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# Zero Mach number limit of the compressible Hall-magnetohydrodynamic equations

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**Abstract.** In this paper, we study the low Mach number limit of the compressible Hall-magnetohydrodynamic equations. It is justified rigorously that, for the well-prepared initial data, the classical solutions of the compressible Hall-magnetohydrodynamic equations converge to that of the incompressible Hall-magnetohydrodynamic equations as the Mach number tends to zero.

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**Keywords.** Compressible Hall-magnetohydrodynamic equations · Incompressible Hall-magnetohydrodynamic equations · Low Mach number limit.

## 1. Introduction

In this paper, we study the following compressible Hall-magnetohydrodynamic (Hall-MHD) equations (see [1]):

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = (\nabla \times H) \times H + \nu \operatorname{div}\left(\nabla u + (\nabla u)^\top\right), \tag{1.2}$$

$$\partial_t H + \nabla \times \left( H \times u + \frac{(\nabla \times H) \times H}{\rho} \right) = -\mu \nabla \times (\nabla \times H), \quad \operatorname{div} H = 0 \tag{1.3}$$

with the initial data

$$(\rho, u, H)(x, 0) = (\rho_0, u_0, H_0)(x), \quad x \in \Omega. \tag{1.4}$$

Here the unknowns are $\rho$, $u = (u_1, u_2, u_3) \in \mathbb{R}^3$, $H = (H_1, H_2, H_3) \in \mathbb{R}^3$ denoting the density of the fluid, the fluid velocity field and the magnetic field, respectively. The pressure $P(\rho) := a\rho^\gamma$ with positive constants $a$ and $\gamma \geq 1$ for simplicity. The spatial domain $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$, a periodic domain in $\mathbb{R}^3$. The parameter $\nu > 0$ denotes the viscous coefficient and $\mu > 0$ the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field. For simplicity, here we assume that both $\nu$ and $\mu$ are positive constants, independent of the magnitude and direction of the magnetic field.

The Hall-MHD equtions can be used to describe many physical phenomena, for example, magnetic reconnection in space plasmas, star formation, neutron stars, and geo-dynamo [24]. Recently, the existence of local strong solutions with positive density, and the global existence and time decay rate of smooth solutions to the system (1.1)–(1.3) were obtained in [11].

If the Hall effect term $\frac{(\nabla \times H) \times H}{\rho}$ in (1.3) is neglected, then the system (1.1)–(1.3) reduces to the well-known compressible isentropic MHD equations and there are a lot of results on it, see [6,13–17,21–23,25] and the references cited therein. The local strong solution was established in [22], the global existence of smooth solution was obtained in [21] and the global weak solution was given in [14] for large initial data and in [25] for small initial data. The zero Mach number limit to the compressible isentropic MHD equations was studied in [6,13,15–17,23] under different situations.

To the author's best knowledge, there are no results on the low Mach number limit of the compressible Hall-MHD system (1.1)–(1.3). In this paper, we shall study this topic by applying the methods developed in [18]. To begin with, we need to introduce some scaling transformations on the unknowns. Denoting $\epsilon$ the (scaled) Mach number, introducing the scales

$$\rho(x,t) = \rho^\epsilon(x,\epsilon t), \quad u(x,t) = \epsilon u^\epsilon(x,\epsilon t), \quad H(x,t) = \epsilon H^\epsilon(x,\epsilon t),$$

and utilizing the identities

$$\begin{aligned} \nabla\left(|H|^2\right) &= 2(H\cdot\nabla)H + 2H\times(\nabla\times H), \\ \nabla\times(\nabla\times H) &= \nabla \operatorname{div} H - \Delta H, \\ \nabla\times(u\times H) &= u(\operatorname{div} H) - H(\operatorname{div} u) + (H\cdot\nabla)u - (u\cdot\nabla)H, \end{aligned}$$

we can rewrite the system (1.1)–(1.3) as

$$\partial_t\rho^\epsilon + \operatorname{div}\,(\rho^\epsilon u^\epsilon) = 0, \tag{1.5}$$

$$\rho^\epsilon\left(\partial_t u^\epsilon + u^\epsilon\cdot\nabla u^\epsilon\right) + \frac{a\nabla(\rho^\epsilon)^\gamma}{\epsilon^2} = (\nabla\times H^\epsilon)\times H^\epsilon - \nu\nabla\cdot\left(\nabla u^\epsilon + (\nabla u^\epsilon)^\top\right), \tag{1.6}$$

$$\partial_t H^\epsilon + \nabla\times\left(H^\epsilon\times u^\epsilon + \frac{(\nabla\times H^\epsilon)\times H^\epsilon}{\rho^\epsilon}\right) = -\mu\nabla\times(\nabla\times H^\epsilon), \quad \operatorname{div} H^\epsilon = 0. \tag{1.7}$$

We shall study the limit of the solution to the system (1.5)–(1.7) as $\epsilon\to 0$. We restrict ourselves to the small density variations, i.e.,

$$\rho^\epsilon := 1 + \epsilon q^\epsilon. \tag{1.8}$$

Putting (1.8) into the system (1.5)–(1.7), then we can rewrite it as

$$\partial_t q^\epsilon + u^\epsilon\cdot\nabla q^\epsilon + \frac{(1+\epsilon q^\epsilon)}{\epsilon}\operatorname{div} u^\epsilon = 0, \tag{1.9}$$

$$\partial_t u^\epsilon + u^\epsilon\cdot\nabla u^\epsilon + \frac{a\gamma\left(1+\epsilon q^\epsilon\right)^{\gamma-1}\nabla q^\epsilon}{\epsilon\left(1+\epsilon q^\epsilon\right)} = \frac{1}{1+\epsilon q^\epsilon}\left((\nabla\times H^\epsilon)\times H^\epsilon + \nu\nabla\cdot\left(\nabla u^\epsilon + (\nabla u^\epsilon)^{\mathrm{T}}\right)\right), \tag{1.10}$$

$$\partial_t H^\epsilon + \nabla\times\left(H^\epsilon\times u^\epsilon + \frac{(\nabla\times H^\epsilon)\times H^\epsilon}{1+\epsilon q^\epsilon}\right) = -\mu\nabla\times(\nabla\times H^\epsilon), \quad \operatorname{div} H^\epsilon = 0. \tag{1.11}$$

The system (1.9)–(1.11) is equipped with the initial data

$$(q^\epsilon, u^\epsilon, H^\epsilon)|_{t=0} = \left(q_0^\epsilon(x), u_0^\epsilon(x), H_0^\epsilon(x)\right), \qquad x\in\Omega. \tag{1.12}$$

Formally if we take the limit $\epsilon\to 0$ in (1.9)–(1.11), we then obtain the following incompressible Hall-MHD equations [suppose that $(u^\epsilon, H^\epsilon)$ converges to $(v,B)$ as $\epsilon\to 0$].

$$\partial_t v + v\cdot\nabla v + \nabla\pi = (\nabla\times B)\times B + \nu\Delta v, \tag{1.13}$$

$$\partial_t B + \nabla\times(B\times v + (\nabla\times B)\times B) = \mu\Delta B, \tag{1.14}$$

$$\operatorname{div} v = 0, \quad \operatorname{div} B = 0. \tag{1.15}$$

The system (1.13)–(1.15) is supplemented with initial data

$$(v,B)|_{t=0} = \left(v_0(x), B_0(x)\right), \quad x\in\Omega. \tag{1.16}$$

In this paper, we shall establish the above limit rigorously. Moreover we shall show that for sufficiently small Mach number, the compressible Hall-MHD system (1.9)–(1.11) admits a smooth solution on the time interval where the smooth solution of the incompressible Hall-MHD equations (1.13)–(1.15) exists.

Before stating our main results, we recall some known results on the incompressible Hall-MHD equations (1.13)–(1.15). The existence of global weak solutions was first obtained in [1], see also [8]. The local existence of smooth solutions was established in [3]. The temporal decay estimates for weak solutions was obtained in [4]. The well-posedness for the axisymmetric incompressible viscous Hall-MHD equations was studied in [9]. Many authors studied the singularity formations [5,7] and the regularity criteria [2,10,12] of the incompressible Hall-MHD equations.

Below we first recall the local existence result on the equations (1.13)–(1.16).

**Proposition 1.1.** (see [3]). *Let $s > 7/2$ be an integer. Assume that the initial data $(v_0(x), B_0(x))$ satisfy $v_0, B_0 \in H^s(\Omega)$, and $\operatorname{div} v_0 = 0$, $\operatorname{div} B_0 = 0$. Then there exist a $T^* \in (0, \infty]$ and a unique solution $(v, B) \in L^\infty([0, T^*), H^s)$ to the incompressible Hall-MHD equations (1.13)–(1.16) satisfying, for any $0 < T < T^*$, $\operatorname{div} v = 0$, $\operatorname{div} B = 0$, and*

$$\sup_{0\le t\le T}\{\|(v,B)(t)\|_{H^s} + \|(\partial_t v, \partial_t B)(t)\|_{H^{s-2}} + \|\nabla\pi(t)\|_{H^{s-2}}\} \le C_T. \tag{1.17}$$

The main result of the present paper is the following.

**Theorem 1.1.** *Let $s > 7/2$ be an integer. Let $(v, B, \pi)$ be a smooth solution to the system (1.13)–(1.15) with the initial data $(v_0, B_0)$ obtained in* Proposition 1.1. *Suppose that the initial data $(q_0^\epsilon, u_0^\epsilon, H_0^\epsilon)$ belong to $H^s$ and satisfy*

$$\|(q_0^\epsilon, u_0^\epsilon - v_0, H_0^\epsilon - B_0)\|_s = O(\epsilon). \tag{1.18}$$

*Then there exists a constant $\epsilon_0 > 0$ such that, for all $\epsilon \in (0, \epsilon_0]$, the problem (1.9)–(1.12) has a unique smooth solution $(q^\epsilon, u^\epsilon, H^\epsilon) \in C([0, T], H^s)$ for any $0 < T < T^*$. Moreover there exists a positive constant $K > 0$, independent of $\epsilon$, such that, for all $\epsilon \le \epsilon_0$ and any $0 < T < T^*$,*

$$\sup_{t\in[0,T]}\left\|\left(q^\epsilon - \frac{\epsilon}{a\gamma}\pi, u^\epsilon - v, H^\epsilon - B\right)(t)\right\|_s \le K\epsilon. \tag{1.19}$$

The proof of Theorem 1.1 is based on the methods developed in [18]. The key point is to obtain the uniform estimates of the error system and apply convergence-stability lemma for general hyperbolic–parabolic system [18] and Growall-type inequality. Compared with the compressible MHD equations, the compressible Hall-MHD equations (1.9)–(1.12) are more complicated and more refined analysis are needed in our arguments. The appearance of the Hall effect term $\frac{\nabla\times H\times H}{\rho}$ brings us a lot of trouble. For example, in the error estimates, we need to finely divide $Q_3$ into seven parts. We fortunately observe that $I_5 = 0$, which is critical in the uniform estimates. Meanwhile, we need to deal with $I_6$ and $I_7$ skillfully, and the other terms also need to be treated very carefully. We shall explain these in detail later.

Before ending the introduction, we give the notations used throughout the current paper. We use the letter $C$ to denote various positive constants independent of $\epsilon$. For convenience, we denote by $H^l \equiv H^l(\Omega)$ ($l \in \mathbb{R}$) the standard Sobolev spaces and write $\|\cdot\|_l$ for the standard norm of $H^l$ and $\|\cdot\|$ for $\|\cdot\|_0$.

In next section, we reformulate our problem in vector form and a convergence-stability lemma. In Sect. 3, we present the proof of Theorem 1.1.

## 2. Reformulation of our problem

Now we begin our proof of Theorem 1.1. Setting $U^\epsilon = (q^\epsilon, u^\epsilon, H^\epsilon)^\top$, we can rewrite the system (1.9)–(1.11) in the vector form:

$$A_0\left(U^\epsilon\right)\partial_t U^\epsilon + \sum_{j=1}^{3} A_j(U^\epsilon)\partial_j U^\epsilon = Q\left(U^\epsilon\right), \tag{2.1}$$

where the matrices $A_j(U^\epsilon)$ $(0 \le j \le 3)$ are given by

$$A_0\left(U^\epsilon\right) = \operatorname{diag}\left(1, 1+\epsilon q^\epsilon, 1+\epsilon q^\epsilon, 1+\epsilon q^\epsilon, 1, 1, 1\right),$$

$$A_1\left(U^\epsilon\right) = \begin{pmatrix} u_1^\epsilon & \frac{1+\epsilon q^\epsilon}{\epsilon} & 0 & 0 & 0 & 0 & 0 \\ \frac{a\gamma(1+\epsilon q^\epsilon)^{\gamma-1}}{\epsilon} & u_1^\epsilon\left(1+\epsilon q^\epsilon\right) & 0 & 0 & 0 & H_2^\epsilon & H_3^\epsilon \\ 0 & 0 & u_1^\epsilon\left(1+\epsilon q^\epsilon\right) & 0 & 0 & -H_1^\epsilon & 0 \\ 0 & 0 & 0 & u_1^\epsilon\left(1+\epsilon q^\epsilon\right) & 0 & 0 & -H_1^\epsilon \\ 0 & 0 & 0 & 0 & u_1^\epsilon & 0 & 0 \\ 0 & H_2^\epsilon & -H_1^\epsilon & 0 & 0 & u_1^\epsilon & 0 \\ 0 & H_3^\epsilon & 0 & -H_1^\epsilon & 0 & 0 & u_1^\epsilon \end{pmatrix},$$

$$A_2(U^\epsilon) = \begin{pmatrix} u_2^\epsilon & 0 & \frac{1+\epsilon q^\epsilon}{\epsilon} & 0 & 0 & 0 & 0 \\ 0 & u_2^\epsilon\left(1+\epsilon q^\epsilon\right) & 0 & 0 & -H_2^\epsilon & 0 & 0 \\ \frac{a\gamma(1+\epsilon q^\epsilon)^{\gamma-1}}{\epsilon} & 0 & u_2^\epsilon\left(1+\epsilon q^\epsilon\right) & 0 & H_1^\epsilon & 0 & H_3^\epsilon \\ 0 & 0 & 0 & u_2^\epsilon\left(1+\epsilon q^\epsilon\right) & 0 & 0 & -H_2^\epsilon \\ 0 & -H_2^\epsilon & H_1^\epsilon & 0 & u_2^\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_2^\epsilon & 0 \\ 0 & 0 & H_3^\epsilon & -H_2^\epsilon & 0 & 0 & u_2^\epsilon \end{pmatrix},$$

$$A_3(U^\epsilon) = \begin{pmatrix} u_3^\epsilon & 0 & 0 & \frac{1+\epsilon q^\epsilon}{\epsilon} & 0 & 0 & 0 \\ 0 & u_3^\epsilon\left(1+\epsilon q^\epsilon\right) & 0 & 0 & -H_3^\epsilon & 0 & 0 \\ 0 & 0 & u_3^\epsilon\left(1+\epsilon q^\epsilon\right) & 0 & 0 & -H_3^\epsilon & 0 \\ \frac{a\gamma(1+\epsilon q^\epsilon)^{\gamma-1}}{\epsilon} & 0 & 0 & u_3^\epsilon\left(1+\epsilon q^\epsilon\right) & H_1^\epsilon & H_2^\epsilon & 0 \\ 0 & -H_3^\epsilon & 0 & H_1^\epsilon & u_3^\epsilon & 0 & 0 \\ 0 & 0 & -H_3^\epsilon & H_2^\epsilon & 0 & u_3^\epsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_3^\epsilon \end{pmatrix},$$

and

$$Q\left(U^\epsilon\right) = \left(0, \nu\left(\Delta u^\epsilon + \nabla \operatorname{div} u^\epsilon\right), \mu\Delta H^\epsilon - \nabla\times\left(\frac{(\nabla\times H^\epsilon)\times H^\epsilon}{1+\epsilon q^\epsilon}\right)\right)^\top.$$

It is easy to check that the matrices $A_j(U^\epsilon)$ $(0 \le j \le 3)$ can be symmetrized by choosing the symmetrizier

$$\hat{A}_0\left(U^\epsilon\right) = \operatorname{diag}\left(\frac{a\gamma\left(1+\epsilon\phi^\epsilon\right)^{\gamma-1}}{1+\epsilon q^\epsilon}, 1,\ 1,\ 1,\ 1,\ 1,\ 1\right).$$

Moreover for $U^\epsilon \in \bar{G}_1 \subset\subset G$ with $G$ being the state space for the system (2.1), $\hat{A}_0(U^\epsilon)$ is a positive definite symmetric matrix for sufficiently small $\epsilon$.

Assume that the initial data

$$U^\epsilon(0,x) = U_0^\epsilon(x) := (q_0^\epsilon(x), u_0^\epsilon(x), H_0^\epsilon(x))^\top \in H^s$$

and $U_0^\epsilon(x) \in G_0$, $\bar{G}_0 \subset\subset G$.

First, following the proof of the local existence theory for the initial value problem of symmetrizable hyperbolic–parabolic systems by Volpert and Hudjaev [26], we obtain that there exists a time interval $[0,T]$ with $T>0$, so that the system (2.1) with the initial data $U_0^\epsilon(x)$ has a unique classical solution $U^\epsilon(t,x) \in C([0,T], H^s)$ and $U^\epsilon(t,x) \in G_2$ with $\bar{G}_2 \subset\subset G$. We remark that the crucial step in the proof of local existence result is to prove the uniform boundedness of the solutions.

Now we define

$$T_\epsilon = \sup\left\{T>0 : U^\epsilon(t,x) \in C\left([0,T], H^s\right), U^\epsilon(t,x) \in G_2, \quad \forall (t,x) \in [0,T]\times \Omega\right\}.$$

Note that $T_\epsilon$ depends on $\epsilon$ and may tend to zero as $\epsilon$ goes to 0.

With the aid of the convergence-stability lemma for general hyperbolic–parabolic system [18], we shall show that $\liminf_{\epsilon\to 0} T_\epsilon > 0$. Similar to [18], for the (2.1), we have the following convergence-stability lemma.

**Lemma 2.1.** *Let $s > 3/2+2$. Suppose that $U_0^\epsilon(x) \in G_0, \bar{G}_0 \subset\subset G$, and $U_0^\epsilon(x) \in H^s$, and the following convergence assumption $(A)$ holds.*

*$(A)$ For each $\epsilon$, there exists $T_\star > 0$ and $U_\epsilon \in L^\infty(0,T_\star; H^s)$ satisfying*

$$\bigcup_{x,t,\epsilon} \{U_\epsilon(t,x)\} \subset\subset G,$$

*such that, for $t \in [0, \min\{T_\star, T_\epsilon\})$,*

$$\sup_{x,t} |U^\epsilon(t,x) - U_\epsilon(t,x)| = o(1), \quad \sup_t \|U^\epsilon(t,x) - U_\epsilon(t,x)\|_s = O(1), \quad \text{as } \epsilon \to 0.$$

*Then there exist an $\bar{\epsilon} > 0$ such that, for all $\epsilon \in (0,\bar{\epsilon}]$, it holds that*

$$T_\epsilon > T_\star.$$

In order to apply Lemma 2.1 to our problem (2.1), we need to structure the approximation $U_\epsilon = (q_\epsilon, v_\epsilon, B_\epsilon)^\top$ with $q_\epsilon = \epsilon\pi/a\gamma, v_\epsilon = v, B_\epsilon = B$, where $(v,B,\pi)$ is the smooth solution to the system (1.13)–(1.15). We easily verify that $U_\epsilon$ satisfies

$$\partial_t q_\epsilon + v_\epsilon \cdot \nabla q_\epsilon + \frac{(1+\epsilon q_\epsilon)}{\epsilon} \operatorname{div} v_\epsilon = \frac{\epsilon}{a\gamma}\left(\pi_t + v\cdot\nabla\pi\right), \tag{2.2}$$

$$\begin{aligned}(1+\epsilon q^\epsilon)\left(\partial_t v_\epsilon + v_\epsilon\cdot\nabla v_\epsilon\right) &+ \frac{1}{\epsilon} a\gamma \left(1+\epsilon q^\epsilon\right)^{\gamma-1} \nabla q^\epsilon - (\nabla\times B_\epsilon)\times B_\epsilon \\ &= \nu\Delta v_\epsilon + \left[\left(1+\frac{\epsilon^2}{a\gamma}\pi\right)^{\gamma-1} - 1\right]\nabla\pi + \frac{\epsilon^2}{a\gamma}\pi\left(\partial_t v + v\cdot\nabla v\right),\end{aligned} \tag{2.3}$$

$$\partial_t B_\epsilon + \nabla\times(B_\epsilon\times v_\epsilon) = \mu\Delta B_\epsilon - \nabla\times\left((\nabla\times B_\epsilon)\times B_\epsilon\right), \quad \operatorname{div} B_\epsilon = 0. \tag{2.4}$$

Similar to the system (1.9)–(1.11), we can rewrite the system (2.2)–(2.4) in the following vector form

$$A_0(U_\epsilon)\partial_t U_\epsilon + \sum_{j=1}^3 A_j(U_\epsilon)\partial_j U_\epsilon = S(U_\epsilon) + R \tag{2.5}$$

with

$$S(U_\epsilon) = \left(0, \nu\Delta v_\epsilon, \mu\Delta B_\epsilon - \nabla\times\left((\nabla\times B_\epsilon)\times B_\epsilon\right)\right)^\top,$$

$$R=\begin{pmatrix}\frac{\epsilon}{a\gamma}\left(\pi_t+v\cdot\nabla\pi\right)\\ \frac{\epsilon^2}{a\gamma}\pi\left(v_t+v\cdot\nabla v\right)+\left[\left(1+\frac{\epsilon^2}{a\gamma}\pi\right)^{\gamma-1}-1\right]\nabla\pi\\ 0\end{pmatrix}.$$

With the help of the Moser-type calculus inequalities in Sobolev spaces [19] and the regularity assumptions on $(v,\pi)$ in Theorem 1.1, we can get that there exists some constant $C$ such that, for any $t\in[0,T^*]$,

$$\begin{aligned}\left\|\frac{\epsilon}{a\gamma}\left(\pi_t+v\cdot\nabla\pi\right)\right\|_s&\le C\epsilon,\\ \left\|\frac{\epsilon^2}{a\gamma}\pi\left(v_t+v\cdot\nabla v\right)\right\|_s&\le C\epsilon,\\ \left\|\left[\left(1+\frac{\epsilon^2}{a\gamma}\pi\right)^{\gamma-1}-1\right]\nabla\pi\right\|_s&=\frac{\epsilon^2}{a\gamma}\left\|f'\left(1+\lambda\frac{\epsilon^2}{a\gamma}\pi\right)\pi\nabla\pi\right\|_s\\ &\le C\epsilon^2\left(\|\nabla\pi\|_{s-1}+\|\nabla\pi\|_{s-1}^s\right)\|\pi\nabla\pi\|_s\\ &\le C\epsilon,\end{aligned}$$

where $f(x)=x^{\gamma-1}$ and $0\le\lambda\le1$. Thus we obtain that

$$\max_{t\in[0,T^*]}\|R(t)\|_s\le C\epsilon. \tag{2.6}$$

## 3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Thanks to Lemma 2.1, it suffices to establish the error estimate (1.7) for $t\in[0,\min\{T^*,T_\epsilon\})$. Introducing

$$E=U^\epsilon-U_\epsilon\quad\text{and}\quad\mathcal{A}_j(U)=A_0^{-1}(U)A_j(U),$$

and using (2.1) and (2.5), we have

$$\begin{aligned}E_t+\sum_{j=1}^3\mathcal{A}_j\left(U^\epsilon\right)E_{x_j}=&\sum_{j=1}^3\left(\mathcal{A}_j\left(U_\epsilon\right)-\mathcal{A}_j\left(U^\epsilon\right)\right)U_{\epsilon x_j}+A_0^{-1}\left(U^\epsilon\right)Q\left(U^\epsilon\right)\\ &-A_0^{-1}\left(U_\epsilon\right)\left(S(U_\epsilon)+R\right).\end{aligned}\tag{3.1}$$

Applying the operator $D^\alpha$ to (3.1) for any multi-index $\alpha$ ($|\alpha|\le s$), we obtain that

$$\partial_tD^\alpha E+\sum_{j=1}^3\mathcal{A}_j\left(U^\epsilon\right)\partial_{x_j}D^\alpha E=P_1^\alpha+P_2^\alpha+Q^\alpha+R^\alpha \tag{3.2}$$

with

$$\begin{aligned}P_1^\alpha&=\sum_{j=1}^3\left\{\mathcal{A}_j\left(U^\epsilon\right)\partial_{x_j}D^\alpha E-D^\alpha\left(\mathcal{A}_j\left(U^\epsilon\right)\partial_{x_j}E\right)\right\},\\ P_2^\alpha&=\sum_{j=1}^3D^\alpha\left\{\left(\mathcal{A}_j\left(U_\epsilon\right)-\mathcal{A}_j\left(U^\epsilon\right)\right)U_{\epsilon x_j}\right\},\\ Q^\alpha&=D^\alpha\left\{A_0^{-1}\left(U^\epsilon\right)Q\left(U^\epsilon\right)-A_0^{-1}\left(U_\epsilon\right)S\left(U_\epsilon\right)\right\},\\ Y^\alpha&=D^\alpha\left\{A_0^{-1}\left(U_\epsilon\right)R\right\}.\end{aligned}$$

As in [18], we define the canonical energy by

$$\|E\|_{\mathrm{e}}^2 := \int \left\langle \tilde{A}_0\left(U^\epsilon\right) E, E\right\rangle \mathrm{d}x,$$

where

$$\tilde{A}_0\left(U^\epsilon\right) = \operatorname{diag}\left(\frac{a\gamma\left(1+\epsilon q^\epsilon\right)^{\gamma-1}}{\left(1+\epsilon q^\epsilon\right)^2}, 1, 1, 1, \frac{1}{1+\epsilon q^\epsilon}, \frac{1}{1+\epsilon q^\epsilon}, \frac{1}{1+\epsilon q^\epsilon}\right).$$

We remark that $\tilde{A}_0(U^\epsilon)$ is a positive definite symmetric matrix and $\tilde{A}_0(U^\epsilon)\mathcal{A}_j(U^\epsilon)$ is symmetric. Now, multiplying (3.2) with $\tilde{A}_0(U^\epsilon)$ and taking the inner product between the resulting system and $D^\alpha E$, we obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|D^\alpha E\|_{\mathrm{e}}^2 &= \int \left\langle \mathit{\Gamma} D^\alpha E, D^\alpha E\right\rangle \mathrm{d}x \\ &\quad + 2\int \left(D^\alpha E\right)^{\mathrm{T}} \tilde{A}_0\left(U^\epsilon\right)\left(P_1^\alpha + P_2^\alpha + Q^\alpha + Y^\alpha\right)\mathrm{d}x,\end{aligned} \tag{3.3}$$

where $\mathit{\Gamma}$ is defined as follows:

$$\mathit{\Gamma} := \left(\partial_t, \nabla\right)\cdot\left(\tilde{A}_0, \tilde{A}_0\left(U^\epsilon\right)\mathcal{A}_1\left(U^\epsilon\right), \tilde{A}_0\left(U^\epsilon\right)\mathcal{A}_2\left(U^\epsilon\right), \tilde{A}_0\left(U^\epsilon\right)\mathcal{A}_3\left(U^\epsilon\right)\right).$$

Next, we estimate every term on the right-hand side of (3.3). We point that our estimates only need to be done for $t \in [0, \min\{T^*, T_\epsilon\})$ where both $U^\epsilon$ and $U_\epsilon$ are regular enough and take values in a convex compact subset of the state space. Thus we get

$$C^{-1}\int |D^\alpha E|^2 \mathrm{d}x \le \|D^\alpha E\|_{\mathrm{e}}^2 \le C\int |D^\alpha E|^2 \mathrm{d}x \tag{3.4}$$

for some $C > 0$ and

$$\left|\left(D^\alpha E\right)^\top \tilde{A}_0\left(U^\epsilon\right)\left(P_1^\alpha + P_2^\alpha + Y^\alpha\right)\right| \le C\left(\left|D^\alpha E\right|^2 + \left|P_1^\alpha\right|^2 + \left|P_2^\alpha\right|^2 + \left|Y^\alpha\right|^2\right). \tag{3.5}$$

We first estimate the term $\mathit{\Gamma}$. We can write $\mathcal{A}_j(U^\epsilon) = u_j^\epsilon I_7 + \bar{\mathcal{A}}_j(U^\epsilon)$. Noticing that $\bar{\mathcal{A}}_j(U^\epsilon)$ depends only on $q^\epsilon$ and $H^\epsilon$ and thanks to (1.9), we have

$$\begin{aligned}\mathit{\Gamma} &= \partial_t\tilde{A}_0 + \sum_{j=1}^3 \partial_j\left(\tilde{A}_0\left(U^\epsilon\right)\mathcal{A}_j\left(U^\epsilon\right)\right) \\ &= \partial_t\tilde{A}_0 + \sum_{j=1}^3 \partial_j\left(\tilde{A}_0\left(U^\epsilon\right)u_j^\epsilon\right) + \sum_{j=1}^3 \partial_j\left(\tilde{A}_0\left(U^\epsilon\right)\bar{\mathcal{A}}_j\left(U^\epsilon\right)\right) \\ &= \partial_t\tilde{A}_0 + u^\epsilon\cdot\nabla\tilde{A}_0 + \tilde{A}_0 \operatorname{div} u^\epsilon + \sum_{j=1}^3 \partial_j\left(\tilde{A}_0\left(U^\epsilon\right)\bar{\mathcal{A}}_j\left(U^\epsilon\right)\right) \\ &= -\tilde{A}_0'\left(1+\epsilon q^\epsilon\right)\operatorname{div} u^\epsilon + \tilde{A}_0 \operatorname{div} u^\epsilon + \sum_{j=1}^3 \partial_j\left(\tilde{A}_0\left(U^\epsilon\right)\bar{\mathcal{A}}_j\left(U^\epsilon\right)\right).\end{aligned}$$

Here the symbol $\tilde{A}_0'$ denotes the differentiation of $\tilde{A}_0$ with respect to $\rho^\epsilon$. Therefore, using Sobolev's embedding theorem and the fact that $s > 3/2 + 2$, we have

$$\begin{aligned}|\mathit{\Gamma}| &\le C + C\left(|\nabla q^\epsilon| + |\nabla u^\epsilon| + |\nabla H^\epsilon| + |\nabla q^\epsilon|^2 + |H^\epsilon|^2\right) \\ &\le C + C\left(|\nabla E| + |\nabla E|^2 + |E|^2 + |U_\epsilon|^2 + |\nabla U_\epsilon| + |\nabla U_\epsilon|^2\right) \\ &\le C\left(1 + \|E\|_s^2\right).\end{aligned} \tag{3.6}$$

For the term $P_1^\alpha$, we rewrite it as

$$\begin{aligned}P_1^\alpha &= \sum_{j=1}^{3} \mathcal{A}_j\left(U^\epsilon\right)\partial_{x_j} D^\alpha E - D^\alpha\left(\mathcal{A}_j\left(U^\epsilon\right)\partial_{x_j} E\right)\\
&= -\sum_{j=1}^{3}\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta \mathcal{A}_j\left(U^\epsilon\right)\partial^{\alpha-\beta} E_{x_j}\\
&= -\sum_{j=1}^{3}\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta\left[u_j^\epsilon I_7 + \bar{A}_j\left(U^\epsilon\right)\right]\partial^{\alpha-\beta} E_{x_j}.\end{aligned}$$

Thanks to the Moser-type calculus inequalities in Sobolev spaces [19], we obtain that

$$\begin{aligned}\|P_1^\alpha\| &\le C\left\{\left(1+\|(u^\epsilon,H^\epsilon)\|_s\right)\left\|E_{x_j}\right\|_{|\alpha|-1} + \epsilon^{-1}\left\|\partial^\beta g\left(q^\epsilon\right)\partial^{\alpha-\beta}E_{x_j}\right\|\right\}\\
&\quad + C\left\|\partial^\beta\left[\left(1+\epsilon q^\epsilon\right)^{-1}\left(H^\epsilon - B_\epsilon\right) + \left(\left(1+\epsilon q^\epsilon\right)^{-1} - \left(1+\epsilon q_\epsilon\right)^{-1}\right)B_\epsilon\right]E_{x_j}\right\| + \left\|\left(1+\epsilon q_\epsilon\right)^{-1}B_\epsilon E_{x_j}\right\|\\
&\le C\left(1+\|E\|_s^{s+1}\right)\|E\|_\alpha,\end{aligned} \tag{3.7}$$

where $g(q^\epsilon) := (1+\epsilon q^\epsilon) + a\gamma(1+\epsilon q^\epsilon)^{\gamma-1}$.

For the term $P_2^\alpha$, with the help of the uniform boundedness of $\|U_\epsilon\|_{s+1}$, we have

$$\begin{aligned}\|P_2^\alpha\| &\le C\left\|U_{\epsilon x_j}\right\|_s\left\|\mathcal{A}_j\left(U_\epsilon\right) - \mathcal{A}_j\left(U^\epsilon\right)\right\|_{|\alpha|}\\
&\le C\left\|\left(u_j^\epsilon - v_{\epsilon j}\right)\right\|_{|\alpha|} + \left\|\bar{A}_j\left(U^\epsilon\right) - \bar{A}_j\left(U_\epsilon\right)\right\|_{|\alpha|}\\
&\le C\left(1+\|u^\epsilon - v_\epsilon\|_{|\alpha|} + \|H^\epsilon - B_\epsilon\|_{|\alpha|}\right) + C\left\|\epsilon^{-1}\left(g\left(q^\epsilon\right) - g\left(q_\epsilon\right)\right)\right\|_{|\alpha|}\\
&\quad + C\left\|\frac{1}{1+\epsilon q^\epsilon}\left(H^\epsilon - B_\epsilon\right)\right\|_{|\alpha|} + C\left\|\left(\frac{1}{1+\epsilon q^\epsilon} - \frac{1}{1+\epsilon q_\epsilon}\right)B_\epsilon\right\|_{|\alpha|}\\
&\le C\left(1+\|q_\epsilon + \theta\left(q^\epsilon - q_\epsilon\right)\|_s^s\right)\|E\|_{|\alpha|}\\
&\le C\left(1+\|E\|_s^s\right)\|E\|_{|\alpha|},\end{aligned} \tag{3.8}$$

where $0 \le \theta \le 1$ is constant.

Finally, we deal with the term $\int (D^\alpha E)^\top \tilde{A}_0(U^\epsilon) Q^\alpha \mathrm{d}x$. Since

$$Q^\alpha = \begin{pmatrix} 0 \\ \nu D^\alpha\left(\frac{1}{1+\epsilon q^\epsilon}\left(\Delta u^\epsilon + \nabla \operatorname{div} u^\epsilon\right) - \frac{1}{1+\epsilon q_\epsilon}\Delta v_\epsilon\right) \\ \mu D^\alpha\left(\Delta H^\epsilon - \Delta B_\epsilon\right) - D^\alpha \nabla\times\left[\left(\frac{(\nabla\times H^\epsilon)\times H^\epsilon}{1+\epsilon q^\epsilon}\right) - \left(\left(\nabla\times B_\epsilon\right)\times B_\epsilon\right)\right]\end{pmatrix},$$

we can rewrite $\int (D^\alpha E)^\top \tilde{A}_0(U^\epsilon) Q^\alpha \mathrm{d}x$ as follows:

$$\begin{aligned}&\int \left(D^\alpha E\right)^\top \tilde{A}_0\left(U^\epsilon\right) Q^\alpha \mathrm{d}x\\
&= \nu \int D^\alpha\left(u^\epsilon - v_\epsilon\right) D^\alpha\left(\frac{\Delta u^\epsilon + \nabla \operatorname{div} u^\epsilon}{1+\epsilon q^\epsilon} - \frac{\Delta v_\epsilon}{1+\epsilon q_\epsilon}\right)\mathrm{d}x\\
&\quad + \mu \int \frac{1}{1+\epsilon q^\epsilon} D^\alpha\left(H^\epsilon - B_\epsilon\right) D^\alpha \Delta\left(H^\epsilon - B_\epsilon\right)\mathrm{d}x\\
&\quad - \int \frac{1}{1+\epsilon q^\epsilon} D^\alpha\left(H^\epsilon - B_\epsilon\right) D^\alpha \nabla\times\left[\frac{\left(\nabla\times H^\epsilon\right)\times H^\epsilon}{1+\epsilon q^\epsilon} - \left(\nabla\times B_\epsilon\right)\times B_\epsilon\right]\mathrm{d}x\\
&=: Q_1 + Q_2 + Q_3.\end{aligned}$$

The terms $Q_1$ and $Q_2$ can be estimated in the same way as that in [20], so we state the following estimates directly and omit their detailed arguments:

$$
\begin{aligned}
Q_1 \leq & -C\nu \int \left|D^\alpha \nabla\left(u^\epsilon-v_\epsilon\right)\right|^2 \mathrm{d}x - C\nu \int \left|D^\alpha \operatorname{div}\left(u^\epsilon-v_\epsilon\right)\right|^2 \mathrm{d}x \\
& + C\epsilon \left\|D^\alpha \nabla\left(u^\epsilon-v_\epsilon\right)\right\|^2 + C\left(\|E\|_{|\alpha|}^2+\|E\|_{|s|}^4+\|E\|_{|\alpha|}^2\|E\|_{|s|}^s\right), \qquad (3.9)\\
Q_2 \leq & -C\mu \int \left|D^\alpha \nabla\left(H^\epsilon-B_\epsilon\right)\right|^2 \mathrm{d}x \\
& + C\epsilon \left\|D^\alpha \nabla\left(u^\epsilon-v_\epsilon\right)\right\|^2 + C\left(\|E\|_{|\alpha|}^2+\|E\|_{|s|}^4\right). \qquad (3.10)
\end{aligned}
$$

Below we deal with $Q_3$. By means of the following formulation:

$$
\begin{gathered}
\int g \cdot \nabla \times h \,\mathrm{d}x = \int h \cdot \nabla \times g \,\mathrm{d}x, \\
\nabla \times (\varphi F) = \varphi \nabla \times F + \nabla \varphi \times F.
\end{gathered}
$$

We can rewrite $Q_3$ as follows:

$$
\begin{aligned}
Q_3 = & -\int \frac{1}{1+\epsilon q^\epsilon} D^\alpha\left(H^\epsilon-B_\epsilon\right) D^\alpha \nabla \times\left[\frac{\left(\nabla \times H^\epsilon\right) \times H^\epsilon}{1+\epsilon q^\epsilon}-\left(\nabla \times B_\epsilon\right) \times B_\epsilon\right] \mathrm{d}x \\
= & -\int \nabla \frac{1}{1+\epsilon q^\epsilon} \times D^\alpha\left(H^\epsilon-B_\epsilon\right) D^\alpha\left[\frac{\left(\nabla \times H^\epsilon\right) \times H^\epsilon}{1+\epsilon q^\epsilon}-\left(\nabla \times B_\epsilon\right) \times B_\epsilon\right] \mathrm{d}x \\
& -\int \frac{1}{1+\epsilon q^\epsilon} D^\alpha \nabla \times\left(H^\epsilon-B_\epsilon\right) D^\alpha\left[\frac{\left(\nabla \times H^\epsilon\right) \times H^\epsilon}{1+\epsilon q^\epsilon}-\left(\nabla \times B_\epsilon\right) \times B_\epsilon\right] \mathrm{d}x \\
= & \,\epsilon \int \frac{\nabla q^\epsilon \times D^\alpha\left(H^\epsilon-B_\epsilon\right)}{\left(1+\epsilon q^\epsilon\right)^2} \cdot D^\alpha\left[\frac{\nabla \times\left(H^\epsilon-B_\epsilon\right) \times H^\epsilon}{1+\epsilon q^\epsilon}\right] \mathrm{d}x \\
& + \epsilon \int \frac{\nabla q^\epsilon \times D^\alpha\left(H^\epsilon-B_\epsilon\right)}{\left(1+\epsilon q^\epsilon\right)^2} \cdot D^\alpha\left[\nabla \times B_\epsilon \times\left(\frac{H^\epsilon}{1+\epsilon q^\epsilon}-B_\epsilon\right)\right] \mathrm{d}x \\
& -\int \frac{1}{1+\epsilon q^\epsilon} D^\alpha \nabla \times\left(H^\epsilon-B_\epsilon\right) D^\alpha\left[\frac{\nabla \times\left(H^\epsilon-B_\epsilon\right) \times H^\epsilon}{1+\epsilon q^\epsilon}\right] \mathrm{d}x, \\
& -\int \frac{1}{1+\epsilon q^\epsilon} D^\alpha \nabla \times\left(H^\epsilon-B_\epsilon\right) D^\alpha\left[\nabla \times B_\epsilon \times\left(\frac{H^\epsilon}{1+\epsilon q^\epsilon}-B_\epsilon\right)\right] \mathrm{d}x \\
= & \,\epsilon \int \frac{\nabla q^\epsilon \times D^\alpha\left(H^\epsilon-B_\epsilon\right)}{\left(1+\epsilon q^\epsilon\right)^2} \cdot D^\alpha \nabla \times\left(H^\epsilon-B_\epsilon\right) \times \frac{H^\epsilon}{1+\epsilon q^\epsilon} \mathrm{d}x \\
& + \epsilon \sum_{0<\beta \leq \alpha} \int \frac{\nabla q^\epsilon \times D^\alpha\left(H^\epsilon-B_\epsilon\right)}{\left(1+\epsilon q^\epsilon\right)^2} \cdot D^{\alpha-\beta} \nabla \times\left(H^\epsilon-B_\epsilon\right) \times D^\beta\left(\frac{H^\epsilon}{1+\epsilon q^\epsilon}\right) \mathrm{d}x \\
& + \epsilon \int \frac{\nabla q^\epsilon \times D^\alpha\left(H^\epsilon-B_\epsilon\right)}{\left(1+\epsilon q^\epsilon\right)^2} \cdot D^\alpha\left[\nabla \times B_\epsilon \times\left(\frac{H^\epsilon}{1+\epsilon q^\epsilon}-B_\epsilon\right)\right] \mathrm{d}x \\
& + \epsilon \int \frac{1}{1+\epsilon q^\epsilon} D^\alpha \nabla \times\left(H^\epsilon-B_\epsilon\right) \cdot D^\alpha\left[\nabla \times B_\epsilon \times \frac{q^\epsilon}{1+\epsilon q^\epsilon} B_\epsilon\right] \mathrm{d}x \\
& -\int \frac{1}{1+\epsilon q^\epsilon} D^\alpha \nabla \times\left(H^\epsilon-B_\epsilon\right) \cdot D^\alpha \nabla \times\left(H^\epsilon-B_\epsilon\right) \times \frac{H^\epsilon}{1+\epsilon q^\epsilon} \mathrm{d}x \\
& -\sum_{0<\beta \leq \alpha} \int \frac{1}{1+\epsilon q^\epsilon} D^\alpha \nabla \times\left(H^\epsilon-B_\epsilon\right) \cdot D^{\alpha-\beta} \nabla \times\left(H^\epsilon-B_\epsilon\right) \times D^\beta \frac{H^\epsilon}{1+\epsilon q^\epsilon} \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&- \int \frac{1}{1+\epsilon q^\epsilon} D^\alpha \nabla \times (H^\epsilon - B_\epsilon) \cdot D^\alpha \left[ \nabla \times B_\epsilon \times \frac{H^\epsilon - B_\epsilon}{1+\epsilon q^\epsilon} \right] \mathrm{d}x \\
=: &\sum_{i=1}^{7} I_i.
\end{aligned}
$$

By means of integration by parts, the Moser-type calculus inequalities, Sobolev's embedding, Holder's inequality and the regularity of $(q_\epsilon, v_\epsilon, B_\epsilon)$, the seven terms $I_i (1 \le i \le 7)$ can be controlled as follows:

$$
\begin{aligned}
|I_1| &\le C\epsilon \left\| \nabla q^\epsilon \times D^\alpha (H^\epsilon - B_\epsilon) \right\| \left\| H^\epsilon \times D^\alpha \nabla \times (H^\epsilon - B_\epsilon) \right\| \\
&\le C\epsilon \left\| \nabla q^\epsilon \right\|_{L^\infty} \left\| D^\alpha (H^\epsilon - B_\epsilon) \right\| \left\| H^\epsilon \right\|_{L^\infty} \left\| D^\alpha \nabla \times (H^\epsilon - B_\epsilon) \right\| \\
&\le C\epsilon \left(1 + \|E\|_s\right)^2 \|E\|_{|\alpha|} \left\| D^\alpha \nabla \times (H^\epsilon - B_\epsilon) \right\| \\
&\le C\epsilon \left(1 + \|E\|_s\right)^2 \|E\|_{|\alpha|}^2 + C\epsilon \left\| D^\alpha \nabla \times (H^\epsilon - B_\epsilon) \right\|^2, \\
|I_2| &\le C\epsilon \left\| \nabla q^\epsilon \times D^\alpha (H^\epsilon - B_\epsilon) \right\| \left\| D^\beta \left( \frac{H^\epsilon}{1+\epsilon q^\epsilon} \times D^{\alpha-\beta} \right) \nabla \times (H^\epsilon - B_\epsilon) \right\| \\
&\le C\epsilon \left(1 + \|E\|_s\right) \|E\|_{|\alpha|} \left\| \frac{H^\epsilon}{1+\epsilon q^\epsilon} \right\|_s \left\| \nabla \times (H^\epsilon - B_\epsilon) \right\|_{|\alpha|-1} \\
&\le C\epsilon \left(1 + \|E\|_s\right)^2 \|E\|_{|\alpha|} \left(1 + \|E\|_s^s\right) \|E\|_{|\alpha|} \\
&\le C\epsilon \left(1 + \|E\|_{s+2}^s\right) \|E\|_{|\alpha|}^2, \\
|I_3| &\le C\epsilon \left\| \nabla q^\epsilon \times D^\alpha (H^\epsilon - B_\epsilon) \right\| \left\| D^\alpha \left[ \left( \frac{H^\epsilon}{1+\epsilon q^\epsilon} - B_\epsilon \right) \times (\nabla \times B_\epsilon) \right] \right\| \\
&\le C\epsilon \left\| \nabla q^\epsilon \right\|_{L^\infty} \|E\|_{|\alpha|} \left\| \left( \frac{H^\epsilon}{1+\epsilon q^\epsilon} - B_\epsilon \right) \times (\nabla \times B_\epsilon) \right\|_{|\alpha|} \\
&\le C\epsilon \left( \|E\|_s + \epsilon \right) \|E\|_{|\alpha|} \left(1 + \|E\|_s^{s+1}\right) \\
&\le C\epsilon \left(1 + \|E\|_s^{2s+2}\right) \|E\|_s^2 + C\epsilon^2, \\
|I_4| &\le C\epsilon \| D^\alpha \nabla \times (H^\epsilon - B_\epsilon) \| \left\| \frac{q^\epsilon}{1+\epsilon q^\epsilon} B_\epsilon \times (\nabla \times B_\epsilon) \right\|_{|\alpha|} \\
&\le C\epsilon \left\| D^\alpha \nabla \times (H^\epsilon - B_\epsilon) \right\| \left( \left(1 + \|E\|_s^s\right) \|E\|_{|\alpha|} + \epsilon \left(1 + \|E\|_s^s\right) \right) \\
&\le C\epsilon \| D^\alpha \nabla \times (H^\epsilon - B_\epsilon) \|^2 + C \left(1 + \|E\|_s^{2s}\right) \|E\|_{|\alpha|}^2 + C\epsilon^2 \|E\|_s^{2s} + C\epsilon^2, \\
|I_6| &\le C \left\| D^\alpha \nabla \times (H^\epsilon - B_\epsilon) \right\| \left\| D^\beta \left( \frac{H^\epsilon}{1+\epsilon q^\epsilon} \right) \times D^{\alpha-\beta} \nabla \times (H^\epsilon - B_\epsilon) \right\| \\
&\le C \left\| D^\alpha \nabla \times (H^\epsilon - B_\epsilon) \right\| \left\| \frac{H^\epsilon}{1+\epsilon q^\epsilon} \right\|_s \|E\|_{|\alpha|} \\
&\le \eta \left\| D^\alpha \nabla \times (H^\epsilon - B_\epsilon) \right\|^2 + \frac{C}{\eta} \left(1 + \|E\|_s^{2s+2}\right) \|E\|_{|\alpha|}^2, \\
|I_7| &\le C \left\| D^\alpha \nabla \times (H^\epsilon - B_\epsilon) \right\| \left\| \frac{H^\epsilon - B_\epsilon}{1+\epsilon q^\epsilon} \times (\nabla \times B_\epsilon) \right\|_{|\alpha|} \\
&\le C \left\| D^\alpha \nabla \times (H^\epsilon - B_\epsilon) \right\| \left(1 + \|E\|_s^s\right) \|E\|_{|\alpha|} \\
&\le \eta \left\| D^\alpha \nabla \times (H^\epsilon - B_\epsilon) \right\|^2 + \frac{C}{\eta} \left(1 + \|E\|_s^{2s}\right) \|E\|_{|\alpha|}^2,
\end{aligned}
$$

where $\eta$ will be decided below.
According to the equality $F\cdot(F\times G)=0$ for any vectors $F$ and $G$, we obtain that

$$I_5\equiv 0.$$

Collecting the above estimates, we have

$$\begin{aligned}
Q_3 &\le C\epsilon\left\|D^\alpha\nabla\times(H^\epsilon-B_\epsilon)\right\|^2+C\left(1+\|E\|_s^{2s+2}\right)\|E\|_s^2+C\epsilon^2\\
&\quad+2\eta\left\|D^\alpha\nabla\times(H^\epsilon-B_\epsilon)\right\|^2+\frac{2C}{\eta}\left(1+\|E\|_s^{2s+2}\right)\|E\|_s^2,\\
Q_2+Q_3 &\le(-C\mu+2\eta)\int\left|D^\alpha\nabla\times(H^\epsilon-B_\epsilon)\right|^2\mathrm{d}x+\frac{2C}{\eta}\left(1+\|E\|_s^{2s+2}\right)\|E\|_s^2\\
&\quad+C\epsilon\left\|D^\alpha\nabla\times(H^\epsilon-B_\epsilon)\right\|^2+C\epsilon\left\|D^\alpha\nabla\times(u^\epsilon-v_\epsilon)\right\|^2\\
&\quad+C\left(1+\|E\|_s^{2s+2}\right)\|E\|_s^2+C\epsilon^2\\
&\le-\frac{C}{2}\mu\int\left|D^\alpha\nabla\times(H^\epsilon-B_\epsilon)\right|^2\mathrm{d}x+C\epsilon\left\|D^\alpha\nabla\times(H^\epsilon-B_\epsilon)\right\|^2\\
&\quad+C\epsilon\left\|D^\alpha\nabla\times(u^\epsilon-v_\epsilon)\right\|^2+C\left(1+\|E\|_s^{2s+2}\right)\|E\|_s^2+C\epsilon^2.
\end{aligned}$$

Thus by choosing $2\eta=-\frac{C}{2}\mu$, we obtain that

$$\begin{aligned}
\int(D^\alpha E)^\top\tilde{A}_0(U^\epsilon)Q^\alpha\mathrm{d}x &\le-C\nu\int\left|D^\alpha\nabla(u^\epsilon-v_\epsilon)\right|^2\mathrm{d}x-C\nu\int\left|D^\alpha\mathrm{div}\,(u^\epsilon-v_\epsilon)\right|^2\mathrm{d}x\\
&\quad-\frac{C}{2}\mu\int\left|D^\alpha\nabla\times(H^\epsilon-B_\epsilon)\right|^2\mathrm{d}x+C\epsilon\left\|D^\alpha\nabla\times(H^\epsilon-B_\epsilon)\right\|^2\\
&\quad+C\epsilon\left\|D^\alpha\nabla\times(u^\epsilon-v_\epsilon)\right\|^2+C\left(1+\|E\|_s^{2s+2}\right)\|E\|_s^2+C\epsilon^2.
\end{aligned}\tag{3.11}$$

Putting the estimates (3.11) and (3.6)–(3.8) into (3.3) and taking $\epsilon$ small enough, we obtain that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|D^\alpha E\|_{\mathrm{e}}^2+\kappa\int\left[\left|D^\alpha\nabla(u^\epsilon-v_\epsilon)\right|^2+\left|D^\alpha\nabla(H^\epsilon-B_\epsilon)\right|^2\right]\mathrm{d}x\\
&\quad\le C\|Y^\alpha\|^2+C\big(1+\|E\|_s^{2s+2}\big)\|E\|_s^2+C\epsilon^2,
\end{aligned}\tag{3.12}$$

where we have used the following estimate

$$\nu\int\left|D^\alpha\nabla(u^\epsilon-v_\epsilon)\right|^2+\nu\int\left|D^\alpha\mathrm{div}\,(u^\epsilon-v_\epsilon)\right|^2\ge\kappa\int\left|D^\alpha\nabla(u^\epsilon-v_\epsilon)\right|^2$$

for some positive constant $\kappa>0$.

Thanks to (3.4), we can integrate the inequality (3.12) over $(0,t)$ with $t<\min\{T_\epsilon,T^*\}$ to obtain that

$$\begin{aligned}
\left\|D^\alpha E(t)\right\|^2 &\le C\left\|D^\alpha E(0)\right\|^2+C\int_0^t\left\|Y^\alpha(\tau)\right\|^2\mathrm{d}\tau\\
&\quad+C\int_0^t\Big\{\big(1+\|E\|_s^{2s+2}\big)\|E\|_s^2+C\epsilon^2\Big\}(\tau)\mathrm{d}\tau.
\end{aligned}$$

Summing up the above inequality for all $\alpha$ with $|\alpha|\le s$, we arrive at

$$\|E(t)\|_s^2\le C\|E(0)\|_s^2+C\int_0^{T^*}\|R(\tau)\|_s^2\mathrm{d}\tau+C\int_0^t\left\{\left(1+\|E\|_s^{2s+2}\right)\|E\|_s^2\right\}(\tau)\mathrm{d}\tau.$$

With the aid of Gronwall's lemma and the fact that

$$\|E(0)\|_s^2 + \int_0^{T^*} \|R(t)\|_s^2 \, \mathrm{d}t = O\left(\epsilon^2\right),$$

we get

$$\|E(t)\|_s^2 \le C\epsilon^2 \exp\left\{ C \int_0^t \left(1 + \|E(\tau)\|_s^{2s+2}\right) \mathrm{d}\tau \right\} \equiv \Phi(t).$$

It is easy to check that $\Phi(t)$ satisfies

$$\Phi'(t) = C\left(1 + \|E(t)\|_s^{2s+2}\right)\Phi(t) \le C\Phi(t) + C\Phi^{s+2}(t).$$

Thus by employing the nonlinear Gronwall-type inequality, we conclude that there exists a constant $K$, independent of $\epsilon$, such that

$$\|E(t)\|_s \le K\epsilon$$

for all $t \in [0, \min\{T_\epsilon, T^*\})$ provided $\Phi(0) = C\epsilon^2 < \exp\{-CT^*\}$. Hence the proof of Theorem 1.1 is completed. □

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