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TEM wave propagation in a microstrip line on a substrate of circular segment cross section

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Abstract. TEM mode wave propagation in a microstrip transmission line on a dielectric substrate of circular segment cross section is of concern. A suggested design of a microstrip is convenient for manufacturing and allows to miniaturize microstrips since there is no lateral fringing effect of the substrate in the transmitted fundamental TEM mode. By using the bipolar orthogonal coordinates, the plane potential problem is reformulated in the form of trigonometric dual integral equations. The discontinuous integrals containing Legendre functions of complex degree and the Abel integral equation are employed to reduce the dual equations to a Fredholm equation of the second kind. The structure of the Fredholm integral equation allows to obtain a simple approximate solution. Basing on this solution, rigorous approximate formulas for the characteristic impedance are derived. The characteristic impedance of a microstrip transmission line on a silicon substrate is computed and plotted as an example.

Mathematics Subject Classification. Primary 78A50 · Secondary 31A25 · 45F10.

Keywords. Microstrip · TEM wave · Characteristic impedance · Dual integral equations.

1. Introduction

Classical planar microstrips on an infinite layer-shaped dielectric substrate with grounded opposite surface were investigated in detail using various techniques (Cohn [1], Wheeler [2–4], Hammerstad [5], Owens [6], James et al. [7], Tuncer and Neikirk [8], and Edwards and Sreer [9]. In the recent years, attention of numerous researchers is attracted to non-planar microstrips with a conductive strip conformed to a surface of a curvilinear dielectric substrate (for instance, Wong [10]).

In this paper, we employ the bipolar orthogonal coordinate system (Korn and Korn [11])

$$x = \frac{R_0 \sinh\alpha}{\cosh\alpha + \cos\beta}, \quad y = \frac{R_0 \sin\beta}{\cosh\alpha + \cos\beta}, \tag{1}$$

to derive an analytical solution to the quasi-static problem of a microstrip on a dielectric substrate of circular segment cross section. Such a simple design is convenient for manufacturing and allows to miniaturize microstrips since there is no lateral fringing effect of the substrate in the transmitted fundamental TEM mode.

We consider two problems: (1) The planar surface of the substrate is covered with a grounded thin conductive film (Problem I); (2) the planar surface of the substrate is bonded to the grounded infinite conductive sheet (Problem II). The cross-sectional geometry in the bipolar coordinates is given in Fig. 1. The curved surface of the substrate $\beta = \beta_0$, $-\infty < \alpha < \infty$, is the arc of the circle $x^2 + (y + R_0 \cot\beta_0)^2 = R_0^2/\sin^2\beta_0$, $0 < \beta_0 < \pi$. The equations of the grounded planar surface of the substrate are $\beta = 0$, $-\infty < \alpha < \infty$, if $y \to +0$, $|x| \le R_0$; and $\beta = 2\pi$, $-\infty < \alpha < \infty$, if $y \to -0$, $|x| \le R_0$. The equation of the half planes $y = 0$, $|x| > R_0$, is $\beta = \pi$, $-\infty < \alpha < \infty$. The thin conductive strip occupies domain

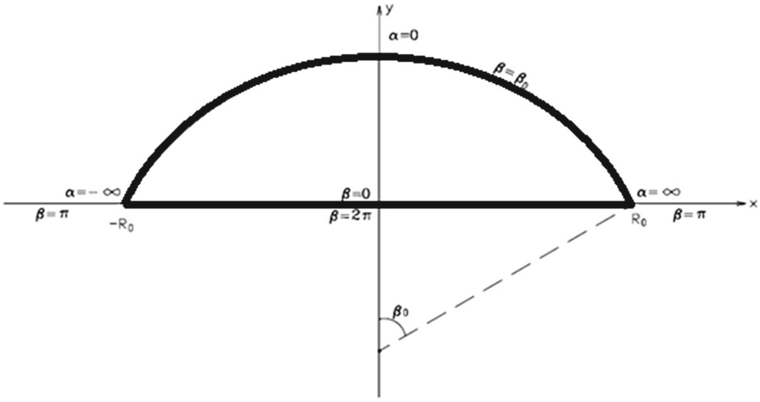

Fig. 1. Geometry of a dielectric substrate in the bipolar coordinates

$|\alpha| \leq \alpha_0$ on the non-planar surface $\beta = \beta_0$ of the substrate. The dielectric material is characterized by the electric permittivity ϵ and the magnetic permeability μ.

The paper is constructed as follows. In Sect. 2, the problem is reformulated in terms of equivalent dual integral equations containing Fourier integrals. In Sect. 3, a certain substitution involving Legendre functions of complex index is employed to reduce the dual integral equations to the Abel integral equation. A Fredholm integral equation of the second kind is obtained on inverting this Abel equation. It is shown that the right part of the Fredholm equation can be taken as an approximate solution. Approximate formulas for the characteristic impedance of the microstrips are derived in Sect. 4.

Note that the dual equations' approach is widely used for solving various problems arising in physics and engineering. Readers can find the state of the art in dual equations' technique and applications in books by Mandal and Mandal [12], Vinogradov at al. [13], Shestopalov at al.[14].

2. Dual integral equations of the problem

Within the framework of the quasi-static model, we have to solve the Laplace equations [15,16]

$$\left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2}\right) \phi_n (\alpha, \beta) = 0, \quad n = 1, 2, \tag{2}$$

where $\phi_2 (\alpha, \beta)$ is the static potential field in the substrate $0 \leq \beta \leq \beta_0$, and $\phi_1 (\alpha, \beta)$ is the static potential field of free space $\beta_0 \leq \beta \leq \widetilde{\beta}$ with $\widetilde{\beta} = 2\pi$ for Problem I and $\widetilde{\beta} = \pi$ for Problem II. These potential fields should be even in the variable α (the magnetic wall at $\alpha = 0$) and satisfy the boundary conditions

$$\phi_1 (\alpha, \beta_0) = \phi_2 (\alpha, \beta_0), \quad |\alpha| < \infty, \tag{3}$$

$$\phi_1 \left(\alpha, \widetilde{\beta}\right) = 0, \quad |\alpha| < \infty, \tag{4}$$

$$\phi_2 (\alpha, 0) = 0, \quad |\alpha| < \infty, \tag{5}$$

$$\phi_2 (\alpha, \beta_0) = V, \quad |\alpha| \leq \alpha_0, \tag{6}$$

$$\frac{\partial \phi_1 (\alpha, \beta_0)}{\partial \beta} = \varepsilon \frac{\partial \phi_2 (\alpha, \beta_0)}{\partial \beta}, \quad \alpha_0 < |\alpha| < \infty, \tag{7}$$

where $\varepsilon = \epsilon/\epsilon_0$, ϵ_0 is the electric permittivity of free space, and V is a certain constant.

The Fourier integrals

$$\phi_1(\alpha,\beta) = -\int\limits_0^\infty A(p)\frac{\sinh(p\beta_0)}{p\Delta(p,\beta_0)}\sinh\left(p\left(\widetilde{\beta}-\beta\right)\right)\cos(p\alpha)\,\mathrm{d}p, \tag{8}$$

$$\phi_2(\alpha,\beta) = -\int\limits_0^\infty A(p)\frac{\sinh\left(p\left(\widetilde{\beta}-\beta_0\right)\right)}{p\Delta(p,\beta_0)}\sinh(p\beta)\cos(p\alpha)\,\mathrm{d}p, \tag{9}$$

$$\Delta(p,\beta_0) = \sinh(p\beta_0)\cosh\left(p\left(\widetilde{\beta}-\beta_0\right)\right) + \varepsilon\cosh(p\beta_0)\sinh\left(p\left(\widetilde{\beta}-\beta_0\right)\right), \tag{10}$$

are seen to be potential functions which are even in α and satisfy the conditions (3), (4), and (5).

Inserting the representations (8) and (9) into (6) and taking into account the evenness of the boundary condition, we obtain the equation

$$\int\limits_0^\infty A(p)\frac{\sinh(p\beta_0)\sinh\left(p\left(\widetilde{\beta}-\beta_0\right)\right)}{p\Delta(p,\beta_0)}\cos(p\alpha)\,\mathrm{d}p = -V, \qquad 0 \le \alpha \le \alpha_0. \tag{11}$$

Another equation follows from (7)

$$\int\limits_0^\infty A(p)\cos(p\alpha)\,\mathrm{d}p = 0, \quad \alpha > \alpha_0. \tag{12}$$

Differentiating (11) and making changes

$$p = \gamma\xi, \quad \gamma\alpha = t, \quad \gamma\alpha_0 = t_0, \quad A(p) = \frac{\sqrt{2}}{\pi}X(\xi), \tag{13}$$

$$\gamma = \pi\frac{(1-\varepsilon)\beta_0 + \varepsilon\widetilde{\beta}}{\beta_0\left(\widetilde{\beta}-\beta_0\right)(1+\varepsilon)}, \tag{14}$$

lead from (11) and (12) to the dual integral equations

$$\frac{\sqrt{2}}{\pi}\int\limits_0^\infty X(\xi)\frac{1-h(\xi)}{\coth(\pi\xi)}\sin(\xi t)\,\mathrm{d}\xi = 0, \quad 0 \le t \le t_0, \tag{15}$$

$$\frac{\sqrt{2}}{\pi}\int\limits_0^\infty X(\xi)\cos(\xi t)\,\mathrm{d}\xi = 0, \quad t > t_0, \tag{16}$$

with the function

$$h(\xi) = 1 - \frac{(1+\varepsilon)\coth(\pi\xi)\tanh(\gamma\beta_0\xi)\tanh\left(\gamma\left(\widetilde{\beta}-\beta_0\right)\xi\right)}{\tanh(\gamma\beta_0\xi) + \varepsilon\tanh\left(\gamma\left(\widetilde{\beta}-\beta_0\right)\xi\right)} \tag{17}$$

possessing the following behavior:

$$\begin{gathered} h(\xi) = O\left(\xi^2\right) \quad \text{as } \xi \to 0, \\ h(\xi) = O(\exp(-2\lambda\xi)), \quad \text{as } \xi \to \infty, \\ \lambda = \min\left(\pi\left(\frac{\widetilde{\beta}}{\beta_0}-1\right), \pi\right). \end{gathered} \tag{18}$$

3. Solving the dual integral equations

Solving the dual integral equations is based on the integral representations of the Legendre functions of the first kind (Lebedev [17]):

$$\frac{P_{-1/2+i\xi}\left(\cosh\theta\right)}{\coth\left(\pi\xi\right)} = \frac{\sqrt{2}}{\pi}\int\limits_{\theta}^{\infty}\frac{\sin\left(\xi t\right)}{\sqrt{\cosh t-\cosh\theta}}\mathrm{d}t, \tag{19}$$

$$P_{-1/2+i\xi}\left(\cosh\theta\right) = \frac{\sqrt{2}}{\pi}\int\limits_{0}^{\theta}\frac{\cos\left(\xi t\right)}{\sqrt{\cosh\theta-\cosh t}}\mathrm{d}t, \tag{20}$$

which give rise to the discontinuous integrals

$$\sqrt{2}\int\limits_{0}^{\infty}\frac{\sin\left(\xi t\right)}{\coth\left(\pi\xi\right)}P_{-1/2+i\xi}\left(\cosh\theta\right)\mathrm{d}\xi = \frac{u\left(t-\theta\right)}{\sqrt{\cosh t-\cosh\theta}}, \tag{21}$$

$$\sqrt{2}\int\limits_{0}^{\infty}P_{-1/2+i\xi}\left(\cosh\theta\right)\cos\left(\xi t\right)\mathrm{d}\xi = \frac{u\left(\theta-t\right)}{\sqrt{\cosh\theta-\cosh t}}, \tag{22}$$

where $u\left(t\right)$ is the unit step function.

The solution to the dual integral equations is sought in the form of the substitution

$$\frac{X\left(\xi\right)}{M} = \int\limits_{0}^{t_0}\varpi\left(s\right)\sinh\left(s\right)P_{-1/2+i\xi}\left(\cosh s\right)\mathrm{d}s + P_{-1/2+i\xi}\left(\cosh t_0\right), \tag{23}$$

in which the unknown function $\varpi\left(s\right)$ and the constant number M to be determined.

Insert (23) into Eq. (16) and interchange the order of integration. Then, by virtue of (22), the second of the dual integral equations is satisfied.

Now, we rewrite (15) in the form

$$\sqrt{2}\int\limits_{0}^{\infty}\frac{X\left(\xi\right)\sin\left(\xi t\right)}{\coth\left(\pi\xi\right)}\mathrm{d}\xi = \sqrt{2}f\left(t\right), \quad 0\le t\le t_0, \tag{24}$$

with

$$f\left(t\right) = \int\limits_{0}^{\infty}\frac{h\left(\xi\right)X\left(\xi\right)}{\coth\left(\pi\xi\right)}\sin\left(\xi t\right)\mathrm{d}\xi.$$

As a result of inserting (23) and interchanging the order of integration, (24) turns due to (21) into the well-known Abel equation

$$M\int\limits_{0}^{t}\frac{\varpi\left(s\right)\sinh\left(s\right)\mathrm{d}s}{\sqrt{\cosh t-\cosh s}} = f\left(t\right) \tag{25}$$

that can be inverted

$$M\varpi\left(t\right) = \frac{2\sqrt{2}f\left(0\right)}{\pi\cosh\left(\frac{t}{2}\right)} + \frac{\sqrt{2}}{\pi}\int\limits_{0}^{t}\frac{f'\left(s\right)\mathrm{d}s}{\sqrt{\cosh t-\cosh s}}. \tag{26}$$

Hence,

$$\varpi(t)=\int_0^\infty \frac{X(\xi)}{M}\tanh(\pi\xi)\,h(\xi)\left(\frac{\sqrt{2}}{\pi}\int_0^t \frac{\cos(\xi s)}{\sqrt{\cosh t-\cosh s}}\mathrm{d}s\right)\mathrm{d}\xi.$$

Finally, using (20) and inserting (23), we derive the Fredholm integral equation of the second kind

$$\varpi(t)-\mathbf{K}(\varpi)=K(t_0,t),\quad 0\le t\le t_0, \tag{27}$$

where $\mathbf{K}$ is the integral operator

$$\mathbf{K}(\varpi)=\int_0^{t_0}\sinh(s)\,\varpi(s)\,K(t,s)\,\mathrm{d}s, \tag{28}$$

and

$$K(t,s)=\int_0^\infty \frac{\xi h(\xi)}{\coth(\pi\xi)}P_{-1/2+i\xi}(\cosh t)\,P_{-1/2+i\xi}(\cosh s)\,\mathrm{d}\xi \tag{29}$$

is a continuous function.

The solution of (27) can be represented by the Neumann series

$$\widetilde{\varpi}(t)=\sum_{n=1}^{\infty}\mathbf{K}^n(K(t_0,t)). \tag{30}$$

that is uniformly convergent. This can be proved as follows. We take the Hilbert space defined by the inner product

$$(f(s),\varpi(s))=\int_0^{t_0}\varpi(s)\,f(s)\sinh s\mathrm{d}s, \tag{31}$$

and make use of the Mehler-Fok integral transform (Lebedev [17])

$$\widehat{f}(\xi)=\int_0^\infty \sinh(\theta)\,f(\theta)\,P_{-\frac{1}{2}+i\xi}(\cosh\theta)\,\mathrm{d}\theta, \tag{32}$$

$$f(\theta)=\int_0^\infty \xi\frac{\widehat{f}(\xi)}{\coth(\pi\xi)}P_{-\frac{1}{2}+i\xi}(\cosh\theta)\,\mathrm{d}\xi. \tag{33}$$

Notice that the Parseval equation

$$\int_0^\infty \xi\frac{\widehat{f}^2(\xi)}{\coth(\pi\xi)}\mathrm{d}\xi=\int_0^\infty \sinh(s)\,f^2(s)\,\mathrm{d}s \tag{34}$$

involves the inequality

$$\|f\|^2\le\int_0^\infty \xi\frac{\widehat{f}^2(\xi)}{\coth(\pi\xi)}\mathrm{d}\xi, \tag{35}$$

which turns into the equality if $f(\theta)=0$ for $t>t_0$. Hence, using (35) and (29), one might obtain $\|K(t,s)\|\le l(t)$ with

$$l^2(t)=\int_0^\infty \xi h^2(\xi)\frac{P_{-1/2+i\xi}(\cosh t)^2}{\coth(\pi\xi)}\mathrm{d}\xi. \tag{36}$$

Then, we can establish with the aid of the Schwartz inequality that

$$\begin{aligned} \left|\mathbf{K}^n\left(K\left(t_0,t\right)\right)\right| &\leq \int_0^{t_0}\left|\mathbf{K}^{n-1}\left(K\left(t_0,s\right)\right)\right|\left|K\left(t,s\right)\right|\sinh\left(s\right)\mathrm{d}s \\ &\leq \left\|K\left(t,s\right)\right\|\sqrt{\int_0^{t_0}\left|\mathbf{K}^{n-1}\left(K\left(t_0,s\right)\right)\right|^2\sinh\left(s\right)\mathrm{d}s} \\ &\leq \left\|\mathbf{K}\right\|^{n-1} l\left(t_0\right) l\left(t\right). \end{aligned}$$

The norm of the operator in the above inequality is readily estimated by means the Parseval equation (34)

$$\begin{aligned} \left\|\mathbf{K}\right\| = \sup_{\|f\|^2=1}\left|\left(\mathbf{K}f,f\right)\right| &= \sup\int_0^{\infty}\xi h\left(\xi\right)\frac{\left(f,P_{-1/2+i\xi}\left(\cosh s\right)\right)^2}{\coth\left(\pi\xi\right)}\mathrm{d}\xi \\ &\leq \sup\left(h\left(\xi\right)\right)\sup_{\|f\|^2=1}\int_0^{\infty}\xi\frac{\left(f,P_{-1/2+i\xi}\left(\cosh s\right)\right)^2}{\coth\left(\pi\xi\right)}\mathrm{d}\xi \\ &= \sup\left(h\left(\xi\right)\right). \end{aligned} \tag{37}$$

The upper estimate for $l\left(\theta\right)$ can be established by means of the inequalities $\left|P_{-1/2+i\xi}\left(\cosh\theta\right)\right| \leq P_{-1/2}\left(\cosh\theta\right) \leq 1$ and $\left|P_{-1/2+i\xi}\left(\cosh\theta\right)\right| \leq \frac{2}{\pi}\coth\left(\pi\xi\right)Q_{-1/2}\left(\cosh\theta\right)$:

$$l\left(t\right) \leq \min\left\{\psi_1\left(\beta_0\right)P_{-1/2}\left(\cosh t\right),\psi_2\left(\beta_0\right)Q_{-1/2}\left(\cosh t\right),\right\}, \tag{38}$$

$$\psi_1^2\left(\beta_0\right) = \int_0^{\infty}\xi\tanh\left(\pi\xi\right)h^2\left(\xi\right)\mathrm{d}\xi,$$

$$\psi_2^2\left(\beta_0\right) = \frac{4}{\pi^2}\int_0^{\infty}\xi\coth\left(\pi\xi\right)h^2\left(\xi\right)\mathrm{d}\xi.$$

Numerical calculations of (37) and (38) show that $\sup h\left(\xi\right)$ and $\sup l\left(t\right)$ are fairly small for both Problem I and Problem II. Finally, we may take the approximate solution

$$\varpi\left(t\right) \approx K\left(t_0,t\right) \tag{39}$$

with an error $\left|\varpi\left(t\right)-K\left(t_0,t\right)\right| \leq l\left(t_0\right)l\left(t\right)/\left(1-\sup\left(h\left(\xi\right)\right)\right)$.

The unknown number M can be found from (11) by inserting (23) and setting $t=0$:

$$M = -V/\Psi, \tag{40}$$

where

$$\Psi = \int_0^{t_0}\varpi\left(s\right)\Psi_0\left(s\right)\sinh\left(s\right)\mathrm{d}s + \Psi_0\left(t_0\right), \tag{41}$$

$$\Psi_0\left(s\right) = \frac{\sqrt{2}}{\pi\left(1+\varepsilon\right)}\left[Q_{-1/2}\left(\cosh s\right)-\Psi_1\left(s\right)\right],$$

$$\Psi_1\left(s\right) = \int_0^{\infty}\frac{h\left(\xi\right)}{\xi\coth\left(\pi\xi\right)}P_{-1/2+i\xi}\left(\cosh s\right)\mathrm{d}\xi.$$

4. Characteristic impedance

The charge $\sigma(\alpha)$ distributed on the conductive strip is even integrable function that can be written as

$$\int_0^\infty A(p)\cos(p\alpha)\,\mathrm{d}p = \begin{cases} \sigma(\alpha)/\epsilon_0, & 0 \le |\alpha| \le t_0 \\ 0, & |\alpha| > 0 \end{cases}.$$

Because the Lame coefficients $g_{\alpha\alpha}$ and $g_{\beta\beta}$ are equal for the bipolar coordinates, we have the following connection of the capacitance C with the total charge

$$CV = \frac{1}{2}\int_{-\alpha_0}^{\alpha_0} \sigma(\alpha)\,\mathrm{d}\alpha = \int_0^{a_0} \sigma(\alpha)\,\mathrm{d}\alpha. \tag{42}$$

Inverting the Fourier integral, we obtain

$$-\frac{\pi\epsilon_0}{2}A(p) = \int_0^{\alpha_0} \sigma(\alpha)\cos(p\alpha)\,\mathrm{d}\alpha. \tag{43}$$

Thus, $2CV = -\pi\epsilon_0 A(0)$. Then, it is seen from (23) and (40) that

$$\frac{\sqrt{2}\Psi}{\epsilon_0}C = \int_0^{t_0} \varpi(s)\sinh(s)\,P_{-1/2}(\cosh s)\,\mathrm{d}s + P_{-1/2}(\cosh t_0). \tag{44}$$

A simple formula for the characteristic impedance ([15,16]) follows from (44)

$$\sqrt{\frac{\epsilon}{\mu}}Z_0 = \frac{\epsilon}{C} = \frac{\sqrt{2}\varepsilon\Psi}{\int_0^{t_0} \varpi(s)\sinh(s)\,P_{-1/2}(\cosh s)\,\mathrm{d}s + P_{-1/2}(\cosh t_0)}. \tag{45}$$

The above equation can be evaluated by taking the approximate solution (39). On neglecting the small term $\int_0^{t_0} K(t_0,s)\sinh(s)\,\Psi_1(s)\,\mathrm{d}s$, it becomes

$$\sqrt{\frac{\epsilon}{\mu}}Z_0 \approx \frac{2\varepsilon}{\pi(1+\varepsilon)}\frac{Q_{-1/2}(\cosh t_0) - \Psi_1(t_0) + \widetilde{\Psi}_0(t_0)}{P_{-1/2}(\cosh t_0) + \widetilde{P}(t_0)} \tag{46}$$

with

$$\widetilde{\Psi}_0(t_0) = \sinh t_0 \int_0^\infty \frac{\widetilde{Q}(\xi,t_0)\,P_{-1/2+i\xi}(\cosh t_0)}{\xi\coth(\pi\xi)}h(\xi)\,\mathrm{d}\xi,$$

$$\widetilde{Q}(\xi,t) = P_{-1/2+i\xi}(\cosh t)\,Q^1_{-1/2}(\cosh t) - P^1_{-1/2+i\xi}(\cosh t)\,Q_{-1/2}(\cosh t),$$

$$\widetilde{P}(t_0) = \sinh t_0 \int_0^\infty \frac{\widetilde{P}(\xi,t_0)\,P_{-1/2+i\xi}(\cosh t_0)}{\xi\coth(\pi\xi)}h(\xi)\,\mathrm{d}\xi,$$

$$\widetilde{P}(\xi,t) = P_{-1/2+i\xi}(\cosh t)\,P^1_{-1/2}(\cosh t) - P^1_{-1/2+i\xi}(\cosh t)\,P_{-1/2}(\cosh t).$$

When $t_0 \gg 1$, the leading terms of the asymptotic expansion for the normalized characteristic impedances can be derived in the same manner as in the paper [18]. On replacing Legendre functions in the expressions for $\Psi_1(t_0), \widetilde{\Psi}_0(t_0)$, and $\widetilde{P}_0(t_0)$ by leading terms of their asymptotic expansions and discarding

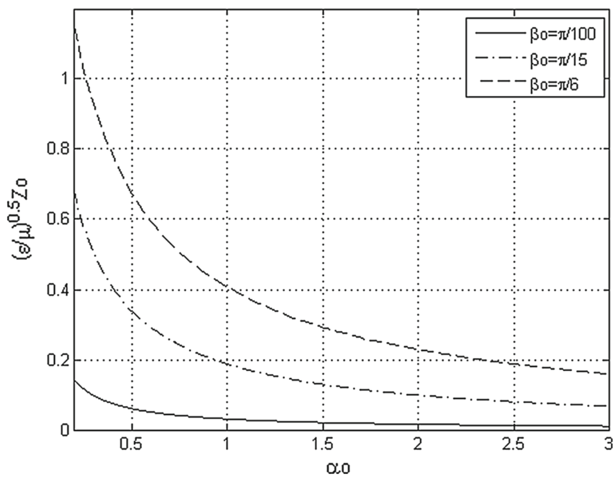

FIG. 2. Normalized characteristic impedance for the silicon substrate versus $0.2 \leq \alpha_0 \leq 3$

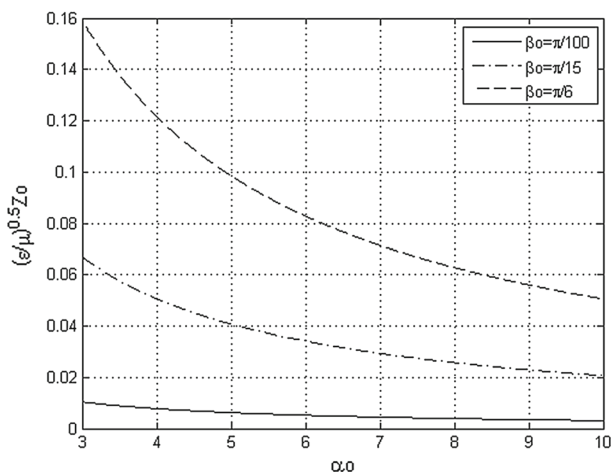

FIG. 3. Normalized characteristic impedance for the silicon substrate versus $3 \leq \alpha_0 \leq 10$

exponentially small quantities, we obtain

$$\sqrt{\frac{\epsilon}{\mu}} Z_0 \approx \frac{\pi\varepsilon}{(1+\varepsilon)\left(t_0 + 2\ln 2 + \psi\left(\varepsilon, \beta_0\right)\right)}, \qquad t_0 \gg 1, \tag{47}$$

$$\psi\left(\varepsilon, \beta_0\right) = \frac{1}{\pi} \int_0^\infty \frac{h\left(\xi\right)}{\xi^2} \mathrm{d}\xi.$$

Normalized characteristic impedances versus α_0 and β_0 computed for the silicon substrate with $\varepsilon = 1.18$ are plotted in Figs. 2 and 3 for Problem I. It shows that the discrepancy between normalized characteristic impedances for different β_0 is significant. It is also seen that for the same shape of a dielectric substrate, one may attain any value of the normalized characteristic impedance by changing the extent of a conductive strip. Calculations for Problem II give results that are very close to those in Problem I. This manifests that for the microstrip geometry studied, the characteristic impedance is rather weakly affected by the extend of the conductive sheet $|x| \leq \widetilde{R}$, $y = 0$, if either $\widetilde{R} = R_0$ or $\widetilde{R} \gg R_0$.

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