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Pointwise estimates of solutions for the multi-dimensional bipolar Euler–Poisson system

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Abstract. In the paper, we consider a multi-dimensional bipolar hydrodynamic model from semiconductor devices and plasmas. This system takes the form of Euler–Poisson with electric field and frictional damping added to the momentum equations. By making a new analysis on Green's functions for the Euler system with damping and the Euler–Poisson system with damping, we obtain the pointwise estimates of the solution for the multi-dimensions bipolar Euler–Poisson system. As a by-product, we extend decay rates of the densities $\rho_i (i = 1, 2)$ in the usual L^2-norm to the L^p-norm with $p \geq 1$ and the time-decay rates of the momentums $m_i (i = 1, 2)$ in the L^2-norm to the L^p-norm with $p > 1$ and all of the decay rates here are optimal.

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1. Introduction

In this paper, we consider the following multi-dimensional bipolar Euler–Poisson system (hydrodynamic model):

$$\begin{cases} \partial_t \rho_1 + \mathrm{div} m_1 = 0, \\ \partial_t m_1 + \mathrm{div}(\frac{m_1 \otimes m_1}{\rho_1}) + \nabla P(\rho_1) = \rho_1 \nabla\phi - m_1, \\ \partial_t \rho_2 + \mathrm{div} m_2 = 0, \\ \partial_t m_2 + \mathrm{div}(\frac{m_2 \otimes m_2}{\rho_2}) + \nabla P(\rho_2) = -\rho_2 \nabla\phi - m_2, \\ \Delta\phi = \rho_1 - \rho_2, \quad x \in \mathbb{R}^d, \ t \geq 0, \end{cases} \tag{1.1}$$

where the unknown functions $\rho_i(x, t), m_i(x, t)$ $(i = 1, 2), \phi(x, t)$ are the charge densities of electrons and ions, momentums and electrostatic potential, respectively. The pressure $P = P(\rho_i)$ is a smooth function with $P'(\rho_i) > 0$ for $\rho_i > 0$. The system can be used to describe charged particle fluids, for example, electrons and holes in semiconductor devices, positively and negatively charged ions in a plasma. This model takes an important role in the fields of applied and computational mathematics, and we can see more details in [13, 24, 27] etc.

Due to their physical importance, mathematical complexity and wide range of applications, there are many studies on well-posedness of the stationary solution, well-posedness and large-time behavior of the non-stationary solution for the multi-dimensional bipolar Euler–Poisson equations (1.1). Li [22] showed existence and some limit analysis of stationary solutions for the multi-dimensional bipolar Euler–Poisson system. Ali and Jüngel [1], Li and Zhang [17] and Peng and Xu [26] studied the global smooth solutions of the Cauchy problem for multi-dimensional bipolar hydrodynamic models in the Sobolev space $H^l(\mathbb{R}^d)(l > 1 + \frac{d}{2})$ and in the Besov space, respectively. Ju [14] discussed the global existence of smooth solutions to the initial boundary value problem for the three-dimensional bipolar Euler–Poisson system. Li and Yang [23] and Wu and Wang [31] showed global existence and L^2 decay rate of the smooth solutions to the three-dimensional bipolar Euler–Poisson systems when the initial data are small perturbation of the constant stationary solution. Huang et al. [8] and Liao and Li [21] showed large-time behavior of solution to n-dimensional bipolar hydrodynamic model for semiconductors when the initial data are near to the planar diffusion waves. Ali and Chen [2] studied the zero-electron-mass limit in the Euler–Poisson system for both well- and ill-prepared initial data. Lattanzio [16] and Li [20] investigated the relaxation limit of the multi-dimensional bipolar isentropic Euler–Poisson model for semiconductors, respectively. Ju et al. [15] discussed the quasi-neutral limit of the two-fluid multi-dimensional Euler–Poisson system. Moreover, it is worth mentioning that there are a lot of references about the one-dimensional bipolar Euler–Poisson equation, and the interesting reader can refer to [3,5–7,9–11,25,28,32,33] and the reference therein.

In this paper, we are interested in the asymptotic behavior of smooth solution to the system (1.1) with the initial data

$$\rho_i(x,0)=\rho_{i0}(x)>0,\ m_i(x,0)=m_{i0}(x),\ i=1,2, \tag{1.2}$$

which satisfy

$$\lim_{|x|\to\infty}\rho_{i0}(x)=\bar\rho>0.$$

The main concern is to deduce the pointwise estimates of the problem (1.1)–(1.2). For stating our results, we first give the following well-posedness Theorem.

Theorem 1.1. (see [31]) *Let $P'(\rho_i)>0 (i=1,2)$ for $\rho_i>0$, and $\bar\rho>0$. Assume that $(\rho_i-\bar\rho,m_{i0},\nabla\phi_0)\in H^3(\mathbb{R}^3)$ for $i=1,2$, with $\epsilon_0=:\|(\rho_{i0}-\bar\rho,m_{i0},\nabla\phi_0)\|_{H^3}$ small. Then there is a unique global classical solution $(\rho_i,m_i,\nabla\phi)$ of the Cauchy problem (1.1)–(1.2) satisfying*

$$\|(\rho_i-\bar\rho,m_i,\nabla\phi)\|^2_{H^3}\le C\epsilon_0. \tag{1.3}$$

Notice that the case of $d\ge4$ for the problem (1.1)–(1.2) can be handled as in Theorem 1.1. On the other hand, since the main goal in the present paper is the large-time behavior of the solution, in order to emphasize the relation between our pointwise estimates and the dimension d, in the following the space dimension will be presented as d with $d\ge3$. Now let us give our main results in the following Theorem.

Theorem 1.2. *Let $P'(\rho_i)>0 (i=1,2)$ for $\rho_i>0$, and $\bar\rho>0$. Assume that $(\rho_i-\bar\rho,m_{i0},\nabla\phi_0)\in H^{s+2}(\mathbb{R}^d)$, $s=[d/2]+1$ and $d\ge3$ for $i=1,2$, with $\epsilon_0=:\|(\rho_{i0}-\bar\rho,m_{i0},\nabla\phi_0)\|_{H^{s+2}}$ small. For $|\alpha'|\le2$*

$$|D_x^{\alpha'}(\rho_{10}+\rho_{20}-2\bar\rho,\rho_{10}-\rho_{20},m_{10},m_{20}),\nabla\phi_0|\le C\epsilon_0(1+|x|^2)^{-r},\ r>\frac{d}{2}, \tag{1.4}$$

we have for $|\alpha|=0$ and $|\alpha|=1$

$$|D_x^{\alpha}(\rho_1+\rho_2-2\bar\rho,\rho_1-\rho_2)|\le C\epsilon_0(1+t)^{-\frac{d+|\alpha|}{2}}B_r(|x|,t),\ r>\frac{d}{2}, \tag{1.5}$$

$$|D_x^{\alpha}(m_1+m_2,m_1-m_2)|\le C\epsilon_0(1+t)^{-\frac{d+1+|\alpha|}{2}}B_{\frac{d}{2}}(|x|,t), \tag{1.6}$$

which imply that

$$|D_x^{\alpha}(\rho_1-\bar\rho,\rho_2-\bar\rho)|\le C\epsilon_0(1+t)^{-\frac{d+|\alpha|}{2}}B_r(|x|,t),\ r>\frac{d}{2}, \tag{1.7}$$

$$|D_x^{\alpha}(m_1,m_2)|\le C\epsilon_0(1+t)^{-\frac{d+1+|\alpha|}{2}}B_{\frac{d}{2}}(|x|,t). \tag{1.8}$$

Here and in the subsequent, $B_k(|x|,t)=(1+\frac{|x|^2}{1+t})^{-k}$ with $k=r,\frac{d}{2},N$.

Furthermore, as a by-product, we have the following optimal L^p-decay rates of the solution.

Corollary 1.3. *Under the assumption in Theorem 1.2, the solution (ρ_1,ρ_2,m_1,m_2) satisfies that*

$$\|D_x^{\alpha}(\rho_1-\bar\rho,\rho_2-\bar\rho)\|_{L^p(\mathbb{R}^d)}\le C(1+t)^{-\frac{d}{2}(1-\frac{1}{p})},\ \text{with } p\ge1,$$

$$\|D_x^{\alpha}(m_1,m_2)\|_{L^p(\mathbb{R}^d)}\le C(1+t)^{-\frac{d+1}{2}(1-\frac{1}{p})},\ \text{with } p>1.$$

The outline of the proof of the main theorems is as follows. Firstly, we rewrite the system (1.1) into the Euler system with damping on the variables $(\rho_1+\rho_2, m_1+m_2)$ and the Euler–Poisson system with damping on the variables $(\rho_1-\rho_2, m_1-m_2)$, which interact each other through the nonlinear inhomogeneous terms on the right-hand side; see (2.5) and (2.6). Then, followed the arguments and ideas in [34,36], we make an analysis on Green's functions of these two linearized systems. Namely, we can directly use the pointwise estimate of Green's functions of the linearized Euler equations with damping and the linearized unipolar Euler–Poisson equations in the middle frequency part and high frequency part. However, in order to overcome the interaction of two particles and make the initial conditions more natural (see Remark 1.4), we need to achieve a refined pointwise estimates of Green's function in low frequency part. More precisely, note the following proposition of the lower frequency part of the Fourier transform for Green's function for linearized Euler equations with damping $(2.4)_{1,2}$:

$$\hat{G}(\xi,t)\sim\begin{pmatrix}1 & \xi^\tau\\ \xi & \xi\xi^\tau\end{pmatrix}e^{-|\xi|^2t}+\begin{pmatrix}1 & \xi^\tau\\ \xi & \frac{\xi\xi^\tau}{|\xi|^2}\end{pmatrix}e^{-at},\ \text{with some constant } a>0,$$

i.e., there is a Calderon–Zygmund operator with symbol $\xi\xi^\tau/|\xi|^2$ in Fourier space for Green's function corresponding to the momentum equation; however, this factor does not appear in the first line of Green's function corresponding to the mass equation. Here and in the subsequent, the symbol τ is used to denote the transpose. Hence, we can expect that the decay of the density with respect to the space variable x (or, it could be called regularity on x) can be faster than that of the momentum. That is, the time-asymptotic shape of the densities is $B_r(|x|,t|)$ with $r>\frac{d}{2}$, and the time-asymptotic shape of the momentums is $B_{\frac{d}{2}}(|x|,t)$. Moreover, from an additional condition $|D_x^\alpha\nabla\phi_0|\le C\epsilon_0(1+|x|^2)^{-r}$ with $r>\frac{d}{2}$ on the potential $\nabla\phi_0$, and the Poisson equation $(1.1)_5$, we can regard the Calderon–Zygmund operator with symbol $\frac{\xi}{|\xi|^2}$ in Green's function of $(2.4)_{3,4}$ as the Calderon–Zygmund operator with symbol $\frac{\xi\xi^\tau}{|\xi|^2}$. From these facts, we can obtain the pointwise estimate of Green's function for linearized Euler system with damping $(2.4)_{1,2}$ and the linearized unipolar Euler–Poisson system $(2.4)_{3,4}$ in low frequency part. Finally, by Duhamel principle and the energy estimate (1.3), we can deduce the pointwise estimates of the variables $\rho_1+\rho_2$, m_1+m_2, $\rho_1-\rho_2$ and m_1-m_2. Consequently, we can immediately obtain the pointwise estimates of the solution (ρ_1,ρ_2,m_1,m_2) to the original problem (1.1)–(1.2).

Remark 1.4. Though Wang and Yang [34] have studied Green's function of the Euler system with damping, we reconsider it here by some new conservations and give a refined result. In fact, they showed that if the initial data $(\rho_0-\bar\rho, u_0)$ (velocity $u=\frac{m}{\rho}$) are small in $H^l(\mathbb{R}^d)$ with $d\ge 3$ and $l\ge[d/2]+3$ and

$$|D_x^{\alpha'}(\rho_0-\bar\rho,u_0)|\le\epsilon_0(1+|x|^2)^{-r},\ r>\frac{3d}{4},\ 0\le|\alpha'|\le l, \tag{1.9}$$

then for $|\alpha|\le\min\{d,l-2\}$

$$|D_x^\alpha(\rho(x,t)-\bar\rho),u(x,t))|\le C(1+t)^{-\frac{d+|\alpha|}{2}}(1+\frac{|x|^2}{1+t})^{-\frac{d}{2}}. \tag{1.10}$$

Nevertheless, (1.10) implies that $\|D_x^\alpha(\rho(x,t)-\bar\rho)\|_{L^2(\mathbb{R}^d)}$ and $\|D_x^\alpha u(x,t))\|_{L^2(\mathbb{R}^d)}$ have the same decay rate on the variable t, which is weaker than those in [29]. Comparing with (1.10), L^2-decay rate with respect to the time t derived from the pointwise estimate (1.6) is the same to that in [29], where they made a Leray project on the velocity field to refine the L^2-decay rate. In the present paper, we make a direct but more subtle analysis on Green's function to achieve a refined pointwise estimates. Moreover, it is worthy mentioning that the condition (1.9) is not good enough in some sense since the initial data are in the unusual $L^p(\mathbb{R}^d)$-space with $p<1$. Our initial condition (1.4) is more natural since the initial data is in the usual L^1 space with respect to space variable x.

Remark 1.5. The time-asymptotic shape $B_{\frac{d}{2}}(|x|,t)$ for $|D_x^\alpha(m_1-m_2)|$, where m_1-m_2 corresponds to the solution of the momentum equation in the Euler–Poisson system with damping $(2.4)_{3,4}$, is better than

$B_{\frac{d-1}{2}}(|x|,t)$ for the velocity (or the momentum) of the Euler–Poisson equations with damping in [36]. This is caused by an additional condition $|D_x^\alpha \nabla\phi_0| \le C\epsilon_0(1+|x|^2)^{-r}$ with $r > \frac{d}{2}$ on the potential $\nabla\phi_0$ in (1.4) than in [36], which together with the Poisson equation $(1.1)_5$ could help us regard the Calderon–Zygmund operator with symbol $\frac{\xi}{|\xi|^2}$ in Green's function of $(2.4)_{3,4}$ as the Calderon–Zygmund operator with symbol $\frac{\xi\xi^\tau}{|\xi|^2}$. See the details in (3.15).

Remark 1.6. Compared with [23,31], we mainly obtain the pointwise estimates of the solution for the multi-dimensions bipolar Euler–Poisson system. Based on them, we show the decay rates of the densities $\rho_i(i=1,2)$ in the L^p-norm with $p \ge 1$ and the time-decay rates of the momentums $m_i(i=1,2)$ in the L^p-norm with $p > 1$ here. Moreover, our results are different from the $L^p(p \in [2,+\infty])$-convergence rate of planar waves in [21]. Finally, we believe that the method in this paper maybe could help us deduce the pointwise results for other bipolar systems, especially for the bipolar Navier–Stokes–Poisson system, bipolar Euler–Maxwell system or bipolar Navier–Stokes–Maxwell system. We refer to the relative articles [4,18,30] and references therein. These are left for the forthcoming future.

Notations. Throughout this paper, we will use C or C_i denotes a positive generic constant (generally large) that may vary at different places. $D^l = \partial_x^l$ with an integer $l \ge 0$ denotes the usual any spatial derivatives of order l. For $1 \le p \le \infty$ and an integer $m \ge 0$, we use L^p and $W^{m,p}$ to denote the usual Lebesgue space $L^p(\mathbb{R}^3)$ and Sobolev spaces $W^{m,p}(\mathbb{R}^d)$ with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{m,p}}$, respectively, and set $H^m = W^{m,2}$ with norm $\|\cdot\|_{H^m}$ when $p=2$. $f \sim g$ means that there exist two positive constants C_1, C_2 such that $C_1|f| \le |g| \le C_2|f|$.

The rest of the paper is arranged as follows. In Sect. 2, we reformulate the original system and then give detailed analysis on Green's functions for the Euler equations with damping and Euler–Poisson equations with damping. The proof of pointwise estimates of the solution will be derived in Sect. 3.

2. Green's functions

2.1. Reformulation of original problem

Assume $\bar\rho = 1$ and $P'(\bar\rho) = 1$ without loss of generality. Then the Cauchy problem (1.1)–(1.2) is reformulated as

$$\begin{cases} \partial_t\rho_1 + \mathrm{div} m_1 = 0, \\ \partial_t m_1 + m_1 + \nabla\rho_1 - \nabla\phi = -\mathrm{div}(\frac{m_1\otimes m_1}{\rho_1}) - \nabla(P(\rho_1)-\rho_1) + (\rho_1-1)\nabla\phi, \\ \partial_t\rho_2 + \mathrm{div} m_2 = 0, \\ \partial_t m_2 + m_2 + \nabla\rho_2 + \nabla\phi = -\mathrm{div}(\frac{m_2\otimes m_2}{\rho_2}) - \nabla(P(\rho_2)-\rho_2) - (\rho_2-1)\nabla\phi, \\ \Delta\phi = \rho_1 - \rho_2, \\ (\rho_1,u_1,\rho_2,u_2)(x,0) = (\rho_{10},u_{10},\rho_{20},u_{20})(x). \end{cases} \tag{2.1}$$

Next, set

$$n_1 = \rho_1+\rho_2-2, \quad n_2 = \rho_1-\rho_2, \quad w_1 = m_1+m_2, \quad w_2 = m_1-m_2, \tag{2.2}$$

which give equivalently

$$\rho_1 = \frac{n_1+n_2}{2}+1, \quad \rho_2 = \frac{n_1-n_2}{2}+1, \quad m_1 = \frac{w_1+w_2}{2}, \quad m_2 = \frac{w_1-w_2}{2}. \tag{2.3}$$

From (2.2) and (2.3), it follows that the Cauchy problem (2.1) can be reformulated into the following Cauchy problem for the unknown $(n_1, w_1, n_2, w_2, \phi)$

$$\begin{cases} \partial_t n_1 + \mathrm{div} w_1 = 0, \\ \partial_t w_1 + w_1 + \nabla n_1 = f_1(n_1, w_1, n_2, w_2), \\ \partial_t n_2 + \mathrm{div} w_2 = 0, \\ \partial_t w_2 + w_2 + \nabla n_2 - 2\nabla\phi = f_2(n_1, w_1, n_2, w_2), \\ \Delta\phi = n_2, \\ (n_1, w_1, n_2, w_2)(x, 0) = (n_{10}, w_{10}, n_{20}, w_{20})(x), \end{cases} \tag{2.4}$$

where $(n_{10}, w_{10}, n_{20}, w_{20}) := (\rho_{10} + \rho_{20} - 2, u_{10} + u_{20}, \rho_{10} - \rho_{20}, u_{10} - u_{20})$, and

$$\begin{aligned} f_1 = &-\mathrm{div}\Big[\frac{(w_1 + w_2) \otimes (w_1 + w_2)}{2(n_1 + n_2) + 4} + \frac{(w_1 - w_2) \otimes (w_1 - w_2)}{2(n_1 - n_2) + 4}\Big] \\ &- \nabla\Big[P(\frac{n_1 + n_2}{2} + 1) - \frac{n_1 + n_2}{2} + P(\frac{n_1 - n_2}{2} + 1) - \frac{n_1 - n_2}{2}\Big] + n_2\nabla\phi, \end{aligned} \tag{2.5}$$

$$\begin{aligned} f_2 = &-\mathrm{div}\Big[\frac{(w_1 + w_2) \otimes (w_1 + w_2)}{2(n_1 + n_2) + 4} - \frac{(w_1 - w_2) \otimes (w_1 - w_2)}{2(n_1 - n_2) + 4}\Big] \\ &- \nabla\Big[P(\frac{n_1 + n_2}{2} + 1) - \frac{n_1 + n_2}{2} - P(\frac{n_1 - n_2}{2} + 1) + \frac{n_1 - n_2}{2}\Big] + n_1\nabla\phi. \end{aligned} \tag{2.6}$$

Hence, the Cauchy problem (2.4) can be formally divided into the Cauchy problem for the Euler equations with damping $(2.4)_{1,2}$ and the Euler–Poisson equations with damping $(2.4)_{3,4,5}$, which interact each other through the nonlinear inhomogeneous terms on the right-hand side.

2.2. Green's function of linearized Euler equations with damping

In this subsection, we shall give the pointwise estimates for Green's function for the linearized Euler equations with damping. A similar analysis can be founded in [34]. For the convenience of the readers, a brief analysis will be also sketched here.

Recall the linearized system on (n_1, w_1) in (2.4)

$$\begin{cases} \partial_t n_1 + \mathrm{div} w_1 = 0, \\ \partial_t w_1 + \nabla n_1 + w_1 = 0, \end{cases} \tag{2.7}$$

which implies

$$\partial_t^2 n_1 + \partial_t n_1 - \Delta n_1 = 0. \tag{2.8}$$

It is obvious that the symbol of the operator in (2.8) is

$$\lambda^2 + \lambda + |\xi|^2 = 0.$$

Here, λ and $\xi^\tau = (\xi_1, \xi_2, \ldots, \xi_d)$ correspond to $\frac{\partial}{\partial t}$ and $(D_{x_1}, \ldots, D_{x_d})$, respectively, where $D_{x_j} = \frac{1}{\sqrt{-1}\partial/\partial_{x_j}}$ with $j = 1, \ldots, d$. It is easy to find that the eigenvalues of (2.7) are $\lambda = \lambda_\pm(\xi) = \frac{-1\pm\sqrt{1-4|\xi|^2}}{2}$. Now we consider Green's function for (2.7), i.e., we study the solution to the following initial value problem:

$$\begin{cases} \left(\frac{\partial}{\partial t} + A(D_x)\right) G(x, t) = 0, \\ G(x, 0) = \delta(x), \end{cases} \tag{2.9}$$

where $\delta(x)$ is the Dirac function, the symbols of operator $A(D_x)$ are

$$A(\xi) = \begin{pmatrix} 0 & \sqrt{-1}\xi^\tau \\ \sqrt{-1}\xi & I_{d\times d} \end{pmatrix}.$$

Applying the Fourier transform with respect to the variable x to (2.9), we get by a direct calculation

$$\begin{aligned}\hat{G}(\xi,t)&=\begin{pmatrix}\hat{G}_{11} & \hat{G}_{12}\\ \hat{G}_{21} & \hat{G}_{22}\end{pmatrix}=\begin{pmatrix}\frac{\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t}}{\lambda_+-\lambda_-} & -\sqrt{-1}\frac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-}\xi^\tau\\ -\sqrt{-1}\frac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-}\xi & e^{-t}I+\left(\frac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-}-e^{-t}\right)\frac{\xi\xi^\tau}{|\xi|^2}\end{pmatrix}\\ &=\begin{pmatrix}\eta_+e^{\lambda_-t}-\eta_-e^{\lambda_+t} & -\sqrt{-1}\eta_0\xi^\tau(-e^{\lambda_+t}+e^{\lambda_-t})\\ -\sqrt{-1}\eta_0\xi(-e^{\lambda_+t}+e^{\lambda_-t}) & e^{-t}(I-\frac{\xi\xi^\tau}{|\xi|^2})+\frac{\xi\xi^\tau}{|\xi|^2}(\eta_+e^{\lambda_+t}-\eta_-e^{\lambda_-t})\end{pmatrix},\end{aligned}$$

where $\eta_0(\xi)=(\lambda_+(\xi)-\lambda_-(\xi))^{-1}, \eta_\pm(\xi)=\lambda_\pm(\xi)\eta_0(\xi)$.

Let

$$\chi_1(\xi)=\begin{cases}1, & |\xi|<\epsilon,\\ 0, & |\xi|>2\epsilon,\end{cases}$$

and

$$\chi_3(\xi)=\begin{cases}1, & |\xi|>K+1,\\ 0, & |\xi|<K,\end{cases}$$

be the smooth cutoff functions with $2\epsilon<K$, and $\chi_2=1-\chi_1-\chi_3$, which correspond to lower, higher and middle frequency parts in the Fourier space. The number of the subscripts on Green's function is large because we will analysis each term in Green's function (a matrix). In order to facilitate the description, here and in the sequel, when we say "$|\xi|$ is sufficiently small" or "the lower frequency part of Green's function," we are actually dealing with the function $\chi_1(\xi)G(\xi,t)$, and when we say "$|\xi|$ is sufficiently large" or "the higher frequency part of Green's function," we are actually dealing with the function $\chi_3(\xi)G(\xi,t)$.

First, we consider the lower frequency part of Green's function, i.e., $|\xi|$ being sufficiently small. In this case, we will write $\hat{G}=\hat{G}^+ + \hat{G}^- + \hat{G}^0$ for $|\xi|$ with

$$\hat{G}^+=\begin{pmatrix}-\eta_- & -\sqrt{-1}\eta_0\xi^\tau\\ -\sqrt{-1}\eta_0\xi & \eta_+(\xi\xi^\tau)/|\xi|^2\end{pmatrix}e^{\lambda_+t}:=\begin{pmatrix}\hat{G}^+_{1,1} & \hat{G}^+_{1,2}\\ \hat{G}^+_{2,1} & \hat{G}^+_{2,2}\end{pmatrix}, \tag{2.10}$$

$$\hat{G}^-=\begin{pmatrix}\eta_+ & \sqrt{-1}\eta_0\xi^\tau\\ \sqrt{-1}\eta_0\xi & 0\end{pmatrix}e^{\lambda_-t}+\begin{pmatrix}0 & 0\\ 0 & -\eta_-(\xi\xi^\tau)/|\xi|^2\end{pmatrix}e^{\lambda_-t}:=\hat{G}^-_1+\hat{G}^-_2, \tag{2.11}$$

and

$$\hat{G}^0:=\hat{G}^0_1+\hat{G}^0_2=\begin{pmatrix}0 & 0\\ 0 & I\end{pmatrix}e^{-t}+\begin{pmatrix}0 & 0\\ 0 & -(\xi\xi^\tau)/|\xi|^2\end{pmatrix}e^{-t}. \tag{2.12}$$

Note that for $|\xi|$ sufficiently small, we have

$$\lambda_+(\xi)=-|\xi|^2+O(|\xi|^4),\quad \lambda_-(\xi)=-1+|\xi|^2+O(|\xi|^4),$$

which implies

$$\eta_-(\xi)=-1+O(|\xi|^2),\quad \eta_+(\xi)=-|\xi|^2+O(|\xi|^4),\quad \eta_0(\xi)=1+O(|\xi|^2).$$

Moreover, we also have

$$e^{\lambda_+(\xi)t}=e^{-|\xi|^2t}(1+O(|\xi|^4)t),\quad e^{\lambda_-(\xi)t}=e^{-t}e^{|\xi|^2t}(1+O(|\xi|^4)t). \tag{2.13}$$

Hence, we have

$$\hat{G}^+(\xi,t) := \begin{pmatrix} \hat{G}^+_{1,1} & \hat{G}^+_{1,2} \\ \hat{G}^+_{2,1} & \hat{G}^+_{2,2} \end{pmatrix}$$
$$= \begin{pmatrix} 1+O(|\xi|^2) & -\frac{\sqrt{-1}}{2}\xi^\tau + O(|\xi|^3) \\ -\sqrt{-1}\xi + O(|\xi|^3) & -\xi\xi^\tau + O(|\xi|^4) \end{pmatrix} e^{-|\xi|^2 t}(1+O(|\xi|^4)t), \tag{2.14}$$
$$\hat{G}^-_1(\xi,t) = \begin{pmatrix} -|\xi|^2 + O(|\xi|^4) & \frac{\sqrt{-1}}{2}\xi^\tau + O(|\xi|^3) \\ \sqrt{-1}\xi + O(|\xi|^3) & 0 \end{pmatrix} e^{-t} e^{|\xi|^2 t}(1+O(|\xi|^4)t). \tag{2.15}$$

By differentiating both sides of (2.14) and (2.15) with respect to ξ, because of the smoothness of $\hat{G}^+(\xi,t)$, $\hat{G}^-_1(\xi,t)$ and $\hat{G}^0_1$, we can immediately obtain the following lemma.

Lemma 2.1. *If $|\xi|$ is sufficiently small, then there exists a constant $b>0$, such that*

$$|D^\beta_\xi(\xi^\alpha \chi_1(\xi)\hat{G}^+_{1,1}(\xi,t))| \le C(|\xi|^{(|\alpha|-|\beta|)_+} + |\xi|^{|\alpha|} t^{|\beta|/2})(1+t|\xi|^2)^{|\beta|+1} e^{-b|\xi|^2 t},$$
$$|D^\beta_\xi(\xi^\alpha \chi_1(\xi)\hat{G}^+_{1,2}(\xi,t))| \le C(|\xi|^{(|\alpha|-|\beta|+1)_+} + |\xi|^{|\alpha|+1} t^{|\beta|/2})(1+t|\xi|^2)^{|\beta|+1} e^{-b|\xi|^2 t},$$
$$|D^\beta_\xi(\xi^\alpha \chi_1(\xi)\hat{G}^+_{2,1}(\xi,t))| \le C(|\xi|^{(|\alpha|-|\beta|+1)_+} + |\xi|^{|\alpha|+1} t^{|\beta|/2})(1+t|\xi|^2)^{|\beta|+1} e^{-b|\xi|^2 t},$$
$$|D^\beta_\xi(\xi^\alpha \chi_1(\xi)\hat{G}^+_{2,2}(\xi,t))| \le C(|\xi|^{(|\alpha|-|\beta|+2)_+} + |\xi|^{|\alpha|+2} t^{|\beta|/2})(1+t|\xi|^2)^{|\beta|+1} e^{-b|\xi|^2 t},$$
$$|D^\beta_\xi(\xi^\alpha \chi_1(\xi)\hat{G}^-_{1}(\xi,t))| \le Ce^{-t}(|\xi|^{(|\alpha|-|\beta|+1)_+} + |\xi|^{|\alpha|} t^{|\beta|/2})(1+t|\xi|^2)^{|\beta|+1} e^{-b|\xi|^2 t},$$
$$|D^\beta_\xi(\xi^\alpha \chi_1(\xi)\hat{G}^0_{1}(\xi,t))| \le Ce^{-t}(|\xi|^{(|\alpha|-|\beta|)_+} + |\xi|^{|\alpha|} t^{|\beta|/2})(1+t|\xi|^2)^{|\beta|+1} e^{-b|\xi|^2 t}.$$

In order to get the pointwise estimate of each term in Green's function in the lower frequency part, we need the following lemma.

Lemma 2.2. *If $\hat{f}(\xi,t)$ has compact support in ξ, and there exists a constant $b>0$, such that if $\hat{f}(\xi,t)$ satisfies*

$$|D^\beta_\xi(\xi^\alpha \hat{f}(\xi,t)| \le C(|\xi|^{(|\alpha|+k-|\beta|)_+} + |\xi|^{|\alpha|+k} t^{|\beta|/2})(1+(t|\xi|^2))^m \exp(-b|\xi|^2 t)$$

for any multi-indexes α,β with $|\beta|\le 2N$ (integer N could be arbitrary large), then

$$|D^\alpha_x f(x,t)| \le C_N t^{-(d+|\alpha|+k)/2} B_N(|x|,t),$$

where k and m are any fixed integers, and $(a)_+ = \max(0,a)$.

Proof. If $|\beta| < k+|\alpha|$, then we have by direct calculation that

$$\begin{aligned}|x^\beta D^\alpha f(x,t)| &= C\Big|\int e^{\sqrt{-1}x\cdot\xi} D^\beta \xi^\alpha \hat{f}(\xi,t)\mathrm{d}\xi\Big| \\ &\le C\int |\xi|^{|\alpha|+k}(|\xi|^{-|\beta|} + t^{|\beta|/2})(1+(t|\xi|^2))^m e^{-b|\xi|^2 t}\mathrm{d}\xi \\ &\le Ct^{-(|\alpha|+d+k-|\beta|)/2}.\end{aligned} \tag{2.16}$$

If $|\beta| \geq k + |\alpha|$, one also can find that

$$
\begin{aligned}
|x^\beta D^\alpha f(x,t)| &= C \left| \int e^{\sqrt{-1}x\cdot\xi} D^\beta \xi^\alpha \hat{f}(\xi,t) \mathrm{d}\xi \right| \\
&\leq C \int \left(|\xi|^{(|\alpha|+k-|\beta|)_+} + |\xi|^{|\alpha|+k} t^{|\beta|/2} \right) (1 + (t|\xi|^2))^m e^{-b|\xi|^2 t} \mathrm{d}\xi \\
&\leq C(1 + t^{-(|\alpha|+k-|\beta|)/2}) t^{-n/2} \\
&\leq C(1+t)^{-(|\alpha|+k-|\beta|)/2}) t^{-n/2} \\
&= C(1+t)^{|\beta|/2} t^{-(|\alpha|+d+k)/2} \left(\frac{t}{1+t} \right)^{(|\alpha|+k)/2} \\
&\leq C(1+t)^{|\beta|/2} t^{-(|\alpha|+d+k)/2}.
\end{aligned} \tag{2.17}
$$

Let $\beta = 0$ when $|x|^2 \leq 1+t$, and $|\beta| = 2N$ when $|x|^2 > 1+t$, we obtain from (2.16) and (2.17) that

$$
|D^\alpha f(x,t)| \leq C t^{-(|\alpha|+d+k)/2} \min \left(1, \left(\frac{1+t}{|x|^2} \right)^N \right).
$$

Since

$$
1 + \frac{|x|^2}{1+t} \leq 2 \begin{cases} 1, & |x|^2 \leq 1+t, \\ \frac{|x|^2}{1+t}, & |x|^2 > 1+t, \end{cases}
$$

we have

$$
\min \left(1, (\frac{1+t}{|x|^2})^N \right) \leq \frac{2^N}{(1 + \frac{|x|^2}{1+t})^N} = 2^N B_N(|x|,t).
$$

This completes the proof. □

Based on Lemma 2.1 and 2.2, we immediately have

Lemma 2.3. *When $|\xi|$ is sufficiently small, we have for any $|\alpha| \geq 0$ and any integer $N > 0$*

$$
\begin{aligned}
&|D_x^\alpha(\chi_1(D_x) G_{1,1}^+(x,t))| \leq C(1+t)^{-(d+|\alpha|)/2} B_N(|x|,t), \\
&|D_x^\alpha(\chi_1(D_x)(G_{1,2}^+(x,t), G_{2,1}^+(x,t)))| \leq C(1+t)^{-(d+1+|\alpha|)/2} B_N(|x|,t), \\
&|D_x^\alpha(\chi_1(D_x) G_{2,2}^+(x,t))| \leq C(1+t)^{-(d+2+|\alpha|)/2} B_N(|x|,t), \\
&|D_x^\alpha(\chi_1(D_x) G_1^-(x,t))| \leq C e^{-c_0 t} B_N(|x|,t), \\
&|D_x^\alpha(\chi_1(D_x) G_1^0(x,t))| \leq C e^{-c_0 t} B_N(|x|,t), \text{ for some constant } c_0 > 0.
\end{aligned}
$$

The proof of Lemma 2.3 is similar as that of Lemma 4.1 in [34], and we can omit the details here.

Remark 2.4. Lemma 2.3 indicates that each term of $\hat{G}$ in lower frequency part has different decay rate, which is an improvement of Lemma 4.1 in [34], where they obtained $|D_x^\alpha(\chi_1(D_x)G(x,t)| \leq C(1+t)^{-(d+|\alpha|)/2} B_N(|x|,t)$. In fact, these estimates in Lemma 2.3 are subtle and critical to get our improved pointwise estimates and L_2-decay rate of the solution.

Now, there are still two terms $\hat{G}_2^-(\xi,t)$ and $\hat{G}_2^0(\xi,t)$ contain a Calderon–Zygmund operator R_{ij} with symbol $\frac{\xi_i \xi_j}{|\xi|^2}$ should be considered in lower frequency part of $\hat{G}(\xi,t)$. That is, we have

Lemma 2.5. *When $|\xi|$ is sufficiently small and $d \geq 3$, we have for some $c_0 > 0$ that*

$$
|D_x^\alpha(\chi_1(D_x) G_2^0(x,t))| + |D_x^\alpha(\chi_1(D_x) G_2^-(x,t))| \leq C e^{-c_0 t} B_{\frac{d}{2}}(|x|,t).
$$

Proof. We only give the proof for $D_x^\alpha(\chi_1(D_x)G_2^-(x,t))$, since the other term can be obtained similarly. For convenience, we denote $D_x^\alpha G_2^-(x,t) = R_{ij} D_x^\alpha H(x,t)$, where R_{ij} is the Calderon–Zygmund operator and the symbol of $H(x,t)$ is $\chi_1(\xi)e^{-t}$. First, recall that

$$D_x^\alpha(\chi_1(D_x)G_2^-(x,t)) = -\mathcal{F}^{-1}\big(\chi_1(\xi)\xi^\alpha \frac{\xi_i\xi_j}{|\xi|^2}e^{-t}\big)$$

by the inverse Fourier transformation and (2.12). When $|\xi|$ is small, as in Lemma 2.1 one can get

$$|D_\xi^\beta(\chi_1(\xi)\xi^\alpha e^{-t}| \le C(\beta)e^{-t} \le Ce^{-\frac{1}{2}t}e^{-|\xi|^2 t}$$

for any $|\beta| \ge 0$, which together with Lemma 2.2 gives that

$$|D_x^\alpha H(x,t)| \le Ce^{-c_1 t}B_N(|x|,t), \text{ for any integer } N > 0 \text{ and some constant } c_1 > 0.$$

Next, the inverse Fourier transformation gives that

$$\begin{aligned}\mathcal{F}^{-1}\big(\tfrac{\xi_i\xi_j}{|\xi|^2}\big) &= -D_{x_i}D_{x_j}\Delta^{-1}\delta(x) = CD_{x_i}D_{x_j}|x|^{2-d}\\ &= CD_{x_i}(x_j|x|^{-d}) = Cx_ix_j|x|^{-(d+2)}.\end{aligned}$$

If $d \ge 3$, we have

$$|D_x^\alpha(\chi_1(D_x)G_2^-(x,t))| \le C\int_{\mathbb{R}}\left|\frac{x_i-y_i}{|x-y|^d}D_y^\alpha D_{y_j}H(y,t)\right|\mathrm{d}y,$$

or

$$|D_x^\alpha(\chi_1(D_x)G_2^-(x,t))| \le C\int_{\mathbb{R}}\left|\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^{d+2}}D_y^\alpha H(y,t)\right|\mathrm{d}y.$$

If $|x|^2 \ge t$ and $|x| \le 2|y|$, it holds that

$$B_N(|y|,t) = B_{\frac{d}{2}}(|y|,t)B_{N-\frac{d}{2}}(|y|,t) \le CB_{\frac{d}{2}}(|x|,t)B_{N-\frac{d}{2}}(|y|,t).$$

Let

$$\Omega_1 = \{x-y; |x-y| \le 1\}, \quad \Omega_2 = \{x-y; |x-y| > 1\}.$$

By Young's inequality,

$$\begin{aligned}\Big|\int_{\mathbb{R}}|x-y|^{-(d-1)}B_{N-\frac{d}{2}}(|y|,t)\mathrm{d}y\Big| &\le C(\||x-y|^{-(d-1)}\|_{L^1(\Omega_1)}\|B_{N-\frac{d}{2}}(|y|,t)\|_{L^\infty(\Omega_1)}\\ &\quad+\||x-y|^{-(d-1)}\|_{L^2(\Omega_2)}\|B_{N-\frac{d}{2}}(|y|,t)\|_{L^2(\Omega_2)}) \le C.\end{aligned}$$

Thus, we have

$$|D_x^\alpha(\chi_1(D_x)G_2^-(x,t))| \le C \le Ce^{-c_1 t}B_{\frac{d}{2}}(|x|,t), \text{ for some constant } c_1 > 0.$$

If $|x|^2 \ge t$ and $|x| > 2|y|$, then

$$\begin{aligned}|D_x^\alpha(\chi_1(D_x)G_2^-(x,t))| &\le C\int_{\mathbb{R}}\Big|\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^{d+2}}D_y^\alpha H(y,t)|\mathrm{d}y\\ &\le C|x|^{-d}\int_{\mathbb{R}}B_N(|y|,t)\mathrm{d}y \le Ce^{-c_1 t}B_{\frac{d}{2}}(|x|,t), \text{ for some constant } c_1 > 0.\end{aligned}$$

If $|x|^2 \le t$, then the fact that $1 \le CB_k(|x|,t)$ for any constant $k \ge 0$, and Young's inequality yield that

$$|D_x^\alpha(\chi_1(D_x)G_2^-(x,t))| \le Ce^{-c_1 t}B_{\frac{d}{2}}(|x|,t).$$

In summary, we deduce the estimate for $D_x^\alpha(\chi_1(D_x)G_2^-(x,t))$. The proof is completed. □

Up to now, all of the terms in Green's function $G(x,t)$ in lower frequency have been estimated. With respect to the pointwise estimates of Green's function in middle frequency and in high frequency, we have

Lemma 2.6. *For fixed ϵ and R defined in the cut-off functions, there exist a positive constant c_0 and C such that*

$$|D_x^\alpha(\chi_2(D_x)G(x,t))| \le Ce^{-c_0 t}B_N(|x|,t), \text{ for any integer } N>0,$$

and

Lemma 2.7. *For $|\xi|$ being sufficiently large, there exists distribution*

$$\chi_3(D_x)K_q(x,t) = \left(\sum_{j=0}^{q-1} L_j(x,t)\right) e^{-t/2},$$

where $L_j(x,t)$ is a Dirac-like function. Then for $q=[\frac{|\alpha|+d+3}{2}]$, we have for some $c_0>0$

$$|D_x^\alpha(\chi_3(D)(G^+ + G^- - K_q(x,t)))| \le Ce^{-c_0 t}B_N(|x|,t).$$

Lemma 2.6 and 2.7 can be found in [34], and we can omit the details here.

In summary, combining the pointwise estimates in the lower, middle and higher frequency parts above, we have the following pointwise estimates for Green's function without the singular term $\chi_3(D_x)K_q(x,t)$.

Proposition 2.8. *For any $|\alpha|\ge 0$, we have*

$$\begin{aligned}
&|D_x^\alpha(G_{1,1}-\chi_3(D_x)K_q)| \le C(1+t)^{-\frac{d+|\alpha|}{2}}B_N(|x|,t),\\
&|D_x^\alpha(G_{1,2}-\chi_3(D_x)K_q)| + |D_x^\alpha(G_{2,1}-\chi_3(D_x)K_q)| \le C(1+t)^{-\frac{d+1+|\alpha|}{2}}B_N(|x|,t),\\
&|D_x^\alpha(G_{2,2}-\chi_3(D_x)K_q)| \le C(1+t)^{-\frac{d+2+|\alpha|}{2}}B_{\frac{d}{2}}(|x|,t),
\end{aligned}$$

where $\chi_3(D_x)K_q(x,t)$ is defined in Lemma 2.7 and integer N could be arbitrary large.

Remark 2.9. Notice that the pointwise estimates in Proposition 2.8 is different from those in [34], where the authors did not give the pointwise estimate for $G^0(x,t)$ with the inverse Fourier transform in (2.12). Thus, when they want to obtain the pointwise estimate for the nonlinear system (2.1), they need an additional lemma (cf. Lemma 3.3 [34]), which directly results in the assumption on the initial data (1.4) in [34]. However, Proposition 2.8 could help us to relax the assumption (1.4) into the following one (belongs to $L^1(\mathbb{R}^d)$)

$$|D_x^\alpha(\rho_0,m_0)| \le C\epsilon_0(1+|x|^2)^{-r}, \ r>\frac{d}{2}.$$

2.3. Green's function of linearized Euler–Poisson equations with damping

In this subsection, we shall make an analysis on Green's function for the linearized Euler–Poisson equations with damping, which have been studied in [36] by us.

Recall the linearized system on (n_2,w_2) in (2.4):

$$\begin{cases} \partial_t n_2 + \mathrm{div} w_2 = 0,\\ \partial_t w_2 + \nabla n_2 + w_2 - 2\nabla\phi = 0, \end{cases} \tag{2.18}$$

which implies

$$\partial_t^2 n_2 + \partial_t n_2 - \Delta n_2 + 2n_2 = 0. \tag{2.19}$$

Then the symbol of the symbol of the operator in (2.19) is

$$\lambda^2 + \lambda + |\xi|^2 + 2 = 0,$$

whose eigenvalues are

$$\lambda = \lambda_{\pm}(\xi) = \frac{-1 \pm \sqrt{-1}\sqrt{7+4|\xi|^2}}{2}.$$

Then we consider Green's function for (2.18), i.e.,

$$\begin{cases} (\frac{\partial}{\partial t} + A(D_x))\mathbb{G}(x,t) = 0, \\ \mathbb{G}(x,0) = \delta(x), \end{cases} \tag{2.20}$$

where $\delta(x)$ is the Dirac function and the symbols of operator $A(D_x)$ are

$$A(\xi) = \begin{pmatrix} 0 & \sqrt{-1}\xi^{\tau} \\ \sqrt{-1}\xi(1+\frac{2}{|\xi|^2}) & I_{d\times d} \end{pmatrix}.$$

Thus by direct calculation, we get

$$\hat{\mathbb{G}}(\xi,t) = \begin{pmatrix} \hat{\mathbb{G}}_{1,1} & \hat{\mathbb{G}}_{1,2} \\ \hat{\mathbb{G}}_{2,1} & \hat{\mathbb{G}}_{2,2} \end{pmatrix}, \tag{2.21}$$

where

$$\hat{\mathbb{G}}_{1,1} = \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-}, \quad \hat{\mathbb{G}}_{1,2} = -\sqrt{-1}\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\xi^{\tau},$$
$$\hat{\mathbb{G}}_{2,1} = -\sqrt{-1}\left(1+\frac{2}{|\xi|^2}\right)\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\xi, \quad \hat{\mathbb{G}}_{2,2} = e^{-t}I + \left[\frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} - e^{-t}\right]\frac{\xi\xi^{\tau}}{|\xi|^2}.$$

For simplicity and completeness, we just state the main results and show the difference of the Green's functions between these two linearized systems in Sects. 2.2 and 2.3. As shown in Section 2 in [36], the main difference is the lower frequency part of Green's function of (2.18) has exponential decay rate on time t. In fact, when $|\xi|$ is sufficiently small, by using Taylor expansion one has

$$\lambda_{\pm} = -\frac{1}{2} \pm \sqrt{-1}\left(\frac{\sqrt{7}}{2} + O(|\xi|^2)\right).$$

On the other hand, for a hyperbolic–parabolic system which satisfies Shizuta–Kawashima condition, the decay rate of the solution mainly depends on the lower frequency part of the Green's function. Thus, we can easily deduce the exponential temporal decay rate of the Green's function (2.20) since the higher frequency part of the Green's function usually has the exponential temporal decay rate. Please see the details in [36].

Thus, we have the pointwise estimates of the Green's function (2.20) in [36] as follows, which are formulated in a more exhaustive way.

Proposition 2.10. *For any $|\alpha| \geq 0$ and $|\beta| \geq 1$, we have for some constant $c_0 > 0$,*

$$|D_x^{\alpha}(\mathbb{G}_{1,1} - \chi_3(D_x)K_q)| + |D_x^{\alpha}(\mathbb{G}_{1,2} - \chi_3(D_x)K_q)| \leq Ce^{-c_0 t}B_N(|x|,t), \tag{2.22}$$
$$|D_x^{\alpha}(\mathbb{G}_{2,2} - \chi_3(D_x)K_q)| \leq Ce^{-c_0 t}B_{\frac{d}{2}}(|x|,t), \tag{2.23}$$
$$|(1-\chi_1(D_x))\mathbb{G}_{2,1} - \chi_3(D_x)K_q)| \leq Ce^{-c_0 t}B_N(|x|,t), \tag{2.24}$$
$$|D_x^{\beta}(\mathbb{G}_{2,1} - \chi_3(D_x)K_q)| \leq Ce^{-c_0 t}B_{\frac{d}{2}}(|x|,t), \tag{2.25}$$

where $\chi_3(D_x)K_q(x,t)$ is defined as in Lemma 2.7 and the integer N could be arbitrary large.

Proof. (2.22) and (2.23) can be deduced as in Sect. 2.2 for the Euler system with damping. On the other hand, note that there is a Calderon–Zygmund operator with the symbol $\frac{\xi}{|\xi|^2}$ in $\mathbb{G}_{2,1}$, which is singular in the lower frequency part and has no difference from the middle and higher frequency parts from $\mathbb{G}_{1,1}$ and $\mathbb{G}_{1,2}$. Obviously, the term in (2.24) does not contain any part with singularity, so it also can be deduced as in Sect. 2.2. Lastly, the estimate (2.25) with $|\beta| \geq 1$ for $\mathbb{G}_{2,1}$ is equivalent to (2.23) with $|\alpha| \geq 0$ for $\mathbb{G}_{2,2}$ containing the Calderon–Zygmund operator with the symbol $\frac{\xi\xi^\tau}{|\xi|^2}$. The proof of Proposition 2.10 is completed. □

Remark 2.11. The estimates (2.24) and (2.25) are critical to help us deduce the asymptotic shape $B_{\frac{d}{2}}(|x|,t)$ in the pointwise estimates, which is an improvement of those in [36] for Euler–Poisson equations with damping, where the asymptotic shape for the velocity (or the momentum) is $B_{\frac{d-1}{2}}(|x|,t)$. In fact, when we deal with convolution $D_x^\beta(\mathbb{G}_{2,1} - \chi_3(D_x)K_q) * n_{2,0}$ with $|\beta| = 0$ in the next section, we can "borrow" a derivation from n_{20} since $n_{20} = \Delta\phi_0 = \mathrm{div}\nabla\phi_0$ (see the details in the proof for the term R_{10}).

3. Pointwise estimates for nonlinear system

In this section, we will study the pointwise estimates of the solution to the nonlinear system (2.4). Firstly, we give several lemmas, which will be used for estimating the convolutions between the Green's function and the nonlinear terms.

Lemma 3.1. *(1) When $n_1, n_2 > \frac{d}{2}$ and $n_3 = \min(n_1, n_2)$, we have*

$$\int_{\mathbb{R}^d} \left(1 + \frac{|x-y|^2}{1+t}\right)^{-n_1} (1+|y|^2)^{-n_2}\, dy \leq C\left(1 + \frac{|x|^2}{1+t}\right)^{-n_3}.$$

(2) When $n_2 > \frac{d}{2}$, we have

$$\int_{\mathbb{R}^d} \left(1 + \frac{|x-y|^2}{1+t}\right)^{-\frac{d}{2}} (1+|y|^2)^{-n_2}\, dy \leq C\left(1 + \frac{|x|^2}{1+t}\right)^{-\frac{d}{2}}.$$

Proof. We only prove (2), since the proof of (1) is similar. First, we have

$$\int_{\mathbb{R}^d} \left(1 + \frac{|x-y|^2}{1+t}\right)^{-\frac{d}{2}} (1+|y|^2)^{-n_2}\mathrm{d}y = \Big(\int_{|x|\geq 2|y|} + \int_{|x|<2|y|}\Big)\left(1 + \frac{|x-y|^2}{1+t}\right)^{-\frac{d}{2}} (1+|y|^2)^{-n_2}\mathrm{d}y.$$

It is easy to obtain

$$\begin{aligned}&\int_{|x|\geq 2|y|} \left(1 + \frac{|x-y|^2}{1+t}\right)^{-\frac{d}{2}} (1+|y|^2)^{-n_2}\mathrm{d}y\\ &\leq C\left(1 + \frac{|x|^2}{1+t}\right)^{-\frac{d}{2}} \int_{\mathbb{R}^d}(1+|y|^2)^{-n_2}\mathrm{d}y \leq C\left(1 + \frac{|x|^2}{1+t}\right)^{-\frac{d}{2}}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\int_{\mathbb{R}^d} \left(1 + \frac{|x-y|^2}{1+t}\right)^{-\frac{d}{2}} (1+|y|^2)^{-n_2}\mathrm{d}y\\ &\leq C(1+|x|^2)^{-\frac{d}{2}} \int_{\mathbb{R}^d}\left(1 + \frac{|x-y|^2}{1+t}\right)^{-\frac{d}{2}} (1+|y|^2)^{-(n_2-\frac{d}{2})}\mathrm{d}y\\ &\leq C\left(1 + \frac{|x|^2}{1+t}\right)^{-\frac{d}{2}},\end{aligned}$$

where we have used the Young's inequality and $n_2 > \frac{d}{2}$. This proves (2) of Lemma 3.1. This completes the proof. □

Lemma 3.2. *Assume $d \geq 3$.*

(a) If functions $F(x,t)$ and $S(x,t)$ satisfy

$$|D_x^\alpha F(x,t)| \leq C(1+t)^{-\frac{d+|\alpha|}{2}} B_{n_1}(|x|,t),$$
$$|D_x^\alpha S(x,t)| \leq C(1+t)^{-\frac{2d+|\alpha|}{2}} B_{n_2}(|x|,t),$$

then, we have

$$I_\alpha =: \left| D_x^\alpha \left(\int_0^t F(t-s) * S(s)\, ds \right) \right| \leq C(1+t)^{-\frac{d+|\alpha|}{2}} B_{n_3}(|x|,t).$$

Here $n_1, n_2 > \frac{d}{2}$ and $n_3 = \min(n_1, n_2)$.

(b) If functions $F(x,t)$ and $S(x,t)$ satisfy

$$|D_x^\alpha F(x,t)| \leq C(1+t)^{-\frac{d+1+|\alpha|}{2}} B_{n_1}(|x|,t),$$
$$|D_x^\alpha S(x,t)| \leq C(1+t)^{-\frac{2d+1+|\alpha|}{2}} B_{n_2}(|x|,t),$$

then, we have

$$I_\alpha =: \left| D_x^\alpha \left(\int_0^t F(t-s) * S(s)\, ds \right) \right| \leq C(1+t)^{-\frac{d+1+|\alpha|}{2}} B_{n_3}(|x|,t).$$

Here $n_1, n_2 > \frac{d}{2}$ and $n_3 = \min(n_1, n_2)$.

(c) If functions $F(x,t)$ and $S(x,t)$ satisfy

$$|D_x^\alpha F(x,t)| \leq C(1+t)^{-\frac{d+|\alpha|}{2}} B_{n_1}(|x|,t), \ n_1 > \frac{d}{2},$$
$$|D_x^\alpha S(x,t)| \leq C(1+t)^{-\frac{2d+|\alpha|}{2}} B_{n_3}(|x|,t), \ n_3 > \frac{2d-1}{2},$$

then, we have

$$I_\alpha =: \left| D_x^\alpha \left(\int_0^t F(t-s) * S(s)\, ds \right) \right| \leq C(1+t)^{-\frac{d+|\alpha|}{2}} B_{n_1}(|x|,t).$$

(d) If functions $F(x,t)$ and $S(x,t)$ satisfy

$$|D_x^\alpha F(x,t)| \leq C(1+t)^{-\frac{d+k+|\alpha|}{2}} B_{\frac{d}{2}}(|x|,t),$$
$$|D_x^\alpha S(x,t)| \leq C(1+t)^{-\frac{2d-1+|\alpha|}{2}} B_{n_3}(|x|,t), \ k \geq 0, n_3 > \frac{2d-1}{2},$$

then, we have

$$I_\alpha =: \left| D_x^\alpha \left(\int_0^t F(t-s) * S(s)\, ds \right) \right| \leq C(1+t)^{-\frac{d+k+|\alpha|}{2}} B_{\frac{d}{2}}(|x|,t).$$

The proofs are similar as those of [34,35] without any new difficulty, and we can omit the details here.

Now, we begin to study the pointwise estimates of the solution (n_1, w_1, n_2, w_2) for (2.4). First, by Duhamel principle, the solution (n_1, w_1) can be expressed as

$$D_x^\alpha \begin{pmatrix} n_1 \\ w_1 \end{pmatrix} = D_x^\alpha \begin{pmatrix} G_{1,1} & G_{1,2} \\ G_{2,1} & G_{2,2} \end{pmatrix} * \begin{pmatrix} n_{1,0} \\ w_{1,0} \end{pmatrix} + \int_0^t D_x^\alpha \begin{pmatrix} G_{1,1} & G_{1,2} \\ G_{2,1} & G_{2,2} \end{pmatrix} (t-s) * \begin{pmatrix} 0 \\ f_1(n_1, w_1, n_2, w_2) \end{pmatrix} (s) \mathrm{d}s, \tag{3.1}$$

and (n_2, w_2) can be expressed as

$$D_x^\alpha \begin{pmatrix} n_2 \\ w_2 \end{pmatrix} = D_x^\alpha \begin{pmatrix} \mathbb{G}_{1,1} & \mathbb{G}_{1,2} \\ \mathbb{G}_{2,1} & \mathbb{G}_{2,2} \end{pmatrix} * \begin{pmatrix} n_{2,0} \\ w_{2,0} \end{pmatrix} + \int_0^t D_x^\alpha \begin{pmatrix} \mathbb{G}_{1,1} & \mathbb{G}_{1,2} \\ \mathbb{G}_{2,1} & \mathbb{G}_{2,2} \end{pmatrix} (t-s) * \begin{pmatrix} 0 \\ f_2(n_1, w_1, n_2, w_2) \end{pmatrix} (s) \mathrm{d}s. \tag{3.2}$$

Without loss of generality, we assume that for $|\alpha| \le 2$

$$|D_x^\alpha (n_{10}, w_{10}, n_{20}, w_{20}, \nabla\phi_0)| \le C\epsilon_0 (1+|x|^2)^{-r}, \ r > \frac{d}{2}. \tag{3.3}$$

Here ϵ_0 is sufficiently small. We first study the pointwise estimates for the solution (n_1, w_1). From (3.1), we have

$$\begin{aligned} D_x^\alpha n_1 &= D_x^\alpha G_{1,1}(t) * n_{1,0} + D_x^\alpha G_{1,2} * w_{1,0} + \int_0^t D_x^\alpha G_{1,2}(t-s) * f_1(n_1, w_1, n_2, w_2)(s) \mathrm{d}s \\ &:= R_1 + R_2 + R_3, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} D_x^\alpha w_1 &= D_x^\alpha G_{2,1}(t) * n_{1,0} + D_x^\alpha G_{2,2}(t) * w_{1,0} + \int_0^t D_x^\alpha G_{2,2}(t-s) * f_1(n_1, w_1, n_2, w_2)(s) \mathrm{d}s \\ &:= R_4 + R_5 + R_6. \end{aligned} \tag{3.5}$$

We rewrite R_1 as

$$R_1 = D_x^\alpha (G_{1,1} - K_q) * n_{1,0} + D_x^\alpha K_q * n_{1,0} := R_1^1 + R_1^2,$$

where $\chi_3(D_x) K_q(x,t)$ is generated from the higher frequency part in Green's function defined in Lemma 2.7. As mentioned in Lemmas 2.6 and 2.7, $\chi_3(D_x) K_q(x,t)$ is a Dirac-like function. Without loss of generality, we shall write $\chi_3(D_x) K_q = e^{-c_0 t}\delta(x) + e^{-c_0 t} g_2(x) := \chi_3(D_x) K_q^1 + \chi_3(D_x) K_q^2$, where $g_2(x)$ satisfies

$$\|g_2\|_{L^1} \le C, \ \text{supp } g_2(x) \subset \{x; |x| < \sigma_0\},$$

with σ_0 being sufficiently small. It is obvious that from (3.3)

$$|D_x^\alpha \chi_3(D_x) K_q^1 * n_{1,0}| = |e^{-c_0 t}\delta(x) * n_{1,0}| \le C e^{-c_0 t} B_r(|x|, t).$$

On the other hand, for $|x-y| \le \sigma_0 \ll 1$, note

$$(1+|y|^2)^{-1} \le C(1+|x|^2)^{-1},$$

then we have

$$\begin{aligned} D_x^\alpha \chi_3(D_x) K_q^2 * n_{1,0} &= \int \chi_3(D_x) K_q^2 (x-y) D_y^\alpha n_{1,0}(y) \mathrm{d}y \le C e^{-c_0 t} (1+|x|^2)^{-r} \\ &\le C e^{-\frac{c_0}{2} t} B_r(|x|, t). \end{aligned} \tag{3.6}$$

For R_1^1, from Proposition 2.8, Lemma 3.1(1) and the assumption (3.3), we have

$$|R_1^1| \le C\epsilon_0(1+t)^{-\frac{d+|\alpha|}{2}} B_r(|x|,t).$$

Hence, we have

$$|R_1| \le C\epsilon_0(1+t)^{-\frac{d+|\alpha|}{2}} B_r(|x|,t). \tag{3.7}$$

Similarly, one can get

$$|R_2| \le C\epsilon_0(1+t)^{-\frac{d+1+|\alpha|}{2}} B_r(|x|,t). \tag{3.8}$$

In the completely same way, using Proposition 2.8, Lemma 3.1 and the assumption (3.3), we have

$$|R_4| \le C\epsilon_0(1+t)^{-\frac{d+1+|\alpha|}{2}} B_r(|x|,t), \tag{3.9}$$

and

$$|R_5| \le C\epsilon_0(1+t)^{-\frac{d+2+|\alpha|}{2}} B_{\frac{d}{2}}(|x|,t). \tag{3.10}$$

Next, we consider the solutions (n_2, w_2). As mentioned above, we mainly consider the lower frequency part of the Green's function $\mathbb{G}$. From the analysis in Sect. 2.3 we know the Fourier transform of the lower frequency part of $\mathbb{G}$ is equivalent to the following form

$$\hat{\mathbb{G}}(\xi,t) = \begin{pmatrix} \hat{\mathbb{G}}_{11} & \hat{\mathbb{G}}_{12} \\ \hat{\mathbb{G}}_{21} & \hat{\mathbb{G}}_{22} \end{pmatrix} \sim \begin{pmatrix} 1 & \xi^\tau \\ \frac{\xi}{|\xi|^2} & \frac{\xi\xi^\tau}{|\xi|^2} \end{pmatrix} e^{-c_0 t} \quad \text{with } c_0 > 0,$$

from which, we know that $\hat{\mathbb{G}}_{12}$ has an additional factor ξ^τ than $\hat{\mathbb{G}}_{11}$ and $\hat{\mathbb{G}}_{22}$ also has an additional factor ξ^τ than $\hat{\mathbb{G}}_{21}$. Thus, we give an assumption on the electric field $\nabla\phi_0(x)$ in (3.3) instead of the usual assumption on the density $\rho_0(x) = \operatorname{div}\nabla\phi_0(x)$ and want to deduce better pointwise estimates for n_2 and w_2. Firstly, we have

$$\begin{aligned} D^\alpha n_2 &= D_x^\alpha \mathbb{G}_{1,1}(t) * n_{2,0} + D_x^\alpha \mathbb{G}_{1,2} * w_{2,0} + \int_0^t D_x^\alpha \mathbb{G}_{1,2}(t-\tau) * f_2(n_1,w_1,n_2,w_2)(\tau)\mathrm{d}\tau \\ &:= R_7 + R_8 + R_9, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} D^\alpha w_2 &= D_x^\alpha \mathbb{G}_{2,1}(t) * n_{2,0} + D_x^\alpha \mathbb{G}_{2,2} * w_{2,0} + \int_0^t D_x^\alpha \mathbb{G}_{2,2}(t-\tau) * f_2(n_1,w_1,n_2,w_2)(\tau)\mathrm{d}\tau \\ &:= R_{10} + R_{11} + R_{12}. \end{aligned} \tag{3.12}$$

We only deduce the pointwise estimate for R_7 and R_{10}. From (2.22), we get

$$|D_x^\alpha(\mathbb{G}_{1,1} - \chi_3(D_x)K_q) * n_{2,0}| \le C\epsilon_0 e^{-c_0 t} B_r(|x|,t), \ r > \frac{d}{2},$$

where we have used Proposition 2.10, Lemma 3.1(1) and the assumption (3.3). The estimate for $|D_x^\alpha \chi_3(D_x) K_q * n_{2,0}|$ can be deduced as R_1^2. Thus, we have

$$|R_7| \le C\epsilon_0 e^{-c_0 t} B_r(|x|,t), \ r > \frac{d}{2}. \tag{3.13}$$

Next, we consider the term R_{10}. When $|\alpha| \ge 1$, (2.25) and Lemma 3.2(d) directly give that

$$\begin{aligned} |R_{10}| &\le |D_x^\alpha(\mathbb{G}_{2,1}(x,t) - \chi_3(D_x)K_q) * n_{2,0}| + |\chi_3(D_x)K_q * n_{2,0}| \\ &\le C\epsilon_0 e^{-c_0 t} B_{\frac{d}{2}}(|x|,t) + C\epsilon_0 e^{-c_0 t} B_r(|x|,t) \\ &\le C\epsilon_0 e^{-c_0 t} B_{\frac{d}{2}}(|x|,t), \end{aligned} \tag{3.14}$$

where the second term $|\chi_3(D_x)K_q) * n_{2,0}|$ is estimated as in R_1^2.

When $|\alpha| = 0$, from the assumption

$$|D_x^\alpha \nabla\phi_0| \leq C\epsilon_0(1+|x|^2)^{-r} \text{ with } r > \frac{d}{2},$$

and the fact $n_{2,0} = \mathrm{div}\nabla\phi_0$, we find that

$$\begin{aligned}
|\mathbb{G}_{2,1} * n_{2,0}| &\leq |\chi_1(D_x)(\mathbb{G}_{2,1} * n_{2,0}| + |(1-\chi_1(D_x))\mathbb{G}_{2,1} - \chi_3(D_x)K_q) * n_{2,0}| + |\chi_3(D_x)K_q * n_{20}| \\
&\leq |\chi_1(D_x)\nabla\mathbb{G}_{2,1} * \nabla\phi_0| + C\epsilon_0 e^{-c_0 t} B_r(|x|,t) \\
&\leq C\epsilon_0 e^{-c_0 t} B_{\frac{d}{2}}(|x|,t) + C\epsilon_0 e^{-c_0 t} B_r(|x|,t) \\
&\leq C\epsilon_0 e^{-c_0 t} B_{\frac{d}{2}}(|x|,t), \qquad (3.15)
\end{aligned}$$

where we have used Lemma 3.2(d) and (2.24) for the first term and (2.25) for the second term, and the third term can be estimated as in R_1^2.

Consequently, we can immediately get

$$|R_{10}| \leq C\epsilon_0 e^{-c_0 t} B_{\frac{d}{2}}(|x|,t). \qquad (3.16)$$

In the same way, one has

$$|R_8| \leq C\epsilon_0 e^{-c_0 t} B_r(|x|,t), \qquad (3.17)$$

and

$$|R_{11}| \leq C\epsilon_0 e^{-c_0 t} B_{\frac{d}{2}}(|x|,t). \qquad (3.18)$$

Up to now, we have obtained the pointwise estimates for the linearized system. Next, we shall deduce those for the nonlinear system (2.4). To this end, we first set

$$\begin{cases}
\psi_1(x,t) = \begin{cases} (1+t)^{\frac{d}{2}+\nu(|\alpha|)} \left(B_r(|x|,t)\right)^{-1}, & |\alpha| \leq h-1, \\ (1+t)^{\frac{d-1}{2}+\nu(|\alpha|)} \left(B_{\frac{d-1}{2}}(|x|,t)\right)^{-1}, & |\alpha| = h, \end{cases} \\
M_1(t) = \sup\limits_{0\leq\tau\leq t, |\alpha|\leq h; x\in\mathbb{R}^d} |D_x^\alpha(n_1,n_2)(x,\tau)|\psi_1(x,\tau), \\
\psi_2(x,t) = \begin{cases} (1+t)^{\frac{d+1}{2}+\nu(|\alpha|)} \left(B_{\frac{d}{2}}(|x|,t)\right)^{-1}, & |\alpha| \leq h-1, \\ (1+t)^{\frac{d-1}{2}+\nu(|\alpha|)} \left(B_{\frac{d-1}{2}}(|x|,t)\right)^{-1}, & |\alpha| = h, \end{cases} \\
M_2(t) = \sup\limits_{0\leq\tau\leq t, |\alpha|\leq h, x\in\mathbb{R}^d} |D_x^\alpha(w_1,w_2)(x,\tau)|\psi_2(x,\tau),
\end{cases} \qquad (3.19)$$

where

$$\nu(|\alpha|) = \begin{cases} |\alpha|/2, & |\alpha| \leq h-1, \\ 0, & |\alpha| = h. \end{cases}$$

It is worthy taking an explanation for the ansatz (3.19). A reasonable ansatz could help us deduce the desired pointwise estimates of the solution to the nonlinear system. Since the nonlinear terms $f_1(n_1,w_1,n_2,w_2)$ and $f_2(n_1,w_1,n_2,w_2)$ of the momentum equations in (2.4) contain the term $n_2\nabla\phi$ and $n_1\nabla\phi$ respectively, we have to firstly obtain the following pointwise estimates for $\nabla\phi$, which will be used to get the pointwise estimates of the solution (n_1,w_1,n_2,w_2).

Lemma 3.3. [35] *If $|\alpha| \leq h$, we have*

$$|D_x^\alpha \nabla\phi| \leq CM_1(1+t)^{-\frac{d-1+|\alpha|}{2}} B_{\frac{d-1}{2}}(|x|,t).$$

By the definition of $M_i(t)$(i=1,2), we have

$$
\begin{aligned}
&|D_y^\alpha f_1(n_1,w_1,n_2,w_2)(y,s)| \\
&\le C\begin{cases}
M_1^2(1+s)^{-\frac{2d+|\alpha|}{2}}B_{2r}(|y|,s)+M_2^2(1+s)^{-\frac{2d+2+|\alpha|}{2}}B_d(|y|,s) \\
\quad +M_1^2(1+s)^{-\frac{2d+2+|\alpha|}{2}}B_{r+\frac{d}{2}}(|y|,s),\ |\alpha|\le h-2, \\
M_1^2(1+s)^{-\frac{2d-1}{2}}B_{r+\frac{d}{2}}(|y|,s)+M_2^2(1+s)^{-d}B_{r+\frac{d}{2}}(|y|,s),\ |\alpha|=h-1, \\
\epsilon_0 M_1(1+s)^{-\frac{d}{2}}B_r(|y|,s)+\epsilon_0 M_2(1+s)^{-\frac{d}{2}}B_{\frac{d}{2}}(|y|,s) \\
\quad +M_1^2(1+s)^{-(d-1)}B_{d-1}(|y|,s),\ |\alpha|=h.
\end{cases}
\end{aligned}
\tag{3.20}
$$

Moreover, we rewrite R_3 of (3.4) as follows:

$$
\begin{aligned}
R_3 &= \int_0^t D_x^\alpha(G_{1,1}-\chi_3(D_x)K_q)(t-\tau)*f_1(n_1,w_1,n_2,w_2)(\tau)\mathrm{d}\tau \\
&\quad + \int_0^t D_x^\alpha \chi_3(D_x)K_q(t-\tau)*f_1(n_1,w_1,n_2,w_2)(\tau)\mathrm{d}\tau \\
&:= R_3^1+R_3^2.
\end{aligned}
$$

Now we consider R_3^1. If $|\alpha|\le h-2$, by Proposition 2.1, Lemma 3.3(c) and (3.20), we know

$$
\begin{aligned}
|R_3^1| &= \left|\int_0^t D_x^\alpha(G_{1,1}-\chi_3(D_x)K_q)(t-\tau)*f_1(n_1,w_1,n_2,w_2)(\tau)\mathrm{d}\tau\right| \\
&\le C(M_1^2+M_2^2)(1+t)^{-\frac{d+1+|\alpha|}{2}}B_r(|x|,t),\ r>\frac{d}{2}.
\end{aligned}
\tag{3.21}
$$

If $h-1\le|\alpha|\le h$, we rewrite

$$
R_3^1=\int_0^t D_x^{\tilde{\alpha}}(D^\beta(G_{1,1}-\chi_3(D_x)K_q))(t-\tau)*f_1(n_1,w_1,n_2,w_2)(\tau)\mathrm{d}\tau,
$$

where $|\tilde{\alpha}|=h-2$ and $|\beta|=|\alpha|-|\tilde{\alpha}|$. Then we replace F by $D^\beta(G_{1,1}-K_q)$, and we can get (3.21) for $h-1\le|\alpha|\le h$.

Next, for R_3^2, we rewrite it as

$$
R_3^2=\int_0^t K_q(t-\tau)*D_x^\alpha f_1(n_1,w_1,n_2,w_2)(\tau)\mathrm{d}\tau.
$$

Then, similar to the proof in (3.6), one can immediately find from (3.20) that for $|\alpha|\le\min\{d-1,2\}$

$$
|R_3^2|\le C\begin{cases}
(M_1^2+M_2^2)(1+t)^{-\frac{d+|\alpha|}{2}}B_d(|x|,t), & |\alpha|\le h-1, \\
\epsilon_0(M_1+M_2)(1+t)^{-\frac{d+|\alpha|}{2}}B_{\frac{d}{2}}(|x|,t), & |\alpha|=h.
\end{cases}
\tag{3.22}
$$

Lastly, repeating the process of the proof for R_3, and noticing the fact in Proposition 2.1 that $G_{2,1}-K_q$ and $G_{2,2}-K_q$ decay faster than $G_{1,1}-K_q$ and $G_{1,2}-K_q$, one can easily obtain for $|\alpha|\le\min\{d-2,2\}$

$$
|R_6|\le C\begin{cases}
(M_1^2+M_2^2)(1+t)^{-\frac{d+1+|\alpha|}{2}}B_d(|x|,t), & |\alpha|\le h-1, \\
\epsilon_0(M_1+M_2)(1+t)^{-\frac{d-1+|\alpha|}{2}}B_{\frac{d-1}{2}}(|x|,t), & |\alpha|=h.
\end{cases}
\tag{3.23}
$$

On the other hand, the estimates of (n_2, w_2) could be deduced similarly. Since the unique difference is the temporal decay rate of the Green's function $\mathbb{G}$ is exponential, which will not bring any new difficulty. In fact, we have for $|\alpha| \le \min\{d-2, 2\}$

$$|R_9| + |R_{12}| \le C \begin{cases} (M_1^2 + M_2^2)(1+t)^{-\frac{d+1+|\alpha|}{2}} B_d(|x|, t), & |\alpha| \le h-1, \\ \epsilon_0(M_1 + M_2)(1+t)^{-\frac{d-1}{2}} B_{\frac{d-1}{2}}(|x|, t), & |\alpha| = h. \end{cases} \tag{3.24}$$

In summary, from (3.7)–(3.10), (3.13), (3.16)–(3.18) and (3.21)–(3.24), we can conclude that

$$M_1 \le C(\epsilon_0 + M_1^2 + M_2^2) \text{ and } M_2 \le C(\epsilon_0 + M_1^2 + M_2^2),$$

which together with the smallness of ϵ_0 imply

$$M_1 + M_2 \le C\epsilon_0.$$

That is, for $|\alpha| \le 1$,

$$|D_x^\alpha(n_1, n_2)| \le C\epsilon_0(1+t)^{-\frac{d+|\alpha|}{2}} B_r(|x|, t), \ r > \frac{d}{2},$$

and

$$|D_x^\alpha(w_1, w_2)| \le C\epsilon_0(1+t)^{-\frac{d+1+|\alpha|}{2}} B_{\frac{d}{2}}(|x|, t).$$

Then, by using the formula (2.3), we have

$$|D_x^\alpha(\rho_1 - 1, \rho_2 - 1)| \le C\epsilon_0(1+t)^{-\frac{d+|\alpha|}{2}} B_r(|x|, t), \ r > \frac{d}{2},$$

and

$$|D_x^\alpha(m_1, m_2)| \le C\epsilon_0(1+t)^{-\frac{d+1+|\alpha|}{2}} B_{\frac{d}{2}}(|x|, t).$$

This proves Theorem 1.2.

As a corollary, we have the following optimal decay rate in $L^p(\mathbb{R}^d)$-space.

Corollary 3.4. *Under the assumption in Theorem 1.2, we have*

$$\|D_x^\alpha(\rho_1 - 1, \rho_2 - 1)\|_{L^p(\mathbb{R}^d)} \le C(1+t)^{-\frac{d+|\alpha|}{2}(1-\frac{1}{p})}, \text{ with } p \ge 1,$$

and

$$\|D_x^\alpha(m_1, m_2)\|_{L^p(\mathbb{R}^d)} \le C(1+t)^{-\frac{d+1+|\alpha|}{2}(1-\frac{1}{p})}, \text{ with } p > 1.$$

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