Z. Angew. Math. Phys. (2016) 67:60

0044-2275/16/030001-18
published online May 4, 2016
DOI 10.1007/s00033-016-0655-x

Zeitschrift für angewandte Mathematik und Physik ZAMP

The beam equation with nonlinear memory

Marcello D'Abbicco and Sandra Lucente

Abstract. In this paper, we study the critical exponent for the beam equation with nonlinear memory, i.e., $u_{tt}+\Delta^2 u = F(t,u)$, where

$$F = \int_0^t f(t-s)N(u)(s,x)\,\mathrm{d}s, \quad N(u) \approx |u|^p.$$

For suitable f and p, we prove the existence of local-in-time solutions and small data global solutions to the Cauchy problem, in homogeneous and nonhomogeneous Sobolev spaces. In some cases, we prove that the local solution cannot be extended to a global one. We also consider the limit case of power nonlinearity, i.e., $F = N(u)$.

Mathematics Subject Classification. 35A01 · 35G25.

1. Introduction

We consider the existence theory for the following Cauchy problem

$$\begin{cases} u_{tt} + \Delta^2 u = \int_0^t f(t-s)N(u)(s,x)\,\mathrm{d}s, & t \geq 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x) \end{cases} \tag{1.1}$$

being $\Delta^2 = \Delta_x \Delta_x$, $N(u)$ a polynomial type nonlinear term and $f(t)$ a convolution kernel with respect to the time variable. In particular, formally setting $f = \delta$, Dirac distribution, we find a standard polynomial type nonlinearity $N(u)(t,x)$. For $f(t) = c_\gamma t^{-\gamma}$, $\gamma \in (0,1)$, where $c_\gamma > 0$, we have a nonlinear memory term representing a (Riemann-Liouville) fractional integral of order $1-\gamma$ of $N(u)(t,x)$. Indeed,

$$\lim_{z\to 1} \frac{s_+^{-z}}{\Gamma(1-z)} = \delta(z),$$

where

$$s_+^{-z} = \begin{cases} s^{-z} & s > 0, \\ 0 & s < 0 \end{cases} \quad \operatorname{Re} z < 1,$$

$$s_+^{-z} = \frac{1}{1-z}\frac{d}{\mathrm{d}s}s^{1-z} \quad 1 \leq \operatorname{Re} z < 2, \quad z \neq 1,$$

in distribution sense and Γ is the Euler Gamma function. This special choice for the kernel $f(t)$ motivated us to study the forward Cauchy problem in this paper, but all of our existence results also apply to the backward Cauchy problem.

The linear operator $\partial_t^2 + \Delta^2$ is known as Germain–Lagrange operator, as well as beam operator, in particular for $n = 1$, or plate operator, in particular for $n = 2$. It models a vibration of an elastic surface,

Birkhäuser

and it is a 2-evolution operator, in the sense of Petrowsky, since its (full) symbol $\tau^2 + |\xi|^4$ has purely imaginary roots $\pm i|\xi|^2$.

The linear beam operator inherits some but not all properties from Schrödinger operator. In particular, we do not have the mass conservation law, since in the beam equation the coefficients are real. On the other hand, the functional representation of the solution contains oscillations like it happens for the wave operator. However, the beam operator is not Kovalesvkian and we do not have the finite speed of propagation.

There has been a growing interest in recent years for memory terms, which may describe hereditary processes. Memories of polynomial type nonlinearities have been considered in [3] for the heat equation and recently for damped wave type equations, which inherit decay properties from the corresponding diffusion equation. Nondamped models require a different approach and, up to our knowledge, no result is known for the beam equation with nonlinear memory. Also, only few results are known for the beam equation with polynomial type nonlinearity (see Sect. 5).

We assume that $N(0) = 0$ and

$$|N(u) - N(v)| \lesssim |u - v| \left(|u|^{p-1} + |v|^{p-1}\right) \quad \text{for some } p > 1. \tag{N0}$$

Typical examples of such nonlinearity are $N(u) = |u|^p$ or $N(u) = \pm|u|^{p-1}u$, $p > 1$.

In this paper, we will consider, in general, mixed space–time norms, according to the following.

Definition 1.1. For any $T \in (0, \infty]$, and for fixed $q \in [1, \infty]$, the mixed space–time norm $\| \cdot \|_{L^q_T X}$ over $[0, T) \times \mathbb{R}^n$ is given by

$$\|u\|_{L^q_T X} := \left(\int_0^T \|u(\tau, \cdot)\|^q_{X(\mathbb{R}^n)} \, d\tau \right)^{1/q}.$$

If $T = \infty$, we may omit the time index writing $L^q X$.

In particular, in Definition 1.1 we will consider homogeneous or nonhomogeneous Sobolev spaces $X = \dot{W}^{s,r}$ or $X = W^{s,r}$, with $s \in \mathbb{R}$, $r \in (1, \infty)$, see Appendix.

We first study the existence of the solution in homogeneous Sobolev spaces.

Definition 1.2. Let $T > 0$ and $s \in [0, 2]$. A $\dot{H}^s$ (mild) local solution to (1.1) is a function $u \in L^\infty_T \dot{H}^s$ such that $u_t \in L^\infty_T \dot{H}^{s-2}$, and

$$u(t, x) = \cos(t\Delta)u_0(x) + \frac{\sin(t\Delta)}{\Delta} u_1(x) + \int_0^t \frac{\sin((t-s)\Delta)}{\Delta} \int_0^s f(s-\tau)\, N(u)(\tau, x) \, \mathrm{d}\tau \, \mathrm{d}s, \tag{1.2}$$

where each term of (1.2) belongs to $L^\infty_T \dot{H}^s$. If $T = \infty$, we say that the $\dot{H}^s$ solution is global.

Theorem 1.1. *Let $n \geq 5$ and $s \in [0, 2]$, or $n \geq 4$ and $s \in (0, 2)$, or $n = 3$ and $s \in (1/2, 3/2)$. Let $f \in L^\theta_{T_0}$, for some $\theta \in [1, 2]$, $T_0 \in (0, \infty)$, and let $N(u)$ satisfy (N0) with*

$$1 + \frac{4}{n - 2s} \leq p < 1 + \frac{4(3 - \theta^{-1})}{n - 2s}. \tag{1.3}$$

Assume $(u_0, u_1) \in \dot{H}^s \times \dot{H}^{s-2}$. Then, there exists $T \in (0, T_0]$ and a unique $\dot{H}^s$ local solution in $[0, T)$ to (1.2).

Theorem 1.2. *Let $n \geq 5$ and $s \in [0, 2]$, or $n \geq 4$ and $s \in (0, 2)$, or $n = 3$ and $s \in (1/2, 3/2)$. Let $f \in L^\theta$, for some $\theta \in [1, 2]$ and let $N(u)$ satisfy (N0) with*

$$p = 1 + \frac{4(3 - \theta^{-1})}{n - 2s}. \tag{1.4}$$

Then, there exists $\varepsilon > 0$ such that if $(u_0, u_1) \in \dot{H}^s \times \dot{H}^{s-2}$ with $\|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-2}} < \varepsilon$, then there exists a unique global $\dot{H}^s$ solution to (1.2). Moreover, it holds $\|u\|_{L^\infty_T \dot{H}^s} < \varepsilon$.

By Theorem 1.2, we derive the following.

Corollary 1.1. *Let $n \geq 3$, $f \in L^\theta$, for some $\theta \in [1, 2]$, and $N(u)$ satisfies (N0) for some*

$$\begin{aligned} 1 + \frac{4(3-\theta^{-1})}{n} \leq p \leq 1 + \frac{4(3-\theta^{-1})}{n-4}, \quad &\text{if } n \geq 5, \\ 1 + \frac{4(3-\theta^{-1})}{n} = 4 - \theta^{-1} < p < \infty, \quad &\text{if } n = 4, \\ 1 + \frac{4(3-\theta^{-1})}{n-1} = 7 - 2\theta^{-1} < p < \infty, \quad &\text{if } n = 3. \end{aligned}$$

Let

$$s = s(p, n) := \frac{n}{2} - \frac{2(3-\theta^{-1})}{p-1}.$$

Then, there exists $\varepsilon > 0$ such that if $(u_0, u_1) \in \dot{H}^s \times \dot{H}^{s-2}$ with $\|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-2}} < \varepsilon$, then there exists a unique global $\dot{H}^s$ solution to (1.2). Moreover, it holds $\|u\|_{L^\infty_T \dot{H}^s} < \varepsilon$.

Now we consider nonhomogeneous Sobolev spaces.

Definition 1.3. Let $T > 0$. A H^s local solution of (1.1), $s > 0$, is a function $u \in L^\infty_T H^s$ such that $u_t \in L^\infty_T \dot{H}^{-2} \cap L^\infty_T \dot{H}^{s-2}$, each term of (1.2) is in $L^\infty_T H^s$ and u satisfies (1.2).

Theorem 1.3. *Let $n \geq 5$ and $s \in (0, 2]$. Let $f \in L^\theta_{T_0}$, for some $\theta \in [1, 2]$ and $T_0 > 0$, and let $N(u)$ satisfy (N0) with*

$$1 + \frac{4}{n} \leq p < 1 + \frac{4(3-\theta^{-1})}{n-2s}. \tag{1.5}$$

Assume $(u_0, u_1) \in H^s \times (\dot{H}^{s-2} \cap \dot{H}^{-2})$. Then, there exists $T \in (0, T_0)$ and a unique $L^\infty_T H^s$ local solution to (1.2).

The restriction $s \leq 2$ in Theorems 1.1–1.3 is consistent with the fact that $\dot{H}^2$ is the energy space of the beam equation.

The previous local existence results may be directly applied to the case in which the nonlinear term represents a fractional integration of a polynomial type nonlinear term.

Example 1.1. Let $f(t) = c_\gamma t^{-\gamma}$, for some $\gamma \in (0, 1)$ and $c_\gamma > 0$, i.e., we are considering a fractional integral of order $1 - \gamma$ in (1.1). If $\gamma \in [1/2, 1)$, then $f \in L^\theta_{\text{loc}}$, for any $1 \leq \theta < \gamma^{-1}$, so that (1.3) and (1.5) reduce, respectively, to the ranges

$$\begin{aligned} 1 + \frac{4}{n-2s} \leq p < 1 + \frac{4(3-\gamma)}{n-2s}, \\ 1 + \frac{4}{n} \leq p < 1 + \frac{4(3-\gamma)}{n-2s}. \end{aligned}$$

On the other hand, if $\gamma \in (0, 1/2)$, then $f \in L^\theta_{\text{loc}}$, for any $\theta \in [1, 2]$, so that (1.3) and (1.5) reduce, respectively, to the ranges

$$\begin{aligned} 1 + \frac{4}{n-2s} \leq p < 1 + \frac{10}{n-2s}, \\ 1 + \frac{4}{n} \leq p < 1 + \frac{10}{n-2s}. \end{aligned}$$

Example 1.2. We cannot apply Theorem 1.2 to kernels related to the fractional integration, since $t^{-\gamma} \notin L^\theta([0,\infty))$, for any $\theta \in [1,2]$, $\gamma \in (0,1)$. However, Theorem 1.2 and, hence, Corollary 1.1, may be applied to nonsingular kernels of type $f(t) = (1+t)^{-\gamma}$, $\gamma > 1/2$, for any $\gamma^{-1} < \theta \leq 2$ if $\gamma \in (1/2,1]$, or for any $\theta \in [1,2]$ if $\gamma > 1$.

If $\gamma \in (1/2,1]$ and $N(u)$ satisfies (N0) for some

$$\begin{aligned} 1+\frac{12-4\gamma}{n} &< p \leq 1+\frac{10}{n-4}, && \text{if } n \geq 5,\\ 1+\frac{12-4\gamma}{n} &= 4-\gamma < p < \infty, && \text{if } n = 4,\\ 1+\frac{12-4\gamma}{n-1} &= 7-2\gamma < p < \infty, && \text{if } n = 3, \end{aligned}$$

we find a global small data solution in $\dot{H}^s$, for any s such that

$$\frac{n}{2}-\frac{5}{p-1} \leq s < \frac{n}{2}-\frac{2(3-\gamma)}{p-1}.$$

If $\gamma > 1$ and $N(u)$ satisfies (N0) for some

$$\begin{aligned} 1+\frac{8}{n} &\leq p \leq 1+\frac{10}{n-4}, && \text{if } n \geq 5,\\ 1+\frac{8}{n} &= 3 < p < \infty, && \text{if } n = 4,\\ 1+\frac{8}{n-1} &= 5 < p < \infty, && \text{if } n = 3, \end{aligned}$$

we find a global small data solution in $\dot{H}^s$, for any s such that

$$\frac{n}{2}-\frac{5}{p-1} \leq s \leq \frac{n}{2}-\frac{4}{p-1}.$$

The plan of the paper is the following:

- in Sect. 2, we summarize some Strichartz estimates for the linear equation, recently obtained by Cordero and Zucco [4], and we fix the auxiliary spaces where we will derive the existence of the solution to the nonlinear problem;
- in Sect. 3, we prove the existence results in homogeneous Sobolev spaces (Theorems 1.1 and 1.2);
- in Sect. 4, we prove the existence results in nonhomogeneous Sobolev spaces (Theorem 1.3);
- in Sect. 5, we extend the previous results to the limit case of power nonlinearity, obtaining the $\dot{H}^s$ and H^s well-posedness for a wider range of (p,s) than the known one. For any $s \in [0,2]$, our range for p is obtained by formally setting $\theta = 1$ into (1.3), (1.4), (1.5);
- in Sect. 6, we apply the test function method to prove the nonexistence of global solutions below a critical exponent; comparing this result with our existence theorems, we find a range of exponents for which the local solution cannot be extended to a global one;
- in Appendix, we recall the definition of Riesz potential, its mapping properties and the definition of homogeneous and nonhomogeneous Sobolev spaces, as well as the related embedding theorems and the fractional Gagliardo–Nirenberg inequality.

2. Preliminary results

The functional representation of the solution to

$$\begin{cases} u_{tt}+\Delta^2 u = F(t,x),\\ u(0,x)=u_0(x), \quad u_t(0,x)=u_1(x) \end{cases}$$

is given by

$$u(t,\cdot) = \cos(t\Delta)u_0 + \frac{\sin(t\Delta)}{\Delta}u_1 + \int_0^t \frac{\sin((t-s)\Delta)}{\Delta}F(s,x)\,\mathrm{d}s, \tag{2.1}$$

where

$$\cos(t\Delta)u_0 := \mathfrak{F}^{-1}\Big(\cos\left(t|\xi|^2\right)\mathfrak{F}(u_0)\Big), \quad \frac{\sin(t\Delta)}{\Delta}u_1 := \mathfrak{F}^{-1}\Big(|\xi|^{-2}\sin\left(t|\xi|^2\right)\mathfrak{F}(u_1)\Big).$$

This equation inherits many properties of the Schrödinger equation, since $\cos(t\Delta)$ and $\sin(t\Delta)$ may be expressed as linear combinations of $\mathrm{e}^{\pm\, \mathrm{i}t\Delta}$. Indeed, the beam operator may be obtained as a composition of Schrödinger operators:

$$\partial_{tt} + \Delta^2 = (\mathrm{i}\partial_t + \Delta)(-\mathrm{i}\partial_t + \Delta),$$

and Strichartz estimates for the beam equation may be derived from the corresponding Schrödinger ones.

Definition 2.1. Given $n \geq 1$, we say that the pair (q,r) is *admissible* if

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}, \quad (q,r) \in [2,\infty], \quad (n,q,r) \neq (2,2,\infty).$$

Lemma 2.1. ([4]) *Let $s \in \mathbb{R}$. Let (q,r) be a admissible couple and (σ',ρ') Hölder conjugate indexes of an admissible couple (σ,ρ). Then, the following estimates hold:*

$$\|\cos(t\Delta)u_0\|_{L^q_T\dot{W}^{s,r}} + \|\Delta\sin(t\Delta)u_0\|_{L^q_T\dot{W}^{s-2,r}} \lesssim \|u_0\|_{\dot{H}^s}, \tag{2.2}$$

$$\left\|\frac{\sin(t\Delta)}{\Delta}u_1\right\|_{L^q_T\dot{W}^{s,r}} + \|\cos(t\Delta)u_1\|_{L^q_T\dot{W}^{s-2,r}} \lesssim \|u_1\|_{\dot{H}^{s-2}}, \tag{2.3}$$

$$\left\|\int_0^t \frac{\sin((t-\tau)\Delta)}{\Delta}F(\tau,\cdot)\,d\tau\right\|_{L^q_T\dot{W}^{s,r}} \lesssim \|F\|_{L^{\sigma'}_T\dot{W}^{s-2,\rho'}}, \tag{2.4}$$

$$\left\|\int_0^t \cos((t-\tau)\Delta)F(\tau,\cdot)\,d\tau\right\|_{L^q_T\dot{W}^{s,r}} \lesssim \|F\|_{L^{\sigma'}_T\dot{W}^{s,\rho'}}. \tag{2.5}$$

There is a huge literature about fixed point results. Our notation is closer to the one in [13].

Lemma 2.2. *Let (Ξ_1,X_1), (Ξ_2,X_2) be Banach spaces, $D:\Xi_1\to\Xi_2$ a linear operator and $\tilde{N}:\Xi_2\to\Xi_1$ a map such that $\tilde{N}(0)=0$. Given $\tilde{u}_0\in\Xi_2$, one considers the equation*

$$u = \tilde{u}_0 + D\tilde{N}(u).$$

Assume that there exist $C_0>0$ and $R>0$, such that

$$X_2(DG) \leq C_0X_1(G), \quad \textit{for any } G\in\Xi_1; \tag{2.6}$$

$$X_1(\tilde{N}(u)-\tilde{N}(v)) \leq \frac{1}{2C_0}X_2(u-v) \tag{2.7}$$

for any $u,v\in\Xi_2$ with $X_2(u)\leq R$, $X_2(v)\leq R$. If $X_2(\tilde{u}_0)\leq R/2$, then there exists a unique $u\in\Xi_2$, solution to $u=\tilde{u}_0+D\tilde{N}(u)$. Moreover, $X_2(u)\leq 2X_2(\tilde{u}_0)$.

Instead of assuming (2.7), one can suppose that there exist $\lambda>0$ and $C_1>0$, such that

$$X_1(\tilde{N}(u)-\tilde{N}(v)) \leq C_1X_2(u-v)(X_2(u)+X_2(v))^\lambda, \quad u,v\in\Xi_2. \tag{2.8}$$

Then, there exists a small $\varepsilon>0$ such that for any $\tilde{u}_0\in\Xi_2$ with $X_2(\tilde{u}_0)\leq\varepsilon/2$, the equation $u=\tilde{u}_0+D\tilde{N}(u)$ admits a unique solution $u\in\Xi_2$ and $X_2(u)\leq\varepsilon$.

The functional spaces in which we will use Lemma 2.2 for the beam equation are given by the closure of the Schwartz space under the following norms:

$$\|G\|_{\mathcal{S}^s(T)} := \sup_{(q,r)\text{ admissible}} \|G\|_{L^q_T \dot{W}^{s,r}},$$
$$\|G\|_{\mathcal{N}^s(T)} := \inf_{(\sigma,\rho)\text{ admissible}} \|G\|_{L^{\sigma'}_T \dot{W}^{s,\rho'}},$$

where $s \in \mathbb{R}$. These norms satisfy

$$\|G\|_{L^q_T \dot{W}^{s,r}} \leq \|G\|_{\mathcal{S}^s(T)}, \quad \|G\|_{\mathcal{N}^s(T)} \leq \|G\|_{L^{\sigma'}_T \dot{W}^{s,\rho'}} \tag{2.9}$$

for any admissible couple (q,r), (σ,ρ).

3. Proof of Theorems 1.1 and 1.2

In order to find a $\dot{H}^s$ local solution, we look for a time $T>0$ for which we can apply Lemma 2.2 with $\Xi_2 = \mathcal{S}^s(T)$ and $\Xi_1 = \mathcal{N}^{s-2}(T)$.

The relation (1.2) can be rewritten as $u = \tilde{u}_0 + D\tilde{N}(u)$ where

$$\tilde{u}_0 = \cos(t\Delta)u_0(x) + \frac{\sin(t\Delta)}{\Delta}u_1(x),$$
$$DG(t,x) = \int_0^t \frac{\sin((t-s)\Delta)}{\Delta} G(s,x)\,\mathrm{d}s,$$
$$\tilde{N}(u)(t,x) = \int_0^t f(t-\tau)\,N(u)(\tau,x)\,\mathrm{d}\tau$$

Strichartz estimates (2.2) and (2.3) imply that $\tilde{u}_0 \in \Xi_2$, whereas Strichartz estimate (2.4) gives (2.6).

It remains to prove (2.7). For any admissible couple (σ,ρ), it holds

$$\|\tilde{N}(u) - \tilde{N}(v)\|_{\mathcal{N}^{s-2}(T)} \leq \|(-\Delta)^{-\alpha/2}(\tilde{N}(u) - \tilde{N}(v))\|_{L^{\sigma'}_T L^{\rho'}} \quad \text{where } \alpha = 2-s.$$

Extending to $f(t)=0$ and $N(u)(t,\cdot)\equiv 0$ for $t<0$, by Young's inequality over $[0,T)$ (see Remark 3.1),

$$\begin{aligned}\|(-\Delta)^{-\alpha/2}(\tilde{N}(u) - \tilde{N}(v))\|_{L^{\sigma'}_T L^{\rho'}} &\leq \left\| \int_0^t |f(t-\tau)|\,\|(-\Delta)^{-\alpha/2}(N(u)-N(v))(\tau,x)\|_{L^{\rho'}}\,\mathrm{d}\tau \right\|_{L^{\sigma'}_T} \\ &= \left\| |f(t)| *_{(t)} \|(-\Delta)^{-\alpha/2}(N(u)-N(v))(t,x)\|_{L^{\rho'}} \right\|_{L^{\sigma'}_T} \\ &\leq \|f\|_{L^\theta_T}\,\|(-\Delta)^{-\alpha/2}(N(u)-N(v))\|_{L^{\rho'}L^{\sigma_0}_T},\end{aligned}$$

where

$$\frac{1}{\sigma_0} = 1 + \frac{1}{\sigma'} - \frac{1}{\theta} = 2 - \frac{1}{\sigma} - \frac{1}{\theta}, \tag{3.1}$$

provided that

$$\frac{1}{\sigma} \geq 1 - \frac{1}{\theta}. \tag{3.2}$$

Remark 3.1. Clearly, taking Young's inequality over $[0,T)$ when $T\in(0,\infty)$, means that we first write

$$\int_0^T \phi * \psi(t)\,\mathrm{d}t = \int_{\mathbb{R}} \chi_{[0,T)}(t) \int_0^t \phi(t-s)\psi(s)\mathrm{d}s\mathrm{d}t$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \big(\chi_{[0,T)}\phi\big)(t-s)\big(\chi_{[0,T)}\psi\big)(s)\mathrm{d}s\mathrm{d}t = \int_{\mathbb{R}} \big(\chi_{[0,T)}\phi\big) * \big(\chi_{[0,T)}\psi\big)(t)\mathrm{d}t,$$

where $\chi_{[0,T)}$ is the characteristic function of the interval $[0,T)$, and ϕ,ψ vanish for $t<0$; then, we apply Young's inequality over $\mathbb{R}$ to the functions $\chi_{[0,T)}\phi$ and $\chi_{[0,T)}\psi$.

Let

$$\sigma \leq \frac{\theta}{\theta-1}, \quad \rho \in [2, n/\alpha), \quad \frac{2}{\sigma} + \frac{n}{\rho} = \frac{n}{2}. \tag{3.3}$$

For $s=2$, i.e., $\alpha=0$, we take $\rho\in[2,\infty)$ in (3.3). Condition (3.3) implies that $n\geq 3$ if $s\in(1/2,2]$, $n\geq 4$ if $s\in(0,1/2]$ and $n\geq 5$ if $s=0$.

Since we assumed $\rho<n/\alpha$ in (3.3), we may apply Lemma 7.1 for $s\in[0,2)$, obtaining

$$\|(-\Delta)^{-\frac{\alpha}{2}}(N(u)-N(v))\|_{L^{\rho'}} \lesssim \|N(u)-N(v)\|_{L^m} \tag{3.4}$$

with

$$\frac{n}{n-\alpha} < \rho' \leq 2, \quad \frac{1}{\rho'} = \frac{1}{m} - \frac{\alpha}{n}. \tag{3.5}$$

Via Hölder inequality, we find

$$\|N(u)-N(v)\|_{L^m} \lesssim \|u(t)-v(t)\|_{L^{m_1}} \left(\|u(t)\|^{p-1}_{L^{m_2(p-1)}} + \|v(t)\|^{p-1}_{L^{m_2(p-1)}}\right) \tag{3.6}$$

with

$$\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{3.7}$$

Now we can manipulate the time norm:

$$\|N(u)-N(v)\|_{L^{\sigma_0}_T L^m} \lesssim T^{\frac{1}{\overline{\sigma}}} \|u-v\|_{L^{\sigma_1}_T L^{m_1}} \left(\|u\|^{p-1}_{L^{\sigma_2(p-1)}_T L^{m_2(p-1)}} + \|v\|^{p-1}_{L^{\sigma_2(p-1)}_T L^{m_2(p-1)}}\right) \tag{3.8}$$

with σ_1, σ_2, $\overline{\sigma}$ in $[\sigma_0,\infty]$, satisfying

$$\frac{1}{\sigma_0} = \frac{1}{\sigma_1} + \frac{1}{\sigma_2} + \frac{1}{\overline{\sigma}}. \tag{3.9}$$

Due to $\Xi_2=\mathcal{S}^s$, as a consequence of the (fractional) Sobolev embeddings, for any admissible couple (τ,r) such that $r\in[2,n/s)$, we may estimate

$$\|G\|_{L^\tau_T L^{r^\sharp}} \leq C\,\|G\|_{\mathcal{S}^s}, \quad r^\sharp = \frac{nr}{n-sr}. \tag{3.10}$$

Then, we may conclude

$$\|N(u)-N(v)\|_{L^{\sigma_0}_T L^m} \lesssim T^{\frac{1}{\overline{\sigma}}} \|u-v\|_{\mathcal{S}^s} \left(\|u\|_{\mathcal{S}^s} + \|v\|_{\mathcal{S}^s}\right)^{p-1}$$

if (σ_1,m_1) and $(\sigma_2(p-1),m_2(p-1))$ are related to Strichartz n-admissible couples (σ_1,q_1) and $(\sigma_2(p-1),q_2)$, i.e., such that

$$\frac{2}{\sigma_1} + \frac{n}{q_1} = \frac{n}{2}, \quad \frac{2}{\sigma_2(p-1)} + \frac{n}{q_2} = \frac{n}{2}, \tag{3.11}$$

as described in (3.10):

$$m_1 = q_1^\sharp = \frac{nq_1}{n-sq_1}, \quad m_2(p-1) = q_2^\sharp = \frac{nq_2}{n-sq_2},$$

for some $q_1,q_2\in[2,n/s)$. That is,

$$q_1 = \frac{nm_1}{n+sm_1}, \quad q_2 = \frac{nm_2(p-1)}{n+sm_2(p-1)}.$$

In particular, the restriction $q_1, q_2 \in [2, n/s)$ implies that $n > 2s$. One has

$$m_1 \geq \frac{2n}{n-2s}, \quad m_2(p-1) \geq \frac{2n}{n-2s}. \tag{3.12}$$

We shall now check whether the bound in (3.12) is consistent with (3.2), (3.5) and (3.7). Indeed, being (σ, ρ) admissible, from (3.2) it follows that

$$\frac{1}{\rho'} = \frac{1}{2} + \frac{2}{n\sigma} \geq \frac{1}{2} + \frac{2}{n}\left(1 - \frac{1}{\theta}\right).$$

On the other hand, by (3.5), (3.7), and (3.12), we get

$$\frac{1}{\rho'} = \frac{1}{m_1} + \frac{1}{m_2} - \frac{\alpha}{n} \leq \frac{n-2s}{2n}\,p - \frac{\alpha}{n} = \frac{n-2s}{2n}\,p - \frac{2-s}{n}.$$

The two inequalities for $1/\rho'$ are compatible if, and only if,

$$\frac{1}{2} + \frac{2}{n}\left(1 - \frac{1}{\theta}\right) \leq \frac{n-2s}{2n}\,p - \frac{2-s}{n},$$

that is,

$$\frac{n-2s}{2n}\,(p-1) \geq \frac{2}{n}\left(2 - \frac{1}{\theta}\right),$$

i.e.,

$$p \geq 1 + \frac{4(2-\theta^{-1})}{n-2s}. \tag{3.13}$$

Due to (3.11), we derive

$$\frac{1}{\sigma_1} = \frac{n}{4} - \frac{n}{2q_1} = \frac{n}{4} - \frac{n+sm_1}{2m_1} = \frac{n}{4} - \frac{n}{2m_1} - \frac{s}{2},$$

as well as

$$\frac{1}{\sigma_2} = \frac{n(p-1)}{4} - \frac{n(p-1)}{2q_2} = \frac{n(p-1)}{4} - \frac{n+sm_2(p-1)}{m_2} = \frac{n(p-1)}{4} - \frac{n}{2m_2} - \frac{s}{2}\,(p-1).$$

Combining these relations with (3.1), (3.3), (3.5), (3.7), (3.9), we get

$$\begin{aligned}
\frac{1}{\bar\sigma} &= 2 - \frac{1}{\theta} - \frac{1}{\sigma} - \frac{1}{\sigma_1} - \frac{1}{\sigma_2} \\
&= 2 - \frac{1}{\theta} - \frac{n}{4} + \frac{n}{2\rho} - \frac{n}{4} + \frac{n}{2m_1} + \frac{s}{2} - \frac{n(p-1)}{4} + \frac{n}{2m_2} + \frac{s}{2}\,(p-1) \\
&= 2 - \frac{1}{\theta} + \frac{s}{2} - \frac{n}{2\rho'} + \frac{n}{2m} - \frac{(n-2s)(p-1)}{4} \\
&= 2 - \frac{1}{\theta} + \frac{s+\alpha}{2} - \frac{(n-2s)(p-1)}{4} = 3 - \frac{1}{\theta} - \frac{(n-2s)(p-1)}{4}.
\end{aligned}$$

Then, $1/\bar\sigma > 0$ if, and only if,

$$p < 1 + \frac{4(3-\theta^{-1})}{n-2s}. \tag{3.14}$$

Therefore, if p satisfies both (3.13) and (3.14), we can conclude that

$$\|\tilde N(u) - \tilde N(v)\|_{\mathcal{N}^{s-2}(T)} \lesssim T^{\frac{1}{\bar\sigma}}\, \|f\|_{L^\theta_T}\, \|u-v\|_{\mathcal{S}^s(T)} \left(\|u\|_{\mathcal{S}^s(T)} + \|v\|_{\mathcal{S}^s(T)}\right)^{p-1}, \tag{3.15}$$

for some positive $1/\bar\sigma > 0$. On the other hand, if p satisfies (1.4), then

$$\|\tilde N(u) - \tilde N(v)\|_{\mathcal{N}^{s-2}(T)} \lesssim \|f\|_{L^\theta_T}\, \|u-v\|_{\mathcal{S}^s(T)} \left(\|u\|_{\mathcal{S}^s(T)} + \|v\|_{\mathcal{S}^s(T)}\right)^{p-1}. \tag{3.16}$$

We are now in a position to prove Theorems 1.1 and 1.2.

Proof of Theorems 1.1 *and* 1.2. Due to the previous discussion, the proof is an application of Lemma 2.2 once we recall that for $n \neq 2$ the couple $(\infty, 2)$ is admissible so that the contraction in $\mathcal{S}^s$ implies the existence in $L^\infty_T \dot{H}^s$.

First, we prove Theorem 1.1. Due to $L^\theta_{T_0} \subset L^{\tilde{\theta}}_{T_0}$ for any $\tilde{\theta} \in [1, \theta]$, being T_0 finite, we may derive (3.15) for any p as in (1.3), due to,

$$\bigcup_{\tilde{\theta} \in [1,\theta]} \left[1 + \frac{4(2 - \tilde{\theta}^{-1})}{n - 2s}, 1 + \frac{4(3 - \tilde{\theta}^{-1})}{n - 2s}\right) = \left[1 + \frac{4}{n - 2s}, 1 + \frac{4(3 - \theta^{-1})}{n - 2s}\right).$$

We can apply the first part of Lemma 2.2. Let $R > 0$ such that $\|u\|_{\mathcal{S}^2} \leq R$ and $\|v\|_{\mathcal{S}^2} \leq R$. Due to the previous discussion, we can take a sufficiently small $T > 0$ in (3.15), so that $2T^{\frac{1}{\sigma}} R^{p-1} \leq (2C_0)^{-1}$ and conclude the contraction argument.

It remains to prove that $u_t \in L^\infty_T \dot{H}^{s-2}$. We can formally derive in time the integral equation (1.2) obtaining

$$u_t(t, \cdot) = -\Delta \sin(t\Delta) u_0 + \cos(t\Delta) u_1 + \int_0^t \cos((t-s)\Delta) \tilde{N}(u)(s, \cdot)\, \mathrm{d}s.$$

In particular,

$$\|u_t\|_{L^\infty_T \dot{H}^{s-2}} \leq \|\Delta \sin(t\Delta) u_0\|_{L^\infty_T \dot{H}^{s-2}} + \|\cos(t\Delta) u_1\|_{L^\infty_T \dot{H}^{s-2}} + \left\| \int_0^t \cos((t-s)\Delta) \tilde{N}(u)\, \mathrm{d}s \right\|_{L^\infty_T \dot{H}^{s-2}},$$

from which, by (2.2), (2.3) and (2.5), we derive

$$\|u_t\|_{L^\infty_T \dot{H}^{s-2}} \lesssim \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-2}} + \|\tilde{N}(u)\|_{\mathcal{N}^{s-2}}.$$

Since $\tilde{N}(0) = 0$, we can take $v = 0$ in (3.15) concluding that $\|u_t\|_{L^\infty_T \dot{H}^{s-2}}$ is finite being

$$\|u_t\|_{L^\infty_T \dot{H}^{s-2}} \lesssim \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-2}} + T^{\frac{1}{\sigma}} \|u\|^p_{\mathcal{S}^s}.$$

The same argument works for showing that the time derivative of (1.2) is indeed not just formally defined. This concludes the proof of Theorem 1.1.

If p satisfies (1.4), by using (3.10) and $f \in L^\theta$, we can apply the second part of Lemma 2.2. This concludes the proof of Theorem 1.2. □

4. Proof of Theorem 1.3

Proof. We want to apply the first part of Lemma 2.2 with $\Xi_2 = \mathcal{S}^0 \cap \mathcal{S}^s$ and $\Xi_1 = \mathcal{N}^{-2} \cap \mathcal{N}^{s-2}$. Since for $n \neq 2$ the couple $(\infty, 2)$ is admissible, the contraction in $\mathcal{S}^0 \cap \mathcal{S}^s$ implies the existence and uniqueness in $L^\infty_T H^s$.

Concerning the initial data, it is sufficient to apply Strichartz estimates (2.2), (2.3): from $(u_0, u_1) \in H^s \times (\dot{H}^{s-2} \cap \dot{H}^{-2})$, we derive that

$$\tilde{u}_0 = \cos(t\Delta) u_0 + \frac{\sin(t\Delta)}{\Delta} u_1 \in \mathcal{S}^0 \cap \mathcal{S}^s.$$

Strichartz estimate (2.4) gives (2.6) with a suitable $C_0 > 0$.

Once we prove (2.7), deriving in time the integral equation and applying (2.7) with $v = 0$, we obtain that each term of (1.2) is in $L^\infty_T H^s$ and $u_t \in L^\infty_T \dot{H}^{-2} \cap L^\infty_T \dot{H}^{s-2}$.

Therefore, it only remains to prove (2.7). Following the argument in Sect. 3, we have

$$X_1(\tilde{N}(u)-\tilde{N}(v)) \lesssim \|f\|_{L^\theta_T} \sum_{\alpha=2,2-s} \|(-\Delta)^{-\frac{\alpha}{2}}(N(u)-N(v))\|_{L^{\sigma_0}_T L^{\rho'}} = \|f\|_{L^\theta_T} \sum_{\alpha=2,2-s} I_\alpha, \tag{4.1}$$

where σ_0 is as in (3.1), and (σ,ρ) are admissible couples satisfying (3.3) with $\alpha=2$ (so that it also satisfies (3.3) with $\alpha=2-s$).

Due to $\Xi_2=\mathcal{S}^0\cap\mathcal{S}^s$, for any admissible couple (τ,r) with $r<n/s$, we may estimate

$$\|G\|_{L^\tau_T L^{\tilde{r}}} \lesssim \|G\|_{\mathcal{S}^0\cap\mathcal{S}^s}, \quad \text{for any } \tilde{r}\in[r,r^\sharp], \text{ where } \quad r^\sharp=\frac{nr}{n-sr}. \tag{4.2}$$

Estimate (4.2) follows by the fractional Gagliardo–Nirenberg inequality (see Proposition 7.2),

$$\|G\|_{L^\tau_T L^{\tilde{r}}} \lesssim \|G\|^\gamma_{L^\tau_T \dot{W}^{s,r}} \|G\|^{1-\gamma}_{L^\tau_T L^r} \lesssim \|G\|_{\mathcal{S}^0\cap\mathcal{S}^s},$$

where γ satisfies

$$\frac{1}{\tilde{r}}=\frac{1}{r}-\frac{\gamma}{n}s.$$

We proceed as in Sect. 3: we have again (3.4) with ρ',m as in (3.5), estimate (3.6) with m_1,m_2 satisfying (3.7), and (3.8), with $\sigma_1,\sigma_2,\overline{\sigma}$ as in (3.9).

Then, we may conclude

$$\|N(u)-N(v)\|_{L^{\sigma_0}_T L^m} \lesssim T^{\frac{1}{\overline{\sigma}}} \|u-v\|_{\mathcal{S}^0\cap\mathcal{S}^s} \left(\|u\|_{\mathcal{S}^0\cap\mathcal{S}^s}+\|v\|_{\mathcal{S}^0\cap\mathcal{S}^s}\right)^{p-1}$$

if (σ_1,m_1) and $(\sigma_2(p-1),m_2(p-1))$ are related to Strichartz n-admissible couples (σ_1,q_1) and $(\sigma_2(p-1),q_2)$ as described in (4.2):

$$m_1\in[q_1,q_1^\sharp], \quad q_1^\sharp=\frac{nq_1}{n-sq_1}, \quad m_2(p-1)\in[q_2,q_2^\sharp], \quad q_2^\sharp=\frac{nq_2}{n-sq_2},$$

for some $q_1,q_2\in[2,n/s)$. That is,

$$\frac{nm_1}{n+m_1s}\le q_1\le m_1, \quad \frac{nm_2(p-1)}{n+m_2(p-1)s}\le q_2\le m_2. \tag{4.3}$$

Due to

$$\frac{2}{\sigma_1}+\frac{n}{q_1}=\frac{n}{2}, \quad \frac{2}{\sigma_2(p-1)}+\frac{n}{q_2}=\frac{n}{2},$$

we derive

$$\frac{1}{\sigma_1}=\frac{n}{4}-\frac{n}{2q_1}\in\left[\frac{n}{4}-\frac{n}{2m_1}-\frac{s}{2},\frac{n}{4}-\frac{n}{2m_1}\right],$$

as well as

$$\frac{1}{\sigma_2}=\frac{n(p-1)}{4}-\frac{n(p-1)}{2q_2}\in\left[\frac{n(p-1)}{4}-\frac{n}{2m_2}-\frac{(p-1)s}{2},\frac{n(p-1)}{4}-\frac{n(p-1)}{2m_2}\right].$$

Let us combine these relations with (3.1), (3.3), (3.5), (3.7) and (3.9). We get

$$\begin{aligned}\max_{q_1,q_2}\frac{1}{\overline{\sigma}} &= 2-\frac{1}{\theta}-\frac{1}{\sigma}-\frac{1}{\sigma_1}-\frac{1}{\sigma_2}\\ &= 2-\frac{1}{\theta}-\frac{n}{4}+\frac{n}{2\rho}-\frac{n}{4}+\frac{n}{2m_1}+\frac{s}{2}-\frac{n(p-1)}{4}+\frac{n}{2m_2}+\frac{(p-1)s}{2}\\ &= 2-\frac{1}{\theta}+\frac{s}{2}-\frac{n}{2\rho'}+\frac{n}{2m_1}+\frac{n}{2m_2}-\frac{(n-2s)(p-1)}{4}\\ &= 2-\frac{1}{\theta}+\frac{s+\alpha}{2}-\frac{(n-2s)(p-1)}{4},\end{aligned}$$

where the maximum is taken over the couples (q_1, q_2), satisfying (4.3). Then, $1/\overline{\sigma} > 0$ for $\alpha = 2, 2-s$ for some choice of q_1, q_2 verifying (4.3), if, and only if,

$$p < 1 + \frac{4(3-\theta^{-1}) + \min\{2s, 0\}}{n-2s} = 1 + \frac{4(3-\theta^{-1})}{n-2s}.$$

On the other hand, for a fixed p, the bound in (3.2) holds for some choice of q_1, q_2 verifying (4.3), if, and only if,

$$\frac{1}{2} + \frac{2}{n}\left(1 - \frac{1}{\theta}\right) \leq \frac{1}{\rho'} = \frac{1}{m} - \frac{\alpha}{n} = \frac{1}{m_1} + \frac{1}{m_2} - \frac{\alpha}{n} \leq \frac{1}{q_1} + \frac{p-1}{q_2} - \frac{\alpha}{n} \leq \frac{p}{2} - \frac{\alpha}{n},$$

for both $\alpha = 2, 2-s$, that is, if, and only if,

$$p \geq 1 + \frac{4(2-\theta^{-1}) - \min\{0, 2s\}}{n} = 1 + \frac{4(2-\theta^{-1})}{n}. \tag{4.4}$$

This concludes the estimate of both I_2 and I_{2-s} in (4.1). Recalling that $L^{\theta}_{T_0} \subset L^{\tilde{\theta}}_{T_0}$ for any $\tilde{\theta} \in [1, \theta]$, the range of p for which (4.4) holds for some $\tilde{\theta} \in [1, \theta]$ is given by $p \geq 1 + 4/n$. The proof of Theorem 1.3 follows. $\square$

5. The case of power nonlinearity

Let us consider

$$\begin{cases} u_{tt} + \Delta^2 u = N(u), \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x), \end{cases} \tag{5.1}$$

which is obtained by formally setting $f = \delta$, Dirac distribution, in (1.1).

The linear case, $N(u) = V(t,x)u$, with V in a suitable Sobolev space, is treated by Cordero and Zucco in [4]. The stability of a very particular nonlinear term in one-dimension case is considered by Fortunato and Benci in [1]. The case of power nonlinearity, with N satisfying (N0), has been studied in [11, 14].

The statements of the existence results for (5.1) are obtained by formally setting $\theta = 1$ in Theorems 1.1–1.3 and in Corollary 1.1 and replacing (1.2) by

$$u(t,x) = \cos(t\Delta)u_0(x) + \frac{\sin(t\Delta)}{\Delta}u_1(x) + \int_0^t \frac{\sin((t-s)\Delta)}{\Delta}N(u)(s,x)\,\mathrm{d}s. \tag{5.2}$$

In particular, we obtain the range of exponents

$$1 + \frac{4}{n-2s} \leq p < 1 + \frac{8}{n-2s},$$

for local solutions in homogeneous Sobolev spaces $\dot{H}^s$, for $n \geq 5$ and $s \in [0, 2]$, for $n = 4$ and $s \in (0, 2)$ and for $n = 3$ and $s \in (1/2, 3/2)$. The corresponding critical exponent is $1 + 8/(n-2s)$ for small data global $\dot{H}^s$ solutions. We obtain the range of exponents

$$1 + \frac{4}{n} \leq p < 1 + \frac{8}{n-2s}.$$

for local solutions in nonhomogeneous Sobolev spaces H^s, for $n \geq 5$ and $s \in (0, 2]$.

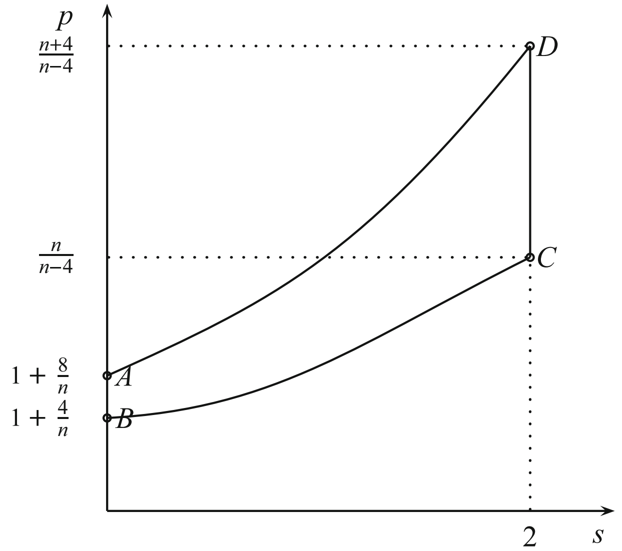
The proof immediately follows by setting $\sigma_0 = \sigma'$ in the proof of Theorems 1.1–1.3, due to the fact that $\tilde{N} = N$; hence, it is no longer necessary to apply Young inequality in time.

We notice that the critical exponent for the small data global solution is consistent with scaling symmetry arguments deriving by the invariance of the equation with respect to the transformation $u(t,x) \to \lambda^{-\frac{4}{p-1}}u(\lambda^{-2}t, \lambda^{-1}x)$ (see [13]).

In the figure below, we summarize the range of exponents p for which the $\dot{H}^s$ local or global existence follows from Theorems 1.1 and 1.2, as s ranges from 0 to 2, taking $n \geq 5$ for the sake of brevity.

The small data critical exponent is found on the critical curve AD where $p = p(s) = 1 + 8/(n - 2s)$. The range for local well-posedness, with arbitrarily large data, is represented by the ABCD closed set to which we subtract the curve AD of critical exponents. Line DC is the energy space line $s = 2$.

Along line DC, some scattering results have been established by Pausader [11] for $N(u) = -|u|^{p-1}u$. More recently, Wang [14] proved the existence of small data global solutions along curve AD, with the exception of the point $A = (0, 1 + 8/n)$, also obtaining partial results for $p = 1 + 8/(n - 2s)$ and $s > 2$.



6. Nonexistence of global weak solutions for beam equation with source term

Following the ideas in [5,10], by using the test function method, we may prove the nonexistence of global (weak) solutions to (1.1) with $f(t) = c_\gamma t^{-\gamma}$, $c_\gamma > 0$, and $N(u) = |u|^p$, for suitable p, and estimate the local existence time from above.

By weak local solution in $[0, T)$, we mean a $L^1_{\mathrm{loc}}([0, T) \times \mathbb{R}^n)$ function u such that the convolution of $t^{-\gamma}$ with $|u|^p$ is in $L^1_{\mathrm{loc}}([0, T) \times \mathbb{R}^n)$ (see Remark 3.1) and

$$
\begin{aligned}
c_\gamma \int_0^T \int_{\mathbb{R}^n} \eta(t)\phi(x) \int_0^t (t-s)^{-\gamma} |u(s,x)|^p \mathrm{d}s\mathrm{d}x\mathrm{d}t
&= \int_0^T \int_{\mathbb{R}^n} u(\eta_{tt}(t)\phi(x) + \eta(t)\Delta^2\phi(x))\mathrm{d}x\mathrm{d}t \\
&\quad - \int_{\mathbb{R}^n} u_1(x)\eta(0)\phi(x)\mathrm{d}x + \int_{\mathbb{R}^n} u_0(x)\eta_t(0)\phi(x)\mathrm{d}x, \qquad (6.1)
\end{aligned}
$$

for any $\phi \in \mathcal{C}^4_c(\mathbb{R}^n)$ and any $\eta \in \mathcal{C}^2([0, T))$, compactly supported. The solution is global if (6.1) holds for any $T > 0$, that is, we may replace $T = \infty$ in (6.1).

Theorem 6.1. *Let $n \geq 1$ and $f(t) = c_\gamma t^{-\gamma}$, with $\gamma \in (0,1)$, $c_\gamma > 0$, and $N(u) = |u|^p$ for some*

$$1 < p \leq 1 + \frac{2(3-\gamma)}{(n+2\gamma-4)_+}. \tag{6.2}$$

Let $u_0 \in L^1_{\mathrm{loc}}$ with $u_0 \geq 0$, $g \in L^1$ be such that

$$\int_{\mathbb{R}^n} g(x)\,dx > 0, \tag{6.3}$$

and let $u_1 = \varepsilon g$, for some $\varepsilon > 0$. Then, there exists no global weak solution $u \in L^p_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^n)$ to (1.1).

Moreover, there exists $C = C(g) > 0$ such that, if the equality does not hold in (6.2), then the existence time T of any local weak solution $u \in L^p_{\mathrm{loc}}([0,T)\times\mathbb{R}^n)$ to (1.1), verifies the following estimate from above:

$$T \leq C\varepsilon^{-\frac{1}{\kappa}}, \quad \kappa = \frac{3-\gamma}{p-1} - \frac{n+2\gamma-4}{2}. \tag{6.4}$$

Remark 6.1. Let $\gamma \in (0,1)$. For any p, such that

$$1 + \frac{4}{n} \leq p \leq 1 + \frac{2(3-\gamma)}{n+2\gamma-4}, \tag{6.5}$$

the H^s local solution, whose existence follows from Theorem 1.3 (Example 1.1), cannot be globally extended to a L^p_{loc} solution, due to Theorem 6.1, provided that the initial data satisfy the assumption therein.

In particular, due to

$$H^s \subset \dot{H}^s \subset L^{\frac{2n}{n-2s}} \subset L^p_{\mathrm{loc}}, \quad p \leq \frac{2n}{n-2s},$$

the H^s local solution cannot be extended to a global one, even if we restrict to $\dot{H}^s$, for $p \leq 2n/(n-2s)$. This latter holds for any $s \in [0,2]$ and for any p in (6.5), if $n \geq 10 - 4\gamma$.

Theorem 6.1 may be extended to the case of polynomial type nonlinearity. Here, by weak solution we mean that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n} \eta(t)\phi(x)|u(t,x)|^p\mathrm{d}x\mathrm{d}t &= \int_0^T\int_{\mathbb{R}^n} u(\eta_{tt}(t)\phi(x) + \eta(t)\Delta^2\phi(x))\mathrm{d}x\mathrm{d}t \\ &\quad - \int_{\mathbb{R}^n} u_1(x)\eta(0)\phi(x)\mathrm{d}x + \int_{\mathbb{R}^n} u_0(x)\eta_t(0)\phi(x)\mathrm{d}x,\end{aligned} \tag{6.6}$$

for any $\phi \in \mathcal{C}^4_c(\mathbb{R}^n)$ and any $\eta \in \mathcal{C}^2([0,T))$, compactly supported. The solution is global if (6.6) holds for any $T > 0$, that is, we may replace $T = \infty$ in (6.6).

Proposition 6.1. *Let $n \geq 1$ and $N(u) = |u|^p$ for some $1 < p \leq 1 + 4/(n-2)_+$. Let $u_1 = \varepsilon g \in L^1$, with g satisfying (6.3), for some $\varepsilon > 0$. Then, there exists no global weak solution $u \in L^p_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^n)$ to (5.1). Moreover, there exists $C = C(g) > 0$ such that if*

$$1 < p < 1 + \frac{4}{(n-2)_+}, \tag{6.7}$$

then the existence time T of any local weak solution $u \in L^p_{\mathrm{loc}}([0,T)\times\mathbb{R}^n)$ to (5.1), verifies the following estimate from above:

$$T \leq C\varepsilon^{-\frac{1}{\kappa}}, \quad \kappa = \frac{2}{p-1} - \frac{n-2}{2}. \tag{6.8}$$

In particular, the nonexistence result in Proposition 6.1 follows by applying Theorem 2.1 in [7] and taking into account of Theorem 2.2 in [6]. Choosing $\eta(t)$ such that η is constant in a neighborhood of $\{t=0\}$, we may remove all the assumptions on the initial data u_0, since the integral containing it vanishes in (6.6). The estimate for the existence time is directly obtained as in the last part of the proof of Theorem 6.1.

Remark 6.1 clearly extends to the case of power nonlinearity formally setting $\gamma=1$; in particular, for

$$1+\frac{4}{n}\leq p\leq 1+\frac{4}{n-2}, \tag{6.9}$$

obtained setting $\gamma=1$ in (6.5), the H^s local solution, whose existence follows from Theorem 1.3, formally setting $\theta=1$, cannot be globally extended to a L^p_{loc} solution, due to Proposition 6.1, provided that the initial data satisfy the assumption therein. Also, H^s or $\dot H^s$ local solution cannot be extended to a global one, for $p\leq 2n/(n-2s)$. This latter holds for any $s\in[0,2]$ and for any p in (6.9), if $n\geq 6$.

Remark 6.2. The nonexistence result in Proposition 6.1 does not contradict the global existence result in Sect. 5, since $p=1+8/(n-2s)$, as in (1.4) with $\theta=1$, cannot verify the following two conditions at the same time:

- $p\leq 1+4/(n-2)$, so that the nonexistence of L^p_{loc} global solutions follows;
- $p\leq 2n/(n-2s)$, so that $\dot H^s\subset L^p_{\mathrm{loc}}$.

Indeed, the first one implies that $n\leq 4-2s$ and the second one that $n\geq 8-2s$.

6.1. Preliminaries

Following [5], for any $\alpha\in(0,1)$ and for a fixed $\tau>0$, we introduce the fractional integral and differential operators $J_{0|t}, D_{0|t}, J_{t|\tau}, D_{t|\tau}$, $t\in[0,\tau]$, defined by:

$$J^\alpha_{0|t}\,h(t):=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{-(1-\alpha)}h(s)\,\mathrm{d}s,\quad D^\alpha_{0|t}:=\partial_t\,J^{1-\alpha}_{0|t},$$
$$J^\alpha_{t|\tau}\,h(t):=\frac{1}{\Gamma(\alpha)}\int_t^\tau (s-t)^{-(1-\alpha)}h(s)\,\mathrm{d}s,\quad D^\alpha_{t|\tau}:=\,-\,\partial_t\,J^{1-\alpha}_{t|\tau}.$$

We have the following properties:

$$\int_0^\tau (D^\alpha_{0|t}\,h_1)(t)\,h_2(t)\mathrm{d}t=\int_0^\tau h_1(t)\,(D^\alpha_{t|\tau}\,h_2)(t)\mathrm{d}t, \tag{6.10}$$
$$D^\alpha_{0|t}J^\alpha_{0|t}\,h(t)=h(t). \tag{6.11}$$

Let us define

$$\omega(t):=\begin{cases}(1-t/\tau) & \text{if } t\in[0,\tau],\\ 0 & \text{if } t>\tau.\end{cases} \tag{6.12}$$

It follows that $\operatorname{supp}\omega=[0,\tau]$ and $\omega(t)^\beta\in\mathcal{C}^k_c([0,\infty))$, $k\geq 0$, for any $\beta>k$. Moreover, we have the following.

Lemma 6.1. (Lemma 4.1 in [5]) *For any $\alpha\in(0,1)$, it follows that*

$$D^\alpha_{t|\tau}\,\omega(t)^\beta=C(\alpha,\beta)\,\tau^{-\alpha}\omega(t)^{\beta-\alpha},\quad \textit{for any } \beta>\alpha, \tag{6.13}$$

where

$$C(\alpha,\beta)=\frac{\Gamma(\beta+1)}{(\beta+2-\alpha)\Gamma(\beta-\alpha)}.$$

6.2. Proof of Theorem 6.1

Proof of Theorem 6.1. We put $\alpha:=1-\gamma$. Let $u\in L^p_{\mathrm{loc}}([0,T)\times\mathbb{R}^n)$ be a weak solution, that is, it satisfies (6.1). Let us fix $\tau\in(0,T)$, and let $\Psi\in\mathcal{C}_c^\infty$ be a radial test function, such that:

- $\operatorname{supp}\Psi=B_1$;
- $\Psi(x)=1$, for any $x\in B_{1/2}$;
- $\Psi(x_1)\geq\Psi(x_2)$ if $|x_1|\leq|x_2|$.

For any $R\geq1$, we denote $\Psi_R(x):=\Psi(x/R)$. Let us fix

$$\beta>(\alpha+2)p',\quad\text{and}\quad \ell>2p',\tag{6.14}$$

where $p':=p/(p-1)$ is the Hölder conjugate of p, and let

$$\Phi_R(t,x):=\omega(t)^\beta\,\Psi_R(x)^\ell,\quad \varphi(t,x):=D^\alpha_{t|\tau}\Phi_R(t,x).$$

Then, by using Lemma 6.1, $\operatorname{supp}\varphi\subset[0,\tau]\times B_R$, for any $\tau,R\geq1$. We may write

$$\begin{aligned}J&:=\int\limits_{B_R}(u_1(x)\varphi(0,x)-u_0(x)\varphi_t(0,x))\,\mathrm{d}x\\&=\int\limits_0^\tau\int\limits_{B_R}u(\varphi_{tt}+\Delta^2\varphi)\,\mathrm{d}x\,\mathrm{d}t-c_\gamma\Gamma(\alpha)\int\limits_0^\tau\int\limits_{B_R}J^\alpha_{0|t}(|u|^p)\varphi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

We remark that $\varphi\geq0$, $\varphi_t\leq0$ and $\varphi_{tt}\geq0$, thanks to (6.13). Due to $u_0\in L^1_{\mathrm{loc}}$ with $u_0\geq0$, it holds

$$J\geq\int\limits_{B_R}u_1(x)\,\varphi(0,x)\,\mathrm{d}x=\varepsilon\int\limits_{B_R}g(x)\,\varphi(0,x)\,\mathrm{d}x\geq c\varepsilon,$$

for $R\geq R_0$, with sufficiently large $R_0=R_0(g)$ and sufficiently small $c=c(g)$.

By virtue of (6.10) and (6.11), we get

$$c_\gamma\Gamma(\alpha)\int\limits_0^\tau\int\limits_{B_R}J^\alpha_{0|t}(|u|^p)\varphi\,\mathrm{d}x\,\mathrm{d}t=c_\gamma\Gamma(\alpha)\int\limits_0^\tau\int\limits_{B_R}|u|^p\Phi_R\,\mathrm{d}x\,\mathrm{d}t.$$

Now, for $\delta\in(0,1/3)$, we use Young inequality to estimate

$$\int\limits_0^\tau\int\limits_{B_R}|u|\varphi_{tt}\,\mathrm{d}x\,\mathrm{d}t\leq\delta c_\gamma\Gamma(\alpha)\int\limits_0^\tau\int\limits_{B_R}|u|^p\Phi_R\mathrm{d}x\,\mathrm{d}t+C_{\delta,\gamma}\int\limits_0^\tau\int\limits_{B_R}{\varphi_{tt}}^{p'}\,\Phi_R^{-\frac{1}{p-1}}\,\mathrm{d}x\,\mathrm{d}t,$$

and, thanks to Lemma 6.1, we have

$$\partial_t^2D^\alpha_{t|\tau}\omega(t)^\beta=\frac{\Gamma(\beta+1)}{(\beta+2-\alpha)\Gamma(\beta-\alpha)}\,\tau^{-\alpha}\partial_t^2\omega(t)^{\beta-\alpha}=\frac{\Gamma(\beta+1)}{(\beta+2-\alpha)\Gamma(\beta-\alpha-2)}\,\tau^{-(\alpha+2)}\omega(t)^{\beta-\alpha-2}.$$

Due to

$$\operatorname{meas}([0,\tau])=\tau,\quad \operatorname{meas}(B_R)\approx R^n,\tag{6.15}$$

we obtain

$$\int_0^\tau \int_{B_R} {\varphi_{tt}}^{p'}\, \Phi_R^{-\frac{1}{p-1}}\, \mathrm{d}x\, \mathrm{d}t = C\tau^{-(\alpha+2)p'} \int_0^\tau \int_{B_R} \omega(t)^{(\beta-\alpha-2)p'-\frac{\beta}{p-1}}\, \Psi_R(x)^\ell\, \mathrm{d}x\, \mathrm{d}t \lesssim \tau^{-(\alpha+2)p'+1}R^n.$$

We remark that the power of $\omega(t)$ in the integral above is positive, by virtue of (6.14). Similarly, we obtain

$$\int_0^\tau \int_{B_R} |u|\Delta^2\varphi\, \mathrm{d}x\, \mathrm{d}t \leq \delta c_\gamma \Gamma(\alpha) \int_0^\tau \int_{B_R} |u|^p \Phi_R \mathrm{d}x\, \mathrm{d}t$$
$$+ C_{\delta,\gamma}\tau^{-\alpha p'} \int_0^\tau \int_{B_R} (D^\alpha_{t|\tau}\omega(t)^\beta |\Delta^2\Psi_R(x)^\ell|)^{p'}\, \Phi_R^{-\frac{1}{p-1}}\, \mathrm{d}x\, \mathrm{d}t.$$

By Glaeser's inequality [8],

$$|\Delta^2\Psi_R(x)^\ell| \leq C_\ell R^{-4}\, \Psi_R(x)^{\ell-2}.$$

Recalling (6.13) and (6.15), we may estimate

$$\tau^{-\alpha p'} R^{-4p'} \int_0^\tau \int_{B_R} \omega(t)^{(\beta-\alpha)p'-\frac{\beta}{p-1}}\, \Psi_R(x)^{(\ell-2)p'-\frac{\ell}{p-1}}\, \mathrm{d}x\, \mathrm{d}t \leq C\tau^{-\alpha p'+1}\, R^{-4p'+n}.$$

We notice that the powers of $\omega(t)$ and $\Psi_R(x)$ in the integral above are positive, by virtue of (6.14).

Summarizing, we obtained:

$$J \leq C_1\, \tau^{-\alpha p'+1} R^n (\tau^{-2p'} + R^{-4p'}) - C_2 \int_0^\tau \int_{B_R} |u|^p \Phi_R\, \mathrm{d}x\, \mathrm{d}t, \tag{6.16}$$

for some $C_2 > 1/3$, due to $\delta < 1/3$. Assuming $\tau \geq R_0^2$, we may fix $R = \sqrt{\tau}$ in (6.16), so that

$$\tau^{-\alpha p'+1} R^n (\tau^{-2p'} + R^{-4p'}) \approx \tau^{-(\alpha+2)p'+\frac{n}{2}+1}.$$

We notice that p verifies the upper bound in (6.2) if, and only if, $(\alpha+2)p' \geq n/2+1$. We first prove that $T < \infty$, by only using that $J \geq 0$ in (6.16). By contradiction, let $T = \infty$. Let $(\alpha+2)p' > n/2+1$; by Beppo Levi's theorem on monotone convergence, being $\Phi_{\sqrt{\tau}} \nearrow 1$ as $\tau \to \infty$, we derive

$$\int_0^\infty \int_{\mathbb{R}^n} |u|^p\, \mathrm{d}x\, \mathrm{d}t = \lim_{\tau\to\infty} \int_0^\infty \int_{\mathbb{R}^n} |u|^p \Phi_{\sqrt{\tau}}\, \mathrm{d}x\, \mathrm{d}t \leq C \lim_{\tau\to\infty} \tau^{-(\alpha+2)p'+\frac{n}{2}+1} = 0\,;$$

hence, $u \equiv 0$. The limit case $(\alpha+2)p' = n/2+1$ may be treated as in [5]: being

$$\int_0^\infty \int_{\mathbb{R}^n} |u|^p\, \mathrm{d}x\, \mathrm{d}t = \lim_{\tau\to\infty} \int_0^\infty \int_{\mathbb{R}^n} |u|^p \Phi_{\sqrt{\tau}}\, \mathrm{d}x\, \mathrm{d}t < \infty,$$

it follows that $u \in L^p$; then, one may prove that $u \equiv 0$ by repeating the previous steps of the proof, employing Hölder inequality instead of Young inequality, to estimate the integrals, and taking advantage that $\operatorname{supp} \Delta^2\Psi_R \subset B_R \backslash B_{R/2}$.

Now, we prove the second part of our statement, for $(\alpha+2)p' > n/2+1$. Due to $J \geq c\varepsilon$, it follows

$$\varepsilon \leq C\tau^{-(\alpha+2)p'+\frac{n}{2}+1} = C\tau^{-\kappa},$$

from which we derive (6.4) if we fix, for instance, $\tau = T/2$. $\square$

Acknowledgments

The first author is supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), Grants 2013/15140-2 and 2014/02713-7, JP-Programa Jovens Pesquisadores em Centros Emergentes, research project *Decay estimates for semilinear hyperbolic equations.* This paper has been realized during a visit of the second author at the Department of Mathematics and Computer Science of University of São Paulo, in Ribeirão Preto, supported by the FAPESP Grant 2013/15140-2. The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Appendix: Riesz potential and homogeneous Sobolev spaces

Following [12], given $\alpha \in \mathbb{R}$, we introduce the Riesz potential

$$(-\Delta)^{-\alpha/2} f = \mathfrak{F}^{-1}\left(|\xi|^{-\alpha}\mathfrak{F}(f)\right),$$

which is meaningful, for example for $f \in \mathcal{S}(\mathbb{R}^n)$.

We recall the Hardy–Littlewood–Sobolev inequality.

Lemma 7.1. *Let* $0 < \alpha < n$, *and* $n/(n-\alpha) < q < \infty$. *Then,*

$$\|(-\Delta)^{-\frac{\alpha}{2}} f\|_q \lesssim \|f\|_p$$

where

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}.$$

For $r \in (1,\infty)$, one defines the homogeneous Sobolev spaces $\dot{W}^{s,r}$ as the space of distributions with the property that $\mathfrak{F}^{-1}(|\xi|^s\mathfrak{F}(f)) \in L^r$ with finite seminorm $\|\mathfrak{F}^{-1}(|\xi|^s\mathfrak{F}(f))\|_r$. In order to have a norm, one takes the distribution in $\mathcal{S}/\mathcal{P}$, where $\mathcal{P}$ is the set of all polynomials, as it is evident for $s \in \mathbb{N}$, due to the fact that $\dot{W}^{s,r} = \{f : \partial_x^\beta f \in L^r, |\beta| = s\}$.

Clearly $\dot{W}^{0,r} = L^r$. We also set $\dot{H}^s = \dot{W}^{s,2}$, for any $s \in \mathbb{R}$, in particular $\dot{H}^0 = L^2$.

When $s > 0$ the definition is well posed; indeed, the distributions whose Fourier transform is supported at the origin give zero under the operation $\mathfrak{F}^{-1}(|\xi|^s\mathfrak{F}(f))$, but the Fourier of these distributions are linear combinations of derivatives of Dirac delta; thus, they are the polynomials and we are working in $\mathcal{S}/\mathcal{P}$.

For $s < 0$, the lack of smoothness of the symbol $|\xi|^s$ requires a proper definition of the distribution $|\xi|^s\hat{f}$. Let $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, such that $\eta = 0$ in $B_1(0)$ and $\eta = 1$ in $\mathbb{R}^n\backslash B_2(0)$. For $s \in \mathbb{R}$ and $f \in \mathcal{S}'/\mathcal{P}$, $\phi \in \mathcal{S}$, we define

$$\left\langle |\xi|^s\hat{f}, \phi\right\rangle = \lim_{\varepsilon\to 0}\left\langle \hat{f}, \eta\left(\varepsilon^{-1}|\xi|\right)|\xi|^s\phi\right\rangle,$$

provided that the limit exists. In such a case, we can compute the norm of $\mathfrak{F}^{-1}(|\xi|^s\mathfrak{F}(f))$.

In particular when $0 < s < n$ we can describe $\dot{W}^{-s,r}$ by means of Riesz potential:

$$\|f\|_{\dot{W}^{-s,r}} = \|(-\Delta)^{-\frac{s}{2}} f\|_{L^r}.$$

These spaces do not have monotonic inclusion with respect to s, but they are interpolation spaces. In particular if $f \in \dot{W}^{s_1,r} \cap \dot{W}^{s_2,r}$, then $f \in \dot{W}^{s,r}$ for any $s_1 \leq s \leq s_2$.

For $s \geq 0$, the nonhomogeneous Sobolev spaces, which norm is

$$\left\|\mathfrak{F}^{-1}(\langle\xi\rangle^s\mathfrak{F}(f))\right\|_r,$$

for any $r \in (1,\infty)$, can be equivalently written as

$$W^{s,r} = L^r \cap \dot{W}^{s,r},$$

if $s \geq 0$. Again, we set $H^s = W^{s,2}$.

We recall the following inequalities in Sobolev spaces, valid for any functions such that the right sides are finite.

Proposition 7.1. (Sobolev embedding [2]) *Let $f \in \mathcal{S}'(\mathbb{R}^n)$, $s \in \mathbb{R}$, $1 < r < \infty$. Then,*

$$\|f\|_{\dot{W}^{s,r}} \lesssim \|f\|_{\dot{W}^{s_1,r_1}}$$

where $s \leq s_1$, $1 < r_1 < \infty$ and $s - \frac{n}{r} = s_1 - \frac{n}{r_1}$.

Proposition 7.2. (Generalized Gagliardo–Nirenberg inequalities [9]) *Let $f \in \mathcal{S}'(\mathbb{R}^n)$, $\tilde{s} \in [0, s)$, $1 < r, \tilde{r} < \infty$. Then,*

$$\|f\|_{\dot{W}^{\tilde{s},\tilde{r}}} \lesssim \|f\|^{\gamma}_{\dot{W}^{s,r}} \|f\|^{1-\gamma}_{L^r}$$

where $\frac{\tilde{s}}{s} \leq \gamma \leq 1$ is given by $\gamma = \frac{n}{s}\left(\frac{1}{r} - \frac{1}{\tilde{r}} + \frac{\tilde{s}}{n}\right)$. As a consequence $r \leq \tilde{r} \leq \frac{rn}{n+r(\tilde{s}-s)}$.

References

1. Benci, V., Fortunato, D.: Existence of solitons in the nonlinear beam equation. J. Fixed Point Theory Appl. **11**, 261–278 (2012)
2. Bergh, J., Löfström, J.: Interpolation Spaces. Grundlehren. Vol. 223, Springer, Berlin (1976)
3. Cazenave, T., Dickstein, F., Weissler, F.B.: An equation whose Fujita critical exponent is not given by scaling. Nonlinear Anal. **68**, 862–874 (2008)
4. Cordero, E., Zucco, D.: Strichartz estimates for the vibrating plate equation. J. Evol. Equ. **11**, 827–845 (2011)
5. D'Abbicco, M.: A wave equation with structural damping and nonlinear memory. Nonlinear Differ. Equ. Appl. **21**(5), 751–773 (2014). doi:10.1007/s00030-014-0265-2
6. D'Abbicco, M., Lucente, S.: A modified test function method for damped wave equations. Adv. Nonlinear Stud. **13**, 867–892 (2013)
7. D'Ambrosio, L., Lucente, S.: Nonlinear Liouville theorems for Grushin and Tricomi operators. J. Differ. Equ. **123**, 511–541 (2003)
8. Glaeser, G.: Racine carrée d'une fonction diffé式etiable. Ann. Inst. Fourier (Grenoble) **13**, 203–210 (1963)
9. Hajaiej, H., Molinet, L., Ozawa, T., Wang, B.: Necessary and sufficient conditions for the fractional Gagliardo–Nirenberg inequalities and applications to Navier–Stokes and generalized boson equations. In: Harmonic analysis and nonlinear partial differential equations, pp 159–175, RIMS Kokyuroku Bessatsu, B26, Res. Inst. Math. Sci. (RIMS), Kyoto (2011)
10. Ikeda, M., Wakasugi, Y.: A note on the lifespan of solutions to the semilinear damped wave equation. Proc. AMS **143**(1), 163–171 (2015)
11. Pausader, B.: Scattering and the Levandosky–Strauss conjecture for fourth-order nonlinear wave equations. J. Differ. Equ. **241**, 237–278 (2007)
12. Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton (1970)
13. Tao T.: Nonlinear Dispersive Equations: Local and Global Analysis. CBMS Regional Conference Series in Mathematics, 106. American Mathematical Society, Providence (2006)
14. Wang, S.: Well-posedness and Ill-posedness for the nonlinear beam equation. arXiv:1306.6411 (2013)

Marcello D'Abbicco
Departamento de Computação e Matemática
Universidade de São Paulo (USP)
FFCLRP, Av. dos Bandeirantes, 3900
Ribeirão Prêto, SP CEP 14040-901, Brazil
e-mail: marcello.dabbicco@uniba.it

Sandra Lucente
Department of Mathematics
University of Bari
Via E. Orabona 4
70125 Bari, Italy

(Received: September 29, 2015; revised: April 7, 2016)