{d}_0\mathrm{d}_1\mathbf{q}_2 - \frac{\delta\Omega^2}{2}\mathbf{B}\mathbf{q}_1(\mathrm{e}^{\mathrm{i}\Omega t_0} + \mathrm{e}^{-\mathrm{i}\Omega t_0}) - \gamma\mathbf{D}(\mathrm{d}_1\mathbf{q}_0 + \mathrm{d}_0\mathbf{q}_1), \\
\varepsilon^2: \quad & \mathbf{M}\mathrm{d}_0^2\mathbf{q}_4 + \mathbf{C}_0\mathbf{q}_4 = -2\mathbf{M}\mathrm{d}_1\mathrm{d}_3\mathbf{q}_0 - \mathbf{M}\mathrm{d}_2^2\mathbf{q}_0 - 2\mathbf{M}\mathrm{d}_0\mathrm{d}_4\mathbf{q}_0 - 2\mathbf{M}\mathrm{d}_0\mathrm{d}_3\mathbf{q}_1 - 2\mathbf{M}\mathrm{d}_1\mathrm{d}_2\mathbf{q}_1 \\
& \qquad -2\mathbf{M}\mathrm{d}_0\mathrm{d}_2\mathbf{q}_2 - \mathbf{M}\mathrm{d}_1^2\mathbf{q}_2 - 2\mathbf{M}\mathrm{d}_0\mathrm{d}_1\mathbf{q}_3 - \Delta p\mathbf{C}_1\mathbf{q}_0 - \frac{\delta\Omega^2}{2}\mathbf{B}\mathbf{q}_2(\mathrm{e}^{\mathrm{i}\Omega t_0} + \mathrm{e}^{-\mathrm{i}\Omega t_0}) \\
& \qquad -\gamma\mathbf{D}(\mathrm{d}_2\mathbf{q}_0 + \mathrm{d}_1\mathbf{q}_1 + \mathrm{d}_0\mathbf{q}_2), \\
& \qquad \ldots
\end{aligned}
\tag{26}
$$

The ε^0-order equation admits the generating solution

$$
\mathbf{q}_0 = A_1(t_1, t_2, \ldots)\mathbf{u}_1 + A_j(t_1, y_2, \ldots)\mathbf{u}_j\mathrm{e}^{\mathrm{i}\omega_j t_0} + \text{c.c..} \tag{27}
$$

which substituted in the equations of $\varepsilon^{1/2}$-order furnishes:

$$
\mathbf{M}\mathrm{d}_0^2\mathbf{q}_1 + \mathbf{C}_0\mathbf{q}_1 = -2\mathrm{i}\omega_j\mathrm{d}_1 A_j\mathbf{M}\mathbf{u}_j\mathrm{e}^{\mathrm{i}\omega_j t_0} + \text{c.c..} \tag{28}
$$

Elimination of secular term requires $\mathrm{d}_1 A_j = 0$, so that:

$$
\mathbf{q}_1 = B_j(t_1, t_2, \ldots)\mathbf{u}_j\mathrm{e}^{\mathrm{i}\omega_j t_0} + \text{c.c.}, \tag{29}
$$

where the arbitrary amplitude B_j has been introduced, differently from the usual procedure adopted in the standard method; in contrast, B_1 has been omitted (see [24] for a discussion on this point).

By substituting $\Omega = \omega_j + \varepsilon^{1/2}\sigma$ for the ε^1-order equations, we obtain:

$$
\begin{aligned}
\mathbf{M}\mathrm{d}_0^2\mathbf{q}_2 + \mathbf{C}_0\mathbf{q}_2 = & -\mathrm{d}_1^2 A_1\mathbf{M}\mathbf{u}_1 - \frac{\delta\Omega^2}{2}\overline{A}_j\mathbf{B}\mathbf{u}_j\mathrm{e}^{\mathrm{i}\sigma t_1} + (-2\mathrm{i}\omega_j(\mathrm{d}_2 A_j + \mathrm{d}_1 B_j)\mathbf{M}\mathbf{u}_j \\
& -\delta\Omega^2 A_1\mathbf{B}\mathbf{u}_1\mathrm{e}^{\mathrm{i}\sigma t_1} - \mathrm{i}\gamma\omega_j\mathbf{D}\mathbf{u}_j\big)\,\mathrm{e}^{\mathrm{i}\omega_j t_0} - \frac{\delta\Omega^2}{2}A_j\mathbf{B}\mathbf{u}_j\mathrm{e}^{\mathrm{i}(\Omega+\omega_j)t_0} + \text{c.c.}
\end{aligned}
\tag{30}
$$

By removing secular terms, it follows:

$$
\begin{aligned}
& -d_1^2 A_1 - \frac{\delta\Omega^2}{4}\left(A_j\mathrm{e}^{-\mathrm{i}\sigma t_1} + \overline{A}_j\mathrm{e}^{\mathrm{i}\sigma t_1}\right)\mathbf{u}_1^T\mathbf{B}\mathbf{u}_j = 0, \\
& -2\mathrm{i}\omega_j(\mathrm{d}_2 A_j + \mathrm{d}_1 B_j) - \delta\Omega^2 A_1\mathrm{e}^{\mathrm{i}\sigma t_1}\mathbf{u}_j^T\mathbf{B}\mathbf{u}_1 - \mathrm{i}\gamma\omega_j A_j\mathbf{u}_j^T\mathbf{D}\mathbf{u}_j = 0,
\end{aligned}
\tag{31}
$$

Starting from this point, the procedure continues by the usual steps (see "Appendix 2"). In particular, the solvability condition at the $\epsilon^{3/2}$-order supplies:

$$
\begin{aligned}
& -2\mathrm{d}_1\mathrm{d}_2 A_1 - \frac{\delta\Omega^2}{4}\left(B_j\mathrm{e}^{-\mathrm{i}\sigma t_1} + \overline{B}_j\mathrm{e}^{\mathrm{i}\sigma t_1}\right)\mathbf{u}_1^T\mathbf{B}\mathbf{u}_j - \gamma\mathrm{d}_1 A_1\mathbf{u}_1^T\mathbf{D}\mathbf{u}_1 = 0, \\
& -2\mathrm{i}\omega_j(\mathrm{d}_3 A_j + \mathrm{d}_2 B_j) - \mathrm{d}_1^2 B_j - \mathrm{i}\gamma\omega_j B_j\mathbf{u}_j^T\mathbf{D}\mathbf{u}_j = 0,
\end{aligned}
\tag{32}
$$

and that at the ϵ^2-order furnishes:

$$\begin{aligned}
&-\mathrm{d}_2^2 A_1 - 2\mathrm{d}_1\mathrm{d}_3 A_1 - (\Delta p \mathbf{u}_1^T \mathbf{C}_1 \mathbf{u}_1 - \frac{\delta^2\Omega^4}{2}\mathbf{u}_1^T \mathbf{B}\mathbf{z}_1)A_1 - \gamma \mathrm{d}_2 A_1 \mathbf{u}_1^T \mathbf{D}\mathbf{u}_1 \\
&\quad + \mathrm{i}\gamma\frac{\delta\Omega^2}{4}\left((\Omega-\omega_j)\mathbf{u}_1^T\mathbf{D}\mathbf{z}_j^- + \omega_j \mathbf{u}_1^T \mathbf{B}\mathbf{v}_j\right)\left(\overline{A}_j \mathrm{e}^{\mathrm{i}\sigma t_1} - A_j \mathrm{e}^{-\mathrm{i}\sigma t_1}\right) = 0, \\
&-2\mathrm{i}\omega_j(\mathrm{d}_4 A_j + \mathrm{d}_3 B_j) - \mathrm{d}_2^2 A_j - 2\mathrm{d}_1\mathrm{d}_2 B_j - \gamma \mathbf{u}_j^T \mathbf{D}\mathbf{u}_j(\mathrm{d}_2 A_j + \mathrm{d}_1 B_j) + \frac{\delta^2\Omega^4}{4}\overline{A}_j \mathbf{u}_j^T \mathbf{B}\mathbf{z}_j^- \mathrm{e}^{2\mathrm{i}\sigma t_1} \\
&\quad + \left(\frac{\delta^2\Omega^4}{4}\mathbf{u}_j^T \mathbf{B}(\mathbf{z}_j^+ + \mathbf{z}_j^-) - \Delta p \mathbf{u}_j^T \mathbf{C}_1 \mathbf{u}_j + \gamma^2\omega_j^2 \mathbf{u}_j^T \mathbf{D}\mathbf{v}_j\right) A_j + \mathrm{i}\gamma\delta\Omega^3 \mathbf{u}_j^T \mathbf{D}\mathbf{z}_1 A_1 \mathrm{e}^{\mathrm{i}\sigma t_1} = 0.
\end{aligned} \tag{33}$$

with the $\mathbf{z}$'s vectors defined in Eq. (73) of "Appendix 2".

To recombine all the previous results, let us introduce the total amplitude $C_j = A_j + \varepsilon^{1/2} B_j$. Then, by using the inverse transformations $\varepsilon\delta \to \delta$, $\varepsilon\gamma \to \gamma$, $\varepsilon^2\Delta p \to \Delta p$, $t_1 \to \varepsilon^{1/2} t$, $\varepsilon^{1/2}\sigma \to \sigma$ and reconstruction of the derivatives as:

$$\begin{aligned}
\ddot{A}_1 &= \varepsilon \mathrm{d}_1^2 A_1 + \varepsilon^{3/2} 2\mathrm{d}_1\mathrm{d}_2 A_1 + \varepsilon^2(\mathrm{d}_2^2 A_1 + 2\mathrm{d}_1\mathrm{d}_3 A_1), \\
\dot{C}_j &= \varepsilon(\mathrm{d}_2 A_j + \mathrm{d}_1 B_j) + \varepsilon^{3/2}(\mathrm{d}_3 A_j + \mathrm{d}_2 B_j) + \varepsilon^2(\mathrm{d}_4 A_j + \mathrm{d}_3 B_j)
\end{aligned} \tag{34}$$

we obtain equations in the original (not-rescaled) quantities, having the form:

$$\begin{aligned}
\ddot{A}_1 &= \alpha_1 \dot{A}_1 + \alpha_2 A_1 + \alpha_3(C_j \mathrm{e}^{-\mathrm{i}\sigma t} + \overline{C}_j \mathrm{e}^{\mathrm{i}\sigma t}) + \mathrm{i}\alpha_4(\overline{C}_j \mathrm{e}^{\mathrm{i}\sigma t} - C_j \mathrm{e}^{-\mathrm{i}\sigma t}), \\
\dot{C}_j &= \alpha_5 \dot{A}_1 \mathrm{e}^{\mathrm{i}\sigma t} + \alpha_6 A_1 \mathrm{e}^{\mathrm{i}\sigma t} + \mathrm{i}\alpha_7 A_1 \mathrm{e}^{\mathrm{i}\sigma t} + \mathrm{i}\alpha_8 C_j + \alpha_9 C_j + \mathrm{i}\alpha_{10}\overline{C}_j \mathrm{e}^{2\mathrm{i}\sigma t},
\end{aligned} \tag{35}$$

where the coefficients α's are defined in Eq. (77) in "Appendix 2". Since $A_1$is real and C_j is complex, the motion occurs on a four-dimensional manifold. Since the equations are coupled, *the static and dynamic components interact* in determining stability.

By substituting $C_j = R\,\mathrm{e}^{\mathrm{i}\sigma t}$, where $R = u + \mathrm{i}v$, we can recast the system (35) in matrix form, similarly to the resonant case $\Omega = \omega/2$ (for which coupling also occurs). The matrix $\mathbf{H}$ now will be:

$$\mathbf{H} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha_2 & \alpha_1 & 2\alpha_3 & 2\alpha_4 \\ \alpha_6 & \alpha_5 & \alpha_9 & \alpha_{10} - \alpha_8 + \sigma \\ \alpha_7 & 0 & \alpha_{10} + \alpha_8 - \sigma & \alpha_9 \end{pmatrix}. \tag{36}$$

The stability conditions have the form of inequalities (17).

5. Sample system analysis

As a sample system we will consider a triple pendulum in its unstable upright equilibrium position (Fig. 2). The system is made of three hinged rigid rods of equal length l, elastically restrained at the hinges by linear springs of equal stiffness c, carrying heavy masses m at the hinges. Hinges are damped by linear viscous devices of equal constants d. The support undergoes a vertical harmonic motion $z = a\cos\Omega t$, with $a \ll l$.

By taking the rotations q_i $(i = 1, 2, 3)$ as Lagrangian coordinates, the equations of motion were derived by using Lagrange equations, with $F = \frac{1}{2}d\left[\dot{q}_1^2 + (\dot{q}_2 - \dot{q}_1)^2 + (\dot{q}_3 - \dot{q}_3)^2\right]$ the dissipation function. After linearization around the vertical position $q_i = 0$, and non-dimensionalization, they were found to assume

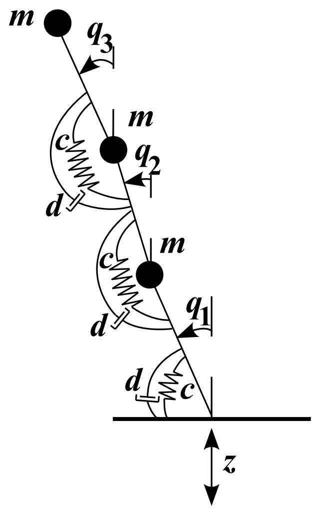

Fig. 2. Inverted triple pendulum under harmonic motion of the support

the form (1), where:

$$\mathbf{M} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{C}_0 = \begin{bmatrix} -2p_0 - 3 & p_0 & 0 \\ p_0 & -2p_0 - 2 & p_0 \\ 0 & p_0 & -p_0 - 1 \end{bmatrix},$$
$$\mathbf{C}_1 = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{37}$$

and, moreover,

$$\delta = \frac{a}{l}, \quad p = -\frac{c}{mgl}, \quad \Omega^* = \Omega\sqrt{\frac{l}{g}}, \quad \gamma = \frac{d}{ml^2}\sqrt{\frac{l}{g}}, \quad t^* = \sqrt{\frac{g}{l}}t \tag{38}$$

with g the gravity acceleration and t^* a nondimensional time (star ahead).

The critical load is found to be $p_0 = -8.1207$. Under this load, the eigenfrequencies are $\omega_i = 0, 3.6389, 8.6722$. The stability boundaries, as supplied by the asymptotic procedure illustrated above, were plotted in the amplitude–frequency plane (δ, Ω), and compared with those determined by a pure numerical approach grounded on the Floquet theory. Results are commented ahead.

Overall view. Figure 3 provides an overall view of the stability regions. The lower curve (plotted in black) which extends over all the Ω-interval is the locus of static bifurcation (damping independent), on which just the zero frequency, statically unstable mode, is involved. The solution, determined as non-resonant, loses validity close to $\Omega = \omega_2, \omega_3$, where a first-order resonance in contrast occurs. The lower bound represents, for each frequency, the lowest amplitude δ able to re-stabilize the equilibrium. Since the curve decreases with Ω, re-stabilization is easier at high, rather than low, frequency. However, close to other resonances as $\Omega = 2\omega_2, 2\omega_3, \omega_2 + \omega_3$, dynamic bifurcations occur, involving the stable modes. In addition to the previous ones, there exist resonances at $\Omega = \frac{1}{2}\omega_2, \frac{1}{2}\omega_3$, but they cannot distinguished in the scale of the figure, so that they will be commented ahead.

Dynamic bifurcations lead to the appearance of instability regions (whose boundaries are plotted in blue) which prevent re-stabilization at lower amplitude and, moreover, constitute upper bounds just out of resonance. When damping is added (red curves), the scenario remains almost unchanged almost everywhere, since modifications are too small to be appreciated on this scale. An exception, however, is

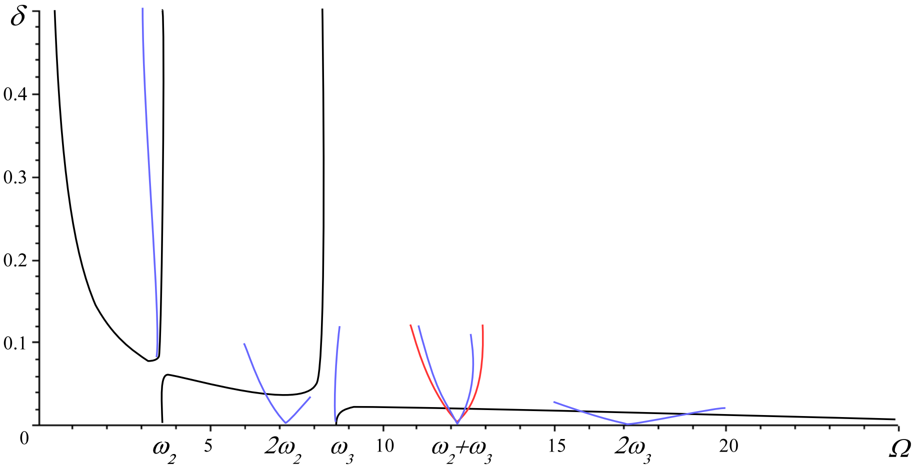

FIG. 3. Overall view of the stability regions; *black curve* static bifurcation; *blue curve* dynamic bifurcation for the undamped system ($\gamma = 0$); *red curve* dynamic bifurcation for a damped system ($\gamma = 0.01$)

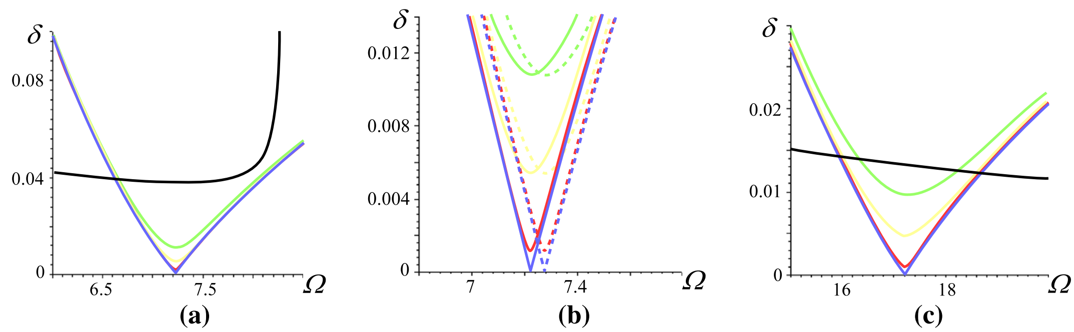

FIG. 4. Stability boundaries around: **a**, **b** $\Omega = 2\omega_2$, **c** $\Omega = 2\omega_3$; *blue curves* undamped system; *red* $\gamma = 0.01$, *yellow* γ=0.05; *green* γ=0.1; *solid lines* second-order asymptotic solution; *dotted lines* first-order asymptotic solution; *black curve* static bifurcation

represented by the resonance zone close to the sum combination resonance $\Omega = \omega_2 + \omega_3$. Here, a small damping $\gamma = 0.01$ produces a *contraction of the stability region*; thus, in spite of the expected beneficial effect, damping has a *destabilizing effect*. This phenomenon is not new, having been observed in some pioneering papers concerning the combination resonance, as resumed in [29] Sect. 5.4.5. The destabilizing effect of the damping, moreover, is well known in autonomous non-conservative systems (see [30,31] for a thorough discussion). Further details on this interesting aspect will be given ahead.

Resonance $\boldsymbol{\Omega = 2\omega_j}$. An enlargement of the stability boundaries around $\Omega = 2\omega_2 = 7.28$ and $\Omega = 2\omega_3 = 17.34$ is plotted in Fig. 4. The asymptotic boundaries evaluated at first-order (dotted lines) and second-order (continuous lines) are plotted for the undamped system (blue lines) and damped system (red, yellow, and green lines). It is seen that second-order corrections are significant (more at lower frequency) and shift on the left the boundary curves. This effect is due to the increment of load Delta p, which has been scaled at second order. Damping has a beneficial effect (as expected at the principal parametric zone, see [29], since it shrinks and raises the boundaries, the more the higher it is).

Resonance $\boldsymbol{\Omega = \frac{1}{2}\omega_j}$. The scenario close to $\Omega = \frac{1}{2}\omega_2 = 1.819$ is depicted in Fig. 5. Here, as we said, interaction between static and dynamic bifurcation occurs. When the system is undamped (Fig. 5a), there

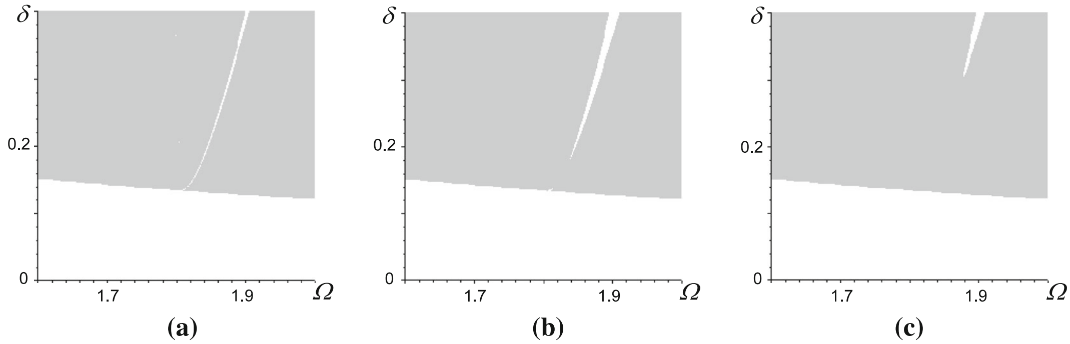

Fig. 5. Stability region around $\Omega = \frac{1}{2}\omega_2$: **a** undamped system, $\gamma = 0$; **b** $\gamma = 0.005$; **c** $\gamma = 0.01$

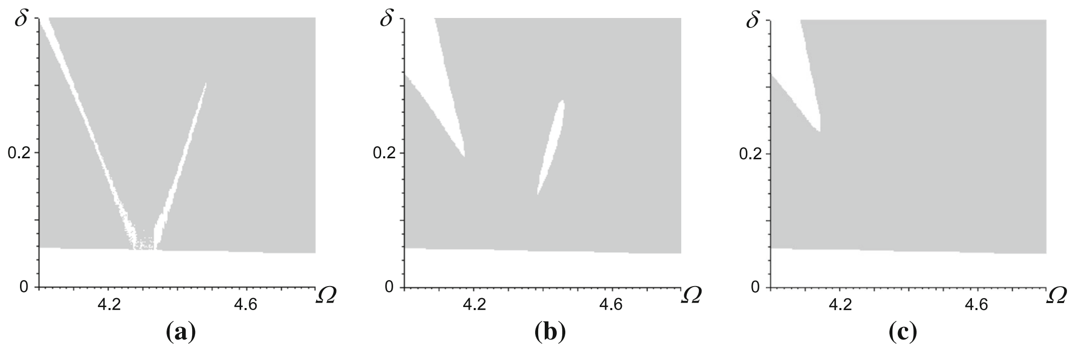

Fig. 6. Stability regions around $\Omega = \frac{1}{2}\omega_3$; **a** undamped system, $\gamma = 0$; **b** $\gamma = 0.005$; **c** $\gamma = 0.01$

is narrow thong of instability; damping (Fig. 5b) reduces and shifts it upward, thus bringing a beneficial effect.

When the region near $\Omega = \frac{1}{2}\omega_3 = 4.336$ is considered (Fig. 6), the behavior is more complex. Indeed, while damping arises the instability zones, it enlarges them, bringing a partially detrimental effect to the re-stabilization.

Combination resonance. Combination resonance leads to instability only in the sum (not in the difference) case, as observed in several cases of literature. The zone close to $\Omega = \omega_2 + \omega_3$, already commented in Fig. 3, is represented in Fig. 7. It appears that damping initially enlarges the instability zone (Fig. 7b), but, increasing, it also arises it (7c). Therefore, damping is detrimental for small values and beneficial for larger values (similarly to what happens for autonomous systems, see, e.g., [32]).

Resonance $\Omega = \omega_j$. The 1:1 resonance case is now addressed that, as we said, also entails a static–dynamic interaction. Figure 8 refers to the case $\Omega = \omega_2$. When damping is absent (Fig. 8a), there exist two regions of instability merging at low amplitude. When a small damping is added, the two regions initially expand (Fig. 8b), so that damping has a detrimental effect; however, when damping is larger (8c) the left region raises, so that a partial beneficial effect of damping can be appreciated.

A similar behavior can be observed close to $\Omega = \omega_3$(see Fig. 9). When the system is undamped (9a), only a narrow stable tongue separates the wide regions of instability. Very small damping values (not shown for brevity) make the tongue extremely thin; however, when a larger damping is considered (9b), the tongue thickens and then merges with the right stable region (9c). As a final result, damping brings an

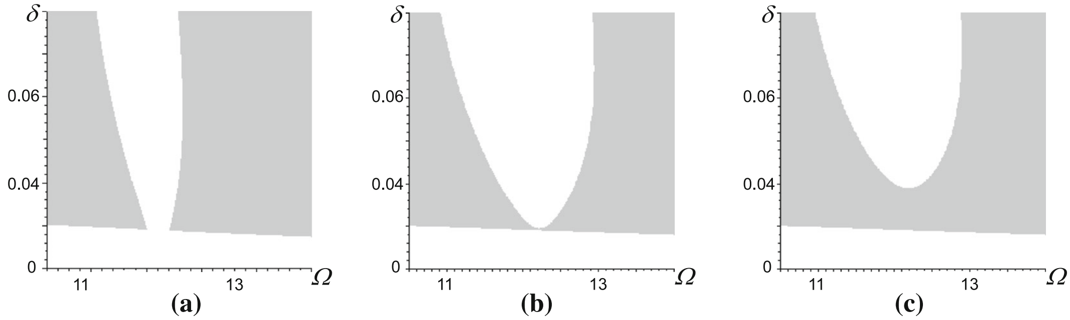

Fig. 7. Stability region around $\Omega = \omega_2 + \omega_3 = 12.311$; **a** undamped system, $\gamma = 0$; **b** $\gamma = 0.05$; **c** $\gamma = 0.1$

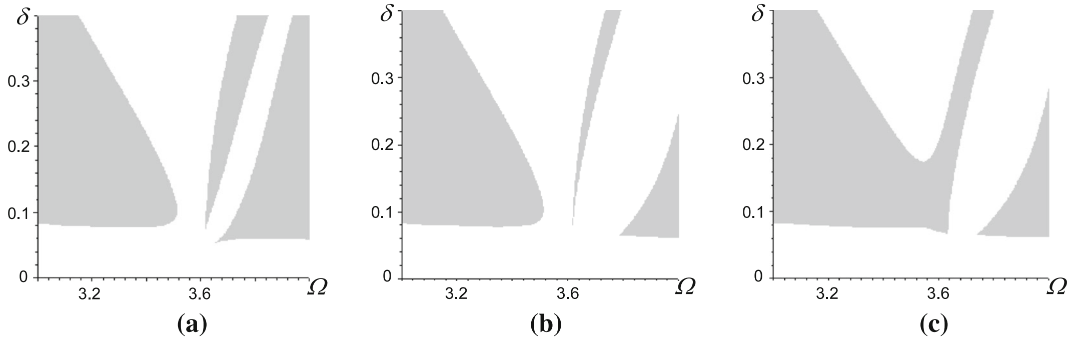

Fig. 8. Stability region around $\Omega = \omega_2 = 3.639$: **a** undamped system, $\gamma = 0$; **b** $\gamma = 0.01$; **c** $\gamma = 0.1$

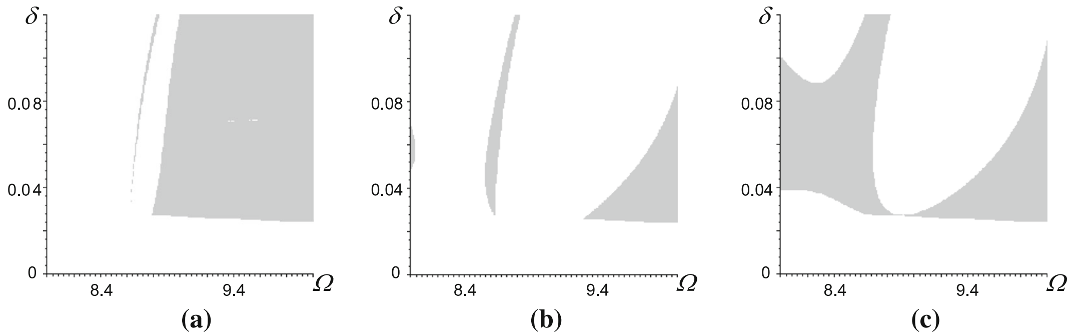

Fig. 9. Stability region around $\Omega = \omega_3 = 8.672$: **a** undamped system, $\gamma = 0$; **b** $\gamma = 0.05$; **c** $\gamma = 0.1$

improvement at lower frequencies and a worsening at higher frequencies, compared with the undamped case.

Comparison with the Floquet theory. The previously illustrated asymptotic results were validated against purely numerical results obtained by evaluating the eigenvalues of the monodromy matrix, according to the Floquet theory. The agreement was found generally good, although less good cases were encountered. Fig. 10 reports some results. Figure 10a refers to the resonance $\Omega = \frac{1}{2}\omega_2$, and it is in excellent agreement with Fig. 5c. Figure 10b concerns the combination resonance $\Omega = \omega_2+\omega_3$, which gives results in reasonable

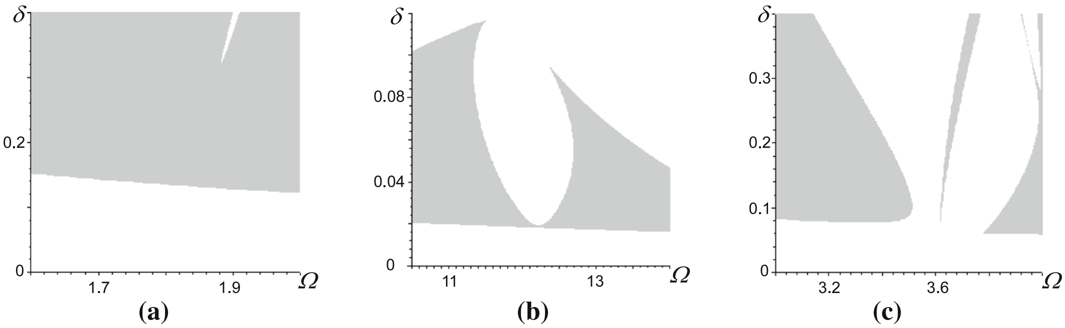

Fig. 10. Stability regions determined via a numerical approach (Floquet theory): **a** around $\Omega = \omega_2/2$ ($\gamma = 0.01$), **b** around $\Omega = \omega_2 + \omega_3$ ($\gamma = 0.05$), **c** around $\Omega = \omega_2$ ($\gamma = 0.01$)

agreement with those in Fig. 7b. Finally, Fig. 10c displays the case $\Omega = \omega_3$, whose results are in good accordance with those in Fig. 8b.

6. Conclusions

The problem of re-stabilization of statically unstable system via parametric excitation has been addressed. The paper extends the results of a previous paper by the authors, by including the effect of damping, so far neglected. The analysis has been carried out via a Multiple Scale analysis, which calls for nonstandard algorithms to be applied: namely integer and fractional power expansions, introduction of the homogeneous solution, reconstitution of amplitude modulation equations of different order, suitable ordering of the parameters, all needed to avoid inconsistencies of the method. Non-resonant and several resonant cases have been studied, including combination resonances. In any cases, a proper set of amplitude modulation equations has been derived, governing the essential dynamics of the multi-degree-of-freedom system, and constituting a reduced-order model. The asymptotic (long-time) dynamics of the system was found to be ruled by the following equations:

1. Far from resonances, the bifurcation is of static type, i.e., only the amplitude of the statically unstable mode is involved, while all the amplitudes of the marginally stable modes, due to the presence of damping, decay. Since damping cannot influence a static bifurcation, it cannot modify the non-resonant boundary curve, which therefore coincides with that of the undamped system. The static bifurcation represents a lower boundary for re-stabilization. It gives the minimum amplitude of the parametric excitation able to re-stabilize the system.
2. Close to the resonances $\Omega = 2\omega_j, \omega_j \pm \omega_k$ the dynamics of the unstable mode is uncoupled by those of the stable modes, and still governed by the non-resonant solution. Here damping enters the amplitude equations, and its effect has to be numerically evaluated.
3. Close to the resonances $\Omega = \frac{1}{2}\omega_j, \omega_j$ the static and dynamics components of motion interacts, since the relevant equations are coupled. Again, the effect of damping calls for a numerical analysis of the reduced-order model.

A triple pendulum has been taken as sample system, and the asymptotic results have been used for drawing the stability boundaries in the excitation amplitude–frequency plane. The following results have been obtained:

1. Damping not always brings a beneficial contribution to re-stabilization. Sometimes it is definitely detrimental, sometimes it is detrimental when it is small and beneficial when it exceeds a threshold value, and sometimes again it brings an improvement on the left side of the resonance but a worsening on the right side. Thus, no general rules can be drawn at this stage.

2. Damping, however, is beneficial close to the principal parametric resonance $\Omega = 2\omega_j$, where the effects of the dynamic instability are strongest.
3. Generally, the modifications of the stability scenario due to damping are weak, since, as it is know, damping works better at low amplitudes, where, however, the system is statically unstable. Therefore, its improvement concerns the upper bounds, which are generally raised.
4. The destabilizing effect of damping on parametric excitation is not new in the literature, since it was discovered in some pioneering works, although confined to the combination resonance. It seems to have similarities with the more-known Ziegler paradox, which concerns autonomous nonconservative systems. The search for possible common aspects between the two phenomenon is an open question that could deserve some future attention.

The study so far carried out concerns any finite-dimensional system admitting a simple-zero eigenvalue and therefore suffering static bifurcation. Of course, a wider class of problems could be studied to check the efficiency of the parametric excitation in restoring stability. In particular, we mention:

1. Lightly damped Hamiltonian systems undergoing multiple (e.g., double)-zero eigenvalues. A prototype for this bifurcation is the Augusti model [33] in the theory of elastic buckling. In the framework of perturbation theory, the relevant treatment is similar to that illustrated here, although quite more complex, since both buckling modes must be included in the generating solution, leading to a reduced-order model of higher dimension.
2. Heavily damped systems, or systems subjected to nonconservative forces (namely, follower or aerodynamic forces) triggering a double-zero (Takens-Bogdanov) bifurcation. Here, re-stabilization is a challenging task, since the bifurcation is of dynamic type, while parametric excitation supplies a constant force and higher harmonics.
3. Nonlinear systems, in which, mainly in the dynamic bifurcation case, the parametric re-stabilization could reduce the limit-cycle amplitude.
4. Continuous (not discretized) systems, for which the procedure illustrated here can still be followed, although with a slightly different mathematical apparatus, as for example illustrated in [34,35].

Appendix 1: The integer asymptotic expansions

Here, we will give details on derivation of the AME by integer series expansions.

The non-resonant case

To remove the appearance of secular terms in the solution of Eq. (8), any harmonic terms in the right-hand members (r.h.m.) of the equation must be rendered orthogonal to the eigenvector of same frequency (solvability condition). It reads:

$$2d_1 A_k + \gamma A_k \mathbf{u}_k^T \mathbf{D}\mathbf{u}_k = 0, \quad k = 2, \ldots, n. \tag{39}$$

while no information is drawn on $\mathrm{d}_1 A_1$. By using this result, Eq. (8) can be rewritten as:

$$\mathbf{M}\mathrm{d}_0^2\mathbf{q}_1 + \mathbf{C}_0\mathbf{q}_1 = -\frac{\delta\Omega^2}{2}\sum_{k=1}^{n} A_k(\mathrm{e}^{\mathrm{i}(\omega_k+\Omega)t_0} + \mathrm{e}^{\mathrm{i}(\omega_k-\Omega)t_0})\mathbf{B}\mathbf{u}_k + \mathrm{i}\gamma\sum_{k=2}^{n}\omega_k A_k \mathbf{V}_k \mathrm{e}^{\mathrm{i}\omega_k t_0} + \mathrm{c.c.}, \tag{40}$$

where the following position holds:

$$\mathbf{V}_k = \mathbf{M}\mathbf{u}_k(\mathbf{u}_k^T\mathbf{D}\mathbf{u}_k) - \mathbf{D}\mathbf{u}_k. \tag{41}$$

Then, the solution for $\mathbf{q}_1$ takes form:

$$\mathbf{q}_1 = -\frac{\delta\Omega^2}{2}\sum_{k=1}^{n} A_k\left(\mathbf{z}_k^+ \mathrm{e}^{\mathrm{i}(\omega_k+\Omega)t_0} + \mathbf{z}_k^- \mathrm{e}^{\mathrm{i}(\omega_k-\Omega)t_0}\right) + \mathrm{i}\gamma\sum_{k=2}^{n}\omega_k A_k \mathbf{v}_k \mathrm{e}^{\mathrm{i}\omega_k t_0} + \mathrm{c.c.}, \tag{42}$$

where $\mathbf{z}_k^{\pm}$ are (unique) solutions to the non-singular algebraic problems:

$$\left(\mathbf{C}_0 - (\omega_k \pm \Omega)^2\mathbf{M}\right)\mathbf{z}_k^{\pm} = \mathbf{B}\mathbf{u}_k, \tag{43}$$

and $\mathbf{v}_k$ are solutions to:

$$\left(\mathbf{C}_0 - \omega_k^2\mathbf{M}\right)\mathbf{v}_k = \mathbf{V}_k, \tag{44}$$

normalized in such a way $\mathbf{u}_k^T\mathbf{M}\mathbf{v}_k = 0$.

By using Eqs. (7) and (42) in the third of equations (6), we get:

$$\begin{aligned}\mathbf{M}\mathrm{d}_0^2\mathbf{q}_2 + \mathbf{C}_0\mathbf{q}_2 = \sum_{k=1}^{n}\Big(&-\gamma\omega_k^2\mathrm{d}_1 A_k\mathbf{M}\mathbf{v}_k - 2\mathrm{i}\omega_k\mathrm{d}_2 A_k\mathbf{M}\mathbf{u}_k - \mathrm{d}_1^2 A_k\mathbf{M}\mathbf{u}_k - \Delta p A_k\mathbf{C}_1\mathbf{u}_k \\ &+\frac{\delta^2\Omega^4}{4}A_k\mathbf{B}(\mathbf{z}_k^+ + \mathbf{z}_k^-) - \gamma\mathrm{d}_1 A_k\mathbf{D}\mathbf{u}_k + \gamma^2\omega_k^2 A_k\mathbf{D}\mathbf{v}_k\Big)\mathrm{e}^{\mathrm{i}\omega_k t_0} + \mathrm{c.c.} + \mathrm{NRT},\end{aligned} \tag{45}$$

where NRT stands for non-resonant terms. To remove secular terms, it needs that frequency-ω_1 terms in the r.h.m. are orthogonal to $\mathbf{u}_1$, and frequency-ω_k terms $(k = 2, \dots)$ are orthogonal to $\mathbf{u}_k$, i.e.:

$$\begin{aligned}&-\mathrm{d}_1^2 A_1 + \left(\frac{\delta^2\Omega^4}{2}\mathbf{u}_1^T\mathbf{B}\mathbf{z}_1 - \Delta p\mathbf{u}_1^T\mathbf{C}_1\mathbf{u}_1\right)A_1 - \gamma\mathbf{u}_1^T\mathbf{D}\mathbf{u}_1\mathrm{d}_1 A_1 = 0, \quad (z_1 = z_1^+ = z_1^-) \\ &-2\mathrm{i}\omega_k\mathrm{d}_2 A_k - \mathrm{d}_1^2 A_k + \left(\frac{\delta^2\Omega^4}{4}\mathbf{u}_k^T\mathbf{B}(\mathbf{z}_k^+ + \mathbf{z}_k^-) - \Delta p\mathbf{u}_k^T\mathbf{C}_1\mathbf{u}_k + \gamma^2\omega_k^2\mathbf{u}_k^T\mathbf{D}\mathbf{v}_k\right)A_k - \gamma\mathbf{u}_k^T\mathbf{D}\mathbf{u}_k\mathrm{d}_1 A_k = 0.\end{aligned} \tag{46}$$

By coming back to the true time and not-rescaled variables $\varepsilon\delta \to \delta$, $\varepsilon\gamma \to \gamma$, $\varepsilon^2\Delta p \to \Delta p$, $\varepsilon\mathrm{d}_1 A_1 \to \dot{A}_1$, $\varepsilon^2\mathrm{d}_1^2 A_1 \to \ddot{A}_1$, $\varepsilon\mathrm{d}_1 A_k + \varepsilon^2\mathrm{d}_2 A_k \to \dot{A}_k$, we finally obtain the AME (9), where the following positions hold:

$$\begin{aligned}&\alpha_1 = -\gamma\mathbf{u}_1^T\mathbf{D}\mathbf{u}_1, \qquad \alpha_2 = \frac{\delta^2\Omega^4}{2}\mathbf{u}_1^T\mathbf{B}\mathbf{z}_1 - \Delta p\mathbf{u}_1^T\mathbf{C}_1\mathbf{u}_1, \\ &\alpha_{3k} = -\frac{\gamma}{2}\mathbf{u}_k^T\mathbf{D}\mathbf{u}_k + \frac{\mathrm{i}}{2\omega_k}\left(\Delta p\mathbf{u}_k^T\mathbf{C}_1\mathbf{u}_k - \frac{\delta^2\Omega^4}{4}\mathbf{u}_k^T\mathbf{B}(\mathbf{z}_k^+ + \mathbf{z}_k^-) - \frac{\gamma^2}{4}(\mathbf{u}_k^T\mathbf{D}\mathbf{u}_k)^2 - \gamma^2\omega_k^2\mathbf{u}_k^T\mathbf{D}\mathbf{v}_k\right)\end{aligned} \tag{47}$$

The resonant case $\Omega = 2\omega_j$

With the generating solution (10), the ε-order perturbation equation (6) reads:

$$\begin{aligned}\mathbf{M}\mathrm{d}_0^2\mathbf{q}_1 + \mathbf{C}_0\mathbf{q}_1 = &-2\mathrm{i}\omega_j\mathrm{d}_1 A_j\mathbf{M}\mathbf{u}_j\mathrm{e}^{\mathrm{i}\omega_j t_0} - \frac{\delta\Omega^2}{2}\overline{A}_j\mathbf{B}\mathbf{u}_j\mathrm{e}^{\mathrm{i}\sigma t_1}\mathrm{e}^{\mathrm{i}\omega_j t_0} \\ &-\frac{\delta\Omega^2}{2}2A_1\mathbf{B}\mathbf{u}_1\mathrm{e}^{\mathrm{i}\Omega t_0} - \frac{\delta\Omega^2}{2}A_j\mathbf{B}\mathbf{u}_j\mathrm{e}^{\mathrm{i}(\Omega+\omega_j)t_0} - \mathrm{i}\gamma\omega_j A_j\mathbf{D}\mathbf{u}_j\mathrm{e}^{\mathrm{i}\omega_j t_0} + \mathrm{c.c.}\end{aligned} \tag{48}$$

Removing secular terms we obtain:

$$-2\mathrm{i}\omega_j\mathrm{d}_1 A_j - \frac{\delta\Omega^2}{2}\mathbf{u}_j^T\mathbf{B}\mathbf{u}_j\overline{A}_j\mathrm{e}^{\mathrm{i}\sigma t_1} - \mathrm{i}\gamma\omega_j\mathbf{u}_j^T\mathbf{D}\mathbf{u}_j A_j = 0, \tag{49}$$

With this result, Eq. (48) transforms into:

$$\begin{aligned}\mathbf{M}\mathrm{d}_0^2\mathbf{q}_1 + \mathbf{C}_0\mathbf{q}_1 = &-\frac{\delta\Omega^2}{2}\left(2A_1\mathbf{B}\mathbf{u}_1\mathrm{e}^{\mathrm{i}\Omega t_0} + A_j\mathbf{B}\mathbf{u}_j\mathrm{e}^{\mathrm{i}(\Omega+\omega_j)t_0}\right) \\ &-\frac{\delta\Omega^2}{2}\overline{A}_j\mathbf{b}_j\mathrm{e}^{\mathrm{i}(\Omega-\omega_j)t_0} + \mathrm{i}\gamma\omega_j A_j\mathbf{V}_j e^{\mathrm{i}\omega_j t_0} + \mathrm{c.c.},\end{aligned} \tag{50}$$

where $\mathbf{V}_j$ is defined by Eq. (41) and, moreover,

$$\mathbf{b}_j = \mathbf{B}\mathbf{u}_j - (\mathbf{u}_j^T\mathbf{B}\mathbf{u}_j)\mathbf{M}\mathbf{u}_j. \tag{51}$$

The solution for $\mathbf{q}_1$ takes the form:

$$\mathbf{q}_1 = -\frac{\delta\Omega^2}{2}\left(2A_1\mathbf{z}_1 \mathrm{e}^{\mathrm{i}\Omega t_0} + A_j\mathbf{z}_j^+ \mathrm{e}^{\mathrm{i}(\Omega+\omega_j)t_0} + \overline{A}_j\mathbf{z}_j^- \mathrm{e}^{\mathrm{i}(\Omega-\omega_j)t_0}\right) + \mathrm{i}\gamma\omega_j A_j\mathbf{v}_j e^{\mathrm{i}\omega_j t_0} + \text{c.c.}, \tag{52}$$

where $\mathbf{v}_j$ is the solution of Eq. (44), $\mathbf{z}_1 = \mathbf{z}_1^+ = \mathbf{z}_1^-$, $\mathbf{z}_j^+$ are solutions of Eq. (43), and $\mathbf{z}_j^-$ satisfies the problem

$$\left(\mathbf{C}_0 - (\Omega-\omega_j)^2\mathbf{M}\right)\mathbf{z}_j^- = \mathbf{b}_j. \tag{53}$$

with the arbitrary component of $\mathbf{z}_j^-$ onto the j-th mode taken as zero.

By substituting solution (52) into the ε^2-equations (6), we obtain:

$$\begin{aligned}
\mathbf{M}\mathrm{d}_0^2\mathbf{q}_2 + \mathbf{C}_0\mathbf{q}_2 = &-\mathrm{d}_1^2A_1\mathbf{M}\mathbf{u}_1 - \Delta p A_1\mathbf{C}_1\mathbf{u}_1 + \frac{\delta^2\Omega^4}{2}A_1\mathbf{B}\mathbf{z}_1 - \gamma\mathrm{d}_1A_1\mathbf{D}\mathbf{u}_1\\
&+\Bigg(-\mathrm{d}_1^2A_j\mathbf{M}\mathbf{u}_j - 2\mathrm{i}\omega_j\mathrm{d}_2A_j\mathbf{M}\mathbf{u}_j - \Delta pA_j\mathbf{C}_1\mathbf{u}_j + \frac{\delta^2\Omega^4}{4}A_j\mathbf{B}(\mathbf{z}_j^+ + \mathbf{z}_j^-)\\
&-\gamma\mathrm{d}_1A_j\mathbf{D}\mathbf{u}_j + \gamma^2\omega_j^2A_j\mathbf{D}\mathbf{v}_j + 2\gamma\omega_j^2\mathrm{d}_1A_j\mathbf{M}\mathbf{v}_j\Bigg)\mathrm{e}^{\mathrm{i}\omega_j t_0}\\
&+\left(\mathrm{i}\delta\Omega^2(\Omega-\omega_j)\mathrm{d}_1\overline{A}_j\mathbf{M}\mathbf{z}_j^- + \mathrm{i}\frac{\delta\Omega^2}{2}\gamma\left(\omega_j\mathbf{B}\mathbf{v}_j + (\Omega-\omega_j)\mathbf{D}\mathbf{z}_j^-\right)\overline{A}_j\right)\\
&\times\mathrm{e}^{\mathrm{i}\omega_j t_0}\mathrm{e}^{\mathrm{i}\sigma t_1} + \text{c.c.} + NRT.
\end{aligned} \tag{54}$$

Elimination of secular terms leads to:

$$\begin{aligned}
&-\mathrm{d}_1^2A_1 - \Delta pA_1\mathbf{u}_1^T\mathbf{C}_1\mathbf{u}_1 + \frac{\delta^2\Omega^4}{2}A_1\mathbf{u}_1^T\mathbf{B}\mathbf{z}_1 - \gamma\mathrm{d}_1A_1\mathbf{u}_1^T\mathbf{D}\mathbf{u}_1 = 0,\\
&-\mathrm{d}_1^2A_j - 2\mathrm{i}\omega_j\mathrm{d}_2A_j - \Delta pA_j\mathbf{u}_j^T\mathbf{C}_1\mathbf{u}_j + \frac{\delta^2\Omega^4}{4}A_j\mathbf{u}_j^T\mathbf{B}(\mathbf{z}_j^+ + \mathbf{z}_j^-) - \gamma\mathrm{d}_1A_j\mathbf{u}_j^T\mathbf{D}\mathbf{u}_j\\
&\quad+\gamma^2\omega_j^2A_j\mathbf{u}_j^T\mathbf{D}\mathbf{v}_j + \mathrm{i}\frac{\delta\Omega^2}{2}\gamma\left(\omega_j\mathbf{u}_j^T\mathbf{B}\mathbf{v}_j + (\Omega-\omega_j)\mathbf{u}_j^T\mathbf{D}\mathbf{z}_j^-\right)\overline{A}_j\mathrm{e}^{\mathrm{i}\sigma t_1} = 0.
\end{aligned} \tag{55}$$

To obtain amplitude equations in the real (not-rescaled) quantities, the inverse transformations $\varepsilon\delta \to \delta$, $\varepsilon\gamma \to \gamma$, $\varepsilon^2\Delta p \to \Delta p$, $t_1 \to \varepsilon t$, $\varepsilon\sigma \to \sigma$ are used, together with the reconstruction of the derivatives:

$$\begin{aligned}
&\dot{A}_1 = \varepsilon\mathrm{d}_1A_1, \quad \ddot{A}_1 = \varepsilon^2\mathrm{d}_1^2A_1,\\
&\dot{A}_j = \varepsilon\mathrm{d}_1A_j + \varepsilon^2\mathrm{d}_2A_j.
\end{aligned} \tag{56}$$

Here, $\mathrm{d}_1^2A_1$ is given by the first of Eq. (55), d_1A_j by Eq. (49), and d_2A_j by the second of Eq. (55), once $\mathrm{d}_1^2A_j$ has been evaluated by differentiation of Eq. (49), with the help of Eq. (49) itself. Thus, Eq. (11) are found, where the coefficients α_1 and α_2 are defined by the first line of equations (47) (holding in the non-resonant case), and, moreover,

$$\begin{aligned}
\alpha_3 &= -\frac{\gamma}{2}\mathbf{u}_j^T\mathbf{D}\mathbf{u}_j,\\
\alpha_4 &= \frac{1}{2\omega_j}\left(\Delta p\mathbf{u}_j^T\mathbf{C}_1\mathbf{u}_j + \frac{\delta^2\Omega^4}{16\omega_j^2}(\mathbf{u}_j^T\mathbf{B}\mathbf{u}_j)^2 - \frac{\delta^2\Omega^4}{4}\mathbf{u}_j^T\mathbf{B}(\mathbf{z}_j^+ + \mathbf{z}_j^-) - \frac{\gamma^2}{4}(\mathbf{u}_j^T\mathbf{D}\mathbf{u}_j)^2 - \gamma^2\omega_j^2\mathbf{u}_j^T\mathbf{D}\mathbf{v}_j\right),\\
\alpha_5 &= \gamma\frac{\delta\Omega^2}{4\omega_j}\left(\omega_j\mathbf{u}_j^T\mathbf{B}\mathbf{v}_j + (\Omega-\omega_j)\mathbf{u}_j^T\mathbf{D}\mathbf{z}_j^-\right),\\
\alpha_6 &= \frac{\delta\Omega^2}{4\omega_j}\mathbf{u}_j^T\mathbf{B}\mathbf{u}_j\left(1 - \frac{\sigma}{2\omega_j}\right),
\end{aligned} \tag{57}$$

The resonant case $\Omega = \omega_j/2$

In this case we have $\Omega = \omega_j/2 + \varepsilon\sigma$. Since this is a second-order resonance, the equations (39) and the first-order solution Eq. (42) still hold. By substituting these solutions (having only two terms for $k = 1$ and j) into the ε^2-order equations (6), and accounting for the closeness of Ω to $\omega_j/2$, we obtain

$$\begin{aligned}\mathbf{M}\mathrm{d}_0^2\mathbf{q}_2 + \mathbf{C}_0\mathbf{q}_2 = & -\mathrm{d}_1^2 A_1\mathbf{M}\mathbf{u}_1 - \Delta p A_1\mathbf{C}_1\mathbf{u}_1 + \frac{\delta^2\Omega^4}{2}A_1\mathbf{B}\mathbf{z}_1 - \gamma\mathrm{d}_1 A_1\mathbf{D}\mathbf{u}_1 + \frac{\delta^2\Omega^4}{4}\overline{A}_j\mathbf{B}\mathbf{z}_j^- \mathrm{e}^{2\mathrm{i}\sigma t_1}\\ & + \left(2\gamma\omega_j^2\mathrm{d}_1 A_j\mathbf{M}\mathbf{v}_j - 2\mathrm{i}\omega_j\mathrm{d}_2 A_j\mathbf{M}\mathbf{u}_j - \mathrm{d}_1^2 A_j\mathbf{M}\mathbf{u}_j - \Delta p A_j\mathbf{C}_1\mathbf{u}_j + \frac{\delta^2\Omega^4}{4}A_j\mathbf{B}(\mathbf{z}_j^+ + \mathbf{z}_j^-)\right.\\ & \left. -\gamma\mathrm{d}_1 A_j\mathbf{D}\mathbf{u}_j + \gamma^2\omega_j^2 A_j\mathbf{D}\mathbf{v}_j + \frac{\delta^2\Omega^4}{2}A_1\mathbf{B}\mathbf{z}_1\mathrm{e}^{2\mathrm{i}\sigma t_1}\right)\mathrm{e}^{\mathrm{i}\omega_j t_0} + \text{c.c.} + NRT. \end{aligned} \tag{58}$$

Removal of secular terms requires the frequency-ω_1 terms to be orthogonal to $\mathbf{u}_1$, and frequency-ω_j terms to be orthogonal to $\mathbf{u}_j$. From this condition, by using the same inverse transformations as in previous case, we obtain Eq. (13), where:

$$\begin{aligned} &\alpha_1 = -\gamma\mathbf{u}_1^T\mathbf{D}\mathbf{u}_1, \qquad \alpha_2 = \frac{\delta^2\Omega^4}{2}\mathbf{u}_1^T\mathbf{B}\mathbf{z}_1 - \Delta p\mathbf{u}_1^T\mathbf{C}_1\mathbf{u}_1,\\ &\alpha_3 = \frac{\delta^2\Omega^4}{8}\mathbf{u}_1^T\mathbf{B}\mathbf{z}_j^-, \qquad \alpha_4 = -\frac{\delta^2\Omega^4}{4\omega_j}\mathbf{u}_j^T\mathbf{B}\mathbf{z}_1, \qquad \alpha_5 = -\frac{\gamma}{2}\mathbf{u}_j^T\mathbf{D}\mathbf{u}_j,\\ &\alpha_6 = \frac{1}{2\omega_j}\left(\Delta p\mathbf{u}_j^T\mathbf{C}_1\mathbf{u}_j - \frac{\delta^2\Omega^4}{4}\mathbf{u}_j^T\mathbf{B}_1(\mathbf{z}_j^+ + \mathbf{z}_j^-) - \frac{\gamma^2}{4}(\mathbf{u}_j^T\mathbf{D}\mathbf{u}_j)^2 - \gamma^2\omega_j^2\mathbf{u}_j^T\mathbf{D}\mathbf{v}_j\right). \end{aligned} \tag{59}$$

The sum combination resonance $\Omega = \omega_j + \omega_i$

With the generating solution (18), Eq. (8) reads:

$$\begin{aligned}\mathbf{M}\mathrm{d}_0^2\mathbf{q}_1 + \mathbf{C}_0\mathbf{q}_1 = & -\left(2\mathrm{i}\omega_i\mathrm{d}_1 A_i\mathbf{M}\mathbf{u}_i + \frac{\delta\Omega^2}{2}\overline{A}_j\mathbf{B}\mathbf{u}_j\mathrm{e}^{\mathrm{i}\sigma t_1} + \mathrm{i}\gamma\omega_i A_i\mathbf{D}\mathbf{u}_i\right)\mathrm{e}^{\mathrm{i}\omega_i t_0}\\ & -\left(2\mathrm{i}\omega_j\mathrm{d}_1 A_j\mathbf{M}\mathbf{u}_j + \frac{\delta\Omega^2}{2}\overline{A}_i\mathbf{B}\mathbf{u}_i\mathrm{e}^{\mathrm{i}\sigma t_1} + \mathrm{i}\gamma\omega_j A_j\mathbf{D}\mathbf{u}_j\right)\mathrm{e}^{\mathrm{i}\omega_j t_0}\\ & -\delta\Omega^2 A_1\mathbf{B}\mathbf{u}_1\mathrm{e}^{\mathrm{i}\Omega t_0} - \frac{\delta\Omega^2}{2}\sum_{k=i,j} A_k\mathbf{B}\mathbf{u}_k\mathrm{e}^{\mathrm{i}(\omega_k+\Omega)t_0} + \text{c.c.} \end{aligned} \tag{60}$$

After removing secular terms:

$$\begin{aligned} &-2\mathrm{i}\omega_i\mathrm{d}_1 A_i - \frac{\delta\Omega^2}{2}\overline{A}_j\mathbf{u}_i^T\mathbf{B}\mathbf{u}_j\mathrm{e}^{\mathrm{i}\sigma t} - \mathrm{i}\gamma\omega_i A_i\mathbf{u}_i^T\mathbf{D}\mathbf{u}_i = 0,\\ &-2\mathrm{i}\omega_j\mathrm{d}_1 A_j - \frac{\delta\Omega^2}{2}\overline{A}_i\mathbf{u}_j^T\mathbf{B}\mathbf{u}_i\mathrm{e}^{\mathrm{i}\sigma t} - \mathrm{i}\gamma\omega_j A_j\mathbf{u}_j^T\mathbf{D}\mathbf{u}_j = 0, \end{aligned} \tag{61}$$

which, when used in Eq. (60), transforms this latter into:

$$\begin{aligned}\mathbf{M}\mathrm{d}_0^2\mathbf{q}_1 + \mathbf{C}_0\mathbf{q}_1 = & \sum_{k=i,j}\left(-\frac{\delta\Omega^2}{2}A_k\mathbf{B}\mathbf{u}_k\mathrm{e}^{\mathrm{i}(\Omega+\omega_k)t_0} - \frac{\delta\Omega^2}{2}\overline{A}_k\mathbf{b}_k\mathrm{e}^{\mathrm{i}(\Omega-\omega_k)t_0} + \mathrm{i}\gamma\omega_k A_k\mathbf{V}_k\mathrm{e}^{\mathrm{i}\omega_k t_0}\right)\\ & -\delta\Omega^2 A_1\mathbf{B}\mathbf{u}_1\mathrm{e}^{\mathrm{i}\Omega t_0} + \text{c.c.}, \end{aligned} \tag{62}$$

where $\mathbf{V}_k$ is defined by Eq. (41) and:

$$\mathbf{b}_i = \mathbf{B}\mathbf{u}_i - (\mathbf{u}_j^T\mathbf{B}\mathbf{u}_i)\mathbf{M}\mathbf{u}_j, \quad \mathbf{b}_j = \mathbf{B}\mathbf{u}_j - (\mathbf{u}_i^T\mathbf{B}\mathbf{u}_j)\mathbf{M}\mathbf{u}_i. \tag{63}$$

The solution for $\mathbf{q}_1$ takes the form:

$$\mathbf{q}_1 = -\frac{\delta\Omega^2}{2}\sum_{k=1,i,j}\left(A_k\mathbf{z}_k^{+}\mathrm{e}^{\mathrm{i}(\Omega+\omega_k)t_0} + \overline{A}_k\mathbf{z}_k^{-}\mathrm{e}^{\mathrm{i}(\Omega-\omega_k)t_0}\right) + \mathrm{i}\gamma\sum_{k=i,j}\omega_k A_k\mathbf{v}_k\mathrm{e}^{\mathrm{i}\omega_k t_0} + \text{c.c.}, \tag{64}$$

where $\mathbf{z}_1 = \mathbf{z}_1^{+} = \mathbf{z}_1^{-}$, $\mathbf{z}_j^{+}$, $\mathbf{z}_i^{+}$ are the solutions of equations (43) when $k = 1, j, i$, respectively; $\mathbf{v}_j$, $\mathbf{v}_i$ are the solutions of equations (44) when $k = j, i$; finally $\mathbf{z}_i^{-}$, $\mathbf{z}_j^{-}$ are the solutions of the equation (53), where j is replaced either by i, j.

With these results, the ε^2-equations (6) becomes:

$$\begin{aligned}\mathbf{M}\mathrm{d}_0^2\mathbf{q}_2 + \mathbf{C}_0\mathbf{q}_2 = &\sum_{k=1,i,j}\Big(-\mathrm{d}_1^2A_k\mathbf{M}\mathbf{u}_k - 2\mathrm{i}\omega_k\mathrm{d}_2A_k\mathbf{M}\mathbf{u}_k - \Delta pA_k\mathbf{C}_1\mathbf{u}_k + \frac{\delta^2\Omega^4}{4}A_k\mathbf{B}(\mathbf{z}_k^{+} + \mathbf{z}_k^{-})\\ &-\gamma\mathrm{d}_1A_k\mathbf{D}\mathbf{u}_k + \gamma^2\omega_k^2A_k\mathbf{D}\mathbf{v}_k + 2\gamma\omega_k^2\mathrm{d}_1A_k\mathbf{M}\mathbf{v}_k\Big)\,\mathrm{e}^{\mathrm{i}\omega_k t_0}\\ &+\left(\mathrm{i}\delta\Omega^2(\Omega-\omega_j)\mathrm{d}_1\overline{A}_j\mathbf{M}\mathbf{z}_j^{-} + \mathrm{i}\frac{\delta\Omega^2}{2}\gamma\left(\omega_j\mathbf{B}\mathbf{v}_j + (\Omega-\omega_j)\mathbf{D}\mathbf{z}_j^{-}\right)\overline{A}_j\right)\mathrm{e}^{\mathrm{i}\omega_i t_0}\mathrm{e}^{\mathrm{i}\sigma t_1}\\ &+\left(\mathrm{i}\delta\Omega^2(\Omega-\omega_i)\mathrm{d}_1\overline{A}_i\mathbf{M}\mathbf{z}_i^{-} + \mathrm{i}\frac{\delta\Omega^2}{2}\gamma\left(\omega_i\mathbf{B}\mathbf{v}_i + (\Omega-\omega_i)\mathbf{D}\mathbf{z}_i^{-}\right)\overline{A}_i\right)\\ &\times\mathrm{e}^{\mathrm{i}\omega_j t_0}\mathrm{e}^{\mathrm{i}\sigma t_1} + \text{c.c.} + NRT.\end{aligned} \tag{65}$$

Elimination of secular terms leads to:

$$\begin{aligned}&-\mathrm{d}_1^2A_i - 2\mathrm{i}\omega_i\mathrm{d}_2A_i - \Delta pA_i\mathbf{u}_i^T\mathbf{C}_1\mathbf{u}_i + \frac{\delta^2\Omega^4}{4}A_i\mathbf{u}_i^T\mathbf{B}(\mathbf{z}_i^{+} + \mathbf{z}_i^{-}) - \gamma\mathrm{d}_1A_i\mathbf{u}_i^T\mathbf{D}\mathbf{u}_i\\ &\quad+\gamma^2\omega_i^2A_i\mathbf{u}_i^T\mathbf{D}\mathbf{v}_i + \mathrm{i}\frac{\delta\Omega^2}{2}\gamma\left(\omega_j\mathbf{u}_i^T\mathbf{B}\mathbf{v}_j + (\Omega-\omega_j)\mathbf{u}_i^T\mathbf{D}\mathbf{z}_j^{-}\right)\overline{A}_j\mathrm{e}^{\mathrm{i}\sigma t_1} = 0,\\ &-\mathrm{d}_1^2A_j - 2\mathrm{i}\omega_j\mathrm{d}_2A_j - \Delta pA_j\mathbf{u}_j^T\mathbf{C}_1\mathbf{u}_j + \frac{\delta^2\Omega^4}{4}A_j\mathbf{u}_j^T\mathbf{B}(\mathbf{z}_j^{+} + \mathbf{z}_j^{-}) - \gamma\mathrm{d}_1A_j\mathbf{u}_j^T\mathbf{D}\mathbf{u}_j\\ &\quad+\gamma^2\omega_j^2A_j\mathbf{u}_j^T\mathbf{D}\mathbf{v}_j + \mathrm{i}\frac{\delta\Omega^2}{2}\gamma\left(\omega_i\mathbf{u}_j^T\mathbf{B}\mathbf{v}_i + (\Omega-\omega_i)\mathbf{u}_j^T\mathbf{D}\mathbf{z}_i^{-}\right)\overline{A}_i\mathrm{e}^{\mathrm{i}\sigma t_1} = 0.\end{aligned} \tag{66}$$

The equation for A_1 coincides with the first of equations (55). By using the inverse transformations and reconstruction (56), we obtain the Eq. (19), where:

$$\begin{aligned}\beta_{1i} &= -\frac{\gamma}{2}\mathbf{u}_i^T\mathbf{D}\mathbf{u}_i,\\ \beta_{2i} &= \frac{1}{2\omega_i}\left(\Delta p\mathbf{u}_i^T\mathbf{C}_1\mathbf{u}_i + \frac{\delta^2\Omega^4}{16\omega_i\omega_j}(\mathbf{u}_i^T\mathbf{B}\mathbf{u}_j)(\mathbf{u}_j^T\mathbf{B}\mathbf{u}_i) - \frac{\delta^2\Omega^4}{4}\mathbf{u}_i^T\mathbf{B}(\mathbf{z}_i^{+} + \mathbf{z}_i^{-}) - \frac{\gamma^2}{4}(\mathbf{u}_i^T\mathbf{D}\mathbf{u}_i)^2 - \gamma^2\omega_i^2\mathbf{u}_i^T\mathbf{D}\mathbf{v}_i\right),\\ \beta_{3i} &= \gamma\frac{\delta\Omega^2}{4\omega_i}\left(\omega_j\mathbf{u}_i^T\mathbf{B}\mathbf{v}_j + (\Omega-\omega_j)\mathbf{u}_i^T\mathbf{D}\mathbf{z}_j^{-} + \frac{1}{4\omega_i}\mathbf{u}_i^T\mathbf{B}\mathbf{u}_j(\mathbf{u}_j^T\mathbf{D}\mathbf{u}_j - \mathbf{u}_i^T\mathbf{D}\mathbf{u}_i)\right),\\ \beta_{4i} &= \frac{\delta\Omega^2}{4\omega_i}\mathbf{u}_i^T\mathbf{B}\mathbf{u}_j\left(1 - \frac{\sigma}{2\omega_i}\right)\end{aligned} \tag{67}$$

and β_{1j}, β_{2j}, β_{3j}, β_{4j} are defined by replacing the indexes i to j and j to i in Eq. (67).

The difference combination resonance $\boldsymbol{\Omega = \omega_j - \omega_i}$

When the generating solution (18) is used into the ε^1- order perturbation equation (6), this latter reads:

$$\begin{aligned} \mathbf{M}\mathrm{d}_0^2\mathbf{q}_1 + \mathbf{C}_0\mathbf{q}_1 &= -\frac{\delta\Omega^2}{2}\left(2A_1\mathbf{B}\mathbf{u}_1\mathrm{e}^{\mathrm{i}\Omega t_0} + \overline{A}_i\mathbf{B}\mathbf{u}_i e^{\mathrm{i}(\Omega-\omega_i)t_0} + A_j\mathbf{B}\mathbf{u}_j e^{\mathrm{i}(\Omega+\omega_j)t_0}\right) \\ &- \left(2\mathrm{i}\omega_i\mathrm{d}_1A_i\mathbf{M}\mathbf{u}_i + \frac{\delta\Omega^2}{2}A_j\mathbf{B}\mathbf{u}_j\mathrm{e}^{-\mathrm{i}\sigma t_1} + \mathrm{i}\gamma\omega_iA_i\mathbf{D}\mathbf{u}_i\right)\mathrm{e}^{\mathrm{i}\omega_i t_0} \\ &- \left(2\mathrm{i}\omega_j\mathrm{d}_1A_j\mathbf{M}\mathbf{u}_j + \frac{\delta\Omega^2}{2}A_i\mathbf{B}\mathbf{u}_i\mathrm{e}^{\mathrm{i}\sigma t_1} + \mathrm{i}\gamma\omega_jA_j\mathbf{D}\mathbf{u}_j\right)\mathrm{e}^{\mathrm{i}\omega_j t_0} + \text{c.c.} \end{aligned} \tag{68}$$

After eliminating the secular terms:

$$\begin{aligned} -2\mathrm{i}\omega_i\mathrm{d}_1A_i - \frac{\delta\Omega^2}{2}A_j\mathbf{u}_i^T\mathbf{B}\mathbf{u}_j\mathrm{e}^{-\mathrm{i}\sigma t} - \mathrm{i}\gamma\omega_iA_i\mathbf{D}\mathbf{u}_i = 0, \\ -2\mathrm{i}\omega_j\mathrm{d}_1A_j - \frac{\delta\Omega^2}{2}A_i\mathbf{u}_j^T\mathbf{B}\mathbf{u}_i\mathrm{e}^{\mathrm{i}\sigma t} - \mathrm{i}\gamma\omega_jA_j\mathbf{D}\mathbf{u}_j = 0, \end{aligned} \tag{69}$$

The solution for $\mathbf{q}_1$ will have the form (64), where $\mathbf{z}_1 = \mathbf{z}_1^+ = \mathbf{z}_1^-$, $\mathbf{z}_i^-$ and $\mathbf{z}_j^+$ are solutions of equations (43), $\mathbf{v}_i$ and $\mathbf{v}_j$ are the solutions of equations (44) and $\mathbf{z}_i^+$ and $\mathbf{z}_j^-$ are the solutions of the following problems:

$$\left(\mathbf{C}_0 - (\Omega+\omega_i)^2\mathbf{M}\right)\mathbf{z}_i^+ = \mathbf{b}_i, \qquad \left(\mathbf{C}_0 - (\Omega-\omega_j)^2\mathbf{M}\right)\mathbf{z}_j^- = \mathbf{b}_j. \tag{70}$$

Here $\mathbf{b}_i$ and $\mathbf{b}_j$ are defined by Eq. (63).

After substitution the expressions for $\mathbf{q}_0$ and $\mathbf{q}_1$ to ε^2-equations (6) and elimination of secular terms, by using the inverse transformations and reconstruction (56), we finally get Eq. (21), where:

$$\begin{aligned} &\beta_{1i} = -\frac{\gamma}{2}\mathbf{u}_i^T\mathbf{D}\mathbf{u}_i, \quad \beta_{1j} = -\frac{\gamma}{2}\mathbf{u}_j^T\mathbf{D}\mathbf{u}_j, \\ &\beta_{2i} = \frac{1}{2\omega_i}\left(\Delta p\mathbf{u}_i^T\mathbf{C}_1\mathbf{u}_i - \frac{\delta^2\Omega^4}{16\omega_i\omega_j}(\mathbf{u}_i^T\mathbf{B}\mathbf{u}_j)(\mathbf{u}_j^T\mathbf{B}\mathbf{u}_i) - \frac{\delta^2\Omega^4}{4}\mathbf{u}_i^T\mathbf{B}(\mathbf{z}_i^+ + \mathbf{z}_i^-) - \frac{\gamma^2}{4}(\mathbf{u}_i^T\mathbf{D}\mathbf{u}_i)^2 - \gamma^2\omega_i^2\mathbf{u}_i^T\mathbf{D}\mathbf{v}_i\right), \\ &\beta_{2j} = \frac{1}{2\omega_j}\left(\Delta p\mathbf{u}_j^T\mathbf{C}_1\mathbf{u}_j - \frac{\delta^2\Omega^4}{16\omega_i\omega_j}(\mathbf{u}_i^T\mathbf{B}\mathbf{u}_j)(\mathbf{u}_j^T\mathbf{B}\mathbf{u}_i) - \frac{\delta^2\Omega^4}{4}\mathbf{u}_j^T\mathbf{B}(\mathbf{z}_j^+ + \mathbf{z}_j^-) - \frac{\gamma^2}{4}(\mathbf{u}_j^T\mathbf{D}\mathbf{u}_j)^2 - \gamma^2\omega_j^2\mathbf{u}_j^T\mathbf{D}\mathbf{v}_j\right), \\ &\beta_{3i} = -\gamma\frac{\delta\Omega^2}{4\omega_i}\left(\omega_j\mathbf{u}_i^T\mathbf{B}\mathbf{v}_j + (\Omega-\omega_j)\mathbf{u}_i^T\mathbf{D}\mathbf{z}_j^- - \frac{1}{4\omega_i}\mathbf{u}_i^T\mathbf{B}\mathbf{u}_j(\mathbf{u}_j^T\mathbf{D}\mathbf{u}_j - \mathbf{u}_i^T\mathbf{D}\mathbf{u}_i)\right), \\ &\beta_{3j} = \gamma\frac{\delta\Omega^2}{4\omega_j}\left(-\omega_i\mathbf{u}_j^T\mathbf{B}\mathbf{v}_i + (\Omega+\omega_i)\mathbf{u}_j^T\mathbf{D}\mathbf{z}_i^+ - \frac{1}{4\omega_j}\mathbf{u}_j^T\mathbf{B}\mathbf{u}_i(\mathbf{u}_j^T\mathbf{D}\mathbf{u}_j - \mathbf{u}_i^T\mathbf{D}\mathbf{u}_i)\right), \\ &\beta_{4i} = \frac{\delta\Omega^2}{4\omega_i}\mathbf{u}_i^T\mathbf{B}\mathbf{u}_j\left(1+\frac{\sigma}{2\omega_i}\right), \quad \beta_{4j} = \frac{\delta\Omega^2}{4\omega_j}\mathbf{u}_j^T\mathbf{B}\mathbf{u}_i\left(1-\frac{\sigma}{2\omega_j}\right), \end{aligned} \tag{71}$$

Appendix 2: The fractional asymptotic expansions

Here, we will explain in more detail how to get AME in the fractional expansion case. By solving Eq. (30) with the help of Eq. (31), we get:

$$\mathbf{q}_2 = -\frac{\delta\Omega^2}{2}\sum_{k=1,j}\left(A_k\mathbf{z}_k^+\mathrm{e}^{\mathrm{i}(\Omega+\omega_k)t_0} + \overline{A}_k\mathbf{z}_k^-\mathrm{e}^{\mathrm{i}(\Omega-\omega_k)t_0}\right) + \mathrm{i}\gamma\omega_jA_j\mathbf{v}_je^{\mathrm{i}\omega_jt_0} + \text{c.c.} \tag{72}$$

Above, $\mathbf{z}_1^- = \mathbf{z}_1^+ = \mathbf{z}_1$ and $\mathbf{z}_j^-$ are the solutions of the problems:

$$\begin{aligned} \left(\mathbf{C}_0 - \Omega^2\mathbf{M}\right)\mathbf{z}_1 = \mathbf{B}\mathbf{u}_1 - (\mathbf{u}_j^T\mathbf{B}\mathbf{u}_1)\mathbf{M}\mathbf{u}_j, \\ \left(\mathbf{C}_0 - (\Omega-\omega_j)^2\mathbf{M}\right)\mathbf{z}_j^- = \mathbf{B}\mathbf{u}_j - (\mathbf{u}_1^T\mathbf{B}\mathbf{u}_j)\mathbf{M}\mathbf{u}_1 \end{aligned} \tag{73}$$

and $\mathbf{z}_j^+$ and $\mathbf{v}_j$ are the solutions of equations (43) and (44). Note that $\mathbf{u}_j^T\mathbf{M}\mathbf{z}_1 = 0$, $\mathbf{u}_1^T\mathbf{M}\mathbf{z}_j^- = 0$ and $\mathbf{u}_j^T\mathbf{M}\mathbf{v}_j = 0$.

In the next step we obtain the equation for $\mathbf{q}_3$:

$$\begin{aligned}\mathbf{M}\mathrm{d}_0^2\mathbf{q}_3+\mathbf{C}_0\mathbf{q}_3 =& -2\mathrm{d}_1\mathrm{d}_2A_1\mathbf{M}\mathbf{u}_1-\frac{\delta\Omega^2}{2}\overline{B}_j\mathbf{B}\mathbf{u}_j\mathrm{e}^{\mathrm{i}\sigma t_1}-\gamma\mathrm{d}_1A_1\mathbf{D}\mathbf{u}_1-\frac{\delta\Omega^2}{2}B_j\mathbf{B}\mathbf{u}_j\mathrm{e}^{\mathrm{i}(\Omega+\omega_j)t_0}\\ &+\left(-2\mathrm{i}\omega_j(\mathrm{d}_3A_j+\mathrm{d}_2B_j)\mathbf{M}\mathbf{u}_j-\mathrm{d}_1^2B_j\mathbf{M}\mathbf{u}_j+2\mathrm{i}\delta\Omega^3\mathrm{d}_1A_1\mathbf{M}\mathbf{z}_1\mathrm{e}^{\mathrm{i}\sigma t_1}-\mathrm{i}\gamma\omega_jB_j\mathbf{D}\mathbf{u}_j\right)\\ &\times\mathrm{e}^{\mathrm{i}\omega_jt_0}+\mathrm{c.c.},\end{aligned}\tag{74}$$

from which, by removing secular terms, we get Eq. (32). Then, we have the solution:

$$\begin{aligned}\mathbf{q}_3=&-\frac{\delta\Omega^2}{2}\left(B_j\mathbf{z}_j^+\mathrm{e}^{\mathrm{i}(\Omega+\omega_j)t_0}+\overline{B}_j\mathbf{z}_j^-\mathrm{e}^{\mathrm{i}(\Omega-\omega_j)t_0}\right)+2\mathrm{i}\delta\Omega^3\mathrm{d}_1A_1\mathbf{g}_1\mathrm{e}^{\mathrm{i}\Omega t_0}\\ &+\mathrm{i}\gamma\omega_jB_j\mathbf{v}_je^{\mathrm{i}\omega_jt_0}+\gamma\mathrm{d}_1A_1\mathbf{v}_1+\mathrm{c.c.},\end{aligned}\tag{75}$$

where $\mathbf{g}_1$ is the solution of equation $\left(\mathbf{C}_0-\Omega^2\mathbf{M}\right)\mathbf{g}_1=\mathbf{M}\mathbf{z}_1$ and $\mathbf{u}_j^T\mathbf{M}\mathbf{g}_1=0$ and $\mathbf{v}_1$ is the solution of equation $\mathbf{C}_0\mathbf{v}_1=(\mathbf{u}_1^T\mathbf{D}\mathbf{u}_1)\mathbf{M}\mathbf{u}_1-\mathbf{D}\mathbf{u}_1$.

The equations of the ε^2-order will be treated next:

$$\begin{aligned}\mathbf{M}\mathrm{d}_0^2\mathbf{q}_4+\mathbf{C}_0\mathbf{q}_4=&-\left(2\mathrm{d}_1\mathrm{d}_3A_1+\mathrm{d}_2^2A_1\right)\mathbf{M}\mathbf{u}_1-\Delta pA_1\mathbf{C}_1\mathbf{u}_1+\frac{\delta^2\Omega^4}{2}A_1\mathbf{B}\mathbf{z}_1-\gamma\mathrm{d}_2A_1\mathbf{D}\mathbf{u}_1\\ &+\mathrm{i}\delta\Omega^2(\Omega-\omega_j)(\mathrm{d}_2\overline{A}_j+\mathrm{d}_1\overline{B}_j)\mathbf{M}\mathbf{z}_j^-\mathrm{e}^{\mathrm{i}\sigma t_1}+\mathrm{i}\gamma\frac{\delta\Omega^2}{2}\left((\Omega-\omega_j)\mathbf{D}\mathbf{z}_j^-+\omega_j\mathbf{B}\mathbf{v}_j\right)\overline{A}_j\mathrm{e}^{\mathrm{i}\sigma t_1}\\ &+\Bigg(-\left(2\mathrm{i}\omega_j(\mathrm{d}_4A_j+\mathrm{d}_3B_j)+\mathrm{d}_2^2A_j+2\mathrm{d}_1\mathrm{d}_2B_j\right)\mathbf{M}\mathbf{u}_j-\Delta pA_j\mathbf{C}_1\mathbf{u}_j+\frac{\delta^2\Omega^4}{4}A_j\mathbf{B}(\mathbf{z}_j^++\mathbf{z}_j^-)\\ &+\frac{\delta^2\Omega^4}{4}\overline{A}_j\mathbf{B}\mathbf{z}_j^-\mathrm{e}^{2\mathrm{i}\sigma t_1}+\delta\Omega^2\left((\mathrm{d}_1^2A_1+2\mathrm{i}\Omega\mathrm{d}_2A_1)\mathbf{M}\mathbf{z}_1+4\Omega^2\mathrm{d}_1^2A_1\mathbf{M}\mathbf{g}_1\right)\mathrm{e}^{\mathrm{i}\sigma t_1}\\ &+\gamma(\mathrm{d}_2A_j+\mathrm{d}_1B_j)(2\omega_j^2\mathbf{M}\mathbf{v}_j-\mathbf{D}\mathbf{u}_j)+\mathrm{i}\gamma\delta\Omega^3A_1\mathbf{D}\mathbf{z}_1\mathrm{e}^{\mathrm{i}\sigma t_1}+\gamma^2\omega_j^2A_j\mathbf{D}\mathbf{v}_j\Bigg)\mathrm{e}^{\mathrm{i}\omega_jt_0}\\ &+\mathrm{NRT}+c.c.\end{aligned}\tag{76}$$

Elimination of secular terms leads to Eq. (33).

Recombination finally supplies Eq. (35), where the following positions hold:

$$\begin{aligned}&\alpha_1=-\gamma\mathbf{u}_1^T\mathbf{D}\mathbf{u}_1,\qquad\alpha_2=\frac{\delta^2\Omega^4}{2}\mathbf{u}_1^T\mathbf{B}\mathbf{z}_1-\Delta p\mathbf{u}_1^T\mathbf{C}_1\mathbf{u}_1,\qquad\alpha_3=-\frac{\delta\Omega^2}{4}\mathbf{u}_1^T\mathbf{B}\mathbf{u}_j,\\ &\alpha_4=\gamma\frac{\delta\Omega^2}{4}\left(\sigma\mathbf{u}_1^T\mathbf{D}\mathbf{z}_j^-+\omega_j\mathbf{u}_1^T\mathbf{B}\mathbf{v}_j\right),\qquad\alpha_5=-\frac{\delta\Omega^2}{4\omega_j^2}\mathbf{u}_j^T\mathbf{B}\mathbf{u}_1\left(1-\frac{\sigma}{\omega_j}\right),\\ &\alpha_6=\gamma\frac{\delta\Omega^2}{2\omega_j}\left(\Omega\mathbf{u}_j^T\mathbf{D}\mathbf{z}_1-\frac{(\mathbf{u}_j^T\mathbf{D}\mathbf{u}_j)(\mathbf{u}_j^T\mathbf{B}\mathbf{u}_1)}{4\omega_j}\right),\qquad\alpha_7=\frac{\delta\Omega^2}{2\omega_j}\mathbf{u}_j^T\mathbf{B}\mathbf{u}_1\left(1-\frac{\sigma}{2\omega_j}+\frac{\sigma^2}{4\omega_j^2}\right),\\ &\alpha_8=\frac{1}{2\omega_j}\left(\frac{\delta^2\Omega^4}{16\omega_j^2}(\mathbf{u}_j^T\mathbf{B}\mathbf{u}_1)(\mathbf{u}_1^T\mathbf{B}\mathbf{u}_j)+\Delta p\mathbf{u}_j^T\mathbf{C}_1\mathbf{u}_j-\frac{\delta^2\Omega^4}{4}\mathbf{u}_j^T\mathbf{B}(\mathbf{z}_j^++\mathbf{z}_j^-)-\gamma^2\omega_j^2\mathbf{u}_j^T\mathbf{D}\mathbf{v}_j-\frac{\gamma^2}{4}(\mathbf{u}_j^T\mathbf{D}\mathbf{u}_j)^2\right),\\ &\alpha_9=-\frac{\gamma}{2}\mathbf{u}_j^T\mathbf{D}\mathbf{u}_j,\qquad\alpha_{10}=\frac{\delta^2\Omega^4}{8\omega_j}\left(\frac{1}{4\omega_j^2}(\mathbf{u}_j^T\mathbf{B}\mathbf{u}_1)(\mathbf{u}_1^T\mathbf{B}\mathbf{u}_j)-\mathbf{u}_j^T\mathbf{B}\mathbf{z}_j^-\right).\end{aligned}\tag{77}$$

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