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Heteroclinic bifurcation in a class of planar piecewise smooth systems with multiple zones

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Abstract. We discuss heteroclinic bifurcation in a class of periodically excited planar piecewise smooth systems with discontinuities on finitely many smooth curves intersecting at the origin. Assume that the unperturbed system has a hyperbolic saddle in each subregion, and those saddles are connected by a heteroclinic cycle that crosses every switching curve transversally exactly once. We present a method of Melnikov type to derive sufficient conditions under which the perturbed stable and unstable manifolds intersect transversally. Such transversal intersections imply that the corresponding Poincaré map has a transverse heteroclinic cycle. As applications, we present examples with 2 and 4 switching curves respectively. Our numerical simulations suggest that such transversal intersections result in the appearance of chaotic motions in those example systems.

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1. Introduction

An important topic in the theory of nonlinear dynamical systems is to investigate the appearance of chaos. For many smooth systems, a typical route to chaos is via homoclinic bifurcation. The Smale–Birkhoff Homoclinic Theorem and the Melnikov method are two powerful tools for studying the occurrence of chaos in an autonomous system with a homoclinic orbit under periodic perturbation [4,16,22,26,27,37,42]. In real applications, such as organized vortex structures in planar fluid flows, there are also systems with multiple heteroclinic saddle connections. In 1988, Bertozzi [11] extended the Smale–Birkhoff Homoclinic Theorem and the Melnikov method to the case of heteroclinic bifurcations, enabling us to study chaos arising from such saddle connections, e.g., [3,43].

In recent years, the study of bifurcation phenomena in piecewise smooth (PWS) dynamical systems has become a hotspot subject of research in scientific community because those systems can be used to model many problems from mechanics, control theory and electrical engineering. It is well known that PWS systems often undergo chaotic motions through discontinuity-induced bifurcations, such as grazing, sliding, border collision and chattering. See, for example, [2,10,18,21,22,32,34,36,41] and the references therein.

Earlier works on piecewise linear systems [17,39] suggest that, like for smooth systems, homoclinic bifurcation is also an important route to chaos for PWS systems. Naturally we ask whether the Melnikov method established for smooth systems can be extended to PWS systems. This problem has been widely studied. Many works focused on the case when the unperturbed homoclinic or periodic orbit intersects the discontinuity surface transversally [2,5,9,14,20,31–33,40]. The more interesting and difficult cases are

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bifurcations of sliding and grazing homoclinic orbits. In [1,6–8,22], Battelli, Fečkan, Awrejcewicz et al. extended the Melnikov method to bifurcation of sliding homoclinic orbits of general n-dimensional PWS systems. Furthermore, they show, for the first time, rigorously the existence of Smale horseshoe-type chaos in these systems. Grazing homoclinic bifurcation in a nonlinear impact inverted pendulum under external periodic excitation was studied in [19]. Calamai and Franca [13] presented the Melnikov method to homoclinic bifurcations in discontinuous systems with the critical point lies on the discontinuity set. Homoclinic bifurcation in a quasiperiodically excited impact inverted pendulum was considered in [23]. As pointed out by Kunze [32], to extend known bifurcation methods such as the Melnikov method for smooth systems to PWS systems is by no means a trivial task.

Although big progress has been made in the study of homoclinic bifurcation and chaos in PWS systems, few attentions have been paid to heteroclinic bifurcations in these systems. Bruhn and Koch [12] investigated heteroclinic bifurcations in a simple model of rigid block motion under external perturbations. Hogan [29] considered heteroclinic bifurcations in a piecewise linear system modeling the rocking motion of a slender rigid block with damping. Due to the piecewise linear nature of the system, he was able to compute the gap between the perturbed stable and unstable manifolds exactly without using perturbation methods. A more general nonlinear model of slender rigid block was studied by Lenci and Rega [35]. Granados et al. [24] developed the Melnikov method for heteroclinic and subharmonic bifurcations in a periodically excited piecewise Hamiltonian system defined in two zones separated by a straight line. Then in [25], they extended the results to a non-autonomous system formed by coupling two planar PWS systems of the form considered in [24].

In real applications, discontinuities may occur on multiple lines or even on nonlinear curves or surfaces and the system is not necessarily piecewise Hamiltonian. Motivated by the works [12,24,29,35], in this paper we study heteroclinic bifurcation in a class of periodically excited planar PWS systems with discontinuities on finitely many smooth curves intersecting at the origin. We assume that the unperturbed system has a hyperbolic saddle in each subregion and those saddles are connected by a heteroclinic cycle that crosses every switching curve transversally exactly once. We present a method of Melnikov type to derive sufficient conditions under which the perturbed stable and unstable manifolds intersect transversally. Such transversal intersections imply that the corresponding Poincaré map has a transverse heteroclinic cycle. As applications, we present examples with 2 and 4 switching curves respectively.

It is worth mentioning that the Heteroclinic Theorem of Berttozzi requires the corresponding Poincaré map to be differentiable [11], which in general is not satisfied by PWS systems. Thus it is not applicable to the PWS system studied in this paper. Nevertheless, our numerical simulations on concrete examples suggest that the transversal intersections of the perturbed stable and unstable manifolds of a heteroclinic orbit of PWS systems may also result in the appearance of chaotic motions. Thus we think that it is very important to investigate if Berttozzi's theorem can be extended to PWS systems.

Our presentation is organized as follows. Basic assumptions and the main results are given in Sect. 2. In Sect. 3 we prove the main results by deriving formulae for the computations of the first-order Melnikov functions. In Sect. 4 we present two examples of piecewise smooth systems with two zones. A concrete nonlinear piecewise smooth system with four zones is presented in Sect. 5.

2. Preliminaries and main results

We first introduce some notations. For any $a = (a_1, a_2)^T, b = (b_1, b_2)^T \in \mathbb{R}^2$, $\langle a, b\rangle$, $\|a\|$, $a \wedge b$ and $a^\perp$ are defined by $\langle a, b\rangle = a^T b$, $\|a\| = \sqrt{\langle a, a\rangle}$, $a \wedge b = a_1 b_2 - a_2 b_1$ and $a^\perp = (-a_2, a_1)^T$ respectively. For $x \in \mathbb{R}^2$, the gradient of a smooth scalar function $f(x)$ is denoted by ∇f, and the divergence and the Jacobian matrix of a smooth map $X : \mathbb{R}^2 \to \mathbb{R}^2$ are denoted by $\text{div}X$ and DX, respectively. Clearly, for any $a, b, c \in \mathbb{R}^2$, we have

$$\langle a, b\rangle c - \langle a, c\rangle b = \det[b, c]a^\perp. \tag{2.1}$$

Let $K > 0$ be a constant and $\Omega := \{x \in \mathbb{R}^2 : \ \|x\| < K\} \subseteq \mathbb{R}^2$ be an open disk. Let $m \geq 2$ be an integer and $\mathcal{J} = \{1, 2, \ldots, m\}$. Assume that Ω is split into m disjoint regions by m disjoint smooth curves $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$, where for each $k \in \mathcal{J}$, $\mathcal{C}_k$ starts at the origin and is given by the equation $h_k(x) = 0$ for $x \in \Omega$, and $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ are numbered in the order of their appearance when counted counterclockwise. The open subregion of Ω between $\mathcal{C}_k$ and $\mathcal{C}_{k+1(\text{mod}\, m)}$ is denoted by Ω_k and let $\bar{\Omega}_k$ be its closure. Suppose that

(H1) For each $k \in \mathcal{J}$, $h_k \in C^2(\Omega, \mathbb{R})$ with $h_k(0) = 0$, $\nabla h_k(x) \neq (0, 0)^T$, $|h_k(x)|$ is strictly increasing as $\|x\|$ increases for $x \in \Omega$.

Now consider the following PWS system defined on Ω:

$$\dot{x} = f_k(x) + \varepsilon g_k(x, t), \quad x \in \Omega_k, \quad k \in \mathcal{J}, \tag{2.2}$$

where $|\varepsilon| \leq \varepsilon_0 \ll 1$ for some $\varepsilon_0 > 0$, $f_k \in C^2(\bar{\Omega}_k, \mathbb{R}^2)$, $g_k \in C^2(\bar{\Omega}_k \times \mathbb{R}, \mathbb{R}^2)$ and are T-periodic in t. When $\varepsilon = 0$, the unperturbed system of (2.2) has the following form:

$$\dot{x} = f_k(x), \quad x \in \Omega_k, \quad k \in \mathcal{J}. \tag{2.3}$$

Let the following assumptions hold:

(H2) For each $k \in \mathcal{J}$, the unperturbed system (2.3) has a hyperbolic saddle $P_k \in \Omega_k$. System (2.3) has a heteroclinic cycle Γ which consists of $2m$ branches $\Gamma_k^s := \{\gamma_k^s(t) : \ t \in [\tau_k^s, +\infty)\} \subset \bar{\Omega}_k$, $\Gamma_k^u := \{\gamma_k^u(t) : \ t \in (-\infty, -\tau_k^u]\} \subset \bar{\Omega}_k$ $(k \in \mathcal{J})$ such that

$$\Gamma = \bigcup_{k=1}^{m} \left(\Gamma_k^u \bigcup \{P_k\} \bigcup \Gamma_k^s \right),$$

where for each $k \in \mathcal{J}$, $\tau_k^{u,s} > 0$ are constants, $\gamma_k^{u,s}(t)$ are solutions of (2.3) in Ω_k, and $\gamma_k^u(-\tau_k^u) = \gamma_{k+1(\text{mod}\, m)}^s(\tau_{k+1(\text{mod}\, m)}^s) := Q_{k+1(\text{mod}\, m)} \in \mathcal{C}_{k+1(\text{mod}\, m)}$. Furthermore,

$$\lim_{t \to +\infty} \gamma_k^s(t) = \lim_{t \to -\infty} \gamma_k^u(t) = P_k.$$

(H3) The heteroclinic cycle Γ crosses $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ counterclockwise and intersects $\mathcal{C}_k$ transversally at exactly one point $Q_k \in \mathcal{C}_k$ for each $k \in \mathcal{J}$.

Here the assumption that Γ crosses $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ counterclockwise in (H3) is not essential because if Γ crosses $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ clockwise, one can reverse the time to satisfy (H3). A heteroclinic cycle Γ of the unperturbed system (2.3) with $m = 4$ is shown in Fig. 1. By (H3), for $k \in \mathcal{J}$, we have

$$\langle \nabla h_k(Q_k), f_k(Q_k) \rangle \neq 0. \tag{2.4}$$

System (2.2) is equivalent to the following suspended system

$$\begin{cases} \dot{x} = f_k(x) + \varepsilon g_k(x, \theta), & x \in \Omega_k, \quad k \in \mathcal{J}, \\ \dot{\theta} = 1, \end{cases} \tag{2.5}$$

where $\theta = t(\text{mod}\, T)$. Let $S^1 = \mathbb{R}(\text{mod}\, T)$ be the unit circle of period T. For $\theta \in S^1$, let

$$\Sigma^\theta := \{(x, t) : \ x \in \mathbb{R}^2, t = \theta\} \subset \mathbb{R}^2 \times S^1$$

be the global cross section at time θ for the suspended system (2.5). The time-T Poincaré return map $\Pi_\varepsilon^\theta : \Sigma^\theta \to \Sigma^\theta$ is given by the flow of system (2.5). By Lemma 4.5.1 in [27], for sufficiently small $|\varepsilon|$ and for each $k \in \mathcal{J}$, the map Π_ε^θ has a unique hyperbolic saddle point in $\Sigma^\theta \cap (\Omega_k \times S^1)$, which corresponds to a unique hyperbolic periodic orbit $\gamma_{k,\varepsilon}(t) = P_k + O(\varepsilon)$ of the subsystem of (2.5) in Ω_k. By the hyperbolicity, each $\gamma_{k,\varepsilon}(t)$ $(k \in \mathcal{J})$ has a stable manifold $W_{k,\varepsilon}^s := W^s(\gamma_{k,\varepsilon}(t))$ and an unstable manifold $W_{k,\varepsilon}^u := W^u(\gamma_{k,\varepsilon}(t))$.

We are interested in the question: *under what conditions, $W_{k,\varepsilon}^u$ and $W_{k+1(\text{mod}\, m),\varepsilon}^s$ intersect transversally for $k \in \mathcal{J}$*? Clearly, if this condition is satisfied, then the Poincaré map $\Pi_\varepsilon^\theta : \Sigma^\theta \to \Sigma^\theta$ possesses a

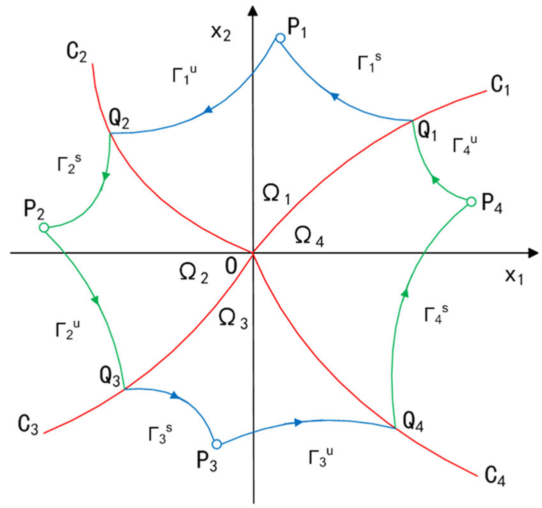

FIG. 1. A heteroclinic cycle Γ of the unperturbed system (2.3) with $m = 4$

transverse heteroclinic cycle. In order to overcome the discontinuities at $\mathcal{C}_k$, by contrast to the classical approach, we proceed as in [12,14,24,29] and study heteroclinic connections at the sections $\mathcal{C}_k$ for $k \in \mathcal{J}$.

Let $\theta \in S^1$ be fixed and $W^u_{0,\varepsilon} = W^u_{m,\varepsilon}$. For $k \in \mathcal{J}$, let $(A^u_{k,\varepsilon}(\theta), \theta)$ and $(A^s_{k,\varepsilon}(\theta), \theta)$ be the intersections of $W^u_{k-1,\varepsilon}$ and $W^s_{k,\varepsilon}$ with $\mathcal{C}_k \times \{\theta\}$ respectively. As θ varies in S^1, $(A^u_{k,\varepsilon}(\theta), \theta)$ and $(A^s_{k,\varepsilon}(\theta), \theta)$ draw two curves in the cylinder $\mathcal{C}_k \times S^1$. The distance between these two curves are given by

$$\Delta_{k,\varepsilon}(\theta) = A^u_{k,\varepsilon}(\theta) - A^s_{k,\varepsilon}(\theta), \quad \theta \in S^1, \quad k \in \mathcal{J}. \tag{2.6}$$

By definition, if all of $\Delta_{1,\varepsilon}(\theta), \ldots, \Delta_{m,\varepsilon}(\theta)$ have simple zeros in S^1 (these simple zeros may be different), then $W^u_{k,\varepsilon}$ and $W^s_{k+1(\mathrm{mod}\, m),\varepsilon}$ intersect transversally for $k \in \mathcal{J}$.

Usually it is impossible to find a closed form of $\Delta_{k,\varepsilon}(\theta)$ for $k \in \mathcal{J}$. It has to be approximated by perturbation methods. By our assumptions, for $k \in \mathcal{J}$, $W^u_{k,\varepsilon}$ and $W^s_{k,\varepsilon}$ are C^2 in ε, implying that both $A^u_{k,\varepsilon}(\theta)$ and $A^s_{k,\varepsilon}(\theta)$ are all C^2 in ε from the way they are defined. Consequently, $\Delta_{k,\varepsilon}(\theta)$ are all C^2 in ε. In fact, we have the following result:

Theorem 2.1. *Suppose that the assumptions* (H1–H3) *hold and let the notations be given above. Let $k \in \mathcal{J}$ be fixed. For $\theta \in S^1$, define*

$$b^u_k(t) = \exp\left(-\int_{-\tau^u_k}^{t} \mathrm{div} f_k(\gamma^u_k(s))\mathrm{d}s\right),$$

$$b^s_k(t) = \exp\left(-\int_{\tau^s_k}^{t} \mathrm{div} f_k(\gamma^s_k(s))\mathrm{d}s\right),$$

$$M^u_{k,1}(\theta) = \int_{-\infty}^{-\tau^u_k} \left(f_k(\gamma^u_k(s)) \wedge g_k(\gamma^u_k(s), s+\theta)\right) b^u_k(s)\mathrm{d}s,$$

$$M^s_{k,1}(\theta) = \int_{\tau^s_k}^{+\infty} \left(f_k(\gamma^s_k(s)) \wedge g_k(\gamma^s_k(s), s+\theta)\right) b^s_k(s)\mathrm{d}s.$$

Then for $\theta \in S^1$, we have

$$\Delta_{k,\varepsilon}(\theta) = \varepsilon \frac{M_{k,1}(\theta)}{\langle \nabla h_k(Q_k), f_k(Q_k)\rangle} \{\nabla h_k(Q_k)\}^{\perp} + O(\varepsilon^2),$$

where

$$M_{k,1}(\theta) = \frac{\langle \nabla h_k(Q_k), f_k(Q_k)\rangle}{\langle \nabla h_k(Q_k), f_{k-1}(Q_k)\rangle} M^u_{k-1,1}(\theta) + M^s_{k,1}(\theta),$$

where we set $f_0(Q_1) = f_m(Q_1)$ and $M^u_{0,1}(\theta) = M^u_{m,1}(\theta)$.

By (2.4), for each $k \in \mathcal{J}$, $M_{k,1}(\theta)$ and $\Delta_{k,\varepsilon}(\theta)$ are well-defined for $\theta \in S^1$. We call $M_{1,1}(\theta), \dots, M_{m,1}(\theta)$ the *first-order Melnikov functions*. We have the following result.

Theorem 2.2. *Suppose that the assumptions* (H1–H3) *hold and let the notations be given above. If for $k \in \mathcal{J}$, $M_{k,1}(\theta_k) = 0$ and $M'_{k,1}(\theta_k) \neq 0$ for some $\theta_k \in S^1$, then for sufficiently small $|\varepsilon| > 0$ the manifolds $W^u_{k,\varepsilon}$ and $W^s_{k+1(\mathrm{mod}\, m),\varepsilon}$ intersect transversally. Consequently, the Poincaré map $\Pi^\theta_\varepsilon : \Sigma^\theta \mapsto \Sigma^\theta$ possesses a transverse heteroclinic cycle.*

Remark. If each subsystem of system (2.3) is Hamiltonian, then $\mathrm{div} f_1 \equiv 0, \dots, \mathrm{div} f_m \equiv 0$. Thus for $k \in \mathcal{J}$, $M^u_{k,1}(\theta)$ and $M^s_{k,1}(\theta)$ in Theorem 2.1 are simplified to

$$M^u_{k,1}(\theta) = \int_{-\infty}^{-\tau^u_k} f_k(\gamma^u_k(s)) \wedge g_k(\gamma^u_k(s), s+\theta)\mathrm{d}s,$$

$$M^s_{k,1}(\theta) = \int_{\tau^s_k}^{+\infty} f_k(\gamma^s_k(s)) \wedge g_k(\gamma^s_k(s), s+\theta)\mathrm{d}s.$$

Assume that each subsystem of system (2.3) is Hamiltonian. It is easy to see that when $m = 2$, $\Omega_1 = \{(x_1, x_2)^T \in \mathbb{R}^2 :\ x_1 > 0\}$ and $\Omega_2 = \{(x_1, x_2)^T \in \mathbb{R}^2 :\ x_1 < 0\}$, one recovers the results given in Theorem 4.1 for the case $r = 1$ of [24].

3. Computation of the Melnikov functions

To apply Theorem 2.2, it is important to compute the first-order Melnikov functions $M_{1,1}(\theta), \dots, M_{m,1}(\theta)$ for $\theta \in S^1$. In this section we prove Theorem 2.1 by deriving the formulae for the calculations of these functions in terms of given functions. In this section we fix $k \in \mathcal{J}$ and compute $M_{k,1}(\theta)$.

From the last section, it is clear that to compute $M_{k,1}(\theta)$, we must estimate $A^u_{k,\varepsilon}(\theta)$ and $A^s_{k,\varepsilon}(\theta)$ respectively. Hence we need to discuss the perturbation of the heteroclinic cycle Γ and estimate the intersections of $W^u_{k-1,\varepsilon}$ and $W^s_{k,\varepsilon}$ with $\mathcal{C}_k \times \{\theta\}$, where we set $W^u_{0,\varepsilon} = W^u_{m,\varepsilon}$. During the computations, for each $j \in \mathcal{J}$, we need to extend the domain of the subsystem of (2.2) (resp. (2.3)) in the region Ω_j to include part of its neighboring regions Ω_{j-1} and $\Omega_{j+1(\mathrm{mod}\, m)}$, where we set $\Omega_0 = \Omega_m$. Thus for technical reasons, we extend f_j and g_j such that $f_j \in C^2(\bar{\Omega}_{j-1} \cup \bar{\Omega}_j \cup \bar{\Omega}_{j+1(\mathrm{mod}\, m)}, \mathbb{R}^2)$, $g_j \in C^2(\bar{\Omega}_{j-1} \cup \bar{\Omega}_j \cup \bar{\Omega}_{j+1(\mathrm{mod}\, m)} \times \mathbb{R}, \mathbb{R}^2)$ and are T-periodic in t. We also C^2-smoothly extend Γ^s_j to $\bar{\Omega}_{j-1} \cup \bar{\Omega}_j$ and Γ^u_j to $\bar{\Omega}_j \cup \bar{\Omega}_{j+1(\mathrm{mod}\, m)}$.

For any fixed $\theta \in S^1$, define

$$\beta_k^u(t) = \exp\left(-\int_{-\tau_k^u+\theta}^{t} \mathrm{div} f_k(\gamma_k^u(s-\theta))\mathrm{d}s\right), \quad t \in (-\infty, -\tau_k^u+\theta],$$

$$\beta_k^s(t) = \exp\left(-\int_{\tau_k^s+\theta}^{t} \mathrm{div} f_k(\gamma_k^s(s-\theta))\mathrm{d}s\right), \quad t \in [\tau_k^s+\theta, +\infty),$$

$$H_k^u(\theta) = \int_{-\infty}^{-\tau_k^u+\theta} \left(f_k(\gamma_k^u(s-\theta)) \wedge g_k(\gamma_k^u(s-\theta), s)\right) \beta_k^u(s)\mathrm{d}s,$$

$$H_k^s(\theta) = \int_{\tau_k^s+\theta}^{+\infty} \left(f_k(\gamma_k^s(s-\theta)) \wedge g_k(\gamma_k^s(s-\theta), s)\right) \beta_k^s(s)\mathrm{d}s.$$

In what follows, we set $H_0^u(\theta) = H_m^u(\theta)$, $\gamma_0^u(-\tau_0^u) = \gamma_m^u(-\tau_m^u) = Q_1$ and $\dot{\gamma}_0^u(-\tau_0^u) = \dot{\gamma}_m^u(-\tau_m^u) = f_m(Q_1)$.

We first compute $A_{k,\varepsilon}^s(\theta)$. We have the following result:

Lemma 3.1. *Let the notations be given above. Then for sufficiently small $|\varepsilon|$, we have*

$$A_{k,\varepsilon}^s(\theta) = Q_k - \varepsilon \frac{H_k^s(\theta)}{\langle \nabla h_k(\gamma_k^s(\tau_k^s)), \dot{\gamma}_k^s(\tau_k^s)\rangle} \left\{\nabla h_k(Q_k)\right\}^{\perp} + O(\varepsilon^2).$$

Proof. Let L_k^s be the normal line of Γ_k^s at $\gamma_k^s(\tau_k^s) = Q_k \in \mathcal{C}_k$. For any fixed $\theta \in S^1$, consider the trajectory of (2.5) starting from a point $Q_{k,\varepsilon}^\theta \in L_k^s$ at the time $t = \tau_k^s + \theta$, denoted by $x_{k,\varepsilon}^s(t; \tau_k^s+\theta, Q_{k,\varepsilon}^\theta)$ such that $(x_{k,\varepsilon}^s(t; \tau_k^s+\theta, Q_{k,\varepsilon}^\theta), t+\tau_k^s+\theta)$ lies in the perturbed stable manifold $W_{k,\varepsilon}^s$. Without loss of generality, we assume that $Q_{k,\varepsilon}^\theta \in \Omega_k$. By Melnikov's result in [37], for each $\theta \in S^1$ the following expansion is valid uniformly with respect to $t \in [\tau_k^s+\theta, +\infty)$:

$$x_{k,\varepsilon}^s(t; \tau_k^s+\theta, Q_{k,\varepsilon}^\theta) = \gamma_k^s(t-\theta) + \varepsilon x_{k,1}^s(t; \tau_k^s+\theta, Q_{k,\varepsilon}^\theta) + O(\varepsilon^2). \tag{3.1}$$

Moreover, there is a constant $G_2 > 0$ such that the inequality

$$\|x_{k,\varepsilon}^s(t; \tau_k^s+\theta, Q_{k,\varepsilon}^\theta)\| \leq G_2 \tag{3.2}$$

holds for all $t \in [\tau_k^s+\theta, +\infty)$.

By the continuous dependency [15, p. 89], for sufficiently small $|\varepsilon|$, there is a unique $t_{k,\varepsilon}^s(\theta) \in \mathbb{R}$ such that the trajectory $x_{k,\varepsilon}^s(t; \tau_k^s+\theta, Q_{k,\varepsilon}^\theta)$ reaches the discontinuity set $\mathcal{C}_k$ at the time $\tau_k^s+\theta+t_{k,\varepsilon}^s(\theta)$, where $t_{k,\varepsilon}^s(\theta)$ is C^2 in (ε, θ) and when $\varepsilon = 0$, $t_{k,\varepsilon}^s(\theta) = 0$ for all $\theta \in S^1$.

From (3.1) it is easy to see that $x_{k,1}^s(t; \tau_k^s+\theta, Q_{k,\varepsilon}^\theta)$ satisfies the following variational equation:

$$\dot{x}_{k,1}^s(t; \tau_k^s+\theta, Q_{k,\varepsilon}^\theta) = Df_k(\gamma_k^s(t-\theta)) x_{k,1}^s(t; \tau_k^s+\theta, Q_{k,\varepsilon}^\theta) + g_k(\gamma_k^s(t-\theta), t). \tag{3.3}$$

Let $\zeta_k^s(t-\theta)$ $(t \in [\tau_k^s+\theta, +\infty))$ be a solution of $\dot{x} = Df_k(\gamma_k^s(t-\theta))x$ such that $\det[\dot{\gamma}_k^s(\tau_k^s), \zeta_k^s(\tau_k^s)] = 1$. As shown in [15, p. 381], we have

$$\begin{aligned} &x_{k,1}^s(t; \tau_k^s+\theta, Q_{k,\varepsilon}^\theta) \\ &= \left(B_k^s - \int_{\tau_k^s+\theta}^{t} \beta_k^s(s)\left(\zeta_k^s(s-\theta) \wedge g_k(\gamma_k^s(s-\theta), s)\right)\mathrm{d}s\right)\dot{\gamma}_k^s(t-\theta) \\ &\quad + \left(\int_{+\infty}^{t} \beta_k^s(s)\left(\dot{\gamma}_k^s(s-\theta) \wedge g_k(\gamma_k^s(s-\theta), s)\right)\mathrm{d}s\right)\zeta_k^s(t-\theta), \end{aligned} \tag{3.4}$$

where

$$B_k^s = \frac{\langle \dot\gamma_k^s(\tau_k^s), \zeta_k^s(\tau_k^s)\rangle}{\|\dot\gamma_k^s(\tau_k^s)\|^2} H_k^s(\theta). \tag{3.5}$$

On the other hand, $t_{k,\varepsilon}^s(\theta)$ is determined by the equation

$$h_k(x_{k,\varepsilon}^s(\tau_k^s + \theta + t_{k,\varepsilon}^s(\theta); \tau_k^s + \theta, Q_{k,\varepsilon}^\theta)) = 0.$$

By (3.1), we have

$$h_k\left(\gamma_k^s(\tau_k^s + t_{k,\varepsilon}^s(\theta)) + \varepsilon x_{k,1}^s(\tau_k^s + \theta + t_{k,\varepsilon}^s(\theta); \tau_k^s + \theta, Q_{k,\varepsilon}^\theta) + O(\varepsilon^2)\right) = 0. \tag{3.6}$$

Note that $h_k(\gamma_k^s(\tau_k^s)) = 0$ and by (3.4), we have

$$x_{k,1}^s(\tau_k^s + \theta + t_{k,\varepsilon}^s(\theta); \tau_k^s + \theta, Q_{k,\varepsilon}^\theta) = B_k^s \dot\gamma_k^s(\tau_k^s) - H_k^s(\theta)\zeta_k^s(\tau_k^s) + O(|\varepsilon|). \tag{3.7}$$

Thus from (3.6), we have

$$\begin{aligned} &\varepsilon \left\{ \langle \nabla h_k(\gamma_k^s(\tau_k^s)), \dot\gamma_k^s(\tau_k^s)\rangle \left.\frac{\partial t_{k,\varepsilon}^s(\theta)}{\partial \varepsilon}\right|_{\varepsilon=0} + \langle \nabla h_k(\gamma_k^s(\tau_k^s)), \dot\gamma_k^s(\tau_k^s)\rangle B_k^s \right. \\ &\left. -\langle \nabla h_k(\gamma_k^s(\tau_k^s)), \zeta_k^s(\tau_k^s)\rangle H_k^s(\theta) \right\} + O(\varepsilon^2) = 0. \end{aligned} \tag{3.8}$$

Since $t_{k,\varepsilon}^s(\theta)$ is C^2 in (ε, θ) and when $\varepsilon = 0$, $t_{k,\varepsilon}^s(\theta) = 0$ for all $\theta \in S^1$, by (3.5) and (3.8), we have

$$t_{k,\varepsilon}^s(\theta) = \varepsilon \left(\left.\frac{\partial t_{k,\varepsilon}^s(\theta)}{\partial \varepsilon}\right|_{\varepsilon=0} \right) + O(\varepsilon^2) = \tilde H_k^s(\theta)\varepsilon + O(\varepsilon^2), \tag{3.9}$$

where

$$\tilde H_k^s(\theta) = -B_k^s + \frac{\langle \nabla h_k(\gamma_k^s(\tau_k^s)), \zeta_k^s(\tau_k^s)\rangle}{\langle \nabla h_k(\gamma_k^s(\tau_k^s)), \dot\gamma_k^s(\tau_k^s)\rangle} H_k^s(\theta).$$

Since $(A_{k,\varepsilon}^s(\theta), \theta)$ is the intersection point of the perturbed stable manifold $W_{k,\varepsilon}^s$ with $\mathcal{C}_k \times \{\theta\}$, it is obvious that $A_{k,\varepsilon}^s(\theta) = x_{k,\varepsilon}^s(\tau_k^s + \theta + t_{k,\varepsilon}^s(\theta); \tau_k^s + \theta, Q_{k,\varepsilon}^\theta)$. Note that $\gamma_k^s(\tau_k^s) = Q_k$, from (3.1), (3.7) and (3.9), we have

$$\begin{aligned} A_{k,\varepsilon}^s(\theta) &= \gamma_k^s(\tau_k^s + t_{k,\varepsilon}^s(\theta)) + \varepsilon x_{k,1}^s(\tau_k^s + \theta + t_{k,\varepsilon}^s(\theta); \tau_k^s + \theta, Q_{k,\varepsilon}^\theta) + O(\varepsilon^2) \\ &= Q_k + \varepsilon \left\{ \left[\tilde H_k^s(\theta) + B_k^s\right] \dot\gamma_k^s(\tau_k^s) - H_k^s(\theta)\zeta_k^s(\tau_k^s) \right\} + O(\varepsilon^2). \end{aligned} \tag{3.10}$$

By the identity (2.1) and the fact that $\det[\dot\gamma_k^s(\tau_k^s), \zeta_k^s(\tau_k^s)] = 1$, we get

$$\begin{aligned} &\left[\tilde H_k^s(\theta) + B_k^s\right] \dot\gamma_k^s(\tau_k^s) - H_k^s(\theta)\zeta_k^s(\tau_k^s) \\ &= -\left\{ \zeta_k^s(\tau_k^s) - \frac{\langle \nabla h_k(\gamma_k^s(\tau_k^s)), \zeta_k^s(\tau_k^s)\rangle}{\langle \nabla h_k(\gamma_k^s(\tau_k^s)), \dot\gamma_k^s(\tau_k^s)\rangle} \dot\gamma_k^s(\tau_k^s) \right\} H_k^s(\theta) \\ &= -\frac{H_k^s(\theta)}{\langle \nabla h_k(\gamma_k^s(\tau_k^s)), \dot\gamma_k^s(\tau_k^s)\rangle} \left\{\nabla h_k(Q_k)\right\}^\perp. \end{aligned} \tag{3.11}$$

The proof is completed by substituting (3.11) into (3.10). □

By exactly the same method we can compute $A_{k,\varepsilon}^u(\theta)$. The result is as following:

Lemma 3.2. *Let the notations be given above. Then for sufficiently small $|\varepsilon|$, we have*

$$A_{k,\varepsilon}^u(\theta) = Q_k + \varepsilon \frac{H_{k-1}^u(\theta)}{\langle \nabla h_k(\gamma_{k-1}^u(-\tau_{k-1}^u)), \dot\gamma_{k-1}^u(-\tau_{k-1}^u)\rangle} \left\{\nabla h_k(Q_k)\right\}^\perp + O(\varepsilon^2).$$

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. By Lemmas 3.1 and 3.2, we have

$$\begin{aligned}\Delta_{k,\varepsilon}(\theta) &= A^u_{k,\varepsilon}(\theta) - A^s_{k,\varepsilon}(\theta)\\ &= \varepsilon \frac{M_{k,1}(\theta)}{\langle \nabla h_k(\gamma^s_k(\tau^s_k)), \dot{\gamma}^s_k(\tau^s_k)\rangle} \{\nabla h_k(Q_k)\}^{\perp} + O(\varepsilon^2),\end{aligned}$$

where

$$M_{k,1}(\theta) = \frac{\langle \nabla h_k(\gamma^s_k(\tau^s_k)), \dot{\gamma}^s_k(\tau^s_k)\rangle}{\langle \nabla h_k(\gamma^u_{k-1}(-\tau^u_{k-1})), \dot{\gamma}^u_{k-1}(-\tau^u_{k-1})\rangle} H^u_{k-1}(\theta) + H^s_k(\theta).$$

Note that $\gamma^u_{k-1}(-\tau^u_{k-1}) = \gamma^s_k(\tau^s_k) = Q_k$, $\dot{\gamma}^s_k(\tau^s_k) = f_k(Q_k)$, $\dot{\gamma}^u_{k-1}(-\tau^u_{k-1}) = f_{k-1}(Q_k)$. By a simple change of variables in the integrations, we can obtain the expressions for $\Delta_{k,\varepsilon}(\theta)$ and $M_{k,1}(\theta)$ as given in Theorem 2.1. The proof is complete. □

4. Applications to systems with two zones

In this section we apply our result to two concrete piecewise smooth systems with two zones.

4.1. The linearized slender rocking block model

The slender rocking block model was first proposed by Housner [30] to analyze the effect of an earthquake on a free-standing tall, slender structure. Its dynamics have been extensively studied, see, for example [12, 24, 28–30]. In particular, by directly computing the gap between the perturbed stable and unstable manifolds exactly, heteroclinic bifurcations of the linearized slender rocking block model with damping was studied by Hogan [29].

To show the effectiveness of our method, we consider the linearized slender rocking block model without damping given by

$$\ddot{x} - x + \mathrm{sign}(x) = -\varepsilon \cos \omega t, \tag{4.1}$$

$$\dot{x} \longmapsto r\dot{x} \text{ as } x = 0, \tag{4.2}$$

where (4.2) is the impact rule, $r \in [0, 1]$ is the coefficient of restitution. Let $r = 1$ be fixed. Let $x_1 = x$, $x_2 = \dot{x}$. We further reverse the time of (4.1) by $t = -\tau$ so that its flows cross the discontinuity line counterclockwise. Then we replace τ by t, (4.1–4.2) is transformed to the following piecewise smooth system with two zones:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{cases}\begin{pmatrix}-x_2\\ -x_1+1\end{pmatrix} + \varepsilon\begin{pmatrix}0\\ \cos\omega t\end{pmatrix}, & \text{if } x_1 > 0,\\ \begin{pmatrix}-x_2\\ -x_1-1\end{pmatrix} + \varepsilon\begin{pmatrix}0\\ \cos\omega t\end{pmatrix}, & \text{if } x_1 < 0.\end{cases} \tag{4.3}$$

When $\varepsilon = 0$, the unperturbed system of (4.3) has two hyperbolic saddles $P_1 = (1, 0)^T \in \Omega_1$ and $P_2 = (-1, 0)^T \in \Omega_2$, where $\Omega_1 = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 > 0\}$, $\Omega_2 = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 < 0\}$. Furthermore, the unperturbed system of (4.3) has a heteroclinic cycle Γ which consists of 4 branches $\Gamma^{u,s}_k \subset \bar{\Omega}_k$ $(k = 1, 2)$ such that

$$\Gamma = \bigcup_{k=1}^{2} \left(\Gamma^u_k \bigcup \{P_k\} \bigcup \Gamma^s_k\right).$$

Here for each $k = 1, 2$, $\Gamma^{u,s}_k$ are given by $\gamma^{u,s}_k(t)$, where $\gamma^{u,s}_k(t)$ are solutions of the unperturbed system of (4.3) in Ω_k given by

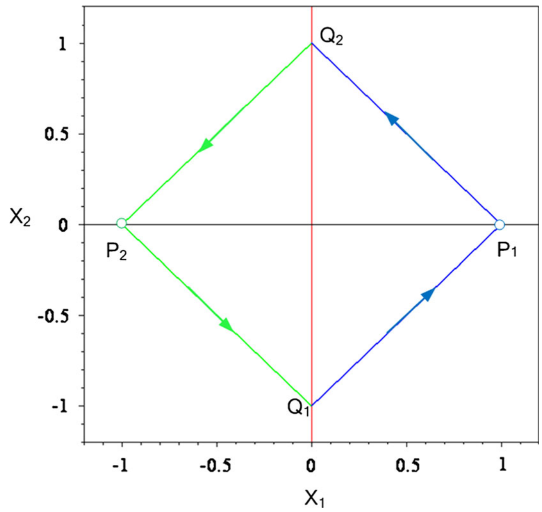

FIG. 2. The heteroclinic cycle of the unperturbed system of (4.3) (i.e. $\varepsilon = 0$)

$$\begin{aligned}
\gamma_1^u(t) &= \left(-e^{t+1}+1, e^{t+1}\right), \quad t \in (-\infty, -1], \\
\gamma_1^s(t) &= \left(-e^{-t+1}+1, -e^{-t+1}\right), \quad t \in [1, +\infty), \\
\gamma_2^u(t) &= \left(e^{t+1}-1, -e^{t+1}\right), \quad t \in (-\infty, -1], \\
\gamma_2^s(t) &= \left(e^{-t+1}-1, e^{-t+1}\right), \quad t \in [1, +\infty).
\end{aligned}$$

See Fig. 2 for the heteroclinic cycle Γ. Clearly assumptions (H1–H3) are all satisfied.

Let $T = \frac{2\pi}{\omega}$. By Theorem 2.1, we have $M_{1,1}(\theta) = -M_{2,1}(\theta)$ for $\theta \in S^1$, where

$$M_{1,1}(\theta) = \frac{2}{1+\omega^2}(\cos\omega - \omega\sin\omega)\cos\omega\theta.$$

It can be shown that $M_{1,1}(\theta)$ and $M_{2,1}(\theta)$ are equivalent to the gap function $\Delta(t_0)$ given by formula (3.15) in [29], only differ by a constant factor due to the time reversing and the time shifting of the heteroclinic orbit. It is worth noting that the result we obtained here is only applicable to the case of $r = 1$. For the more general case of $r < 1$, please see [24,29].

Clearly, for each $\omega > 0$, both $M_{1,1}(\theta)$ and $M_{2,1}(\theta)$ have two simple zeros $\theta_1^* = \frac{\pi}{2\omega}$ and $\theta_2^* = \frac{3\pi}{2\omega}$ in S^1. By Theorem 2.2, the corresponding Poincaré map has a transverse heteroclinic cycle for sufficiently small $|\varepsilon|$. Let $\varepsilon = 0.1$ be fixed. In Figs. 3 and 4 we show the stroboscopic Poincaré maps of system (4.3) with $\omega = 4.8$ and $\omega = 7.8$, respectively, suggesting that system (4.3) is chaotic in both cases.

4.2. A nonlinear compliant oscillator

A class of the most studied PWS systems is impact systems. See, for example, [10,32] for many interesting examples of impact systems. Many examples for rigid impacts assume instantaneous jump in velocity. However, as pointed out in [10, p. 26], this is unrealistic in practice as it would require an infinite force. It is natural to replace the rigid impact by a highly stiff, elastic deformation that takes a short but finite time which leads to models of compliant oscillators. The simplest type of compliant oscillators is bilinear oscillators, which has been extensively studied [38]. In what follows we consider a nonlinear compliant oscillator with positive linear stiffness and sinusoidal forcing given by

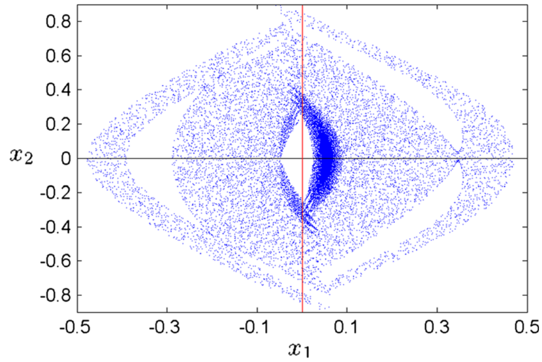

FIG. 3. The stroboscopic Poincaré map of (4.3) with $\varepsilon = 0.1$ and $\omega = 4.8$

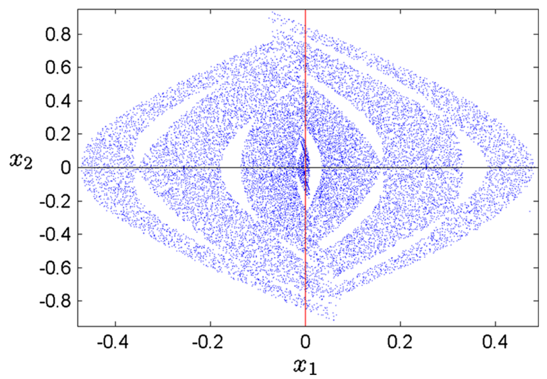

FIG. 4. The stroboscopic Poincaré map of (4.3) with $\varepsilon = 0.1$ and $\omega = 7.8$

$$\ddot{x} + \varepsilon\delta\dot{x} + 2x - 2x^3 = \varepsilon\gamma\sin\omega t, \quad x < 0, \tag{4.4a}$$

$$\ddot{x} + \varepsilon\delta\dot{x} + 8x - 32x^3 = \varepsilon\gamma\sin\omega t, \quad x > 0, \tag{4.4b}$$

where $\varepsilon, \delta \geq 0$ and $\gamma, \omega > 0$. System (4.4) is a piecewise smooth Duffing equation. Let $x_1 = x$, $x_2 = \dot{x}$. We further reverse the time of (4.4) by $t = -\tau$ so that its flows cross the discontinuity line counterclockwise. Then we replace τ by t, (4.4) is transformed to the following piecewise smooth system with two zones:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} -x_2 \\ 2x_1 - 2x_1^3 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ \delta x_2 + \gamma\sin\omega t \end{pmatrix}, & \text{if } x_1 < 0, \\ \begin{pmatrix} -x_2 \\ 8x_1 - 32x_1^3 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ \delta x_2 + \gamma\sin\omega t \end{pmatrix}, & \text{if } x_1 > 0. \end{cases} \tag{4.5}$$

When $\varepsilon = 0$, the unperturbed system of (4.5) has two hyperbolic saddles $P_1 = (-1, 0)^T \in \Omega_1 := \{(x_1, x_2)^T \in \mathbb{R}^2 : \ x_1 < 0\}$ and $P_2 = (\frac{1}{2}, 0)^T \in \Omega_2 := \{(x_1, x_2)^T \in \mathbb{R}^2 : \ x_1 > 0\}$. Furthermore, the unperturbed system of (4.5) has a heteroclinic cycle Γ which consists of 4 branches $\Gamma_k^{u,s} \subset \bar{\Omega}_k$ $(k = 1, 2)$ such that

$$\Gamma = \bigcup_{k=1}^{2} \left(\Gamma_k^u \bigcup \{P_k\} \bigcup \Gamma_k^s \right).$$

Here for each $k = 1, 2$, $\Gamma_k^{u,s}$ are given by $\gamma_k^{u,s}(t)$, where $\gamma_k^{u,s}(t)$ are solutions of the unperturbed system of (4.5) in Ω_k given by

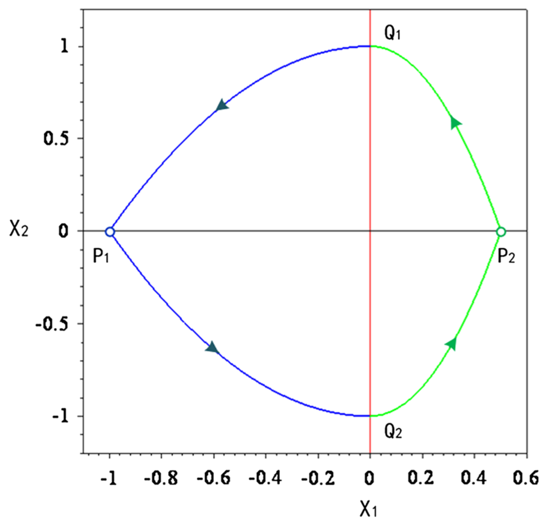

FIG. 5. The heteroclinic cycle of the unperturbed system of (4.5) (i.e. $\varepsilon = 0$)

$$\begin{aligned}
\gamma_1^u(t) &= \left(\tanh(t+1), -\mathrm{sech}^2(t+1)\right), \quad t \in (-\infty, -1],\\
\gamma_1^s(t) &= \left(-\tanh(t-1), \mathrm{sech}^2(t-1)\right), \quad t \in [1, +\infty),\\
\gamma_2^u(t) &= \left(-\frac{1}{2}\tanh\left(2(t+1)\right), \mathrm{sech}^2\left(2(t+1)\right)\right), \quad t \in (-\infty, -1],\\
\gamma_2^s(t) &= \left(\frac{1}{2}\tanh\left(2(t-1)\right), -\mathrm{sech}^2\left(2(t-1)\right)\right), \quad t \in [1, +\infty).
\end{aligned}$$

The heteroclinic cycle Γ is shown in Fig. 5.

Let $T = \frac{2\pi}{\omega}$, $\Lambda_c^{\pm}(\omega) = I_c(\omega) \pm \frac{1}{2} I_c(\frac{\omega}{2})$ and $\Lambda_s^{\pm}(\omega) = I_s(\omega) \pm \frac{1}{2} I_s(\frac{\omega}{2})$, where

$$I_c(\omega) = \int_0^{+\infty} \mathrm{sech}^2(t)\cos\omega t dt, \quad I_s(\omega) = \int_0^{+\infty} \mathrm{sech}^2(t)\sin\omega t dt.$$

Then by Theorem 2.2, for $\theta \in S^1$, we have

$$\begin{aligned}
M_{1,1}(\theta) &= -\delta - \gamma\left[\Pi_1(\omega)\cos\omega\theta + \Pi_2(\omega)\sin\omega\theta\right],\\
M_{2,1}(\theta) &= -\delta - \gamma\left[\Pi_1(\omega)\cos\omega\theta - \Pi_2(\omega)\sin\omega\theta\right],
\end{aligned}$$

where $\Pi_1(\omega) = \Lambda_s^-(\omega)\cos\omega + \Lambda_c^-(\omega)\sin\omega$, $\Pi_2(\omega) = \Lambda_c^+(\omega)\cos\omega - \Lambda_s^+(\omega)\sin\omega$. Hence when $\delta < \gamma\sqrt{[\Pi_1(\omega)]^2 + [\Pi_2(\omega)]^2}$, both $M_{1,1}(\theta)$ and $M_{2,1}(\theta)$ have two distinct simple zeros for $\theta \in S^1$. For example, when $\omega = 4.8, \gamma = 3.6, \delta = 0.001$, $M_{1,1}(\theta)$ has two simple zeros $\theta_{1,1}^1 \approx 0.6214352661$ and $\theta_{1,1}^2 \approx 1.275697866$ in S^1, $M_{2,1}(\theta)$ has two simple zeros $\theta_{2,1}^1 \approx 0.03329907324$ and $\theta_{2,1}^2 \approx 0.6875616729$ in S^1. When $\omega = 5.8, \gamma = 2.8, \delta = 0.001$, $M_{1,1}(\theta)$ has two simple zeros $\theta_{1,1}^1 \approx 0.5405536845$ and $\theta_{1,1}^2 \approx 1.081666741$ in S^1, $M_{2,1}(\theta)$ has two simple zeros $\theta_{2,1}^1 \approx 0.001641070902$ and $\theta_{2,1}^2 \approx 0.5427541271$ in S^1. By Theorem 2.2, the corresponding Poincaré map has a transverse heteroclinic cycle for sufficiently small $|\varepsilon|$. Let $\varepsilon = 0.1$ be fixed. In Fig. 6 and 7 we show the stroboscopic Poincaré maps of system (4.5) with $\omega = 4.8, \gamma = 3.6, \delta = 0.001$ and $\omega = 5.8, \gamma = 2.8, \delta = 0.001$ respectively, suggesting that system (4.5) is chaotic in both cases.

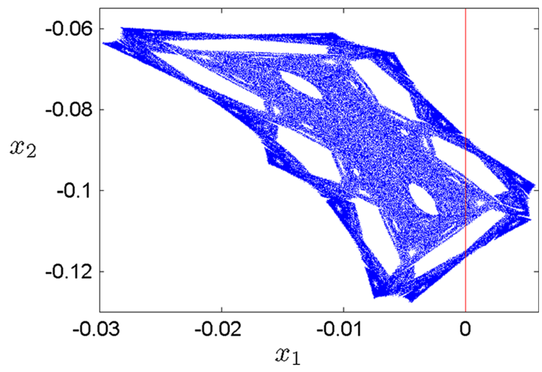

Fig. 6. The stroboscopic Poincaré map of (4.5) with $\varepsilon = 0.1$, $\omega = 4.8$, $\gamma = 3.6$, $\delta = 0.001$

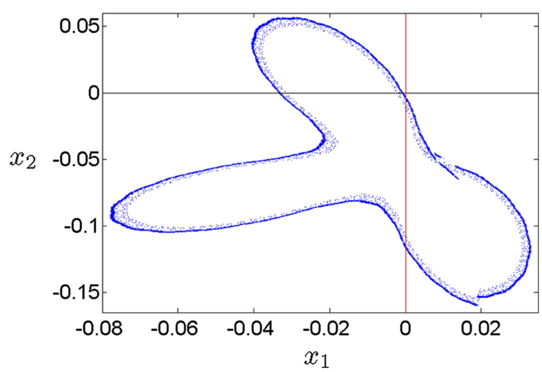

Fig. 7. The stroboscopic Poincaré map of (4.5) with $\varepsilon = 0.1$, $\omega = 5.8$, $\gamma = 2.8$, $\delta = 0.001$

5. Application to a conewise switching system with four zones

To further show the effectiveness of our results, in this section we consider a conewise switching system with four zones.

For each $A > 0$, it is easy to see that the equation

$$\frac{\xi}{2} = A\cos\left(\frac{\xi}{2}\right)$$

has a unique solution $\xi_*(A) \in [0, \pi]$. Let $\tau_*(A) = \frac{1}{A}\operatorname{arcosh}\left(\frac{2A}{\xi_*(A)}\right)$. Assume that $\mathbb{R}^2$ is split into 4 disjoint regions Ω_k by straight lines $\mathcal{C}_k$ starting from the origin for $k = 1, \ldots, 4$, where

$$\begin{aligned}
\mathcal{C}_1 &= \left\{(x_1, x_2)^T \in \mathbb{R}^2 : \ x_2 = x_1, x_1 > 0\right\}, \\
\mathcal{C}_2 &= \left\{(x_1, x_2)^T \in \mathbb{R}^2 : \ x_2 = -x_1, x_1 < 0\right\}, \\
\mathcal{C}_3 &= \left\{(x_1, x_2)^T \in \mathbb{R}^2 : \ x_2 = x_1, x_1 < 0\right\}, \\
\mathcal{C}_4 &= \left\{(x_1, x_2)^T \in \mathbb{R}^2 : \ x_2 = -x_1, x_1 > 0\right\}.
\end{aligned}$$

Let $A_1, A_2 > 0$, $\alpha = \xi_*(A_1)$, $\beta = \xi_*(A_2)$, $x = (x_1, x_2)^T \in \mathbb{R}^2$. Consider the following piecewise smooth system with four zones:

$$\dot{x} = \begin{cases} \begin{pmatrix} (\beta^2-\alpha^2)x_1 + 2\alpha\beta(x_2-\alpha-\beta) \\ 2\alpha\beta x_1 + (\alpha^2-\beta^2)(x_2-\alpha-\beta) \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ \sin\omega t \end{pmatrix}, & \text{if } x \in \Omega_1, \\ \begin{pmatrix} -x_2 \\ A_2^2 \sin x_1 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ \sin\omega t \end{pmatrix}, & \text{if } x \in \Omega_2, \\ \begin{pmatrix} (\alpha^2-\beta^2)x_1 + 2\alpha\beta(x_2+\alpha+\beta) \\ 2\alpha\beta x_1 + (\beta^2-\alpha^2)(x_2+\alpha+\beta) \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ \sin\omega t \end{pmatrix}, & \text{if } x \in \Omega_3, \\ \begin{pmatrix} -x_2 \\ A_1^2 \sin x_1 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ \sin\omega t \end{pmatrix}, & \text{if } x \in \Omega_4. \end{cases} \tag{5.1}$$

When $\varepsilon = 0$, the unperturbed system of (5.1) has four hyperbolic saddles $P_1 = (0, \alpha+\beta)^T \in \Omega_1$, $P_2 = (-\pi, 0)^T \in \Omega_2$, $P_3 = (0, -\alpha-\beta)^T \in \Omega_3$ and $P_4 = (\pi, 0)^T \in \Omega_4$. Furthermore, the unperturbed system of (5.1) has a heteroclinic cycle Γ which consists of 8 branches $\Gamma_k^{u,s} \subset \bar{\Omega}_k$ $(k = 1, \ldots, 4)$ such that

$$\Gamma = \bigcup_{k=1}^{4} \left(\Gamma_k^u \bigcup \{P_k\} \bigcup \Gamma_k^s \right).$$

Here for each $k = 1, \ldots, 4$, $\Gamma_k^{u,s}$ are given by $\gamma_k^{u,s}(t)$, where $\gamma_k^{u,s}(t)$ are solutions of the unperturbed system of (5.1) in Ω_k given by

$$\begin{aligned} \gamma_1^u(t) &= \left(-\beta e^{(\alpha^2+\beta^2)(t+1)}, \alpha+\beta-\alpha e^{(\alpha^2+\beta^2)(t+1)}\right), \quad t \in (-\infty, -1], \\ \gamma_1^s(t) &= \left(\alpha e^{-(\alpha^2+\beta^2)(t-1)}, \alpha+\beta-\beta e^{-(\alpha^2+\beta^2)(t-1)}\right), \quad t \in [1, +\infty), \\ \gamma_2^u(t) &= \left(2\arcsin(\tanh(A_2 t)), -2A_2 \mathrm{sech}(A_2 t)\right), \quad t \in (-\infty, -\tau_*(A_2)], \\ \gamma_2^s(t) &= \left(-2\arcsin(\tanh(A_2 t)), 2A_2 \mathrm{sech}(A_2 t)\right), \quad t \in [\tau_*(A_2), +\infty), \\ \gamma_3^u(t) &= \left(\alpha e^{(\alpha^2+\beta^2)(t+1)}, -\alpha-\beta+\beta e^{(\alpha^2+\beta^2)(t+1)}\right), \quad t \in (-\infty, -1], \\ \gamma_3^s(t) &= \left(-\beta e^{-(\alpha^2+\beta^2)(t-1)}, -\alpha-\beta+\alpha e^{-(\alpha^2+\beta^2)(t-1)}\right), \quad t \in [1, +\infty), \\ \gamma_4^u(t) &= \left(-2\arcsin(\tanh(A_1 t)), 2A_1 \mathrm{sech}(A_1 t)\right), \quad t \in (-\infty, -\tau_*(A_1)], \\ \gamma_4^s(t) &= \left(2\arcsin(\tanh(A_1 t)), -2A_1 \mathrm{sech}(A_1 t)\right), \quad t \in [\tau_*(A_1), +\infty). \end{aligned}$$

Let $T = \frac{2\pi}{\omega}$. By Theorem 2.2, for $\theta \in S^1$, we obtain

$$\begin{aligned} M_{1,1}(\theta) = -(\alpha^2+\beta^2) \Bigg\{ & \frac{2A_1(\alpha+\beta)}{\alpha + A_1^2 \sin\alpha} \int_{-\infty}^{-\tau_*(A_1)} \mathrm{sech}(A_1 s) \sin\omega(s+\theta) \mathrm{d}s \\ & + \alpha \int_1^{+\infty} e^{-(\alpha^2+\beta^2)(s-1)} \sin\omega(s+\theta) \mathrm{d}s \Bigg\}, \\ M_{2,1}(\theta) = -\frac{\beta(\beta + A_2^2 \sin\beta)}{\alpha+\beta} & \int_{-\infty}^{-1} e^{(\alpha^2+\beta^2)(s+1)} \sin\omega(s+\theta) \mathrm{d}s \\ & -2A_2 \int_{\tau_*(A_2)}^{+\infty} \mathrm{sech}(A_2 s) \sin\omega(s+\theta) \mathrm{d}s, \end{aligned}$$

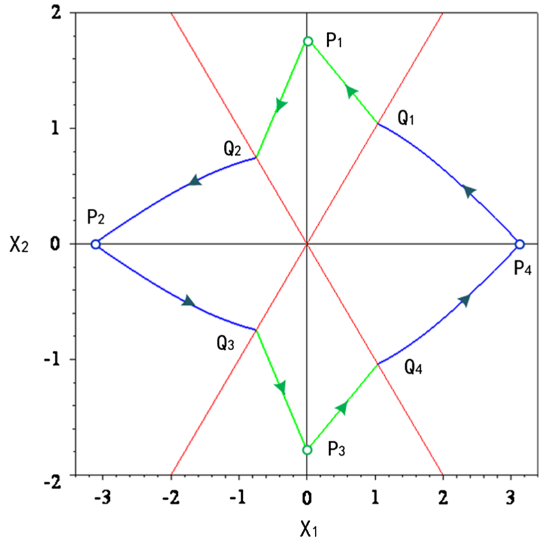

Fig. 8. The heteroclinic cycle of the unperturbed system of (5.1) (i.e. $\varepsilon = 0$) with $A_1 = 0.6$ and $A_2 = 0.4$

$$M_{3,1}(\theta) = (\alpha^2+\beta^2)\left\{\frac{2A_2(\alpha+\beta)}{\beta + A_2^2\sin\beta}\int_{-\infty}^{-\tau_*(A_2)} \operatorname{sech}(A_2 s)\sin\omega(s+\theta)\mathrm{d}s + \beta\int_1^{+\infty} e^{-(\alpha^2+\beta^2)(s-1)}\sin\omega(s+\theta)\mathrm{d}s\right\},$$

$$M_{4,1}(\theta) = \frac{\alpha(\alpha + A_1^2\sin\alpha)}{\alpha+\beta}\int_{-\infty}^{-1} e^{(\alpha^2+\beta^2)(s+1)}\sin\omega(s+\theta)\mathrm{d}s + 2A_1\int_{\tau_*(A_1)}^{+\infty}\operatorname{sech}(A_1 s)\sin\omega(s+\theta)\mathrm{d}s.$$

(1) Take $\omega = 2.5$, $A_1 = 0.6$ and $A_2 = 0.4$. Then we get $\alpha \approx 1.041065278$, $\beta \approx 0.745118992$, $\tau_*(A_1) \approx 0.909614667$ and $\tau_*(A_2) \approx 0.953724379$. The corresponding heteroclinic cycle Γ is shown in Fig. 8. By direct computation, we find that in S^1, $M_{1,1}(\theta)$ has two simple zeros $\theta_1^1 \approx 0.08850095558$ and $\theta_1^2 \approx 1.345138017$, $M_{2,1}(\theta)$ has two simple zeros $\theta_2^1 \approx 1.072200042$ and $\theta_2^2 \approx 2.328837103$, $M_{3,1}(\theta)$ has two simple zeros $\theta_3^1 \approx 0.1844370194$ and $\theta_3^2 \approx 1.441074081$, $M_{4,1}(\theta)$ has two simple zeros $\theta_4^1 \approx 1.168136106$ and $\theta_4^2 \approx 2.424773167$. By Theorem 2.2, the corresponding Poincaré map has a transverse heteroclinic cycle for sufficiently small $|\varepsilon|$. Let $\varepsilon = 0.1$ be fixed. In Fig. 9 we show the stroboscopic Poincaré map of system (5.1), suggesting that system (5.1) is chaotic in this case.

(2) Take $\omega = 4.5$, $A_1 = 0.8$ and $A_2 = 0.4$. Then we get that $\alpha \approx 1.282268566$, $\beta \approx 0.7451189917$, $\tau_*(A_1) \approx 0.8627390845$ and $\tau_*(A_2) \approx 0.9537243791$. The corresponding heteroclinic cycle Γ is shown in Fig. 10. We find that in S^1, $M_{1,1}(\theta)$ has two simple zeros $\theta_1^1 \approx 0.6457341386$ and $\theta_1^2 \approx 1.343865839$, $M_{2,1}(\theta)$ has two simple zeros $\theta_2^1 \approx 0.03796567756$ and $\theta_2^2 \approx 0.7360973784$, $M_{3,1}(\theta)$ has two simple zeros $\theta_3^1 \approx 0.6601660232$ and $\theta_3^2 \approx 1.358297724$, $M_{4,1}(\theta)$ has two simple zeros $\theta_4^1 \approx 0.05239756225$ and $\theta_4^2 \approx 0.7505292630$. By Theorem 2.2, the corresponding Poincaré map has a transverse heteroclinic cycle for sufficiently small $|\varepsilon|$. Let $\varepsilon = 0.1$ be fixed. In Fig. 11 we show the stroboscopic Poincaré map of system (5.1), suggesting that system (5.1) is chaotic in this case.

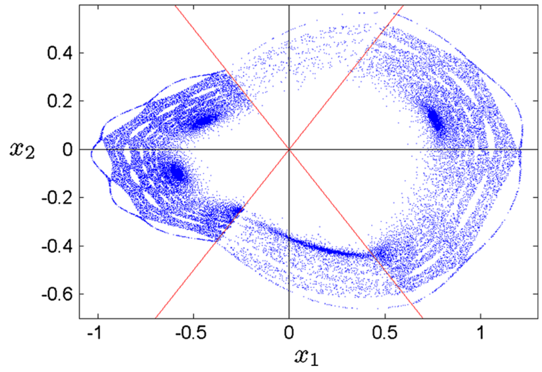

FIG. 9. The stroboscopic Poincaré map of (5.1) with $\varepsilon = 0.1$, $A_1 = 0.6$, $A_2 = 0.4$ and $\omega = 2.5$

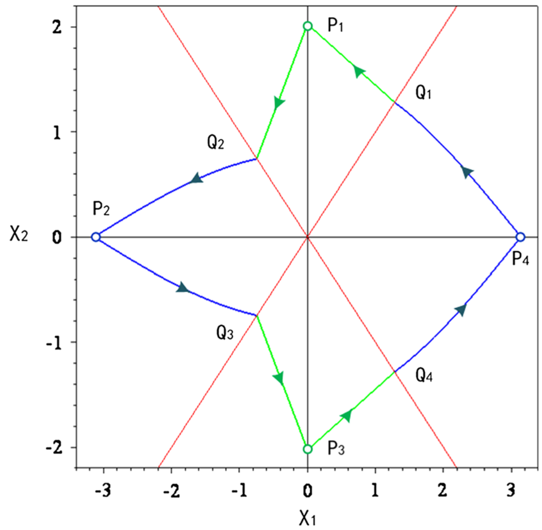

FIG. 10. The heteroclinic cycle of the unperturbed of (5.1) (i.e. $\varepsilon = 0$) with $A_1 = 0.8$ and $A_2 = 0.4$

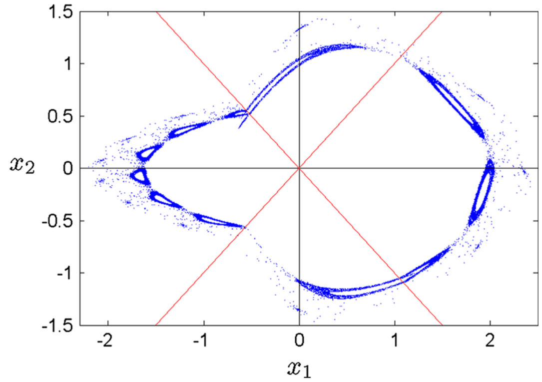

FIG. 11. The stroboscopic Poincaré map of (5.1) with $\varepsilon = 0.1$, $A_1 = 0.8$, $A_2 = 0.4$ and $\omega = 4.5$

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