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The mass concentration phenomenon for L^2-critical constrained problems related to Kirchhoff equations

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Abstract. In this paper, we study the concentration behavior of critical points with a minimax characterization to the following functional

$$I(u) = \frac{a}{2}\int\limits_{\mathbb{R}^N} |\nabla u|^2 + \frac{b}{4}\left(\int\limits_{\mathbb{R}^N} |\nabla u|^2\right)^2 - \frac{N}{2N+8}\int\limits_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}}$$

constrain on $S_c = \{u \in H^1(\mathbb{R}^N)|\ |u|_2 = c, c > 0\}$ when $c \to (c^*)^+$, where $c^* = \left(2^{-1}b|Q|_2^{\frac{8}{N}}\right)^{\frac{N}{8-2N}}$, $N = 1, 2, 3$, and Q is up to translations, the unique positive solution of $-2\Delta Q + \left(\frac{4}{N} - 1\right) Q = |Q|^{\frac{8}{N}} Q$ in $\mathbb{R}^N$.
As such constraint problem is L^2-critical, it seems impossible to benefit from natural constraints $V_c = \left\{u \in S_c|\ a\int\limits_{\mathbb{R}^N} |\nabla u|^2 + b\left(\int\limits_{\mathbb{R}^N} |\nabla u|^2\right)^2 = \frac{2N}{N+4}\int\limits_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}}\right\}$. We show that the mountain pass energy level $\gamma(c) = \inf\limits_{u \in M_c} I(u)$ for some submanifold $M_c \subset V_c$ and then prove the strict monotonicity of $\gamma(c)$ on $(c^*, +\infty)$. We obtained that the critical point u_c behaves like

$$u_c(x) \approx \left(\frac{a^2}{2b(c^*)^2[(\frac{c}{c^*})^{\frac{8-2N}{N}} - 1]^2}\right)^{\frac{N}{8}} Q\left(\left(\frac{a}{b(c^*)^2[(\frac{c}{c^*})^{\frac{8-2N}{N}} - 1]}\right)^{\frac{1}{2}} (x - y_c)\right)$$

for some $y_c \in \mathbb{R}^N$ as c approaches c^* from above.

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1. Introduction and main result

The following nonlinear Kirchhoff equation

$$-\left(a + b\int\limits_{\mathbb{R}^N} |\nabla u|^2\right)\Delta u - |u|^{p-2}u = \lambda u,\ x \in \mathbb{R}^N,\ \lambda \in \mathbb{R} \tag{1.1}$$

has attracted considerable attention, where $N = 1, 2, 3$, a, $b > 0$ are constants and $p \in (2, 2^*)$, $2^* = 6$ if $N = 3$ and $2^* = +\infty$ if $N = 1, 2$.

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Equation (1.1) is a nonlocal one as the appearance of the term $\int_{\mathbb{R}^N} |\nabla u|^2$ implies that (1.1) is not a pointwise identity. This causes some mathematical difficulties which make the study of (1.1) particularly interesting, see [1,2,5–7,12,19,23] and the references therein. The first line to study (1.1) is to consider the case where λ is a fixed and assigned parameter, see, e.g., [9,11,16–18,20,26,29,30]. In such direction, the critical point theory is used to look for nontrivial solutions; however, nothing can be given a priori on the L^2-norm of the solutions. Recently, since the physicists are often interested in "normalized solutions," solutions with prescribed L^2-norm are considered. To state the main results, for $a > 0$ fixed, we introduce an equivalent norm on $H^1(\mathbb{R}^N)$:

$$\|u\| = \left(\int_{\mathbb{R}^N} (a|\nabla u|^2 + u^2) \right)^{\frac{1}{2}}, \quad \forall\, u \in H^1(\mathbb{R}^N),$$

which is induced by the corresponding inner product on $H^1(\mathbb{R}^N)$. Then such solutions are obtained by looking for critical points of the following C^1 functional

$$I_p(u) = \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{b}{4} \left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^2 - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p$$

constrained on the L^2-spheres in $H^1(\mathbb{R}^N)$:

$$S_c = \{u \in H^1(\mathbb{R}^N) |\ |u|_2 = c,\ c > 0\}.$$

The parameter λ is not fixed any longer but appears as an associated Lagrange multiplier. By the L^2-preserving scaling and the well-known Gagliardo–Nirenberg inequality with the best constant [27]: Let $p \in [2, \frac{2N}{N-2})$ if $N \geq 3$ and $p \geq 2$ if $N = 1, 2$, then

$$|u|_p^p \leq \frac{p}{2|Q_p|_2^{p-2}} |\nabla u|_2^{\frac{N(p-2)}{2}} |u|_2^{p - \frac{N(p-2)}{2}}, \tag{1.2}$$

with equality only for $u = Q_p$, where Q_p is, up to translations, the unique positive least energy solution of

$$-\frac{N(p-2)}{4} \Delta Q + \left(1 + \frac{p-2}{4}(2-N)\right) Q = |Q|^{p-2} Q, \quad x \in \mathbb{R}^N, \tag{1.3}$$

it is showed in [31] that $p = \frac{2N+8}{N}$ is the L^2-critical exponent for constrained minimization problems $I_{p,c^2} = \inf\limits_{u \in S_c} I_p(u)$, namely, for all $c > 0$, $I_p(u)$ is bounded from below and coercive on S_c if $p \in \left(2, \frac{2N+8}{N}\right)$ and is not bounded from below on S_c if $p \in \left(\frac{2N+8}{N}, 2^*\right)$. When $p = \frac{2N+8}{N}$, there exists

$$c^* = \left(2^{-1} b |Q_{\frac{2N+8}{N}}|_2^{\frac{8}{N}}\right)^{\frac{N}{8-2N}} \tag{1.4}$$

such that $I_{\frac{2N+8}{N},c^2} = \begin{cases} 0, & 0 < c \leq c^* \\ -\infty, & c > c^* \end{cases}$ and $I_{\frac{2N+8}{N},c^2}$ has no minimizer for all $c > 0$.

Thus for $2 < p < \frac{2N+8}{N}$, normalized solutions are obtained by using the concentration compactness principle to prove that I_{p,c^2} is attained. For $p \geq \frac{2N+8}{N}$, the minimization problems cannot work. To obtain normalized solutions, it is proved in [31,32] that $I_p(u)$ has a critical point restricted to S_c with a mountain pass geometry, i.e., there exists $K_p(c) > 0$ such that

$$\gamma_p(c) = \inf_{h \in \Gamma_p(c)} \max_{\tau \in [0,1]} I_p(h(\tau)) > \max_{h \in \Gamma_p(c)} \{\max\{I_p(h(0)), I_p(h(1))\}\} \tag{1.5}$$

holds in the set

$$\Gamma_p(c) = \{h \in C([0,1], S_c) |\ h(0) \in B_{K_p(c)}, I_p(h(1)) < 0\} \tag{1.6}$$

and $B_{K_p(c)} = \left\{u \in S_c \mid |\nabla u|_2^2 \le K_p(c)\right\}$. In particular, for $p = \frac{2N+8}{N}$, we have the following existing results:

Lemma 1.1. ([31,32], *Theorems* 1.2 *and* 1.3)

(1) *$I_{\frac{2N+8}{N}}(u)$ has no critical point on the constraint S_c for all $0 < c \le c^*$.*

(2) *For any $c > c^*$, there exists at least one couple $(u_c, \lambda_c) \in S_c \times \mathbb{R}_-$ solution of the following problem:*

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\right)\Delta u - |u|^{\frac{8}{N}}u = \lambda_c u,\ \ x \in \mathbb{R}^N \tag{1.7}$$

with $I_{\frac{2N+8}{N}}(u_c) = \gamma_{\frac{2N+8}{N}}(c)$.

For the L^2-supercritical case $p > \frac{2N+8}{N}$, it is shown in [31] that the following two properties are essential to get normalized solutions at the level $\gamma_p(c)$:

$$\text{the function } c \mapsto \gamma_p(c) \text{ is strictly decreasing on } (0, +\infty) \tag{1.8}$$

and

$$\gamma_p(c) = \inf_{\{u \in S_c \mid G_p(u) = 0\}} I_p(u), \tag{1.9}$$

where

$$G_p(u) := a\int_{\mathbb{R}^N} |\nabla u|^2 + b\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 - \frac{N(p-2)}{2p}\int_{\mathbb{R}^N} |u|^p.$$

However, for $p = \frac{2N+8}{N}$, as $\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2$ and $\int_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}}$ behave at the same way under L^2-preserving scaling of u, it seems impossible to show that for any $u \in S_c$, there exists $t = t(u) > 0$ such that $G_{\frac{2N+8}{N}}(u^t) = 0$, where $u^t(x) = t^{\frac{N}{2}}u(tx)$. Then the property (1.9) no longer holds, which results in that the strict monotonicity of the function $c \mapsto \gamma_{\frac{2N+8}{N}}(c)$ is still a question. Indeed, properties (1.8) and (1.9) are also necessary in looking for normalized solutions to other problems with L^2-supercritical nonlinearities, e.g., Schrödinger equations (see [10]), Schrödinger–Poisson system (see [3]).

In this paper, we try to study properties of the function $c \mapsto \gamma_{\frac{2N+8}{N}}(c)$ and try to get some similar properties to (1.8)(1.9). As far as we know, there is no paper on this aspect. For simplicity, in what follows, we use $I(u)$, Q, $\gamma(c)$, $\Gamma(c)$, $K(c)$ and $G(u)$ to denote $I_{\frac{2N+8}{N}}(u)$, $Q_{\frac{2N+8}{N}}$, $\gamma_{\frac{2N+8}{N}}(c)$, $\Gamma_{\frac{2N+8}{N}}(c)$, $K_{\frac{2N+8}{N}}(c)$ and $G_{\frac{2N+8}{N}}(u)$ given above, respectively.

We set

$$E_c = \left\{u \in S_c \mid b\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 < \frac{2N}{N+4}\int_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}}\right\},$$

then $E_c \neq \varnothing$ since $u_c \in E_c$. Define

$$M_c = \{u \in E_c \mid G(u) = 0\},$$

then we have the following main results:

Theorem 1.2.

$$\gamma(c) = \inf_{u \in M_c} I(u) = \inf_{u \in E_c} \max_{t > 0} I(u^t),$$

where $u^t(x) = t^{\frac{N}{2}}u(tx)$.

Theorem 1.3. *Let*

$$m_c = \inf_{u \in M_c} I(u),$$

then each minimizer of m_c *is a solution of problem* (1.7).

Proposition 1.4. *(1) The function* $c \mapsto \gamma(c)$ *is continuous on* $(c^*, +\infty)$;
(2) The function $c \mapsto \gamma(c)$ *is strictly decreasing on* $(c^*, +\infty)$;
(3) $\lim_{c\to+\infty} \gamma(c) = 0$ *and* $\lim_{c\to(c^*)^+} \gamma(c) = +\infty$.

We also concern the behavior of solutions u_c obtained in Lemma 1.1 as c approaches the critical value c^* from above. Our main result is as follows:

Theorem 1.5. *For any* $c > c^*$, *let* (u_c, λ_c) *be the couple of solution obtained in Lemma 1.1. Then*

(1) $\begin{cases} |\nabla u_c|_2 \to 0, \\ \lambda_c \to 0, \end{cases}$ *as* $c \to +\infty$.

(2) $\begin{cases} |\nabla u_c|_2 \to +\infty, \\ \lambda_c \to -\infty, \end{cases}$ *as* $c \to (c^*)^+$.

(3) For any sequence $\{c_k\} \subset (c^*, +\infty)$ *satisfying that* $c_k \to (c^*)^+$ *as* $k \to +\infty$, *there exists a subsequence of* $\{c_k\}$ *(still denoted by* $\{c_k\}$*) and a sequence* $\{y_k\} \subset \mathbb{R}^N$ *such that*

$$\left(\frac{4b}{a^2}\right)^{\frac{N}{8}} \left(\left(\frac{c_k}{c^*}\right)^{\frac{8-2N}{N}} - 1\right)^{\frac{N}{4}} u_{c_k}\left(\sqrt[4]{\frac{4b}{a^2}}\left(\left(\frac{c_k}{c^*}\right)^{\frac{8-2N}{N}} - 1\right)^{\frac{1}{2}} x + y_k\right) \to \left(\frac{\sqrt{2}}{c^*}\right)^{\frac{N}{4}} Q\left(\frac{\sqrt{2}}{b^{\frac{1}{4}} c^*} x\right)$$

in $L^p(\mathbb{R}^N)$ *for all* $2 \le p < 2^*$.

Our mass concentration result Theorem 1.5 is new, which has not appeared in other autonomous problems.

Let us underline the main idea in proving Theorems 1.2–1.5. As mentioned above, we cannot benefit from the natural manifold $\{u \in S_c |\ G(u) = 0\}$ since it may occur that $I(u^t)$ is strictly increasing with respect to t on $(0, +\infty)$ for some $u \in S_c$. So we need to exclude the interference of the functions, satisfying that $b|\nabla u|_2^4 \ge \frac{2N}{N+4} \int_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}}$, which is the reason why the set E_c is introduced. We can show that for any $u \in E_c$, there exists a unique $t(u) > 0$ such that

$$u^{t(u)} \in M_c \text{ and } \gamma(c) \le I(u^{t(u)}) = \max_{t>0} I(u^t). \tag{1.10}$$

Then M_c can be viewed as a suitable submanifold, and hence, Theorems 1.2–1.3 and Proposition 1.4 (1) and (2) can be proved. Moreover, by using (1.10), we indeed show that

$$\gamma(c) = \frac{a^2}{4b} \frac{1}{(\frac{c}{c^*})^{\frac{8-2N}{N}} - 1}, \tag{1.11}$$

which shows Proposition 1.3 (3) and Theorem 1.5 (1)(2). Our method to show the accurate value of $\gamma(c)$ can be also used to the classical nonlinear Schrödinger equation case $-\Delta u - |u|^{p-2}u = \lambda u$ in $\mathbb{R}^N$ with $\frac{2N+4}{N} < p < 2^*$, which will be discussed in Sect. 5.

To prove Theorem 1.5 (3), since $G(u_c) = 0$, we see that $\frac{\frac{2N}{N+4}\int_{\mathbb{R}^N} |u_c|^{\frac{2N+8}{N}}}{b\left(\int_{\mathbb{R}^N} |\nabla u_c|^2\right)^2} \to 1$ as $c \to (c^*)^+$. We succeeded in proving the theorem by choosing a suitable L^2-preserving scaling and translation as follows:

$$v_c(x) = \varepsilon_c^{\frac{N}{2}} u_c(\varepsilon_c x + \varepsilon_c y_c),$$

where $\varepsilon_c = \sqrt[4]{\frac{4b}{a^2}}\left((\frac{c}{c^*})^{\frac{8-2N}{N}} - 1\right)^{\frac{1}{2}}$ and $\{y_c\}$ is derived from the vanishing lemma. Indeed, since $\{v_c\}$ is uniformly bounded in $H^1(\mathbb{R}^N)$, $v_c \rightharpoonup v$ in $H^1(\mathbb{R}^N)$ for some $v \ne 0$ as $c \to (c^*)^+$. Using the Euler–Lagrange equation satisfied by v_c, we can show that v is a nontrivial solution of

$$-\Delta v+\frac{4-N}{N\sqrt{b}(c^*)^2}v=\frac{1}{2\sqrt{b}}|v|^{\frac{8}{N}}v,\quad x\in\mathbb{R}^N. \tag{1.12}$$

If v is positive, then by a rescaling argument and the uniqueness of positive solutions (up to translations) of (1.3), the theorem is proved. Indeed, once Theorem 1.3 is proved, we can show that v is positive.

We finally consider the relationship between $\gamma(c)$ and the least energy among all solutions of problem (1.7), i.e.,

$$d_c=\inf\{F_c(u)|\ u\in H^1(\mathbb{R}^N)\text{ is a nontrivial solution of (1.7)}\},$$

where $F_c : H^1(\mathbb{R}^N)\to\mathbb{R}$ is the functional corresponding to (1.7) defined as

$$F_c(u)=\frac{1}{2}\int\limits_{\mathbb{R}^N}(a|\nabla u|^2-\lambda_c|u|^2)+\frac{b}{4}\left(\int\limits_{\mathbb{R}^N}|\nabla u|^2\right)^2-\frac{N}{2N+8}\int\limits_{\mathbb{R}^N}|u|^{\frac{2N+8}{N}}.$$

Recall that it has been proved in [9] that problem (1.7) has at least one positive least energy solution $w\in H^1(\mathbb{R}^N)$, satisfying that $F_c(w)=d_c$.

Theorem 1.6.

$$d_c=\frac{4}{N}\gamma(c)+\frac{4(4-N)b}{Na^2}\gamma^2(c).$$

Moreover,

(1) u_c is a positive least energy solution of problem (1.7)*;*
(2) each least energy solution w of problem (1.7) *is a critical point of I constrained on S_c with $I(w)=\gamma(c)$.*

Throughout this paper, we use standard notations. For simplicity, we write $\int_\Omega h$ to mean the Lebesgue integral of $h(x)$ over a domain $\Omega\subset\mathbb{R}^N$. $L^p:=L^p(\mathbb{R}^N)$ $(1\le p<+\infty)$ is the usual Lebesgue space with the standard norm $|\cdot|_p$. We use "$\to$" and "$\rightharpoonup$" to denote the strong and weak convergence in the related function space, respectively. C will denote a positive constant unless specified. We use "$:=$" to denote definitions and $B_r(x):=\{y\in\mathbb{R}^N|\,|x-y|<r\}$. We denote a subsequence of a sequence $\{u_n\}$ as $\{u_n\}$ to simplify the notation unless specified.

The paper is organized as follows: In Sect. 2, we prove Theorems 1.2 and 1.3. In Sect. 3, we prove Proposition 1.4 and Theorem 1.5. In Sect. 4, we prove Theorem 1.6. In Sect. 5, we discuss the nonlinear Schrödinger equation case.

2. Proof of Theorems 1.2 and 1.3

In what follows, for any $u\in S_c$ and $t\ge 0$, we denote for simplicity

$$u^t(x):=t^{\frac{N}{2}}u(tx).$$

Then $u^t\in S_c$.

Lemma 2.1. *Suppose that $u\in H^1(\mathbb{R}^N)$ is a weak solution of problem* (1.7)*, then we have $G(u)=0$, where*

$$G(u):=a\int\limits_{\mathbb{R}^N}|\nabla u|^2+b\left(\int\limits_{\mathbb{R}^N}|\nabla u|^2\right)^2-\frac{2N}{N+4}\int\limits_{\mathbb{R}^N}|u|^p.$$

Proof. Since u is a weak solution of (1.7), u satisfies the following Pohozaev identity (e.g., [16], Lemma 2.1):

$$P_c(u) := \frac{N-2}{2}\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\right) - \frac{N}{2}\lambda_c \int_{\mathbb{R}^N} u^2 - \frac{N^2}{2N+8}\int_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}} = 0.$$

Then $\frac{N}{2}\langle F_c'(u), u\rangle - P_c(u) = 0$, i.e., $G(u) = 0$. □

We easily get the following lemma.

Lemma 2.2. *Let $(u_c, \lambda_c) \in S_c \times \mathbb{R}_-$ be the couple of solution to (1.7) obtained in Lemma 1.1. Then*
(1) $G(u_c) = 0$;
(2) $\gamma(c) = I(u_c) = \frac{a}{4}\int_{\mathbb{R}^N} |\nabla u_c|^2$;
(3) $\lambda_c = -\frac{2(4-N)}{Nc^2}[\gamma(c) + \frac{4b}{a^2}\gamma^2(c)]$.

Set

$$E_c := \left\{ u \in S_c \,\middle|\, b\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 < \frac{2N}{N+4}\int_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}} \right\}, \tag{2.1}$$

then $E_c \neq \varnothing$ since $u_c \in E_c$.

Lemma 2.3. *For any $u \in E_c$, there exists a unique $\tilde{t} = t(u) > 0$ such that $G(u^{\tilde{t}}) = 0$; moreover, $I(u^{\tilde{t}}) = \max_{t>0} I(u^t)$.*

Proof. For any $t > 0$, we consider

$$h(t) := I(u^t) = \frac{at^2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{t^4}{4}\left[\frac{2N}{N+4}\int_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}} - b\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2\right]. \tag{2.2}$$

Since $u \in E_c$, we easily see that h has a unique critical point $\tilde{t} > 0$ corresponding to its maximum, i.e., $h'(\tilde{t}) = 0$ and $h(\tilde{t}) = \max_{t>0} h(t)$. So

$$a\tilde{t}^2\int_{\mathbb{R}^N} |\nabla u|^2 + b\tilde{t}^4\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 = \tilde{t}^4\frac{2N}{N+4}\int_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}},$$

i.e., $G(u^{\tilde{t}}) = 0$. □

Define

$$M_c = \{u \in E_c |\ G(u) = 0\},$$

then we see from Lemma 2.3 that $M_c \neq \varnothing$. For any $u \in M_c$, we have $I(u) = I(u) - \frac{1}{4}G(u) = \frac{a}{4}\int_{\mathbb{R}^N} |\nabla u|^2 > 0$, hence

$$m_c := \inf_{u \in M_c} I(u)$$

is well defined.

Lemma 2.4. *For any $u \in E_c$, we have*

$$\gamma(c) \le \frac{1}{4}\frac{a^2\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2}{\frac{2N}{N+4}\int_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}} - b\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2}.$$

Proof. For any $u \in E_c$, we consider the function $h(t)$ given as in (2.2). Then there exists $t_0 > 0$ large such that $h(t_0) < 0$. As $|\nabla u^t|_2 \to 0$ as $t \to 0^+$, there exists $t_1 > 0$ small such that $u^{t_1} \in B_{K(c)}$, where $B_{K(c)}$ was defined in (1.6). Then $u^{(1-t)t_1+tt_0}|_{t\in[0,1]}$ is a path in $\Gamma(c)$. So by Lemma 2.3, we have

$$\gamma(c) \le \max_{t\in[0,1]} I(u^{(1-t)t_1+tt_0}) \le \max_{t>0} I(u^t) = \frac{1}{4}\frac{a^2(\int_{\mathbb{R}^N}|\nabla u|^2)^2}{\frac{2N}{N+4}\int_{\mathbb{R}^N}|u|^{\frac{2N+8}{N}} - b(\int_{\mathbb{R}^N}|\nabla u|^2)^2}.$$

□

Corollary 2.5. *If $u \in M_c$, then $\gamma(c) \le \frac{a}{4}\int_{\mathbb{R}^N}|\nabla u|^2$.*

Recall it has been proved in Lemma 3.1 and Remark 3.2 of [32] that

$$0 < K(c) < \frac{a}{b}\left(\frac{c^*}{c}\right)^{\frac{8-2N}{N}} \tag{2.3}$$

and

$$\sup_{u\in B_{K(c)}} I(u) < \frac{\gamma(c)}{2}, \tag{2.4}$$

where $B_{K(c)}$ was defined as in (1.6).

Lemma 2.6. *For any $u \in B_{K(c)}$, we have $G(u) > 0$.*

Proof. For any $u \in S_c$, by (1.2) we see that

$$G(u) - b|\nabla u|_2^4 \ge a|\nabla u|_2^2 - \left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}} b|\nabla u|_2^4 \ge 0 \text{ if } |\nabla u|_2^2 \le \frac{a}{b}\left(\frac{c^*}{c}\right)^{\frac{8-2N}{N}},$$

i.e., $G(u) > 0$ for all $0 < |\nabla u|_2^2 \le \frac{a}{b}\left(\frac{c^*}{c}\right)^{\frac{8-2N}{N}}$. Then the lemma follows from (2.3). □

2.1. Proof of Theorem 1.2

Proof. By Lemma 2.3, we conclude that

$$\inf_{u\in E_c}\max_{t>0} I(u^t) = \inf_{u\in M_c} I(u).$$

For any $u \in M_c$, we have $u \in E_c$ and $G(u) = 0$. Hence $I(u) = \frac{a}{4}\int_{\mathbb{R}^N}|\nabla u|^2$. So by Corollary 2.5, we see that

$$\gamma(c) \le \frac{a}{4}\int_{\mathbb{R}^N}|\nabla u|^2 = I(u),$$

which implies that $\gamma(c) \le m_c$.

On the other hand, for any $g \in \Gamma(c)$, by Lemma 2.6 we see that $G(g(0)) > 0$. $I(g(1)) < 0$ implies that $G(g(1)) < 0$. Then there exists $t_0 \in (0,1)$ such that $G(g(t_0)) = 0$. Moreover, $G(g(t_0)) = 0$ implies that $g(t_0) \in E_c$, so $g(t_0) \in M_c$. Then

$$m_c \le I(g(t_0)) \le \max_{t\in[0,1]} I(g(t)).$$

By the arbitrary of g, we see that $m_c \le \gamma(c)$. So we complete the proof. □

We recall from [28] that for any $c>0$, S_c is a submanifold of $H^1(\mathbb{R}^N)$ with codimension 1 and the tangent space at $u\in S_c$ is defined as $T_u=\{v\in H^1(\mathbb{R}^N)|\ \int_{\mathbb{R}^N} uv=0\}$. Let $\Phi\in C^1(H^1(\mathbb{R}^N),\mathbb{R})$ and $u\in S_c$. The norm of the derivative of $\Phi|_{S_c}$ is defined by

$$\|(\Phi|_{S_c})'(u)\|_*=\sup_{v\in T_u,\|v\|=1}\langle\Phi'(u),v\rangle.$$

For any $l\in\mathbb{R}$, set $\Phi^l=\{u\in S_c|\ \Phi(u)\le l\}$. The following lemma is a direct sequence of Lemma 5.15 in [28].

Lemma 2.7. *Let $\Phi\in C^1(H^1(\mathbb{R}^N),\mathbb{R})$, $K\subset S_c$ and $m\in\mathbb{R}$. If there exist $\varepsilon,\delta>0$ such that*

$$\forall\ u\in\Phi^{-1}([m-2\varepsilon,m+2\varepsilon])\cap K_{2\delta}\Rightarrow\|(\Phi|_{S_c})'(u)\|_*\ge\frac{8\varepsilon}{\delta},$$

then there exists $\eta\in C([0,1]\times S_c,S_c)$ such that

(i) $\eta(t,u)=u$ if $t=0$ or $u\notin\Phi^{-1}([m-2\varepsilon,m+2\varepsilon])\cap K_{2\delta}$,

(ii) $\eta(1,\Phi^{m+\varepsilon}\cap K)\subset\Phi^{m-\varepsilon}$,

(iii) $\Phi(\eta(\cdot,u))$ is nonincreasing for all $u\in S_c$.

2.2. Proof of Theorem 1.3

Proof. The idea of the proof comes from that in [3], but with some changes.

Suppose that $v_c\in M_c$ is a minimizer of m_c, then by the Lagrange multiplier theory, it is enough to prove that v_c is a critical point of $I(u)$ constrained on S_c. In fact, if $(I|_{S_c})'(v_c)=0$, then there exists some μ_c such that $I'(u_c)-\mu_c v_c=0$. Since $G(v_c)=0$ and $I(v_c)=m_c=\gamma(c)$, by Lemmas 2.1 and 2.2 we see that

$$\mu_c=-\frac{2(4-N)}{Nc^2}[\gamma(c)+\frac{4b}{a^2}\gamma^2(c)]=\lambda_c,$$

which implies that v_c is a solution of (1.7).

By contradiction, we just assume that $\|(I|_{S_c})'(v_c)\|_*\ne 0$. Then there exist $\delta,\varepsilon>0$ such that

$$\|(I|_{S_c})'(u)\|_*\ge\frac{8\varepsilon}{\delta}$$

for all $u\in\{u\in S_c|\ \|u-v_c\|\le 2\delta\}\cap I^{-1}([\gamma(c)-2\varepsilon,\gamma(c)+2\varepsilon])$.

We may furthermore assume that $\varepsilon<\frac{\gamma(c)}{4}$. Then by Lemma 2.7, there exists a deformation $\eta: S_c\times[0,1]\to S_c$ such that

(i) $\eta(1,u)=u$ if $u\notin I^{-1}([\gamma(c)-2\varepsilon,\gamma(c)+2\varepsilon])\cap\{u\in S_c|\ \|u-v_c\|\le 2\delta\}$,

(ii) $\eta(1,v_c)\subset I^{\gamma(c)-\varepsilon}$,

(iii) $I(\eta(1,u))\le I(u)$ for all $u\in S_c$.

Since $v_c\in M_c$, similarly to the proof of Lemma 2.4, there exist $t_0>1$ large, $t_1>0$ small such that $g(t):=v_c^{(1-t)t_1+tt_0}|_{t\in[0,1]}\in\Gamma(c)$. By (i)(iii) we have $\eta(1,g(t))|_{t\in[0,1]}\in\Gamma(c)$. Indeed, by (iii) we have $I(\eta(1,g(1))\le I(g(1))<0$. Since (2.4) implies that $I(g(0))<\frac{\gamma(c)}{2}$, then by (i) we see that $\eta(1,g(0))=g(0)\in B_{K(c)}$.

Then by (iii), Lemma 2.3 and Theorem 1.2, we see that

$$\gamma(c)\le\max_{t\in[0,1]}I(\eta(1,g(t))\le\max_{t\in[0,1]}I(g(t))\le\max_{t>0}I(v_c^t)=I(v_c)=m_c=\gamma(c),$$

which implies that

$$\gamma(c)=\max_{t\in[0,1]}I(\eta(1,g(t)))=I(\eta(1,v_c)),\tag{2.5}$$

which is a contradiction with (ii). $\square$

3. Proof of Proposition 1.4 and Theorem 1.5

In this section, we study the properties of the function $c \mapsto \gamma(c)$.

Lemma 3.1. *The function $c \mapsto \gamma(c)$ is strictly decreasing on $(c^*, +\infty)$.*

Proof. For any $c_2 > c_1 > c^*$, by Lemmas 1.1 and 2.2, there exist $u_{c_i} \in S_{c_i}$ such that

$$I(u_{c_i}) = \gamma(c_i) = \frac{a}{4}\int_{\mathbb{R}^N} |\nabla u_{c_i}|^2 \text{ and } G(u_{c_i}) = 0 \ (i = 1, 2). \tag{3.1}$$

Moreover, $u_{c_i} \in M_{c_i}$. Since $c_2 > c_1$, we see that $\frac{c_2}{c_1} u_{c_1} \in E_{c_2}$. Then by Lemma 2.4 and (3.1), we have

$$\begin{aligned}
\gamma(c_2) &\le \frac{1}{4}\frac{a^2\left(\int_{\mathbb{R}^N}|\nabla u_{c_1}|^2\right)^2}{\frac{2N}{N+4}\left(\frac{c_2}{c_1}\right)^{\frac{8-2N}{N}}\int_{\mathbb{R}^N}|u_{c_1}|^{\frac{2N+8}{N}} - b\left(\int_{\mathbb{R}^N}|\nabla u_{c_1}|^2\right)^2}\\
&< \frac{1}{4}\frac{a^2\left(\int_{\mathbb{R}^N}|\nabla u_{c_1}|^2\right)^2}{\frac{2N}{N+4}\int_{\mathbb{R}^N}|u_{c_1}|^{\frac{2N+8}{N}} - b\left(\int_{\mathbb{R}^N}|\nabla u_{c_1}|^2\right)^2}\\
&= \frac{a}{4}\int_{\mathbb{R}^N}|\nabla u_{c_1}|^2 = \gamma(c_1),
\end{aligned}$$

so the lemma is proved. □

Lemma 3.2. *The function $c \mapsto \gamma(c)$ is continuous on $(c^*, +\infty)$.*

Proof. By Lemma 3.1, to prove this lemma is equivalent to show that for any $c > c^*$ and any sequence $c_n \to c^-$, we have

$$\lim_{c_n \to c^-} \gamma(c_n) \le \gamma(c). \tag{3.2}$$

By Lemmas 1.1 and 2.2, there exists $u_c \in S_c$ such that

$$\gamma(c) = I(u_c) = \frac{a}{4}\int_{\mathbb{R}^N}|\nabla u_c|^2 \text{ and } G(u_c) = 0. \tag{3.3}$$

Since $\frac{c_n}{c} \to 1^-$, for n large enough we see that

$$b\left(\int_{\mathbb{R}^N}|\nabla u_c|^2\right)^2 < \left(\frac{c_n}{c}\right)^{\frac{8-2N}{N}}\frac{2N}{N+4}\int_{\mathbb{R}^N}|u_c|^{\frac{2N+8}{N}}.$$

Set $v_n = \frac{c_n}{c}u_c$, then $v_n \in E_{c_n}$. So by Lemma 2.4, we have

$$\gamma(c_n) \le \frac{1}{4}\frac{a^2\left(\int_{\mathbb{R}^N}|\nabla u_c|^2\right)^2}{\left(\frac{c_n}{c}\right)^{\frac{8-2N}{N}}\frac{2N}{N+4}\int_{\mathbb{R}^N}|u_c|^{\frac{2N+8}{N}} - b\left(\int_{\mathbb{R}^N}|\nabla u_c|^2\right)^2}.$$

Then we conclude from (3.3) that (3.2) holds. □

To obtain the concentration behavior of u_c as $c \to (c^*)^+$, we need the following lemmas:

Lemma 3.3. *For each $c > c^*$ and let $(u_c, \lambda_c) \in S_c \times \mathbb{R}_-$ be the couple of solution obtained in Lemma 1.1. Then u_c is positive.*

Proof. By Lemma 2.2, we see that $u_c \in M_c$. Since $|\nabla|u_c||_2 \leq |\nabla u_c|_2$, we have $|u_c| \in E_c$ and $G(|u_c|) \leq 0$. By Lemma 2.3, there exists a unique $t \in (0,1]$ such that $G(|u_c|^t) = 0$, i.e., $|u_c|^t \in M_c$. Hence by Corollary 2.5, we have

$$\gamma(c) \leq \frac{at^2}{4} \int_{\mathbb{R}^N} |\nabla|u_c||^2 \leq \frac{a}{4} \int_{\mathbb{R}^N} |\nabla u_c|^2 = I(u_c) = \gamma(c),$$

which implies that $t = 1$ and

$$|\nabla|u_c||_2 = |\nabla u_c|_2 \text{ and } I(|u_c|) = \gamma(c). \tag{3.4}$$

Then $|u_c|$ is a minimizer of $I(u)$ on M_c. So by (3.4) and Theorem 1.3, we know that $(|u_c|, \lambda_c)$ also satisfies the Eq. (1.7). So we may assume that u_c does not change sign, i.e., $u_c \geq 0$. By using the strong maximum principle and standard arguments, see, e.g., [4,14,22,24,25], we obtain that $u_c(x) > 0$ for all $x \in \mathbb{R}^N$. □

Lemma 3.4. *([28], Vanishing Lemma) Let $r > 0$ and $2 \leq q < 2^*$. If $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and*

$$\sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |u_n|^q \to 0, \ n \to +\infty,$$

then $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$.

3.1. Proof of Proposition 1.4 and Theorem 1.5

Proof. The proof of Proposition 1.4 (i) (ii) has been given above. We complete the rest proof in three steps.

Step 1. $\begin{cases} \gamma(c) \to 0, \\ |\nabla u_c|_2 \to 0, \quad \text{as } c \to +\infty. \\ \lambda_c \to 0, \end{cases}$

By Lemma 2.2, it is enough to prove that $\gamma(c) \to 0$ as $c \to +\infty$. Recall that $\gamma(c) > 0$ for each $c > c^*$. For Q given in (1.3), we have

$$\int_{\mathbb{R}^N} |\nabla Q|^2 = \int_{\mathbb{R}^N} |Q|^2 = \frac{N}{N+4} \int_{\mathbb{R}^N} |Q|^{\frac{2N+8}{N}}$$

and $\frac{c}{|Q|_2} Q \in E_c$. Then by Lemma 2.4, we see that

$$\gamma(c) \leq \frac{a^2}{4b} \frac{1}{(\frac{c}{c^*})^{\frac{8-2N}{N}} - 1} \to 0$$

as $c \to +\infty$. So $\gamma(c) \to 0$ as $c \to +\infty$.

Step 2. $\begin{cases} \gamma(c) \to +\infty, \\ |\nabla u_c|_2 \to +\infty, \quad \text{as } c \to (c^*)^+. \\ \lambda_c \to -\infty, \end{cases}$

For any $c > c^*$ and any $u \in M_c$, by (1.2) we see that

$$a \int_{\mathbb{R}^N} |\nabla u|^2 \leq \left[(\frac{c}{c^*})^{\frac{8-2N}{N}} - 1 \right] b \left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^2,$$

which implies that $\int_{\mathbb{R}^N}|\nabla u|^2 \geq \frac{a}{b[(\frac{c}{c^*})^{\frac{8-2N}{N}}-1]}$. Hence

$$I(u)=I(u)-\frac{1}{4}G(u)=\frac{a}{4}\int\limits_{\mathbb{R}^N}|\nabla u|^2 \geq \frac{a^2}{4b}\frac{1}{(\frac{c}{c^*})^{\frac{8-2N}{N}}-1}.$$

By the arbitrary of $u \in M_c$ and Theorem 1.2, we see that $\lim\limits_{c\to(c^*)^+}\gamma(c)=+\infty$. Since $u_c \in M_c$, $\lim\limits_{c\to(c^*)^+}|\nabla u_c|_2=+\infty$. By Lemma 2.2 again, we see that $\lambda_c \to -\infty$ as $c \to (c^*)^+$.

Step 3. The concentration of $\{u_c\}$ as $c \to (c^*)^+$.

For any sequence $\{c_k\} \subset (c^*, +\infty)$ with $c_k \to (c^*)^+$ as $k \to +\infty$, by Lemmas 1.1 and 2.2, there exists a sequence $\{u_{c_k}\} \subset S_{c_k}$ such that $\gamma(c_k)=\frac{a}{4}\int_{\mathbb{R}^N}|\nabla u_{c_k}|^2$ and

$$G(u_{c_k})=0, \tag{3.5}$$

By Step 1 and Step 2, we see that

$$\frac{a}{4}\int\limits_{\mathbb{R}^N}|\nabla u_{c_k}|^2=\gamma(c_k)=\frac{a^2}{4b}\frac{1}{(\frac{c_k}{c^*})^{\frac{8-2N}{N}}-1} \to +\infty \text{ as } k \to +\infty.$$

Then (3.5) implies that

$$\lim_{k\to+\infty}\frac{\frac{N}{2N+8}\int \_\mathbb{R}^N|u_{c_k}|^{\frac{2N+8}{N}}}{\frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla u_{c_k}|^2\right)^2}=1.$$

Let

$$\varepsilon_{c_k}:=\sqrt[4]{\frac{4b}{a^2}}\left(\left(\frac{c_k}{c^*}\right)^{\frac{8-2N}{N}}-1\right)^{\frac{1}{2}} \to 0$$

as $k \to +\infty$. Set $\tilde{v}_{c_k}(x):=\varepsilon_{c_k}^{\frac{N}{2}}u_{c_k}(\varepsilon_{c_k}x)$, then $\tilde{v}_{c_k} \in S_{c_k}$ and

$$\int\limits_{\mathbb{R}^N}|\nabla \tilde{v}_{c_k}|^2=\frac{2}{\sqrt{b}} \text{ and } \frac{N}{2N+8}\int\limits_{\mathbb{R}^N}|\tilde{v}_{c_k}|^{\frac{2N+8}{N}} \to 1 \text{ as } k \to +\infty. \tag{3.6}$$

So $\{\tilde{v}_{c_k}\}$ is uniformly bounded in $H^1(\mathbb{R}^N)$.

Let $\delta=\lim\limits_{k\to+\infty}\sup\limits_{y\in\mathbb{R}^N}\int_{B_1(y)}|\tilde{v}_{c_k}|^2$. If $\delta=0$, then the Vanishing Lemma 3.4, $\tilde{v}_{c_k} \to 0$ in $L^{\frac{2N+8}{N}}(\mathbb{R}^N)$, which is a contradiction with (3.6). So $\delta>0$. Then there exists a sequence $\{y_{c_k}\} \subset \mathbb{R}^N$ such that $\int_{B_1(y_{c_k})}|\tilde{v}_{c_k}|^2 \geq \frac{\delta}{2}>0$. Set

$$v_{c_k}(x)=\tilde{v}_{c_k}(x+y_{c_k})=\varepsilon_{c_k}^{\frac{N}{2}}u_{c_k}(\varepsilon_{c_k}x+\varepsilon_{c_k}y_{c_k}),$$

then $\|v_{c_k}\|=\|\tilde{v}_{c_k}\|$ and

$$\int\limits_{B_1(0)}|v_{c_k}|^2 \geq \frac{\delta}{2}.$$

So there exists $v \in H^1(\mathbb{R}^N)\backslash\{0\}$ such that

$$v_{c_k} \rightharpoonup v \text{ in } H^1(\mathbb{R}^N) \text{ and } v_{c_k}(x) \to v(x) \text{ a.e. in } \mathbb{R}^N. \tag{3.7}$$

Since u_{c_k} satisfies the following equation:

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u_{c_k}|^2\right)\Delta u_{c_k}-\lambda_{c_k}u_{c_k}=|u_{c_k}|^{\frac{8}{N}}u_{c_k},\ x \in \mathbb{R}^N,$$

where by Lemma 2.2 $\lambda_{c_k} = -\frac{4-N}{2Nc_k^2}[a\int_{\mathbb{R}^N} |\nabla u_{c_k}|^2 + b(\int_{\mathbb{R}^N} |\nabla u_{c_k}|^2)^2]$, v_{c_k} is a solution of the equation:

$$-a\varepsilon_{c_k}^2 \Delta v_{c_k} - 2\sqrt{b}\Delta v_{c_k} - \lambda_{c_k}\varepsilon_{c_k}^4 v_{c_k} = |v_{c_k}|^{\frac{8}{N}} v_{c_k},\ x \in \mathbb{R}^N. \tag{3.8}$$

By the definition of ε_{c_k}, we conclude that $\lim\limits_{k\to+\infty} \lambda_{c_k}\varepsilon_{c_k}^4 = -\frac{2(4-N)}{N(c^*)^2}$. Let $k \to +\infty$ in (3.8), then v is a nontrivial solution of

$$-\Delta v + \frac{4-N}{N\sqrt{b}(c^*)^2} v = \frac{1}{2\sqrt{b}}|v|^{\frac{8}{N}} v,\ x \in \mathbb{R}^N. \tag{3.9}$$

By Lemma 3.3, we have u_{c_k} is positive, and then, by (3.7) we see that $v(x) \geq 0$ for all $x \in \mathbb{R}^N$. Then by the maximum principle, v is a positive solution of (3.9). So by a rescaling together with the uniqueness of positive solutions of (1.3) (up to translations), we conclude that

$$v(x) = \left(\frac{\sqrt{2}}{c^*}\right)^{\frac{N}{4}} Q\left(\frac{\sqrt{2}}{b^{\frac{1}{4}}c^*}x\right).$$

Then by the definition of c^*, we have $|v|_2 = (c^*)^{\frac{N}{4}} \left(\frac{b}{2}\right)^{\frac{N}{8}} |Q|_2 = c^*$. So $v_{c_k} \to v$ in $L^2(\mathbb{R}^N)$. Hence by the Gagliardo–Nirenberg inequality (1.2) we see that

$$\varepsilon_{c_k}^{\frac{N}{2}} u_{c_k}(\varepsilon_{c_k}x + \varepsilon_{c_k}y_{c_k}) \to \left(\frac{\sqrt{2}}{c^*}\right)^{\frac{N}{4}} Q\Big(\frac{\sqrt{2}}{b^{\frac{1}{4}}c^*}x\Big)$$

in $L^p(\mathbb{R}^N)$ for all $2 \leq p < 2^*$. □

4. Proof of Theorem 1.6

Proof. Suppose that w is a least energy solution of problem (1.7), by Lemma 2.1 we have $G(w) = 0$. Hence

$$-\lambda_c \int\limits_{\mathbb{R}^N} |w|^2 = \frac{4-N}{2N}\left[a\int\limits_{\mathbb{R}^N} |\nabla w|^2 + b\left(\int\limits_{\mathbb{R}^N} |\nabla w|^2\right)^2\right]. \tag{4.1}$$

So

$$d_c = F_c(w) = \frac{a}{N}\int\limits_{\mathbb{R}^N} |\nabla w|^2 + \frac{(4-N)b}{4N}\left(\int\limits_{\mathbb{R}^N} |\nabla w|^2\right)^2. \tag{4.2}$$

For the couple $(u_c, \lambda_c) \in S_c \times \mathbb{R}_-$ solution of (1.7) obtained in Lemma 1.1, then $F_c'(u_c) = 0$. Hence by Lemma 2.2 (2)(3), we see that

$$d_c \leq F_c(u_c) = I(u_c) - \frac{\lambda_c c^2}{2} = \frac{4}{N}\gamma(c) + \frac{4(4-N)b}{Na^2}\gamma^2(c). \tag{4.3}$$

Moreover, by Lemma 2.2 (2) again, we have

$$d_c \leq \frac{a}{N}\int\limits_{\mathbb{R}^N} |\nabla u_c|^2 + \frac{(4-N)b}{4N}\left(\int\limits_{\mathbb{R}^N} |\nabla u_c|^2\right)^2,$$

which and (4.2) imply that

$$\int\limits_{\mathbb{R}^N} |\nabla w|^2 \leq \int\limits_{\mathbb{R}^N} |\nabla u_c|^2, \tag{4.4}$$

where we have used the fact that the function $h(t) = \frac{a}{N}t + \frac{(4-N)b}{4N}t^2$ is increasing on $[0, +\infty)$. Then by (4.1) and Lemma 2.2 (3), we conclude that $|w|_2 \leq c$. Furthermore, we conclude from Lemma 1.1 (1) that it has to be $|w|_2 > c^*$. So $w \in M_{|w|_2}$. Therefore, it follows from Theorem 1.2 that

$$\gamma(|w|_2) \leq I(w) = I(w) - \frac{1}{4}G(w) = \frac{a}{4}\int\limits_{\mathbb{R}^N} |\nabla w|^2.$$

By (4.4), we have

$$\begin{aligned}\gamma(|w|_2) - \frac{\lambda_c c^2}{2} &\leq \frac{a}{4}\int\limits_{\mathbb{R}^N} |\nabla w|^2 - \frac{\lambda_c c^2}{2} \\ &\leq \frac{a}{4}\int\limits_{\mathbb{R}^N} |\nabla u_c|^2 - \frac{\lambda_c c^2}{2} = \gamma(c) - \frac{\lambda_c c^2}{2},\end{aligned} \tag{4.5}$$

i.e., $\gamma(|w|_2) \leq \gamma(c)$, which implies that $|w|_2 \geq c$. So $w \in S_c$. By (4.5) and (4.2), we see that

$$\gamma(c) - \frac{\lambda_c c^2}{2} \leq \frac{a}{4}\int\limits_{\mathbb{R}^N} |\nabla w|^2 - \frac{\lambda_c}{2}\int\limits_{\mathbb{R}^N} |w|^2 = F_c(w) = d_c,$$

which and (4.3) imply

$$d_c = \frac{4}{N}\gamma(c) + \frac{4(4-N)b}{Na^2}\gamma^2(c) = F_c(u_c)$$

and $I(w) = \gamma(c)$. Therefore, the proof is completed. □

5. Comparison with the nonlinear Schrödinger case

In [10], it is proved that when $\frac{2N+4}{N} < p < \frac{2N}{N-2}$ if $N \geq 3$ and $\frac{2N+4}{N} < p$ if $N = 1, 2$, for any $c > 0$,

$$\widetilde{I}(u) = \frac{1}{2}\int\limits_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{p}\int\limits_{\mathbb{R}^N} |u|^p$$

has at least one critical point $\widetilde{u}_c$ restricted to S_c at the mountain pass level $\widetilde{\gamma}(c)$ with $\widetilde{I}(\widetilde{u}_c) = \widetilde{\gamma}(c)$, where

$$\widetilde{\gamma}(c) = \inf_{g\in\widetilde{\Gamma}(c)} \max_{t\in[0,1]} \widetilde{I}(g(t)) > \max_{g\in\widetilde{\Gamma}(c)} \{\max\{\widetilde{I}(g(0)), \widetilde{I}(g(1))\}\}$$

and $\widetilde{\Gamma}(c) = \{g \in C([0,1], S_c)|\ g(0) \in \widetilde{B}_{K(c)},\ \widetilde{I}(g(1)) < 0\}$ for some $\widetilde{B}_{k(c)} > 0$. Moreover, there exists $\widetilde{\lambda}_c < 0$ such that $(\widetilde{u}_c, \widetilde{\lambda}_c)$ satisfies the following equation:

$$-\Delta u - |u|^{p-2}u = \widetilde{\lambda}_c u,\ x \in \mathbb{R}^N. \tag{5.1}$$

Since $p > \frac{2N+4}{N}$, it is shown in [3] that

$$\widetilde{\gamma}(c) = \inf_{u\in\widetilde{M}_c} \widetilde{I}(u) = \inf_{u\in S_c} \max_{t>0} \widetilde{I}(u^t), \tag{5.2}$$

where $\widetilde{M}_c$ is a natural constraint defined as $\widetilde{M}_c = \{u \in S_c|\ \widetilde{G}(u) = 0\}$ with

$$\widetilde{G}(u) = \int\limits_{\mathbb{R}^N} |\nabla u|^2 - \frac{N(p-2)}{2p}\int\limits_{\mathbb{R}^N} |u|^p.$$

Based on the above results, we now prove

Lemma 5.1. *(1)* $\widetilde{G}(\widetilde{u}_c) = 0$;
(2) $\widetilde{\gamma}(c) = \frac{N(p-2)-4}{2N(p-2)}\int_{\mathbb{R}^N} |\nabla \widetilde{u}_c|^2$;

(3) $\widetilde{\lambda}_c = -\frac{4N-2p(N-2)}{[N(p-2)-4]c^2}\widetilde{\gamma}(c)$.

Proof. By using the Pohozaev identity, the proof of this lemma is similar to that of Lemma 2.2. So we omit it. □

Lemma 5.2. *For any $c > 0$,*

$$\widetilde{\gamma}(c) = \frac{N(p-2)-4}{2N(p-2)}\left(\frac{4}{N(p-2)}\right)^{\frac{4}{N(p-2)-4}} \frac{|Q_p|_2^{\frac{4(p-2)}{N(p-2)-4}}}{c^{\frac{4N-2p(N-2)}{N(p-2)-4}}}.$$

Proof. The proof is similar to that of Theorem 1.5 but with more complicated calculations.

For any $c > 0$, let $u := \frac{c}{|Q_p|_2}Q_p$, where Q_p was given in (1.3), then $u \in S_c$ and by (1.2) we have

$$\int_{\mathbb{R}^N} |\nabla u|^2 = c^2, \quad \int_{\mathbb{R}^N} |u|^p = \frac{pc^p|Q_p|_2^{2-p}}{2}. \tag{5.3}$$

Since $p > \frac{2N+4}{N}$, there exists a unique $\widetilde{t} = \widetilde{t}(u) > 0$ such that $u^{\widetilde{t}} \in \widetilde{M}_c$, i.e.,

$$\widetilde{t}^{\frac{N(p-2)-4}{2}} = \frac{4}{N(p-2)}\left(\frac{|Q_p|_2}{c}\right)^{p-2}. \tag{5.4}$$

Then we see from (1.2)(1.3) and (5.2)–(5.4) that

$$\widetilde{\gamma}(c) \le \widetilde{I}(u^{\widetilde{t}}) = \widetilde{I}(u^{\widetilde{t}}) - \frac{2}{N(p-2)}\widetilde{G}(u^{\widetilde{t}}) = \frac{N(p-2)-4}{2N(p-2)}\left(\frac{4}{N(p-2)}\right)^{\frac{4}{N(p-2)-4}} \frac{|Q_p|_2^{\frac{4(p-2)}{N(p-2)-4}}}{c^{\frac{4N-2p(N-2)}{N(p-2)-4}}}.$$

On the other hand, there exists $\widetilde{u}_c \in S_c$ such that $\widetilde{I}(\widetilde{u}_c) = \widetilde{\gamma}(c)$. Moreover, Lemma 5.1 shows that $\widetilde{u}_c \in \widetilde{M}_c$, hence by (1.2) we see that

$$|\nabla \widetilde{u}_c|_2^2 \le \frac{N(p-2)}{4}\frac{c^{p-\frac{N(p-2)}{2}}}{|Q_p|_2^{p-2}}|\nabla \widetilde{u}_c|_2^{\frac{N(p-2)}{2}}.$$

So

$$\gamma(c) = \frac{N(p-2)-4}{2N(p-2)}\int_{\mathbb{R}^N} |\nabla \widetilde{u}_c|^2 \ge \frac{N(p-2)-4}{2N(p-2)}\left(\frac{4}{N(p-2)}\right)^{\frac{4}{N(p-2)-4}} \frac{|Q_p|_2^{\frac{4(p-2)}{N(p-2)-4}}}{c^{\frac{4N-2p(N-2)}{N(p-2)-4}}}.$$

Then the lemma is proved. □

Lemma 5.3. *$\widetilde{u}_c$ is positive.*

Proof. The proof is simpler than that of Lemma 3.3 since it is proved in [3] that each minimizer of $\widetilde{I}|_{\widetilde{M}_c}$ is a critical point of $\widetilde{I}|_{S_c}$. □

Lemma 5.4. *Up to translations, $\widetilde{u}_c$ is the unique positive least energy solution of* (5.1) *and*

$$\widetilde{u}_c(x) = \left(\frac{-4\widetilde{\lambda}_c}{2N-(N-2)p}\right)^{\frac{1}{p-2}} Q_p\left(\sqrt{\frac{-N(p-2)\widetilde{\lambda}_c}{2N-(N-2)p}}x\right),$$

where $\widetilde{\lambda}_c$ is given in (5.1).

Proof. By Lemma 5.3, $\widetilde{u}_c$ is positive. Then by [8,13,15,21], $\widetilde{u}_c$ is the unique (up to translations) positive least energy solution of (5.1).

Let $v(x) := \theta\widetilde{u}_c(\rho x)$, where $\theta = \left(\frac{-4\widetilde{\lambda}_c}{2N-(N-2)p}\right)^{-\frac{1}{p-2}}$ and $\rho = (\frac{-N(p-2)\widetilde{\lambda}_c}{2N-(N-2)p})^{-\frac{1}{2}}$, then by direct calculation, we see that v is a positive solution of the Eq. (1.3). Hence up to translations, $v = Q_p$. □

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