Z. Angew. Math. Phys. 66 (2015), 3357–3376

0044-2275/15/063357-20
published online October 8, 2015
DOI 10.1007/s00033-015-0589-8

Zeitschrift für angewandte Mathematik und Physik ZAMP

Asymptotic analysis of random boundary layers between two incompressible viscous fluid flows

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Abstract. The asymptotic analysis of boundary layers of random thinness and of higher Reynolds number separating two interacting incompressible viscous fluid flows is described using Γ-convergence methods. An asymptotic interfacial contact law is derived, which involves the jumps of the velocity and of the pressures of the fluids through an ergodic coefficient.

Mathematics Subject Classification. 76A20 · 76D05.

Keywords. Interacting fluid flows · Random boundary layers · Asymptotic behavior · Γ-convergence · Interfacial boundary conditions.

1. Introduction

The main purpose of this work is to study the interaction between two viscous incompressible fluids which flow along a thin layer with random boundary. This problem has important applications in many branches of science and engineering such as biology, chemical engineering, combustion and geophysics.

Surface tension gradients across a fluid–fluid interface provoke strong convective activity called Marangoni flow. The Marangoni flow, which proceeds from regions of lower surface tension to those of higher surface tension, is responsible for driving shear flow instabilities which take place within oscillatory boundary layers located near the interface between the two fluids. Within such layers, the fluid velocity changes rapidly, which implies a steep gradient of the shearing stress.

The boundary layer theory was first developed by Prandtl in 1904 [9] for a fluid in the close vicinity of a surrounding surface. The thickness of the boundary layers depends on Reynolds number, which increases when the viscosity effects become smaller: the higher the Reynolds number is, the thinner is the thickness of the boundary layers (see, for instance, [6] and [10]).

The interaction between two viscous incompressible fluids which flow along a thin viscous boundary layer of higher Reynolds number whose surface boundaries are defined through locally Lipschitz continuous functions, including periodic and self-similar cases, has been recently considered in [4]. In this paper, a physical situation was considered in which random fluctuations occur within thin viscous boundary layers of higher Reynolds number between two interacting viscous incompressible fluid flows. Indeed, a boundary layer of higher Reynolds number or more specifically a turbulent boundary layer contains a variety of coherent structures over a range of length scales, from small structures to larger structures, which essentially form random locations over the surface (see [8]). In the present paper, we consider a random microscopic cellular structure of the boundary layers. Each cell is supposed to have a random behavior in the thickness direction. The mixing scale in the boundary layers is related to the so-called turbulence viscosity or eddy viscosity, which has no precise expression, as far as we know. In the present paper, we suppose that Reynolds number is in the boundary layer of order $O\left(\varepsilon^{-\gamma}\right)$, with $\gamma > 0$, where $\varepsilon > 0$ is the layer thickness.

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We consider a bounded open subset $\Omega \subset \mathbb{R}^3$ with Lipschitz continuous boundary $\partial\Omega$, such that $\Omega = \Omega^+ \cup \Sigma \cup \Omega^-$, where Ω^+ and Ω^- are two nonempty open subsets separated by the smooth surface Σ. For simplicity, we suppose that Σ is contained in the plane $\{x_3 = 0\}$. We suppose that the boundary $\partial\Omega^+\backslash\Sigma$ (resp. $\partial\Omega^-\backslash\Sigma$) can be represented by a smooth and positive (resp. negative) function $x' = (x_1, x_2) \mapsto h^+(x')$ (resp. $x' \mapsto h^-(x')$).

Let (Π, Υ, P) be some probability space and $(G(x,y))_{(x,y)\in\mathbb{R}^2}$ be a group of transformations on (Π, Υ) that is satisfying for every (x,y), (x_1,y_1), (x_2,y_2) of $\mathbb{R}^2$

$$\begin{cases} G(0,0) = Id_\Pi, \\ G((x_1,y_1)+(x_2,y_2)) = G(x_1,y_1) \circ G(x_2,y_2), \\ P\left(G^{-1}(x,y)A\right) = P(A), \quad \forall A \in \Upsilon, \end{cases} \tag{1}$$

where Id_Π is the identity map on Π, and such that the set

$$\left\{(x,y,\omega) \in \mathbb{R}^2 \times \Pi \mid G(x,y)\,\omega \in A\right\}$$

is $\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}P$ measurable for every $A \in \Upsilon$. We suppose that G is ergodic (or metrically transitive) in the sense that every set $A \in \Upsilon$ such that $G(x,y)A = A$, for every $(x,y) \in \mathbb{R}^2$, has a probability $P(A)$ equal to 0 or 1.

For the construction of the boundary layer, we introduce two random processes q and r defined on $\mathbb{R}^2 \times \Pi$ and satisfying the following conditions:

1. q is a stationary random process, that is, for every positive integer n, for every couples $(x_1,y_1), \ldots, (x_n,y_n) \in \mathbb{R}^2$, and for every $B \in \mathcal{B}(\mathbb{R})$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$, one has
$$\begin{aligned} &P\left(\{\omega \mid q((x+x_1, y+y_2),\omega), \ldots, q((x+x_n, y+y_n),\omega) \in B\}\right) \\ &\quad = P\left(\{\omega \mid q((x_1,y_2), G(x,y)\,\omega), \ldots, q((x_n,y_n), G(x,y)\,\omega) \in B\}\right). \end{aligned} \tag{2}$$
As G preserves the measure P (see $(1)_3$) the above equality implies that the joint distribution of $\{\omega \mid q((x_1,y_2),\omega), \ldots, q((x_n,y_n),\omega)\}$ is the same as that of $\{q((x+x_1,y+y_2),\omega), \ldots, q((x+x_n,y+y_n),\omega)\}$ for every $(x,y) \in \mathbb{R}^2$.
2. The partial derivatives $\frac{\partial q}{\partial x_\alpha}$ and $\frac{\partial r}{\partial x_\alpha}$, $\alpha = 1,2$, exist, and there exist nonrandom positive constants c_1, c_2 and c_3 such that the following bounds hold true with probability 1
$$\begin{cases} 0 < c_1 \le q((x,y),\omega) \le c_2 < 1 & \forall (x,y) \in \mathbb{R}^2, \forall \omega \in \Pi, \\ |r((x,y),\omega)| \le q((x,y),\omega) & \forall (x,y) \in \mathbb{R}^2, \forall \omega \in \Pi, \\ \left|\dfrac{\partial q}{\partial x_\alpha}\right|, \ \left|\dfrac{\partial r}{\partial x_\alpha}\right| \le c_3 & \alpha = 1,2, \\ \left|\dfrac{\partial (r-q)}{\partial x_\alpha}((x,y),\omega)\right| \le \dfrac{C}{\|(x,y)\|} & \|(x,y)\| \to \infty,\ \alpha = 1,2. \end{cases} \tag{3}$$
Notice that we do not assume that the random process r is a stationary one.
From the properties of G and q, we deduce the ergodic property (see [7])
$$\langle q(0,0)\rangle = \lim_{\mathcal{T}\longrightarrow\infty} \frac{1}{\mathcal{T}^2} \int_{-\mathcal{T}}^{\mathcal{T}} \int_{-\mathcal{T}}^{\mathcal{T}} q((x,y),\omega)\,\mathrm{d}x\mathrm{d}y, \tag{4}$$
almost surely, where the symbol $\langle . \rangle$ stands for the mathematical expectation with respect to the measure P.

Let $(\alpha_{1i}(\omega))_{i\in\mathbb{Z}}$, $(\alpha_{2i}(\omega))_{i\in\mathbb{Z}}$, $(\beta_{1i}(\omega))_{i\in\mathbb{Z}}$ and $(\beta_{2i}(\omega))_{i\in\mathbb{Z}}$ be sequences of random variables satisfying

$$|\alpha_{li}(\omega)|, \ |\beta_{li}(\omega)| \le c_4, \ \forall i \in \mathbb{Z}, \text{ and } \quad \text{for } l = 1,2, \tag{5}$$

with probability 1, where c_4 is a nonrandom constant.

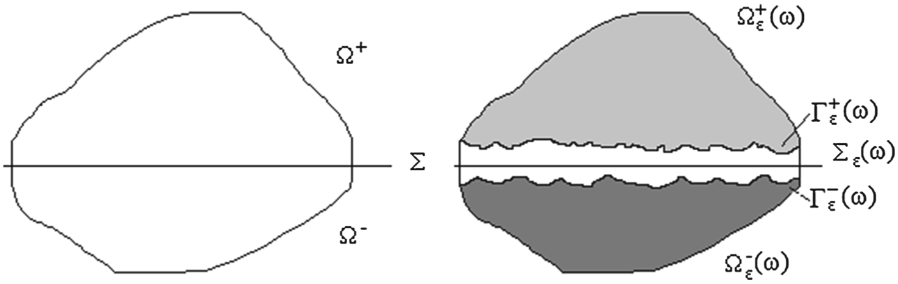

Fig. 1. The domains Ω and Ω_ε

We define the 2D unit reference cell $Y = \left]-\frac{1}{2}, \frac{1}{2}\right[^2$; then, for every $\varepsilon \in]0,1[$ the 2D ε-cell

$$Y_{ij}^{\varepsilon} = \left]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right[^2 + (i\varepsilon, j\varepsilon), \quad \forall i, j \in \mathbb{Z}$$

and the set $I_\varepsilon \subset \mathbb{Z}^2$ as

$$I_\varepsilon = \left\{(i,j) \in \mathbb{Z}^2 \mid Y_{ij}^{\varepsilon} \subset \Sigma\right\}.$$

For an arbitrary realization ω from a set of full measure on Π, for which the conditions (2)–(5) are satisfied, we define the 3D random cell $Z_{ij}^{\varepsilon}(\omega)$, $\forall (i,j) \in I_\varepsilon$, through

$$Z_{ij}^{\varepsilon}(\omega) = \left\{(x_1, x_2, x_3) \mid (x_1, x_2) \in Y_{ij}^{\varepsilon},\, x_3 \in \left]\varepsilon a_{ij,\varepsilon}^{-}, \varepsilon a_{ij,\varepsilon}^{+}\right[\right\}, \tag{6}$$

where $\theta \in \left]1, \frac{3}{2}\right[$ is a given parameter and

$$a_{ij,\varepsilon}^{\pm} = \frac{1}{2}\begin{pmatrix} r\left(\left(\varepsilon^{-\theta}(x_1 - i\varepsilon) + \alpha_{1i}(\omega), \varepsilon^{-\theta}(x_2 - j\varepsilon) + \alpha_{2j}(\omega)\right), \omega\right) \\ \pm q\left(\left(\varepsilon^{-\theta}(x_1 - i\varepsilon) + \beta_{1i}(\omega), \varepsilon^{-\theta}(x_2 - j\varepsilon) + \beta_{2j}(\omega)\right), \omega\right) \end{pmatrix}, \tag{7}$$

(we here omit the dependence with respect to (x_1, x_2) and to the random parameter ω). We suppose that, for every $\varepsilon \in]0,1[$, the layer $\Sigma_\varepsilon(\omega)$ with random thinness which is the union of the cells $Z_{ij}^{\varepsilon}(\omega)$

$$\Sigma_\varepsilon(\omega) = \bigcup_{(i,j)\in I_\varepsilon} Z_{ij}^{\varepsilon}(\omega),$$

is contained in Ω, and we set

$$\Omega_\varepsilon^{\pm}(\omega) = \Omega^{\pm}\backslash\overline{\Sigma}_\varepsilon(\omega),\ \Gamma_\varepsilon^{\pm}(\omega) = \partial\Omega_\varepsilon^{\pm}(\omega) \cap \partial\Sigma_\varepsilon(\omega), \tag{8}$$

according to Fig. 1.

Let ω be an arbitrary realization from a set of full measure on Π, for which the conditions (2)–(5) are satisfied, and $f \in \mathbf{L}^{\infty}\left(\Omega; \mathbb{R}^3\right)$. We consider the following stationary Navier–Stokes problem posed in Ω

$$\begin{cases} -\nu^{+}\Delta u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon + \nabla p_\varepsilon = f & \text{in } \Omega_\varepsilon^{+}(\omega), \\ -\nu^{-}\Delta u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon + \nabla p_\varepsilon = f & \text{in } \Omega_\varepsilon^{-}(\omega), \\ -\varepsilon^{\gamma}\nu^{0}\Delta u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon + \nabla p_\varepsilon = f & \text{in } \Sigma_\varepsilon(\omega), \\ \operatorname{div}(u_\varepsilon) = 0 & \text{in } \Omega, \end{cases} \tag{9}$$

with the transmission and boundary conditions

$$\begin{cases} [u_\varepsilon]_{\Gamma_\varepsilon^{\pm}(\omega)} = 0 & \text{on } \Gamma_\varepsilon^{\pm}(\omega), \\ \nu^{\pm}\dfrac{\partial u_\varepsilon}{\partial n} - \varepsilon^{\gamma}\nu^{0}\dfrac{\partial u_\varepsilon}{\partial n} - [p_\varepsilon]\, n = 0 & \text{on } \Gamma_\varepsilon^{\pm}(\omega), \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

where $[u]_{\Gamma_\varepsilon^{\pm}(\omega)}$ is the jump of u across $\Gamma_\varepsilon^{\pm}$, which is the difference of the two traces of u on the surface $\Gamma_\varepsilon^{\pm}$, and n is the unit normal on $\Gamma_\varepsilon^{\pm}$ outer to $\Sigma_\varepsilon(\omega)$.

The problem (9)–(10) has a unique solution $(u^{\varepsilon}, p^{\varepsilon}) \in \mathbf{V}_{\varepsilon}(\omega)\left(\Omega; \mathbb{R}^3\right) \times L^2(\Omega)/\mathbb{R}$, see [11], where

$$\mathbf{V}_{\varepsilon}(\omega)\left(\Omega; \mathbb{R}^3\right) = \left\{ \begin{array}{l} u \in \mathbf{H}^1\left(\Omega; \mathbb{R}^3\right) \mid \operatorname{div}(u) = 0 \text{ in } \Omega,\ [u]_{\Gamma_{\varepsilon}^{\pm}(\omega)} = 0, \\ \quad u = 0 \text{ on } \partial\Omega \end{array} \right\}. \tag{11}$$

Our purpose is to describe the asymptotic behavior of the solution u_{ε} of (9)–(10) as ε tends to zero. We will use Γ-convergence methods, referring to [1], [5], for instance, for the definition and the properties of this variational convergence. In the case where $\gamma = 1$, we will prove that, with probability 1, the solution u_{ε} of (9)–(10) converges in some topology which is precisely defined in Definition 1 to the solution u_0 of the following nonrandom Navier–Stokes limit problem

$$\begin{cases} -\nu^{\pm}\Delta u_0 + (u_0 \cdot \nabla) u_0 + \nabla p_0^{\pm} = f & \text{in } \Omega^{\pm}, \\ \operatorname{div}\ (u_0) = 0 & \text{in } \Omega^{\pm}, \\ u_0 = 0 & \text{on } \partial\Omega, \\ [(u_0)_3]_{\Sigma} = 0 & \text{on } \Sigma, \\ \nu^{+}\left(\dfrac{\partial (u_0)_3}{\partial x_3}\right)^{+} |_{\Sigma} - \nu^{-}\left(\dfrac{\partial (u_0)_3}{\partial x_3}\right)^{-} |_{\Sigma} = p_0^{+} - p_0^{-} & \text{on } \Sigma, \\ \nu^{+}\left(\dfrac{\partial (u_0)_{\beta}}{\partial x_3}\right)^{+} |_{\Sigma} = \nu^{-}\left(\dfrac{\partial (u_0)_{\beta}}{\partial x_3}\right)^{-} |_{\Sigma} = \dfrac{\nu^0}{\langle q(0,0)\rangle}\left[(u_0)_{\beta}\right]_{\Sigma} & \text{on } \Sigma,\ \beta = 1, 2 \end{cases} \tag{12}$$

The boundary condition $(12)_4$ represents the continuity of the normal velocity through the interface Σ. The relation $(12)_5$ means that the difference of the normal fluxes on the two sides of the interface Σ is equal to the difference of the pressures. In the interfacial law $(12)_6$, the tangential fluxes are equal and are proportional to the jump of the tangential velocities across the interface Σ, through the coefficient of proportionality $\frac{\nu^0}{\langle q(0,0)\rangle}$.

This convergence result shows the importance of the frictionless dynamics and of the random configuration of the boundary layers when trying to control the exchange between the two interacting fluid flows.

The description of the asymptotic behavior of a three-body system composed of a thin layer placed between two pieces and given a different constitutive law in each domain has been treated by many authors in different contexts: scalar case, elastic materials or fluid flows. In the scalar case, Bakhvalov and Panasenko considered in [2, Chapter 9, section 4] the problem

$$\sum_{i=1}^{s} \frac{\partial}{\partial x_i}\left(\overline{K}_{\varepsilon}\left(\frac{x}{\varepsilon}\right)\frac{\partial u}{\partial x_i}\right) = f(x),$$

posed in the layer $\{x \in \mathbb{R}^s \mid x_1 \in (d_1, d_2)\}$ where $d_1 < 0 < d_2$ are multiples of ε and

$$\overline{K}_{\varepsilon}(\xi) = \begin{cases} K_1(\xi) & \text{if } \xi_1 < 0, \\ K_2(\xi) & \text{if } \xi_1 > 1, \\ \varepsilon^{\gamma} K_3(\xi) & \text{if } \xi_1 \in (0,1), \end{cases}$$

the functions K_i being 1-periodic. The boundary condition

$$u\,|_{x_1 = d_1} = g(x_2, \ldots, x_s)$$

was added with g periodic.

In the case where $\gamma = 1$ (called "poorly conductive interlayer"), the authors proved that the limit v_0 of u satisfies

$$\frac{\partial v_0}{\partial \nu^1}\left(:= \sum_{i=1}^{s} \widehat{K}_1^{1j}\left(\frac{x}{\varepsilon}\right)\frac{\partial v_0}{\partial x_i}\,|_{x_1 = (-1)^1 \cdot 0}\right) = \frac{\partial v_0}{\partial \nu^2} = \lambda\,[v_0],$$

where $\lambda = \left\langle K_3 \frac{\partial N^3}{\partial \xi_1} \right\rangle$, N^3 being the solution in $H^1(0,1)^s$ and 1-periodic in $\xi_2, \ldots, \xi_s$ of the auxiliary problem

$$\begin{cases} \sum_{i=1}^{s} \frac{\partial}{\partial \xi_i} \left(K_3(\xi) \frac{\partial N^3}{\partial \xi_i} \right) = 0, \\ N^3 \mid_{\xi_1=0} = 0, \\ N^3 \mid_{\xi_1=1} = 1. \end{cases}$$

In our limit fluid flow problem (12), the difference between the normal derivatives of $(u_0)_3$ is not equal to 0 but to the difference of the pressures. For the other components $(u_0)_\beta$, we get a comparable limit problem than in the above-described scalar case.

In the last part of this work, we will discuss the case where Reynolds number in the boundary layer is of order $O(\varepsilon^{-\gamma})$, with $\gamma > 1$ or $0 < \gamma < 1$.

2. A priori estimates in the case where $\gamma = 1$

Lemma 1. *Let ω be a fixed realization from a set of full measure on Π, for which the conditions (2)–(5) are satisfied. Then, the solution $(u_\varepsilon, p_\varepsilon)$ of (9)–(10) satisfies the following estimates*

$$\sup_\varepsilon \left(\int_{\Omega_\varepsilon^+(\omega) \cup \Omega_\varepsilon^-(\omega)} |\nabla u_\varepsilon|^2 \, dx + \varepsilon \int_{\Sigma_\varepsilon(\omega)} |\nabla u_\varepsilon|^2 \, dx \right) < +\infty,$$

$$\sup_\varepsilon \left(\int_{\Omega_\varepsilon^+(\omega) \cup \Omega_\varepsilon^-(\omega)} |u_\varepsilon|^2 \, dx \right) < +\infty,$$

$$\sup_\varepsilon \left(\frac{1}{\varepsilon} \int_{\Sigma_\varepsilon(\omega)} |u_\varepsilon|^2 \, dx \right) < +\infty. \tag{13}$$

Proof. Let $x' = (x_1, x_2) \in Y_{ij}^\varepsilon$, for some $(i,j) \in I_\varepsilon$. We write, for every x_3 belonging to the interval $\left(\varepsilon a_{ij,\varepsilon}^-, \varepsilon a_{ij,\varepsilon}^+\right)$

$$u_\varepsilon(x', x_3) = u_\varepsilon\left(x', \varepsilon a_{ij,\varepsilon}^-\right) + \int_{\varepsilon a_{ij,\varepsilon}^-}^{x_3} \frac{\partial u_\varepsilon}{\partial x_3}(x', s) \, ds.$$

Using (3) and Cauchy–Schwarz' inequality, we get

$$|u_\varepsilon(x', x_3)|^2 \le 2 \left(\left| u_\varepsilon\left(x', \varepsilon a_{ij,\varepsilon}^-\right) \right|^2 + \varepsilon \int_{\varepsilon a_{ij,\varepsilon}^-}^{\varepsilon a_{ij,\varepsilon}^+} \left| \frac{\partial u_\varepsilon}{\partial x_3} \right|^2 ds \right),$$

from which we deduce, using (3) and (5),

$$\int_{Z_{ij}^\varepsilon(\omega)} |u_\varepsilon(x)|^2 \, dx \le 2 \left(\varepsilon \int_{Y_{ij}^\varepsilon} \left| u_\varepsilon\left(x', \varepsilon a_{ij,\varepsilon}^-\right) \right|^2 dx' + \varepsilon^2 \int_{Z_{ij}^\varepsilon(\omega)} |\nabla u_\varepsilon(x)|^2 \, dx \right)$$

and, summing over I_ε,

$$\int_{\Sigma_\varepsilon(\omega)} |u_\varepsilon(x)|^2 \,\mathrm{d}x \leq C\left(\varepsilon\sum_{i,j}\int_{Y_{ij}^\varepsilon} \left|u_\varepsilon\left(x',\varepsilon a_{ij,\varepsilon}^-\right)\right|^2 \mathrm{d}x' + \varepsilon^2 \int_{\Sigma_\varepsilon(\omega)} |\nabla u_\varepsilon(x)|^2 \,\mathrm{d}x\right). \tag{14}$$

Using a trace theorem in $\Omega_\varepsilon^\pm(\omega)$ and the homogeneous Dirichlet boundary condition $u^\varepsilon = 0$ on $\partial\Omega$, there exists a nonrandom positive constant C independent of ε such that

$$\sum_{i,j}\int_{Y_{ij}^\varepsilon} \left|u_\varepsilon\left(x',\varepsilon a_{ij,\varepsilon}^\pm\right)\right|^2 \mathrm{d}x' \leq C \int_{\Omega_\varepsilon^\pm(\omega)} |\nabla u_\varepsilon(x)|^2 \,\mathrm{d}x. \tag{15}$$

We deduce from (14)–(15) that

$$\int_{\Sigma_\varepsilon(\omega)} |u_\varepsilon(x)|^2 \,\mathrm{d}x \leq C\left(\varepsilon \int_{\Omega_\varepsilon^\pm(\omega)} |\nabla u_\varepsilon(x)|^2 \,\mathrm{d}x + \varepsilon^2 \int_{\Sigma_\varepsilon(\omega)} |\nabla u_\varepsilon(x)|^2 \,\mathrm{d}x\right). \tag{16}$$

Now multiplying $(9)_{1,2,3}$ by u_ε and using Green's formula and Cauchy–Schwarz' inequality, we get, thanks to the boundary conditions (10),

$$\begin{aligned}
&\nu^+ \int_{\Omega_\varepsilon^+(\omega)} |\nabla u_\varepsilon|^2 \,\mathrm{d}x + \nu^- \int_{\Omega_\varepsilon^-(\omega)} |\nabla u_\varepsilon|^2 \,\mathrm{d}x + \varepsilon\nu^0 \int_{\Sigma_\varepsilon(\omega)} |\nabla u_\varepsilon|^2 \,\mathrm{d}x \\
&= \int_\Omega f\cdot u_\varepsilon \mathrm{d}x \leq C\left(\left(\int_{\Omega_\varepsilon^+(\omega)\cup\Omega_\varepsilon^-(\omega)} |u_\varepsilon|^2 \,\mathrm{d}x\right)^{1/2} + \left(\int_{\Sigma_\varepsilon(\omega)} |u_\varepsilon|^2 \,\mathrm{d}x\right)^{1/2}\right).
\end{aligned}$$

Using Poincaré's inequality in $\Omega_\varepsilon^+(\omega)$ and $\Omega_\varepsilon^-(\omega)$ and (16), we obtain

$$\begin{aligned}
&\nu^+ \int_{\Omega_\varepsilon^+(\omega)} |\nabla u_\varepsilon|^2 \,\mathrm{d}x + \nu^- \int_{\Omega_\varepsilon^-(\omega)} |\nabla u_\varepsilon|^2 \,\mathrm{d}x + \varepsilon\nu^0 \int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|^2 \,\mathrm{d}x \\
&\leq C\left(\left(\int_{\Omega_\varepsilon^+(\omega)\cup\Omega_\varepsilon^-(\omega)} |\nabla u_\varepsilon|^2 \,\mathrm{d}x\right)^{1/2} + \left(\varepsilon\int_{\Sigma_\varepsilon} |\nabla u_\varepsilon(x)|^2 \,\mathrm{d}x\right)^{1/2}\right).
\end{aligned} \tag{17}$$

Let now x, y and z be nonnegative real numbers satisfying

$$x^2 + y^2 + z^2 \leq c_1(x+y+z), \tag{18}$$

for some positive constant c_1. We prove the existence of a positive constant c_2 such that $x^2+y^2+z^r \leq c_2$. Otherwise, $\frac{x^2+y^2+z^2}{x+y+z}$ tends to ∞ when x, y or z tends to ∞, which contradicts (18). Thus, (17) implies $(13)_1$.

Using Poincaré's inequality, we deduce $(13)_2$. Using (16), we deduce $(13)_3$, which ends the proof. □

Remark 1. When $\gamma < 1$, we deduce from the preceding computations that the estimates (13) are still true and that we have

$$\sup_{\varepsilon>0} \varepsilon^\gamma \int_{\Sigma_\varepsilon(\omega)} |\nabla u_\varepsilon|^2 \,\mathrm{d}x < +\infty.$$

In order to get estimates on the pressure p_ε, let us first define the zero mean value pressure

$$\overline{p_\varepsilon^\pm} = p_\varepsilon - \frac{1}{\left|\Omega_\varepsilon^\pm(\omega)\right|} \int\limits_{\Omega_\varepsilon^\pm(\omega)} p_\varepsilon \mathrm{d}x.$$

We have the following uniform estimate.

Lemma 2. *One has* $\sup_\varepsilon \left\|\overline{p_\varepsilon^\pm}\right\|_{L^2\left(\Omega_\varepsilon^\pm\right)} < \infty$.

Proof. The problem

$$\begin{cases} \operatorname{div}\left(\psi_\varepsilon^\pm\right) = \overline{p_\varepsilon^\pm} & \text{in } \Omega_\varepsilon^\pm(\omega), \\ \psi_\varepsilon^\pm = 0 & \text{on } \partial\Omega_\varepsilon^\pm(\omega), \end{cases}$$

has a unique solution $\psi_\varepsilon^\pm \in \mathbf{H}_0^1\left(\Omega_\varepsilon^\pm(\omega);\mathbb{R}^3\right)$ satisfying

$$\left\|\nabla\psi_\varepsilon^\pm\right\|_{\mathbf{L}^2\left(\Omega_\varepsilon^\pm(\omega);\mathbb{R}^9\right)} \leq C(\Omega)\left\|\overline{p_\varepsilon^\pm}\right\|_{L^2\left(\Omega_\varepsilon^\pm(\omega)\right)},$$

for a constant $C(\Omega)$ only depending of Ω (see [11]).

Multiplying $(9)_1$ (resp. $(9)_2$) by ψ_ε^+ (resp. ψ_ε^-), using Green's formula and summing, we obtain

$$\begin{aligned} &\nu^+ \int\limits_{\Omega_\varepsilon^+(\omega)} \nabla u_\varepsilon \cdot \nabla\psi_\varepsilon^\pm \mathrm{d}x + \nu^- \int\limits_{\Omega_\varepsilon^-(\omega)} \nabla u_\varepsilon \cdot \nabla\psi_\varepsilon^\pm \mathrm{d}x \\ &\quad + \int\limits_{\Omega_\varepsilon^+(\omega)} (u_\varepsilon\cdot\nabla)\, u_\varepsilon \cdot \psi_\varepsilon^+ \mathrm{d}x + \int\limits_{\Omega_\varepsilon^-(\omega)} (u_\varepsilon\cdot\nabla)\, u_\varepsilon \cdot \psi_\varepsilon^- \mathrm{d}x \\ &= \int\limits_{\Omega_\varepsilon^+(\omega)} f\cdot\psi_\varepsilon^+ \mathrm{d}x + \int\limits_{\Omega_\varepsilon^-(\omega)} f\cdot\psi_\varepsilon^- \mathrm{d}x + \int\limits_{\Omega_\varepsilon^+(\omega)} \left(\overline{p_\varepsilon^+}\right)^2 \mathrm{d}x + \int\limits_{\Omega_\varepsilon^-(\omega)} \left(\overline{p_\varepsilon^-}\right)^2 \mathrm{d}x, \end{aligned}$$

thanks to the boundary conditions satisfied by $\psi_\varepsilon^\pm$ on $\partial\Omega_\varepsilon^\pm(\omega)$. We then compute, using Cauchy–Schwarz' inequality,

$$\begin{aligned} \left|\int\limits_{\Omega_\varepsilon^\pm(\omega)} f\cdot\psi_\varepsilon^\pm \mathrm{d}x\right| &\leq C\left\|\overline{p_\varepsilon^\pm}\right\|_{L^2\left(\Omega_\varepsilon^\pm(\omega)\right)}, \\ \left|\int\limits_{\Omega_\varepsilon^\pm(\omega)} (u_\varepsilon\cdot\nabla)\, u_\varepsilon\cdot\psi_\varepsilon^\pm \mathrm{d}x\right| &\leq C\left\|\nabla\psi_\varepsilon^\pm\right\|_{\mathbf{L}^2\left(\Omega_\varepsilon^\pm(\omega);\mathbb{R}^3\right)} \left\|\nabla u_\varepsilon\right\|^2_{\mathbf{L}^2\left(\Omega_\varepsilon^\pm(\omega);\mathbb{R}^9\right)} \\ &\leq C\left\|\overline{p_\varepsilon^\pm}\right\|_{L^2\left(\Omega_\varepsilon^\pm(\omega)\right)} \left\|\nabla u_\varepsilon\right\|^2_{\mathbf{L}^2\left(\Omega_\varepsilon^\pm(\omega);\mathbb{R}^9\right)}, \\ \left|\int\limits_{\Omega_\varepsilon^\pm(\omega)} \nabla u_\varepsilon\cdot\nabla\psi_\varepsilon^\pm \mathrm{d}x\right| &\leq C\left\|\overline{p_\varepsilon^\pm}\right\|_{L^2\left(\Omega_\varepsilon^\pm(\omega)\right)} \left\|\nabla u_\varepsilon\right\|_{\mathbf{L}^2\left(\Omega_\varepsilon^\pm(\omega);\mathbb{R}^3\right)}, \end{aligned}$$

which leads to the desired estimate. □

From the conditions (3) and (5), we deduce the following construction of operators, which will allow working in fixed domains.

Lemma 3. *Let ω be a fixed event, for which the conditions (2)–(5) are satisfied. Then, there exists an invertible map $T_\varepsilon^+ : \Omega^+ \to \Omega_\varepsilon^+(\omega)$ (resp. $T_\varepsilon^- : \Omega^- \to \Omega_\varepsilon^-(\omega)$) such that for every $u \in \mathbf{H}_0^1\left(\Omega_\varepsilon(\omega);\mathbb{R}^3\right)$*

$$\begin{array}{l} \|u \circ T_\varepsilon^+\|_{\mathbf{H}^1(\Omega^+;\mathbb{R}^3)} \le C\,\|u\|_{\mathbf{H}^1\left(\Omega_\varepsilon^+(\omega);\mathbb{R}^3\right)}, \\ \|u \circ T_\varepsilon^-\|_{\mathbf{H}^1(\Omega^-;\mathbb{R}^3)} \le C\,\|u\|_{\mathbf{H}^1\left(\Omega_\varepsilon^-(\omega);\mathbb{R}^3\right)}, \end{array} \tag{19}$$

where C is a nonrandom positive constant independent of ε.

We define the transformation T^ε from $\Omega\backslash\Sigma$ into $\Omega_\varepsilon(\omega)$ as

$$T^\varepsilon(x) = \begin{cases} T_\varepsilon^+(x) & \text{if } x \in \Omega^+, \\ T_\varepsilon^-(x) & \text{if } x \in \Omega^-. \end{cases}$$

Proof. Referring to the construction (8) of $\Omega_\varepsilon(\omega)$, we define

$$\forall (x', x_3) \in \Omega^\pm : T_\varepsilon^\pm(x) = \begin{cases} x & \text{if } x \in \Omega^\pm \backslash \Sigma_\varepsilon(\omega), \\ \left(x', L_{ij,\varepsilon}^\pm(x)\right) & \text{if } x \in \Omega^\pm \cap Z_{ij}^\varepsilon(\omega),\, \forall (i,j) \in I_\varepsilon, \end{cases}$$

where $L_{ij,\varepsilon}^\pm(x) = \varepsilon a_{ij,\varepsilon}^\pm + \frac{x_3}{h^\pm(x')}\left(h^\pm(x') - \varepsilon a_{ij,\varepsilon}^\pm\right)$. We immediately observe that

$$\begin{array}{c} T_\varepsilon^\pm(x',0) = \left(x', \varepsilon a_{ij,\varepsilon}^\pm\right), \\ T_\varepsilon^\pm(x', h^\pm(x')) = (x', h^\pm(x')). \end{array}$$

We compute the Jacobian of $T_\varepsilon^\pm$, which is the determinant of its gradient: $Jac\,(T_\varepsilon^\pm)(x) = \frac{\partial (T_\varepsilon^\pm)_3}{\partial x_3}(x) = \frac{h^\pm(x') - \varepsilon a_{ij,\varepsilon}^\pm}{h^\pm(x')}$, from which we deduce, according to the properties (3) of the random processes q and r, that $Jac\,(T_\varepsilon^\pm) = 1 + o(\varepsilon) \to_{\varepsilon\to 0} 1$. This proves that $T_\varepsilon^\pm$ is invertible.

The estimate (19) is then a direct consequence of the definition of $T_\varepsilon^\pm$. □

We have the following compactness result.

Proposition 1. *Let ω be an arbitrary realization from a set of full measure on Π, for which the conditions (2)–(5) are satisfied. Let $(u_\varepsilon)_\varepsilon$ be a sequence such that $u_\varepsilon \in \mathbf{V}_\varepsilon(\omega)\left(\Omega;\mathbb{R}^3\right)$ for every ε, which satisfies the estimates (13). Then, with probability 1, there exists a subsequence of $(u_\varepsilon)_\varepsilon$, still denoted in the same way, such that:*

1. *one has the following convergences*

$$\begin{array}{rl} u_\varepsilon \circ T^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} u_0 & \text{in } \mathbf{H}^1\left(\Omega\backslash\Sigma;\mathbb{R}^3\right)\text{-weak}, \\ u_\varepsilon \underset{\varepsilon\to 0}{\rightarrow} u_0 & \text{in } \mathbf{L}^2\left(\Omega;\mathbb{R}^3\right)\text{-strong}. \end{array}$$

2. *u_0 belongs to the space $\mathbf{V}_0\left(\Omega\backslash\Sigma;\mathbb{R}^3\right)$ defined through*

$$\mathbf{V}_0\left(\Omega\backslash\Sigma;\mathbb{R}^3\right) = \left\{ u \in \mathbf{H}^1\left(\Omega\backslash\Sigma;\mathbb{R}^3\right) \mid div\,(u) = 0 \text{ in } \Omega\, [u_3]_\Sigma = 0 \text{ and } u = 0 \text{ on } \partial\Omega \right\}. \tag{20}$$

Proof. 1. From the estimates (13) and Lemma 3, we deduce that the sequence $(u_\varepsilon \circ T^\varepsilon)_\varepsilon$ is bounded in $\mathbf{H}^1\left(\Omega\backslash\Sigma;\mathbb{R}^3\right)$. Up to some subsequence, this sequence $(u_\varepsilon \circ T^\varepsilon)_\varepsilon$ converges to some u_0 in $\mathbf{H}^1\left(\Omega\backslash\Sigma;\mathbb{R}^3\right)$-weak.

From $(13)_{1,2}$, it follows that u_0 belongs to $\mathbf{L}^2\left(\Omega;\mathbb{R}^3\right)$ and, up to some subsequence, we have

$$\mathbf{1}_{\Omega^\pm}\nabla\left(u_\varepsilon \circ T^\varepsilon\right) \underset{\varepsilon\to 0}{\rightharpoonup} \nabla u_0 \text{ in } \mathbf{L}^2\left(\Omega^\pm;\mathbb{R}^9\right)\text{-weak},$$

where $\mathbf{1}_{\Omega^\pm}$ is the characteristic function of $\Omega^\pm$.

We then write

$$\int_{\Omega} |u_\varepsilon - u_0|^2 \,\mathrm{d}x = \int_{\Omega_\varepsilon^+(\omega)\cup\Omega_\varepsilon^-(\omega)} |u_\varepsilon - u_0|^2 \,\mathrm{d}x + \int_{\Sigma_\varepsilon(\omega)} |u_\varepsilon - u_0|^2 \,\mathrm{d}x.$$

Using the above convergences, a Sobolev embedding and $(13)_3$, we prove that $(u_\varepsilon)_\varepsilon$ converges to u_0 in $\mathbf{L}^2\left(\Omega;\mathbb{R}^3\right)$-strong.

2. Let $(u_0)^+$ (resp. $(u_0)^-$) be the trace of $u_0 \in \mathbf{H}^1\left(\Omega^+;\mathbb{R}^3\right)$ (resp. $\mathbf{H}^1\left(\Omega^-;\mathbb{R}^3\right)$) on Σ. Using a trace theorem, we have, up to some subsequence,

$$\lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon^\pm(\omega)} \left| u_\varepsilon - \left((u_0)^\pm \circ \left(T_\varepsilon^\pm\right)^{-1}\right)\right| \mathrm{d}s = \lim_{\varepsilon\to 0} \int_{\Sigma} \left| u_\varepsilon \circ T_\varepsilon^\pm - (u_0)^+ \right| Jac\left(T_\varepsilon^\pm\right) \mathrm{d}x' = 0. \tag{21}$$

As $\operatorname{div}(u_\varepsilon) = 0$ in Ω, we have, for every $\varphi \in C_c^\infty(\Sigma)$,

$$0 = \int_{\Sigma_\varepsilon(\omega)} \operatorname{div}(u_\varepsilon)\,\varphi \mathrm{d}x = -\int_{\Sigma_\varepsilon(\omega)} (u_\varepsilon)_\tau \cdot \nabla_\tau \varphi \mathrm{d}x + \int_{\Gamma_\varepsilon^+(\omega)} u_\varepsilon \cdot n\varphi \mathrm{d}s - \int_{\Gamma_\varepsilon^-(\omega)} u_\varepsilon \cdot n\varphi \mathrm{d}s,$$

where $(u_\varepsilon)_\tau = \left((u_\varepsilon)_1, (u_\varepsilon)_2\right)$ and $\nabla_\tau\varphi = \left(\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}\right)$. Then, passing to the limit using $(13)_3$ and (21), we deduce that $\left[(u_0)_3\right]_\Sigma = 0$.

As $\operatorname{div}(u_\varepsilon) = 0$ in $\Omega_\varepsilon^\pm(\omega)$, we easily deduce that $\operatorname{div}(u_0) = 0$ in $\Omega^\pm$. Thus, u_0 belongs to $\mathbf{V}_0\left(\Omega\backslash\Sigma;\mathbb{R}^3\right)$. □

We have the following result in the layer $\Sigma_\varepsilon(\omega)$.

Lemma 4. *Let ω be an arbitrary realization from a set of full measure on Π, for which the conditions (2)–(5) are satisfied.*

1. *For every $\varphi \in C_0^1\left(\mathbb{R}^3\right)$, we have*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\Sigma_\varepsilon(\omega)} \varphi(x_1,x_2,x_3)\,\mathrm{d}x = \langle q(0,0)\rangle \int_{\Sigma} \varphi(x_1,x_2,0)\,\mathrm{d}x_1\mathrm{d}x_2.$$

2. *Let $(w_\varepsilon)_\varepsilon$ be a sequence in $L^2(\Omega)$ such that $\sup_\varepsilon \frac{1}{\varepsilon}\int_{\Sigma_\varepsilon(\omega)} (w_\varepsilon(x))^2 \,\mathrm{d}x < +\infty$. There exists a subsequence, still denoted in the same way, and $w \in L^2\left(\mathbb{R}^2\right)$ such that, for every $\varphi \in C_0^1\left(\mathbb{R}^3\right)$, we have almost surely*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\Sigma_\varepsilon(\omega)} w_\varepsilon(x)\,\varphi(x)\,\mathrm{d}x = \langle q(0,0)\rangle \int_{\Sigma} w(x_1,x_2)\,\varphi(x_1,x_2,0)\,\mathrm{d}x_1\mathrm{d}x_2.$$

Proof. Let ω be a fixed event, for which the conditions (2)–(5) are satisfied.

1. Assuming that the conditions (3) are satisfied, we compute

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\limits_{\Sigma_\varepsilon(\omega)}\varphi\left(x_1,x_2,x_3\right)\mathrm{d}x\\
&=\lim_{\varepsilon\to 0}\sum_{(i,j)\in I_\varepsilon}\frac{1}{\varepsilon}\int\limits_{Z_{ij}^\varepsilon(\omega)}\varphi\left(x_1,x_2,x_3\right)\mathrm{d}x\\
&=\lim_{\varepsilon\to 0}\sum_{(i,j)\in I_\varepsilon}\frac{1}{\varepsilon}\int\limits_{Y_{ij}^\varepsilon}\int\limits_{\varepsilon a_{ij,\varepsilon}^-}^{\varepsilon a_{ij,\varepsilon}^+}\varphi\left(x_1,x_2,x_3\right)\mathrm{d}x\\
&=\lim_{\varepsilon\to 0}\sum_{(i,j)\in I_\varepsilon}\int\limits_{Y_{ij}^\varepsilon}\int\limits_{-\frac{1}{2}}^{\frac{1}{2}}q_{ij}^\varepsilon\varphi\left(x_1,x_2,\varepsilon q_{ij}^\varepsilon z+\frac{\varepsilon}{2}r_{ij}^\varepsilon\right)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}z\\
&=\lim_{\varepsilon\to 0}\sum_{(i,j)\in I_\varepsilon}\varepsilon^2\varphi\left(i\varepsilon,j\varepsilon,0\right)\int\limits_{Y}q_{ij,\varepsilon}^\theta\left(y_1,y_2\right)\mathrm{d}y_1\mathrm{d}y_2,
\end{aligned}$$

where we have introduced the change in variables $x_3=\varepsilon q_{ij}^\varepsilon z+\frac{\varepsilon}{2}r_{ij}^\varepsilon$ with

$$\begin{aligned}
q_{ij}^\varepsilon &:= q\left(\left(\varepsilon^{-\theta}\left(x_1-i\varepsilon\right)+\beta_{1i}\left(\omega\right),\varepsilon^{-\theta}\left(x_2-j\varepsilon\right)+\beta_{2j}\left(\omega\right)\right),\omega\right),\\
r_{ij}^\varepsilon &:= r\left(\left(\varepsilon^{-\theta}\left(x_1-i\varepsilon\right)+\alpha_{1i}\left(\omega\right),\varepsilon^{-\theta}\left(x_2-j\varepsilon\right)+\alpha_{2j}\left(\omega\right)\right),\omega\right),\\
q_{ij,\varepsilon}^\theta\left(y_1,y_2\right) &:= q\left(\left(\varepsilon^{-(\theta-1)}y_1+\beta_{1i}\left(\omega\right),\varepsilon^{-(\theta-1)}y_2+\beta_{2j}\left(\omega\right)\right),\omega\right)
\end{aligned}\tag{22}$$

(see the definition (7) of $a_{ij,\varepsilon}^\pm$) and then the change in variables defined through

$$\left(y_1=\frac{x_1-i\varepsilon}{\varepsilon},y_2=\frac{x_2-j\varepsilon}{\varepsilon}\right)$$

in the cell Y_{ij}^ε. Introducing $z_1=\varepsilon^{-(\theta-1)}y_1$ and $z_2=\varepsilon^{-(\theta-1)}y_2$, we finally obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\limits_{\Sigma_\varepsilon(\omega)}\varphi\left(x_1,x_2,x_3\right)\mathrm{d}x\\
&=\lim_{\varepsilon\to 0}\sum_{(i,j)\in I_\varepsilon}\varepsilon^2\varphi\left(i\varepsilon,j\varepsilon,0\right)\frac{1}{\varepsilon^{\theta-1}}\int\limits_{-\frac{\varepsilon^{\theta-1}}{2}}^{\frac{\varepsilon^{\theta-1}}{2}}\frac{1}{\varepsilon^{\theta-1}}\int\limits_{-\frac{\varepsilon^{\theta-1}}{2}}^{\frac{\varepsilon^{\theta-1}}{2}}q\left(\left(z_1+\beta_{1i}\left(\omega\right),z_2+\beta_{2j}\left(\omega\right)\right),\omega\right)\mathrm{d}z_1\mathrm{d}z_2\\
&=\left\langle q\left(0,0\right)\right\rangle\int\limits_{\Sigma}\varphi\left(x_1,x_2,0\right)\mathrm{d}x_1\mathrm{d}x_2,
\end{aligned}$$

using the ergodicity property (4).

2. One can deduce from the preceding point that the sequence of measures $\left(\mu_\varepsilon\right)_\varepsilon$ defined through $\mu_\varepsilon=\frac{\mathbf{1}_{\Sigma_\varepsilon(\omega)}(x)\mathrm{d}x}{\varepsilon}$, $\mathbf{1}_A$ being the characteristic function of the set A, converges in the weak sense of measures to the measure $\mu=\left\langle q\left(0,0\right)\right\rangle\mathbf{1}_\Sigma\left(x'\right)\mathrm{d}x'$, when ε goes to 0. Observing that

$$\int\limits_{\mathbb{R}^3}\left|w_\varepsilon\right|d\mu_\varepsilon\le\frac{1}{\varepsilon}\int\limits_{\Sigma_\varepsilon(\omega)}\left(w_\varepsilon\left(x\right)\right)^2\mathrm{d}x$$

and using the hypothesis on $(w_\varepsilon)_\varepsilon$, we deduce that the sequence $(w_\varepsilon \nu_\varepsilon)_\varepsilon$ converges, up to some subsequence, to some measure χ, in the weak sense of measures. For every $\varphi \in C_0^1(\mathbb{R}^3)$, we have, thanks to Fenchel's inequality,

$$2\int_{\mathbb{R}^3} w_\varepsilon \varphi d\mu_\varepsilon - \int_{\mathbb{R}^3} (\varphi)^2 d\mu_\varepsilon \leq \int_{\mathbb{R}^3} (w_\varepsilon)^2 d\mu_\varepsilon.$$

Then, passing to the limit, we get

$$2\langle \chi, \varphi\rangle - \int_{\mathbb{R}^2} \varphi^2(x',0)\, d\mu \leq \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^3} (w_\varepsilon)^2 d\mu_\varepsilon < +\infty.$$

This implies

$$\sup\left\{ \langle \chi, \varphi\rangle \mid \varphi \in C_0^1(\mathbb{R}^3),\ \int_{\mathbb{R}^2} \varphi^2(x',0)\, d\mu < +\infty \right\}.$$

Thus, using Riesz' representation theorem, we can identify χ with $w\mu$, for some $w \in L^2(\mathbb{R}^2)$. □

Lemma 5. *Let ω be an arbitrary realization from a set of full measure on Π, for which the conditions (2)–(5) are satisfied. Let $(u_\varepsilon)_\varepsilon$ be a sequence such that $u_\varepsilon \in \mathbf{V}_\varepsilon(\omega)(\Omega;\mathbb{R}^3)$, for every $\varepsilon > 0$, and satisfying the estimates (13). Then, with probability 1, there exists a subsequence of $(u_\varepsilon)_\varepsilon$, still denoted in the same way, such that*

$$\begin{aligned}
u_\varepsilon \circ T^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} u_0 &\quad \text{in } \mathbf{H}^1(\Omega\backslash\Sigma;\mathbb{R}^3)\text{-weak},\\
u_\varepsilon \underset{\varepsilon\to 0}{\rightarrow} u_0 &\quad \text{in } \mathbf{L}^2(\Omega;\mathbb{R}^3)\text{-strong},\\
\lim_{\varepsilon\to 0} \int_{\Sigma_\varepsilon(\omega)} \nabla (u_\varepsilon)_i(x)\cdot \varphi(x)\,\mathrm{d}x &= \int_\Sigma [(u_0)_i]_\Sigma \varphi_3(x_1,x_2,0)\,\mathrm{d}x_1\mathrm{d}x_2,\\
\liminf_{\varepsilon\to 0} \int_{\Sigma_\varepsilon(\omega)} \varepsilon |\nabla u_\varepsilon(x)|^2 \mathrm{d}x &\geq \frac{1}{\langle q(0,0)\rangle} \int_\Sigma ([u_0]_\Sigma)^2 \mathrm{d}x_1\mathrm{d}x_2,
\end{aligned}$$

for every $\varphi \in \mathbf{C}_0^1(\mathbb{R}^3;\mathbb{R}^3)$ and every $i = 1, 2, 3$.

Proof. The first and second convergences of Lemma 5 have already been proved in Proposition 1 1. Let us now fix an event ω, for which the conditions (2)–(5) are satisfied. Then, as $(u_\varepsilon)_\varepsilon$ satisfies the estimate $(13)_1$, one has

$$\sup_\varepsilon \frac{1}{\varepsilon} \int_{\Sigma_\varepsilon(\omega)} |\varepsilon \nabla u_\varepsilon(x)|^2 \mathrm{d}x < +\infty.$$

Thus, using Lemma 4 2., there exists $\chi \in \mathbf{L}^2(\Sigma;\mathbb{R}^2)$, such that, for every $\varphi \in \mathbf{C}_0^1(\mathbb{R}^3;\mathbb{R}^3)$,

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\Sigma_\varepsilon(\omega)} \varepsilon \nabla (u_\varepsilon)_i(x)\cdot\varphi(x)\,\mathrm{d}x = \langle q(0,0)\rangle \int_\Sigma \chi_i \varphi_3(x_1,x_2,0)\,\mathrm{d}x_1\mathrm{d}x_2,$$

up to some subsequence of $(\varepsilon\nabla u^\varepsilon)_\varepsilon$ still denoted in the same way. On the other hand, using Green's formula, we still have for every $\varphi \in \mathbf{C}_0^1(\mathbb{R}^3;\mathbb{R}^3)$ and every $i = 1, 2, 3$

$$\int_{\Sigma_\varepsilon(\omega)} \nabla (u_\varepsilon)_i(x)\cdot\varphi(x)\,\mathrm{d}x = -\int_{\Sigma_\varepsilon(\omega)} (u_\varepsilon)_i(x)\,\mathrm{div}\varphi(x)\,\mathrm{d}x + \int_{\Gamma_\varepsilon^+} (u_\varepsilon)_i \varphi\cdot n\mathrm{d}s - \int_{\Gamma_\varepsilon^-} (u_\varepsilon)_i \varphi\cdot n\mathrm{d}s,$$

from which we deduce, using $(13)_3$ and (21), that, up to some subsequence,

$$\lim_{\varepsilon\to 0}\int_{\Sigma_\varepsilon(\omega)} \nabla\left(u_\varepsilon\right)_i(x)\cdot\varphi(x)\,\mathrm{d}x=\int_{\Sigma}\left[\left(u_0\right)_i\right]_\Sigma\varphi_3\left(x_1,x_2,0\right)\mathrm{d}x_1\mathrm{d}x_2.$$

Thus, $\chi=\frac{[u_0]_\Sigma}{\langle q(0,0)\rangle}$, and using Fenchel's inequality again, we get

$$\liminf_{\varepsilon\to 0}\int_{\Sigma_\varepsilon(\omega)}\varepsilon\left|\nabla u_\varepsilon(x)\right|^2\mathrm{d}x\geq\frac{1}{\langle q(0,0)\rangle}\int_{\Sigma}\left(\left[u_0\right]_\Sigma\right)^2\mathrm{d}x_1\mathrm{d}x_2,$$

which ends the proof. □

We deduce from the preceding convergences the topology τ which is adapted to the description of the asymptotic behavior of the solution u_ε of (9)–(10).

Definition 1. Let ω be an arbitrary realization from a set of full measure on Π, for which the conditions (2)–(5) are satisfied. Then, with probability 1, a sequence $(u_\varepsilon)_\varepsilon$, with $u_\varepsilon\in\mathbf{V}_\varepsilon(\omega)\left(\Omega;\mathbb{R}^3\right)$ for every ε, where $\mathbf{V}_\varepsilon(\omega)\left(\Omega;\mathbb{R}^3\right)$ is the space defined in (11), converges to $u_0\in\mathbf{V}_0\left(\Omega\backslash\Sigma;\mathbb{R}^3\right)$ (defined in (20)) in the topology τ if it satisfies the estimates (13) and the convergences

$$\begin{cases} u_\varepsilon\underset{\varepsilon\to 0}{\rightarrow}u_0 & \text{in }\mathbf{L}^2\left(\Omega;\mathbb{R}^3\right)\text{-strong},\\ u_\varepsilon\circ T^\varepsilon\underset{\varepsilon\to 0}{\rightharpoonup}u_0 & \text{in }\mathbf{H}^1\left(\Omega\backslash\Sigma;\mathbb{R}^3\right)\text{-weak}.\end{cases}$$

3. Convergence

3.1. Case where $\gamma = 1$

Let us introduce the random sequence of functionals $\left(F_\varepsilon(\omega)\right)_\varepsilon$ associated with the Stokes part of the problem (9), which is defined on the space $\mathbf{L}^2\left(\Omega;\mathbb{R}^3\right)$ through

$$F_\varepsilon(\omega)(u)=\begin{cases}\nu^+\displaystyle\int_{\Omega_\varepsilon^+(\omega)}|\nabla u|^2\,\mathrm{d}x+\nu^-\int_{\Omega_\varepsilon^-(\omega)}|\nabla u|^2\,\mathrm{d}x & \\ +\nu^0\varepsilon\displaystyle\int_{\pm_\varepsilon(\omega\omega)}|\nabla u|^2\,\mathrm{d}x & \text{if } u\in\mathbf{V}_\varepsilon(\omega)\left(\Omega;\mathbb{R}^3\right),\\ +\infty & \text{otherwise}\end{cases}$$

and the functional F_0 defined on the space $\mathbf{L}^2\left(\Omega;\mathbb{R}^3\right)$ through

$$F_0(u)=\begin{cases}\nu^+\displaystyle\int_{\Omega^+}|\nabla u|^2\,\mathrm{d}x+\nu^-\int_{\Omega^-}|\nabla u|^2\,\mathrm{d}x+\frac{\nu^0}{\langle q(0,0)\rangle}\int_\Sigma[u]_\Sigma^2\,\mathrm{d}x' & \text{if } u\in\mathbf{V}_0\left(\Omega\backslash\Sigma;\mathbb{R}^3\right),\\ +\infty & \text{otherwise.}\end{cases}$$

Our main result reads as follows.

Theorem 1. *Let ω be an arbitrary realization from a set of full measure on Π, for which the conditions (2)–(5) are satisfied. Then, with probability 1, we have:*

1. *($\limsup$ inequality) For every $u\in\mathbf{V}_0\left(\Omega\backslash\Sigma;\mathbb{R}^3\right)$, there exists a sequence $\left(u_\varepsilon^0\right)_\varepsilon$ with $u_\varepsilon^0\in\mathbf{V}_\varepsilon(\omega)\left(\Omega;\mathbb{R}^3\right)$, for every $\varepsilon>0$, and such that $\left(u_\varepsilon^0\right)_\varepsilon$ τ-converges to u and*

$$\limsup_{\varepsilon\to 0}F_\varepsilon(\omega)\left(u_\varepsilon^0\right)\leq F_0(u),$$

2. *(*$\liminf$ *inequality) For every sequence* $(u_\varepsilon)_\varepsilon$ *such that* $u_\varepsilon \in \mathbf{V}_\varepsilon(\omega)\left(\Omega;\mathbb{R}^3\right)$*, for every* $\varepsilon > 0$*, and such that* $(u_\varepsilon)_\varepsilon$ τ*-converges to* u*, we have* $u \in \mathbf{V}_0\left(\Omega\backslash\Sigma;\mathbb{R}^3\right)$ *and*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(\omega)(u_\varepsilon) \geq F_0(u).$$

Proof. Let ω be a fixed event, for which the conditions (2)–(5) are satisfied. Let $u : \Omega \to \mathbb{R}^3$ be such that $\mathrm{div}(u) = 0$ in Ω, $u = 0$ on $\partial\Omega$, $u\,|_{\overline{\Omega^\pm}} \in \mathbf{C}^1\left(\overline{\Omega^\pm};\mathbb{R}^3\right)$. Let u^+ (resp. u^-) be the trace of $u\,|_{\overline{\Omega^+}}$ (resp. $u\,|_{\overline{\Omega^-}}$) on Σ, with $(u^+)_3 = (u^-)_3$.

We define the function z_ε in the cell $Z^\varepsilon_{ij}(\omega) \subset \Sigma_\varepsilon(\omega)$, for every $(i,j) \in I_\varepsilon$, through

$$\begin{cases} (z_\varepsilon)_\alpha(x',x_3) = u_\alpha\left(x',\varepsilon a^+_{ij,\varepsilon}(x')\right) + \dfrac{x_3 - \varepsilon a^+_{ij,\varepsilon}(x')}{\varepsilon a^+_{ij,\varepsilon}(x') - \varepsilon a^-_{ij,\varepsilon}(x')} \\ \qquad\qquad \times\left(u_\alpha\left(x',\varepsilon a^+_{ij,\varepsilon}(x')\right) - u_\alpha\left(x',\varepsilon a^-_{ij,\varepsilon}(x')\right)\right), \\ (z_\varepsilon)_3(x',x_3) = u_3(x',x_3), \end{cases}$$

where $a^\pm_{ij,\varepsilon}$ (now indicating its dependence with respect to $x' = (x_1,x_2)$) has been defined in (7) and for $\alpha = 1,2$. $(z_\varepsilon)_\alpha$ connects in an affine way with respect to x_3 the traces $u_\alpha\left(x',\varepsilon a^\pm_{ij,\varepsilon}(x')\right)$ of u_α on the boundaries $\Gamma^\pm_\varepsilon(\omega)$. The definition of $(z_\varepsilon)_3$ makes sense, as u_3 presents no jump across Σ, thus doing $z_\varepsilon \in \mathbf{H}^1\left(\Omega\backslash\Sigma;\mathbb{R}^3\right)$.

We observe that z_ε is not divergence free in $\Sigma_\varepsilon(\omega)$. But it satisfies $\int_{\partial\Sigma_\varepsilon(\omega)} z_\varepsilon \cdot n d\sigma = 0$, as $z_\varepsilon = 0$ on the boundary $\partial\Sigma_\varepsilon(\omega) \setminus \Gamma^\pm_\varepsilon(\omega)$ and $z_\varepsilon = u$, a divergence-free function which vanishes on $\partial\Omega$, on the boundaries $\Gamma^\pm_\varepsilon(\omega)$. Therefore, there exists a solution σ_ε to the problem

$$\begin{cases} \mathrm{div}(\sigma_\varepsilon) = -\mathrm{div}(z_\varepsilon) & \text{in } \Sigma_\varepsilon(\omega), \\ \sigma_\varepsilon = 0 & \text{on } \partial\Sigma_\varepsilon(\omega). \end{cases}$$

Using direct computations on the expressions of $(z_\varepsilon)_\alpha$ or $(z_\varepsilon)_3$, we prove the following estimates on z_ε, the details of the proof being postponed to the Appendix.

Lemma 6. *There exists a nonrandom constant* C *independent of* ε *such that*

$$\int\limits_{\Sigma_\varepsilon(\omega)} |z_\varepsilon|^2(x)\,\mathrm{d}x \leq C\varepsilon,$$

$$\int\limits_{\Sigma_\varepsilon(\omega)} |\nabla(z_\varepsilon)_3|^2(x)\,\mathrm{d}x \leq \frac{C}{\varepsilon},$$

$$\int\limits_{\Sigma_\varepsilon(\omega)} |\nabla(z_\varepsilon)_\alpha|^2(x)\,\mathrm{d}x \leq C\varepsilon^{3-2\theta}.$$

Concerning the function σ_ε, we have the following estimate, whose proof is also postponed to the Appendix.

Lemma 7. *We have*

$$\int\limits_{\Sigma_\varepsilon(\omega)} |\nabla\sigma_\varepsilon|^2(x)\,\mathrm{d}x \underset{\varepsilon\to 0}{\to} 0.$$

We then define the test function u^0_ε as

$$u^0_\varepsilon = \begin{cases} u & \text{in } \Omega \setminus \Sigma_\varepsilon(\omega), \\ z_\varepsilon + \sigma_\varepsilon & \text{in } \Sigma_\varepsilon(\omega). \end{cases} \tag{23}$$

We deduce from its construction that u_ε^0 belongs to $\mathbf{V}_\varepsilon(\omega)\left(\Omega;\mathbb{R}^3\right)$ for every $\varepsilon>0$. Let us prove that the sequence $\left(u_\varepsilon^0\right)_\varepsilon$ τ-converges to u. According to the construction of the sequence $(z_\varepsilon)_\varepsilon$, the sequence $\left(u_\varepsilon^0\right)_\varepsilon$ satisfies the estimates (13) and the convergences

$$\begin{cases} u_\varepsilon^0 \underset{\varepsilon\to 0}{\to} u & \text{in } \mathbf{L}^2\left(\Omega;\mathbb{R}^3\right)\text{-strong},\\ u_\varepsilon^0\circ T^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} u & \text{in } \mathbf{H}^1\left(\Omega\backslash\Sigma;\mathbb{R}^3\right)\text{-weak}. \end{cases}$$

Let us prove that $\lim_{\varepsilon\to 0} F_\varepsilon(\omega)\left(u_\varepsilon^0\right)=F_0(u)$. Observe that if ω is an arbitrary realization from a set of full measure on Π, for which the conditions (2)–(5) are satisfied, we compute

$$\begin{aligned}
\lim_{\varepsilon\to 0}\nu^0\varepsilon\int_{\pm_\varepsilon(\omega)}\left|\nabla u_\varepsilon^0\right|^2\mathrm{d}x &= \lim_{\varepsilon\to 0}\nu^0\varepsilon\int_{\Sigma_\varepsilon(\omega)}\left|\nabla z_\varepsilon\right|^2\mathrm{d}x\\
&=\lim_{\varepsilon\to 0}\nu^0\varepsilon\sum_{\substack{(i,j)\in I_\varepsilon\\ \alpha=1,2}}\int_{Z_{ij}^\varepsilon(\omega)}\left(\frac{u_\alpha\left(x',\varepsilon a_{ij,\varepsilon}^+(x')\right)-u_\alpha\left(x',\varepsilon a_{ij,\varepsilon}^-(x')\right)}{\varepsilon\left(a_{ij,\varepsilon}^+(x')-a_{ij,\varepsilon}^-(x')\right)}\right)^2\mathrm{d}x\\
&=\lim_{\varepsilon\to 0}\nu^0\sum_{\substack{(i,j)\in I_\varepsilon\\ \alpha=1,2}}\int_{Y_{ij}^\varepsilon}\frac{\left(u_\alpha\left(i\varepsilon,j\varepsilon,\varepsilon a_{ij,\varepsilon}^+(x')\right)-u_\alpha\left(i\varepsilon,j\varepsilon,\varepsilon a_{ij,\varepsilon}^-(x')\right)\right)^2}{\varepsilon q_{ij}^\varepsilon(x')}\mathrm{d}x'\\
&=\lim_{\varepsilon\to 0}\nu^0\sum_{\substack{(i,j)\in I_\varepsilon\\ \alpha=1,2}}\varepsilon^2\left(u_\alpha\left(i\varepsilon,j\varepsilon,\varepsilon a_{ij,\varepsilon}^+(x')\right)-u_\alpha\left(i\varepsilon,j\varepsilon,\varepsilon a_{ij,\varepsilon}^-(x')\right)\right)^2\int_Y\frac{\mathrm{d}y_1\mathrm{d}y_2}{q_{ij}^\theta(y_1,y_2)},
\end{aligned}$$

where q_{ij}^ε, r_{ij}^ε and q_{ij}^θ are defined in (22). We deduce from this computation that

$$\lim_{\varepsilon\to 0}\nu^0\varepsilon\int_{\pm_\varepsilon(\omega)}\left|\nabla u_\varepsilon^0\right|^2\mathrm{d}x=\sum_{\alpha=1,2}\frac{\nu^0}{\langle q(0,0)\rangle}\int_{\pm}[u_\alpha]_\Sigma^2\,\mathrm{d}x'. \tag{24}$$

On the other hand, one can easily see that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^\pm(\omega)}\left|\nabla u_\varepsilon^0\right|^2\mathrm{d}x=\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^\pm(\omega)}|\nabla u|^2\,\mathrm{d}x=\int_{\Omega^\pm}|\nabla u|^2\,\mathrm{d}x. \tag{25}$$

Thus, owing to (24) and (25), we get $\lim_{\varepsilon\to 0}F_\varepsilon(\omega)\left(u_\varepsilon^0\right)=F_0(u)$.

Let us now verify the lim sup property of the Γ-convergence in the general case. For every $u\in\mathbf{V}_0\left(\Omega\backslash\Sigma;\mathbb{R}^3\right)$, there exists a sequence $(u_n)_n$ such that $\operatorname{div}(u_n)=0$ in Ω, $u_n=0$ on $\partial\Omega$, $u_n\mid_{\overline{\Omega^\pm}}\in\mathbf{C}^1\left(\overline{\Omega^\pm};\mathbb{R}^3\right)$, $(u_n^+)_3=(u_n^-)_3$, and $(u_n)_n$ converges to u in the strong topology of $\mathbf{V}_0\left(\Omega\backslash\Sigma;\mathbb{R}^3\right)$. Using (23), we build $(u_n)_\varepsilon^0\in\mathbf{V}_\varepsilon(\omega)\left(\Omega;\mathbb{R}^3\right)$ such that $\left((u_n)_\varepsilon^0\right)_\varepsilon$ converges to u_n in the topology τ and $\lim_{\varepsilon\to 0}F_\varepsilon(\omega)\left((u_n)_\varepsilon^0\right)=F_0(u_n)$. The continuity of the functional F_0 with respect to the strong topology of $\mathbf{V}_0\left(\Omega\backslash\Sigma;\mathbb{R}^3\right)$ implies

$$\lim_{n\to+\infty}\lim_{\varepsilon\to 0}F_\varepsilon(\omega)\left((u_n)_\varepsilon^0\right)=F_0(u).$$

The topology τ being metrizable, the diagonalization argument of [1, Corollary 1.18] proves the existence of a subsequence $\left(\left(u_{n(\varepsilon)}\right)_\varepsilon^0\right)_\varepsilon$ which converges to u in the topology τ and satisfies $\limsup_{\varepsilon\to 0}F_\varepsilon(\omega)\left(\left(u_{n(\varepsilon)}\right)_\varepsilon^0\right)\le F_0(u)$. This proves the lim sup property in the general case.

2. Let ω be an arbitrary realization from a set of full measure on Π, for which the conditions (2)–(5) are satisfied. Let $(u_\varepsilon)_\varepsilon$, with $u_\varepsilon \in \mathbf{V}_\varepsilon(\omega)\left(\Omega;\mathbb{R}^3\right)$ for every $\varepsilon > 0$, be a sequence such that $(u_\varepsilon)_\varepsilon$ τ-converges to u. According to Proposition 1 2., u belongs to $\mathbf{V}_0\left(\Omega\backslash\Sigma;\mathbb{R}^3\right)$, and, using Lemma 5, we get

$$\liminf_{\varepsilon\to 0} \int\limits_{\Sigma_\varepsilon(\omega)} \varepsilon\left|\nabla u_\varepsilon(x)\right|^2 \mathrm{d}x \geq \frac{1}{\langle q(0,0)\rangle} \int\limits_{\Sigma} \left([u]_\Sigma\right)^2 \mathrm{d}x_1 \mathrm{d}x_2.$$

As $(u_\varepsilon)_\varepsilon$ τ-converges to u, we have

$$\liminf_{\varepsilon\to 0} \int\limits_{\Omega_\varepsilon^\pm} \left|\nabla u_\varepsilon\right|^2 \mathrm{d}x \geq \int\limits_{\Omega^\pm} \left|\nabla u\right|^2 \mathrm{d}x.$$

Thus, $\liminf_{\varepsilon\to 0} F_\varepsilon(\omega)(u_\varepsilon) \geq F_0(u)$. This completes the proof of the main convergence result. □

Let us now prove some convergence results for the solution u_ε of (9), for the zero mean value pressure and for the associated energies.

Corollary 1. *Let ω be an arbitrary realization from a set of full measure on Π, for which the conditions (2)–(5) are satisfied. Then, with probability 1, the sequence $(u_\varepsilon, p_\varepsilon)_\varepsilon$, where $(u_\varepsilon, p_\varepsilon)$ is the solution of (9)–(10), is such that $(u_\varepsilon)_\varepsilon$ τ-converges to u_0 and*

$$\int\limits_{\Omega_\varepsilon^\pm(\omega)} \left|\overline{p_\varepsilon^\pm}\right|^2 \mathrm{d}x \underset{\varepsilon\to 0}{\to} \int\limits_{\Omega^\pm} \left|p_0^\pm\right|^2 \mathrm{d}x,$$

where $\left(u_0, p_0 = \begin{cases} p_0^+ & \text{in } \Omega^+ \\ p_0^- & \text{in } \Omega^- \end{cases}\right)$ *belongs to* $\mathbf{V}_0\left(\Omega\backslash\Sigma;\mathbb{R}^3\right) \times L^2(\Omega)/\mathbb{R}$ *and is the solution of the limit problem (12).*

Moreover,

$$\lim_{\varepsilon\to 0} F_\varepsilon(\omega)(u_\varepsilon) = F_0(u_0) = \nu^+ \int\limits_{\Omega^+} \left|\nabla u_0\right|^2 \mathrm{d}x + \nu^- \int\limits_{\Omega^-} \left|\nabla u_0\right|^2 \mathrm{d}x + \frac{\nu^0}{\langle q(0,0)\rangle} \int\limits_{\Sigma} \left|[u_0]_\Sigma\right|^2 \mathrm{d}x'.$$

Proof. We first observe that, for every sequence $(v_\varepsilon)_\varepsilon$ τ-converging to v, we have

$$\begin{aligned} \lim_{\varepsilon\to 0} \int\limits_{\Omega_\varepsilon^\pm(\omega)} f\cdot v_\varepsilon \mathrm{d}x &= \int\limits_{\Omega^\pm} f\cdot v \mathrm{d}x, \\ \lim_{\varepsilon\to 0} \int\limits_{\Omega_\varepsilon^\pm(\omega)} (u_\varepsilon\cdot\nabla)\, u_\varepsilon\cdot v_\varepsilon \mathrm{d}x &= \int\limits_{\Omega^\pm} (u_0\cdot\nabla)\, u_0\cdot v \mathrm{d}x. \end{aligned} \tag{26}$$

Take $v:\Omega\to\mathbb{R}^3$ such that $\operatorname{div}(v) = 0$ in Ω, $v = 0$ on $\partial\Omega$, $v\mid_{\overline{\Omega^+}} \in C^1\left(\overline{\Omega^+};\mathbb{R}^3\right)$ and $v\mid_{\overline{\Omega^-}} \in C^1\left(\overline{\Omega^-};\mathbb{R}^3\right)$. We then multiply $(9)_{1,2}$ by v_ε^0 defined in (23) and use Green's formula in order to get

$$\begin{aligned}&\nu^+\int_{\Omega_\varepsilon^+(\omega)}\nabla u_\varepsilon\cdot\nabla v_\varepsilon^0\mathrm{d}x+\nu^-\int_{\Omega_\varepsilon^-(\omega)}\nabla u_\varepsilon\cdot\nabla v_\varepsilon^0\mathrm{d}x\\&\quad+\int_{\Omega_\varepsilon^+(\omega)}(u_\varepsilon\cdot\nabla)\,u_\varepsilon\cdot v_\varepsilon^0\mathrm{d}x+\int_{\Omega_\varepsilon^-(\omega)}(u_\varepsilon\cdot\nabla)\,u_\varepsilon\cdot v_\varepsilon^0\mathrm{d}x\\&\quad+\nu^+\int_{\Gamma_\varepsilon^+(\omega)}\frac{\partial u_\varepsilon}{\partial n}\cdot v_\varepsilon^0d\sigma+\nu^-\int_{\Gamma_\varepsilon^-(\omega)}\frac{\partial u_\varepsilon}{\partial n}\cdot v_\varepsilon^0d\sigma\\&\quad+\int_{\Gamma_\varepsilon^+(\omega)}p_\varepsilon n\cdot v_\varepsilon^0d\sigma+\int_{\Gamma_\varepsilon^-(\omega)}p_\varepsilon n\cdot v_\varepsilon^0d\sigma=\int_{\Omega_\varepsilon^+(\omega)\cup\Omega_\varepsilon^-(\omega)}f\cdot v_\varepsilon^0\mathrm{d}x,\end{aligned}$$

thanks to the boundary conditions (10). Taking into account the boundary conditions $(10)_2$, the preceding equality may be written as

$$\begin{aligned}&\nu^+\int_{\Omega_\varepsilon^+(\omega)}\nabla u_\varepsilon\cdot\nabla v_\varepsilon^0\mathrm{d}x+\nu^-\int_{\Omega_\varepsilon^-(\omega)}\nabla u_\varepsilon\cdot\nabla v_\varepsilon^0\mathrm{d}x\\&\quad+\int_{\Omega_\varepsilon^+(\omega)}(u_\varepsilon\cdot\nabla)\,u_\varepsilon\cdot v_\varepsilon^0\mathrm{d}x+\int_{\Omega_\varepsilon^-(\omega)}(u_\varepsilon\cdot\nabla)\,u_\varepsilon\cdot v_\varepsilon^0\mathrm{d}x\\&\quad+\varepsilon\nu^0\int_{\Gamma_\varepsilon^+(\omega)}\frac{\partial u_\varepsilon}{\partial n}\cdot v_\varepsilon^0d\sigma+\varepsilon\nu^0\int_{\Gamma_\varepsilon^-(\omega)}\frac{\partial u_\varepsilon}{\partial n}\cdot v_\varepsilon^0d\sigma\\&\quad+\int_{\Gamma_\varepsilon^+(\omega)}p_\varepsilon n\cdot v_\varepsilon^0d\sigma+\int_{\Gamma_\varepsilon^-(\omega)}p_\varepsilon n\cdot v_\varepsilon^0d\sigma=\int_{\Omega_\varepsilon^+(\omega)\cup\Omega_\varepsilon^-(\omega)}f\cdot v_\varepsilon^0\mathrm{d}x;\end{aligned}$$

hence,

$$\begin{aligned}&\nu^+\int_{\Omega_\varepsilon^+(\omega)}\nabla u_\varepsilon\cdot\nabla v_\varepsilon^0\mathrm{d}x+\nu^-\int_{\Omega_\varepsilon^-(\omega)}\nabla u_\varepsilon\cdot\nabla v_\varepsilon^0\mathrm{d}x+\varepsilon\nu^0\int_{\Sigma_\varepsilon(\omega)}\nabla u_\varepsilon\cdot v_\varepsilon^0\mathrm{d}x\\&\quad+\int_{\Omega_\varepsilon^+(\omega)}(u_\varepsilon\cdot\nabla)\,u_\varepsilon\cdot v_\varepsilon^0\mathrm{d}x+\int_{\Omega_\varepsilon^-(\omega)}(u_\varepsilon\cdot\nabla)\,u_\varepsilon\cdot v_\varepsilon^0\mathrm{d}x\\&\quad+\int_{\Sigma_\varepsilon(\omega)}(u_\varepsilon\cdot\nabla)\,u_\varepsilon\cdot v_\varepsilon^0\mathrm{d}x=\int_{\Omega_\varepsilon^+(\omega)\cup\Omega_\varepsilon^-(\omega)}f\cdot v_\varepsilon^0\mathrm{d}x.\end{aligned}$$

We then take the limit when ε goes to 0 and get

$$\begin{aligned}&\nu^+\int_{\Omega^+}\nabla u_0\cdot\nabla v\mathrm{d}x+\nu^-\int_{\Omega^-}\nabla u_0\cdot\nabla v\mathrm{d}x\\&\quad+\int_{\Omega^+}(u_0\cdot\nabla)\,u_0\cdot v\mathrm{d}x+\int_{\Omega^-}(u_0\cdot\nabla)\,u_0\cdot v\mathrm{d}x\\&\quad+\nu^0\left\langle q\left(0,0\right)\right\rangle\int_{\Sigma}\left[(u_0)_\alpha\right]_\Sigma\left[v_\alpha\right]_\Sigma\mathrm{d}x'=\int_{\Omega}f\cdot v\mathrm{d}x,\end{aligned}$$

using the construction (23) of v_ε^0 and (26). Thus, u_0 satisfies

$$\forall v \in \mathbf{V}_0\left(\Omega\backslash\Sigma;\mathbb{R}^3\right): \int\limits_{\Omega^+}\left(-\nu^+\Delta u_0 + \left(u_0\cdot\nabla\right)u_0 - f\right)\cdot v\mathrm{d}x$$
$$+\int\limits_{\Omega^-}\left(-\nu^-\Delta u_0 + \left(u_0\cdot\nabla\right)u_0 - f\right)\cdot v\mathrm{d}x$$
$$-\nu^+\int\limits_{\Sigma}\frac{\partial u_0}{\partial x_3}\cdot v\left(x',0^+\right)\mathrm{d}x' - \nu^-\int\limits_{\Sigma}\frac{\partial u_0}{\partial x_3}\cdot v\left(x',0^-\right)\mathrm{d}x'$$
$$+\nu^0\left\langle q\left(0,0\right)\right\rangle\int\limits_{\Sigma}\left[\left(u_0\right)_\alpha\right]_\Sigma\left[v_\alpha\right]_\Sigma\mathrm{d}x' = 0,$$

where the superscript $^+$ (resp. $^-$) corresponds to the trace on Σ seen from Ω^+ (resp. Ω^-). This leads to the limit problem (12).

The second assertion is a direct consequence of the properties of the Γ-convergence. □

Remark 2. Let us consider the case where the functions $a^{\pm}_{ij,\varepsilon}$ are defined through

$$a^+_{ij,\varepsilon} = \frac{q^+\left(\left(\varepsilon^{-\theta}\left(x_1 - i\varepsilon\right) + \alpha_{1i}\left(\omega\right), \varepsilon^{-\theta}\left(x_2 - j\varepsilon\right) + \alpha_{2j}\left(\omega\right)\right), \omega\right)}{2},$$
$$a^-_{ij,\varepsilon} = \frac{-q^-\left(\left(\varepsilon^{-\theta}\left(x_1 - i\varepsilon\right) + \beta_{1i}\left(\omega\right), \varepsilon^{-\theta}\left(x_2 - j\varepsilon\right) + \beta_{2j}\left(\omega\right)\right), \omega\right)}{2},$$

the two positive random processes $q^{\pm}$ satisfying the conditions (2)–(5). We here obtain a limit problem similar to (12) with

$$\left\langle q\left(0,0\right)\right\rangle = \frac{\left\langle q^+\left(0,0\right)\right\rangle + \left\langle q^-\left(0,0\right)\right\rangle}{2}.$$

3.2. Cases where $0 < \gamma < 1$ or $\gamma > 1$

Let us introduce the random sequence of functionals $\left(F_{\varepsilon^\gamma}\left(\omega\right)\right)_\varepsilon$ associated with Reynolds number of order $O\left(\varepsilon^{-\gamma}\right)$ and with the Stokes part of the problem (9), which is defined on the space $\mathbf{L}^2\left(\Omega;\mathbb{R}^3\right)$ through

$$F_{\varepsilon^\gamma}\left(\omega\right)\left(u\right) = \begin{cases} \nu^+\displaystyle\int\limits_{\Omega_\varepsilon^+(\omega)}\left|\nabla u\right|^2\mathrm{d}x + \nu^-\int\limits_{\Omega_\varepsilon^-(\omega)}\left|\nabla u\right|^2\mathrm{d}x & \\ +\nu^0\varepsilon^\gamma\displaystyle\int\limits_{\pm_\varepsilon(\omega\omega)}\left|\nabla u\right|^2\mathrm{d}x & \text{if } u \in \mathbf{V}_\varepsilon\left(\omega\right)\left(\Omega;\mathbb{R}^3\right), \\ +\infty & \text{otherwise} \end{cases}$$

and the functional $F_{0,\lambda}$, with $\lambda > 0$, defined on the space $\mathbf{L}^2\left(\Omega;\mathbb{R}^3\right)$ through

$$F_{0,\lambda}\left(u\right) = \begin{cases} \nu^+\displaystyle\int\limits_{\Omega^+}\left|\nabla u\right|^2\mathrm{d}x + \nu^-\int\limits_{\Omega^-}\left|\nabla u\right|^2\mathrm{d}x + \frac{\lambda\nu^0}{\left\langle q\left(0,0\right)\right\rangle}\int\limits_{\Sigma}\left[u\right]_\Sigma^2\mathrm{d}x' & \text{if } u \in \mathbf{V}_0\left(\Omega\backslash\Sigma;\mathbb{R}^3\right), \\ +\infty & \text{otherwise.} \end{cases}$$

Using comparison principles, we can easily prove the following result.

Proposition 2. *Let ω be an arbitrary realization from a set of full measure on Π, for which the conditions (2)–(5) are satisfied. Then, with probability 1, we have:*

1. *If $\gamma < 1$, $(F_{\varepsilon^\gamma})_\varepsilon$ Γ-converges in the topology τ to the functional $F_{0,\infty}$ defined on the space $\mathbf{L}^2\left(\Omega;\mathbb{R}^3\right)$ through*

$$F_{0,\infty}(u) = \begin{cases} \nu^+ \int\limits_{\Omega^+} |\nabla u|^2 \,\mathrm{d}x + \nu^- \int\limits_{\Omega^-} |\nabla u|^2 \,\mathrm{d}x & \text{if } u \in \mathbf{V}_0\left(\Omega\backslash\Sigma;\mathbb{R}^3\right), [u]_\Sigma = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

2. *If $\gamma > 1$, $(F_{\varepsilon^\gamma})_\varepsilon$ Γ-converges in the topology τ to the functional $F_{0,0}$ defined on the space $\mathbf{L}^2\left(\Omega;\mathbb{R}^3\right)$ through*

$$\mathbf{F}_{0,0}(u) = \begin{cases} \nu^+ \int\limits_{\Omega^+} |\nabla u|^2 \,\mathrm{d}x + \nu^- \int\limits_{\Omega^-} |\nabla u|^2 \,\mathrm{d}x & \text{if } u \in \mathbf{V}_0\left(\Omega\backslash\Sigma;\mathbb{R}^3\right), \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. 1. We observe that, for every $u \in \mathbf{L}^2\left(\Omega;\mathbb{R}^3\right)$ and every $\lambda > 0$, we have

$$F_{\varepsilon^\gamma}(\omega)(u) \geq F_{\lambda\varepsilon}(\omega)(u),$$

for ε sufficiently small. In Theorem 1, we proved that $(F_{\varepsilon^\gamma})_\varepsilon$ Γ-converges in the topology τ to $F_{0,\lambda}$ and, because of the properties of the Γ -convergence, we obtain

$$\Gamma\text{-}\lim F_{\varepsilon^\gamma}(\omega)(u) \geq F_{0,\lambda}(\omega)(u), \quad \forall\lambda > 0,$$

which implies

$$\Gamma\text{-}\lim F_{\varepsilon^a}(\omega)(u) \geq F_{0,\infty}(\omega)(u).$$

As $\Gamma\text{-}\lim F_{\varepsilon^\gamma}(\omega)(u) \leq F_{0,\infty}(\omega)(u)$ we conclude with the equality.

2. We observe that, for every $u \in L^2\left(\Omega;\mathbb{R}^3\right)$ and every $\lambda > 0$, we have

$$F_{\varepsilon^\gamma}(\omega)(u) \leq F_{\lambda\varepsilon}(\omega)(u),$$

for ε sufficiently small. In Theorem 1, we proved that $(F_{\varepsilon^\gamma})_\varepsilon$ Γ-converges in the topology τ to $F_{0,\lambda}$ and because of the properties of the Γ-convergence, we obtain

$$\Gamma\text{-}\lim F_{\varepsilon^\gamma}(\omega)(u) \leq F_{0,\lambda}(\omega)(u), \quad \forall\lambda > 0,$$

which implies

$$\Gamma\text{-}\lim F_{\varepsilon^\gamma}(\omega)(u) \leq F_{0,0}(\omega)(u).$$

As $\Gamma\text{-}\lim F_{\varepsilon^\gamma}(\omega)(u) \leq F_{0,0}(\omega)(u)$ we conclude with the equality. □

4. Appendix

We first prove Lemma 6.

Because $(z_\varepsilon)_\alpha$ connects in an affine way with respect to x_3 the traces on $\Gamma_\varepsilon^\pm(\omega)$ of the smooth function u_α in $\Omega^\pm$, we get

$$\int\limits_{\Sigma_\varepsilon(\omega)} \left((z_\varepsilon)_\alpha\right)^2 \mathrm{d}x \leq C meas\left(\Sigma_\varepsilon(\omega)\right) \leq C\varepsilon.$$

Because $(z_\varepsilon)_3 = u_3$, in $\Sigma_\varepsilon(\omega)$, where u_3 is a smooth function in Ω, we immediately get

$$\int\limits_{\Sigma_\varepsilon(\omega)} \left((z_\varepsilon)_3\right)^2 \mathrm{d}x \leq C meas\left(\Sigma_\varepsilon(\omega)\right) \leq C\varepsilon.$$

Because $\nabla (z_\varepsilon)_3 = \nabla u_3$, we immediately get

$$\int_{\Sigma_\varepsilon(\omega)} |\nabla (z_\varepsilon)_3|^2 \, \mathrm{d}x \le C meas\left(\Sigma_\varepsilon(\omega)\right) \le C\varepsilon,$$

because u_3 is a smooth function in Ω. We then compute

$$\frac{\partial (z_\varepsilon)_\alpha}{\partial x_3}(x', x_3) = \frac{u_\alpha\left(x', \varepsilon a^+_{ij,\varepsilon}(x')\right) - u_\alpha\left(x', \varepsilon a^-_{ij,\varepsilon}(x')\right)}{\varepsilon\left(a^+_{ij,\varepsilon}(x') - a^-_{ij,\varepsilon}(x')\right)},$$

which implies

$$\begin{aligned}\int_{\Sigma_\varepsilon(\omega)} \left|\frac{\partial (z_\varepsilon)_\alpha}{\partial x_3}\right|^2 \mathrm{d}x &\le \frac{C}{\varepsilon^2}\sum_{ij}\int_{Z^\varepsilon_{ij}(\omega)} \frac{\mathrm{d}x}{q^2\left(\left(\varepsilon^{-\theta}(x_1 - i\varepsilon) + \beta_{1i}(\omega), \varepsilon^{-\theta}(x_2 - j\varepsilon) + \beta_{2j}(\omega)\right), \omega\right)}\\ &\le \frac{C}{\varepsilon}\sum_{ij}\int_{Y^\varepsilon_{ij}} \frac{\mathrm{d}x_1\mathrm{d}x_2}{q\left(\left(\varepsilon^{-\theta}(x_1 - i\varepsilon) + \beta_{1i}(\omega), \varepsilon^{-\theta}(x_2 - j\varepsilon) + \beta_{2j}(\omega)\right), \omega\right)}\\ &\le C\varepsilon\sum_{ij}\int_{Y} \frac{\mathrm{d}y_1\mathrm{d}y_2}{q\left(\left(\varepsilon^{-\theta+1}y_1 + \beta_{1i}(\omega), \varepsilon^{-\theta+1}y_2 + \beta_{2j}(\omega)\right), \omega\right)},\end{aligned}$$

where $Z^\varepsilon_{ij}(\omega)$ has been defined in (6) and taking $y_1 = \frac{x_1 - i\varepsilon}{\varepsilon}$ and $y_2 = \frac{x_2 - j\varepsilon}{\varepsilon}$. Then, taking $z_1 = \varepsilon^{-(\theta-1)}y_1$ and $z_2 = \varepsilon^{-(\theta-1)}y_2$, we obtain

$$\int_{\Sigma_\varepsilon(\omega)} \left|\frac{\partial (z_\varepsilon)_\alpha}{\partial x_3}\right|^2 \mathrm{d}x \le C\varepsilon\frac{meas(\Sigma)}{\varepsilon^2\varepsilon^{\theta-1}} \int_{-\frac{\varepsilon^{\theta-1}}{2}}^{\frac{\varepsilon^{\theta-1}}{2}} \frac{1}{\varepsilon^{\theta-1}} \int_{-\frac{\varepsilon^{\theta-1}}{2}}^{\frac{\varepsilon^{\theta-1}}{2}} \frac{\mathrm{d}z_1\mathrm{d}z_2}{q\left(\left(z_1 + \beta_{1i}(\omega), z_2 + \beta_{2j}(\omega)\right), \omega\right)} \le \frac{C}{\varepsilon},$$

using the ergodic result (4).

Finally, we compute, for $\beta = 1, 2$,

$$\begin{aligned}\frac{\partial (z_\varepsilon)_\alpha}{\partial x_\beta}(x', x_3) &= \frac{\partial u_\alpha}{\partial x_\beta}\left(x', \varepsilon a^+_{ij,\varepsilon}(x')\right) + \frac{\partial u_\alpha}{\partial x_3}\left(x', \varepsilon a^+_{ij,\varepsilon}(x')\right)\varepsilon\varepsilon^{-\theta}\frac{\partial a^+_{ij,\varepsilon}}{\partial x_\beta}(x')\\ &+\frac{\begin{array}{c}-\left(a^+_{ij,\varepsilon}(x') - a^-_{ij,\varepsilon}(x')\right)\varepsilon\varepsilon^{-\theta}\dfrac{\partial a^+_{ij,\varepsilon}}{\partial x_\beta}(x')\\ -\left(x_3 - \varepsilon a^+_{ij,\varepsilon}(x')\right)\varepsilon\varepsilon^{-\theta}\dfrac{\partial q}{\partial x_\beta}(x')\end{array}}{\varepsilon\left(a^+_{ij,\varepsilon}(x') - a^-_{ij,\varepsilon}(x')\right)^2}\begin{pmatrix} u_\alpha\left(x', \varepsilon a^+_{ij,\varepsilon}(x')\right)\\ -u_\alpha\left(x', \varepsilon a^-_{ij,\varepsilon}(x')\right)\end{pmatrix}\\ &+\frac{x_3 - \varepsilon a^+_{ij,\varepsilon}(x')}{\varepsilon\left(a^+_{ij,\varepsilon}(x') - a^-_{ij,\varepsilon}(x')\right)}\begin{pmatrix}\dfrac{\partial u_\alpha}{\partial x_\beta}\left(x', \varepsilon a^+_{ij,\varepsilon}(x')\right)\\ +\dfrac{\partial u_\alpha}{\partial x_3}\left(x', \varepsilon a^+_{ij,\varepsilon}(x')\right)\varepsilon\varepsilon^{-\theta}\dfrac{\partial a^+_{ij,\varepsilon}}{\partial x_\beta}(x')\\ -\dfrac{\partial u_\alpha}{\partial x_\beta}\left(x', \varepsilon a^-_{ij,\varepsilon}(x')\right)\\ -\dfrac{\partial u_\alpha}{\partial x_3}\left(x', \varepsilon a^-_{ij,\varepsilon}(x')\right)\varepsilon\varepsilon^{-\theta}\dfrac{\partial a^-_{ij,\varepsilon}}{\partial x_\beta}(x')\end{pmatrix}.\end{aligned}$$

The preponderant terms surely are the first part of the second term and the second and fourth parts of the third term. Thanks to the smoothness of u_α in $\Omega^\pm$ and of r and q, we can estimate this preponderant term as

$$\left|A_{\varepsilon}\left(x',x_3\right)\right| \leq C\varepsilon\varepsilon^{-\theta} + \frac{C\varepsilon^{-\theta}}{\left|a^{+}_{ij,\varepsilon}\left(x'\right) - a^{-}_{ij,\varepsilon}\left(x'\right)\right|}\left|\frac{\partial a^{+}_{ij,\varepsilon}}{\partial x_{\beta}}\left(x'\right)\right|$$

and we compute

$$\begin{aligned}\int_{\Sigma_{\varepsilon}(\omega)} \left|A_{\varepsilon}\right|^2 \mathrm{d}x &\leq C\varepsilon^{3-2\theta} + C\sum_{ij}\int_{Z^{\varepsilon}_{ij}(\omega)} \frac{\mathrm{d}x}{q\left(\left(\varepsilon^{-\theta}\left(x_1 - i\varepsilon\right) + \beta_{1i}\left(\omega\right), \varepsilon^{-\theta}\left(x_2 - j\varepsilon\right) + \beta_{2j}\left(\omega\right)\right), \omega\right)} \\ &\leq C\varepsilon^{3-2\theta} + C\varepsilon,\end{aligned}$$

thanks to the hypotheses (3) on r and q.

For the proof of Lemma 7, we first observe that $\int_{\Sigma_{\varepsilon}(\omega)} \left(\operatorname{div}\left(z_{\varepsilon}\right)\right)^2 \mathrm{d}x \to 0$, when $\varepsilon \to 0$, as we proved in the preceding estimates that the third and the two first terms of $\operatorname{div}\left(z_{\varepsilon}\right)\left(x',x_3\right) = \frac{\partial (z_{\varepsilon})_1}{\partial x_1}\left(x',x_3\right) + \frac{\partial (z_{\varepsilon})_2}{\partial x_2}\left(x',x_3\right) + \frac{\partial (z_{\varepsilon})_3}{\partial x_3}\left(x',x_3\right)$ have a $L^2\left(\Sigma_{\varepsilon}\left(\omega\right)\right)$ -norm which goes to 0 when ε goes to 0. We then use Bogovskii's result in [3].

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(Received: February 4, 2015; revised: September 7, 2015)