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A reaction–diffusion SIS epidemic model in an almost periodic environment

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Abstract. A susceptible–infected–susceptible almost periodic reaction–diffusion epidemic model is studied by means of establishing the theories and properties of the basic reproduction ratio R_0. Particularly, the asymptotic behaviors of R_0 with respect to the diffusion rate D_I of the infected individuals are obtained. Furthermore, the uniform persistence, extinction and global attractivity are presented in terms of R_0. Our results indicate that the interaction of spatial heterogeneity and temporal almost periodicity tends to enhance the persistence of the disease.

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Keywords. Reaction–diffusion · Almost periodicity · Epidemic model · Basic reproduction ratio · Threshold dynamics.

1. Introduction

Reaction–diffusion systems are frequently used to describe the random motion of populations and spatial heterogeneity in epidemic transmissions (see e.g., [2,5,17,19–21,25,33,39]). In order to understand the influence of spatial heterogeneity of environment and movement of individuals on the persistence and extinction of a disease, Allen et al. [2] studied the susceptible–infected–susceptible (SIS) reaction–diffusion epidemic model:

$$\begin{cases} \frac{\partial S}{\partial t} - D_S \Delta S = -\frac{\beta(x)SI}{S+I} + \gamma(x)I, & x \in \Omega,\ t > 0, \\ \frac{\partial I}{\partial t} - D_I \Delta I = \frac{\beta(x)SI}{S+I} - \gamma(x)I, & x \in \Omega,\ t > 0, \\ \frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.1}$$

where the habitat $\Omega \subset \mathbb{R}^n$ $(n \geqslant 1)$ is a bounded domain with smooth boundary $\partial\Omega$ $(n > 1)$, and ν is the outward normal vector on $\partial\Omega$ and $\frac{\partial}{\partial \nu}$ represents the normal derivative along ν on $\partial\Omega$; $S(x,t)$ and $I(x,t)$ denote the density of susceptible and infected individuals at location x and time t, respectively; the constants D_S and D_I are positive diffusion coefficients of the susceptible and infected population; $\beta(x)$ and $\gamma(x)$ are positive Hölder continuous on Ω which denote the rate of disease transmission and disease recovery at location x, respectively. Under the assumption that the total population number is a constant, the results in [2] showed that the disease-free equilibrium of (1.1) is unstable for high-risk domains where the spatial average of the transmission rate of the disease is large than the spatial average of the recovery rate. For low-risk domains, which defined in a reverse way, the disease-free equilibrium is stable. Involving system (1.1), Peng et al. [23,24] studied the asymptotic behavior and global stability of the endemic equilibrium, Allen et al. [1] considered the continuous-time cases, and Huang et al. [14] investigated the global dynamics subject to the Dirichlet boundary conditions.

Mixed factors lead to the outbreak of a disease. As same as the spatial heterogeneity, the temporal heterogeneity is also one of important factors which influences the transmission of a disease. The temporal heterogeneity comes from the seasonality. As noted in [3], seasonal changes are cyclic, largely predictable and arguably represent the strongest and most ubiquitous source of external variation influencing human

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and natural systems. Exploring the role of seasonality, Peng and Zhao [25] assumed that β and γ in (1.1) are spatiotemporal heterogeneous and periodic in time, introduced the definition of the basic reproduction ratio R_0 and its properties and established threshold-type results on the global dynamics in terms of R_0. Particularly, the asymptotic behavior of R_0 with respect to the diffusion rate of the infected individuals was obtained. The results in [25] also suggested that the interaction of spatial heterogeneity and temporal periodicity tends to enhance the persistence of the disease. From an applied perspective, though a periodic epidemic model offers broad insights into understanding the mechanisms of a disease outbreak, the disease transmission rate and recovery rate are not necessary to share a common period. In particular, if the periods of these periodic coefficients have no common integer multiple, then the system is not a periodic system. Mathematically, we can treat such a system as an almost periodic system. In order to describe the transmission of a disease more reasonably, we consider almost periodic system:

$$\begin{cases} \frac{\partial S}{\partial t} - D_S \Delta S = -\frac{\beta(x,t)SI}{S+I} + \gamma(x,t)I, & x \in \Omega,\ t > 0, \\ \frac{\partial I}{\partial t} - D_I \Delta I = \frac{\beta(x,t)SI}{S+I} - \gamma(x,t)I, & x \in \Omega,\ t > 0, \\ \frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ S(x,0) = S_0(x), I(x,0) = I_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where $\beta(x,t)$ and $\gamma(x,t)$ denote the rate of disease transmission and the recovery rate at location x and time t, respectively. In an almost periodic environment, we first make the following assumption:

$(H1)$ $\beta(x,t)$ and $\gamma(x,t)$ are Hölder continuous and nonnegative nontrivial on $\bar{\Omega}\times\mathbb{R}$, and uniformly almost periodic in t.

As noted in [25], since $SI/(S+I)$ is Lipschitz continuous in S and I in the first open quadrant, we can define S and I in the entire first quadrant by defining it to be zero when either $S = 0$ or $I = 0$. Refer to [2] (see also [25]), we assume that

$(H2)$ $\int_\Omega I(x,0)\mathrm{d}x > 0$, with $S_0 \geqslant 0$ and $I_0 \geqslant 0$ for all $x \in \Omega$, and S_0, I_0 are continuous on $\bar{\Omega}$,

which assures that there is a positive number of infected individuals at the initial time.

By the theory for parabolic equations (see [18]), system (1.2) admits a unique classical solution $(S,I) \in C^{2,1}(\bar{\Omega} \times (0,\infty))$. It follows from the strong maximum principle and the Hopf boundary lemma for parabolic equations (see [26]) that both $S(x,t)$ and $I(x,t)$ are positive for all $x \in \bar{\Omega}$ and $t > 0$. As following from [2], let

$$N := \int_\Omega [S_0(x) + I_0(x)]\mathrm{d}x > 0$$

be the total number of individuals in Ω at $t = 0$. It then follows from (1.2) that

$$\frac{\partial}{\partial t}\int_\Omega [S + I]\mathrm{d}x = \int_\Omega \Delta(D_S S + D_I I)\mathrm{d}x = 0, \quad t > 0.$$

Hence, the total population size is constant, i.e.,

$$\int_\Omega (S + I)\mathrm{d}x = N, \quad t \geqslant 0. \tag{1.3}$$

Thus, $\|S(\cdot,t)\|_{L^1(\Omega)}$ and $\|I(\cdot,t)\|_{L^1(\Omega)}$ are bounded in $[0,\infty)$. Let $I \equiv 0$ on $\bar{\Omega} \times \mathbb{R}$ in (1.2), then we obtain the unique disease-free almost periodic solution $(S,0) = (N/|\Omega|, 0)$ (see [2, Lemma 2.1]), where $|\Omega|$ denotes the volume of the domain Ω.

Our analysis in this paper will mainly focus on the persistence and extinction of almost periodic reaction–diffusion system (1.2). Based on the theory of linear almost periodic parabolic equations, and using the idea of the next-generation operator in [8,31] (see also [25,32,34]), we establish the definition and the theories of the basic reproduction ratio R_0 and its computation formulae for almost periodic reaction–diffusion system (1.2). In particular, we present the asymptotic properties of R_0 with respect to the diffusion rate D_I of the infected individuals. Finally, we consider the uniform persistence, extinction and global attractivity in terms of R_0. Our results suggest that the interaction of spatial heterogeneity and temporal almost periodicity tends to enhance the persistence of the disease. Since almost periodic functions are a generalization (see [9]) of periodic functions, we would like to illustrate that, differing from the autonomous case (see [2]) and periodic case (see [25]), the theory of the eigenvalue is not applicable to almost periodic cases. Here, exponential growth bounds and skew-product semiflows are employed.

This paper is organized as follows. In Sect. 2, we present the theory of abstract linear almost periodic parabolic equations. In Sect. 3, we introduce the basic reproduction ratio of (1.2) and obtain its computation formulae and properties. In Sect. 4, we establish a threshold result on uniform persistence and global extinction of (1.2). In Sect. 5, we give the description of the global attractivity for some special cases. In Sect. 6, we discuss biological interpretations of our results briefly.

2. Almost periodic parabolic equations

A function $f \in C(\mathbb{R}, \mathbb{R}^m)$ is said to be *almost periodic* if for any $\epsilon_* > 0$, there exists $l = l(\epsilon_*) > 0$ such that every interval of $\mathbb{R}$ of length l contains at least one point of the set

$$T(f, \epsilon_*) = \{s_* \in \mathbb{R} : |f(t + s_*) - f(t)| < \epsilon_*, \forall t \in \mathbb{R}\},$$

where $|\cdot|$ is the usual Euclidean norm in $\mathbb{R}^m$. Let $D \subset \mathbb{R}^n$. A function $f \in C(D \times \mathbb{R}, \mathbb{R}^m)$ is said to be *uniformly almost periodic* in t if $f(x, \cdot)$ is almost periodic for each $x \in D$, and for any compact set $E \subset D$, f is uniformly continuous on $E \times \mathbb{R}$ (see [6,9]). A special class of almost periodic functions consists of quasi-periodic functions. A function $f \in C(\mathbb{R}, \mathbb{R}^m)$ is said to be *quasi-periodic* if there exist positive numbers $\varpi_1, \ldots, \varpi_p$ with their reciprocals $T_1, \ldots, T_p$ being rationally linearly independent such that $f(t) = F(t, \ldots, t)$ and for each $1 \leqslant i \leqslant p$, $F(t_1, \ldots, t_p)$ is ϖ_i-periodic in t_i.

It follows from [6,9] that if f is an almost periodic function, then there exists a Fourier series $\sum_{j=1}^{\infty} A_j e^{i\lambda_j t}$ associated with the function f, i.e., $f(t) \sim \sum_{j=1}^{\infty} A_j e^{i\lambda_j t}$. We call λ_j, $j = 1, 2, \ldots$, the Fourier exponent of $f(t)$, and A_j the Fourier coefficient of $f(t)$. The module of f, $mod(f)$, is defined as the smallest additive group of real numbers that contains the Fourier exponent of $f(t)$.

For each $\alpha \in (1/2 + n/2p, 1)$, let $X := X_\alpha \subset L^p(\Omega)$ $(p > n)$ be the fractional power space with respect to $-\Delta$ with homogeneous Neumann boundary condition (see [12]). Then X is an ordered Banach space with the cone X_+ defined by $X_+ = \{\varphi \in X : \varphi(x) \geqslant 0, \forall x \in \Omega\}$. Note that the interior of X_+ satisfies $\mathrm{Int}X_+ \neq \emptyset$. Hence, the ordering on X can be defined as follows:

$$\begin{aligned} &\varphi_1 \leqslant \varphi_2 \Leftrightarrow \varphi_1(x) \leqslant \varphi_2(x), \quad \forall x \in \Omega, \\ &\varphi_1 < \varphi_2 \Leftrightarrow \varphi_1 \leqslant \varphi_2,\ \varphi_1 \neq \varphi_2, \\ &\varphi_1 \ll \varphi_2 \Leftrightarrow \varphi_2 - \varphi_1 \in \mathrm{Int}X_+. \end{aligned}$$

We denote the norm in X by $\|\cdot\|$. Let AP be the ordered Banach space consisting of all almost periodic and continuous functions from $\mathbb{R}$ to X with the maximum norm. Hence, the positive cone can be defined by $AP_+ = \{\psi \in AP : \psi(t)(x) \geqslant 0,\ \forall x \in \bar{\Omega}, t \in \mathbb{R}\}$. Here, we use the notation $\psi(t)(x) = \psi(x, t)$.

Consider the linear almost periodic parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - D_u \Delta u = \xi(x, t) u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{2.1}$$

where ξ is Hölder continuous and uniformly almost periodic in t. Define the hull of an almost periodic function $\xi(x, t)$ as

$$H(\xi) = \mathrm{cls}\{\xi_s : s \in \mathbb{R}, \xi_s(\cdot, t) = \xi(\cdot, s + t)\},$$

where the closure is taken in the compact open topology. It then follows that the translation $\sigma : \mathbb{R} \times H(\xi) \to H(\xi)$, $\sigma(t)(\zeta) = \zeta_s = \zeta \cdot s$ with $\zeta \cdot s(t) = \zeta(t + s)$, defines a continuous, compact, almost periodic minimal and distal flow (see [28, Lemma VI.C], which is denoted by $(H(\xi), \sigma, \mathbb{R})$. Then (2.1) generates a skew-product semiflow:

$$\begin{aligned} \Pi_t : X \times H(\xi) &\to X \times H(\xi), \quad t \geqslant 0, \\ (u_0, \zeta) &\mapsto (u(\cdot, t, u_0, \zeta), \zeta_t), \end{aligned} \tag{2.2}$$

where $u(x, t, u_0, \zeta)$ is the solution of

$$\begin{cases} \frac{\partial u}{\partial t} - D_u \Delta u = \zeta(x, t) u, & x \in \Omega, \ t > 0, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0 \end{cases} \tag{2.3}$$

satisfying $u(x, 0, u_0, (\zeta)) = u_0(x)$ $(\zeta \in H(\xi))$.

For any given $\eta \in \mathbb{R}$, define the following linear skew-product semiflow

$$\begin{aligned} \Pi_t^\eta : X \times H(\xi) &\to X \times H(\xi), \quad t \geqslant 0, \\ (u_0, \zeta) &\mapsto (\Phi_\eta(t, \zeta) u_0, \zeta_t), \end{aligned} \tag{2.4}$$

where $\Phi_\eta(t, \zeta) u_0 = e^{-\eta t} u(\cdot, t, u_0, \zeta)$.

Let $\mathcal{Q}$ be a subset of $H(\xi)$. According to [16, Definition 3.4], the linear skew-product semiflow (2.4) admits an *exponential dichotomy* over $\mathcal{Q}$ if there exist $\sigma_1 > 0$, $K_1 > 0$ and continuous projections $P(\zeta) : X \to X$ $(\zeta \in \mathcal{Q})$ such that for any $\zeta \in \mathcal{Q}$, the following hold:

(1) $\Phi_\eta(t, \zeta) P(\zeta) = P(\zeta \cdot t) \Phi_\eta(t, \zeta)$, $\forall t \in \mathbb{R}_+$;

(2) $\Phi_\eta(t, \zeta)|_{R(P(\zeta))} : R(P(\zeta)) \to R(P(\zeta \cdot t))$ is an isomorphism for $t \in \mathbb{R}_+$, where $R(P)$ denotes the range of P (hence, $\Phi_\eta(-t, \zeta) := \Phi_\eta^{-1}(t, \zeta \cdot (-t)) : R(P(\zeta)) \to R(P(\zeta \cdot (-t)))$ is well defined for $t \in \mathbb{R}_+$);

(3)

$$\|\Phi_\eta(t, \zeta)(I - P(\zeta))\| \leqslant K_1 e^{-\sigma_1 t}, \quad \forall t \in \mathbb{R}_+,$$

and

$$\|\Phi_\eta(t, \zeta) P(\zeta)\| \leqslant K_1 e^{\sigma_1 t}, \quad \forall t \in \mathbb{R}_-.$$

The set

$$\Sigma(\mathcal{Q}) := \{\eta \in \mathbb{R} : \text{(2.4) admits no exponential dichotomy over } \mathcal{Q}\}$$

is called the *dynamic* (*Sacker–Sell*) *spectrum* of (2.1) or (2.2) over $\mathcal{Q}$.

For any $\zeta \in H(\xi)$, we define the *Lyapunov exponent* λ_ζ as

$$\lambda_\zeta = \limsup_{t \to \infty} \frac{\ln \|\Phi(t, \zeta)\|}{t},$$

where $\Phi(t, \zeta) = \Phi_0(t, \zeta)$. The number $\lambda^* := \sup_{\zeta \in H(\xi)} \lambda_\zeta$ is called the *upper Lyapunov exponent* of (2.1) or (2.2).

Theorem 2.1. *There exist an almost periodic function $a(t, \zeta)$ and a uniformly almost periodic function $\tilde{u}(\cdot, t, \zeta) \in IntX_+$ such that $u(x, t, \zeta) = e^{\int_0^t a(\tau, \zeta) \mathrm{d}\tau} \tilde{u}(x, t, \zeta)$ is a solution of (2.3). Furthermore,*

$$\lambda^* = \lim_{t \to \infty} \frac{\ln \|u(x, t, \zeta)\|}{t} = \lim_{t \to \infty} \frac{1}{t} \int_0^t a(\tau, \zeta) \mathrm{d}\tau,$$

and it is a strictly monotone decreasing function of $D_u > 0$ if $\xi(\cdot, t)$ changes sign for every t.

Proof. It follows from [16, Lemma 3.3] that the skew-product semiflow (2.2) is strongly monotone in the sense that $u(\cdot, t, u_0, \zeta)) \in \mathrm{Int}X_+$ for any $t > 0$, $\zeta \in H(\xi)$, and $u_0 \in X_+ \setminus \{0\}$. Hence, the skew-product semiflow (2.2) admits a continuous separation ([16, Lemma 3.12]) in the sense that there exist subspaces $\{X_1(\zeta)\}_{\zeta \in H(\xi)}$, $\{X_2(\zeta)\}_{\zeta \in H(\xi)}$ such that the following properties hold:

(1) $X = X_1(\zeta) \oplus X_2(\zeta)$ $(\zeta \in H(\xi))$ and $X_1(\zeta)$, $X_2(\zeta)$ vary continuously in $\zeta \in H(\xi)$;

(2) $X_1(\zeta) = \text{span}\{u(\zeta)\}$, where $u(\zeta) \in \text{Int}X_+$ and $\|u(\zeta)\| = 1$ for $\zeta \in H(\xi)$;
(3) $X_2(\zeta) \cap X_+ = 0$ for each $\zeta \in H(\xi)$;
(4) $\Phi(t,\zeta)X_1(\zeta) = X_1(\zeta \cdot t)$ and $\Phi(t,\zeta)X_2(\zeta) \subset X_2(\zeta \cdot t)$ for all $t \geqslant 0$, $\zeta \in H(\xi)$;
(5) There exist $K_2 > 0$ and $\sigma_2 > 0$ such that for any $\zeta \in H(\xi)$ and $\hat{u}(\zeta) \in X_2(\zeta)$ with $\|\hat{u}(\zeta)\| = 1$, we have

$$\|\Phi(t,\zeta)\hat{u}(\zeta)\| \leqslant K_2 e^{-\sigma_2 t}\|\Phi(t,\zeta)u(\zeta)\|$$

for all $t > 0$.

Let

$$\tilde{u}(\cdot,t,\zeta) = \frac{u(\zeta \cdot t)}{\|u(\zeta \cdot t)\|_2} \in \text{Int}X_+,\ \zeta \in H(\xi)$$

with $\tilde{u}(\zeta) = \tilde{u}(\cdot,0,\zeta)$, where $\|\cdot\|_2$ denotes the norm in $L^2(\Omega)$, and $r(t,\zeta) = \|u(\cdot,t,\tilde{u}(\zeta),\zeta)\|_2$. Observe that $\tilde{u}(x,t,\zeta)$ can also be expressed as

$$\tilde{u}(x,t,\zeta) = \frac{u(x,t,u(\zeta),\zeta)}{\|u(x,t,u(\zeta),\zeta)\|_2}.$$

Then we have $u(x,t,\tilde{u}(\zeta),\zeta) = r(t,\zeta)\tilde{u}(x,t,\zeta)$ and $r(t,\zeta)$ are differentiable in t (see [16, Lemma 3.13]). Hence,

$$r_t(t,\zeta)\tilde{u}(x,t,\zeta) + r(t,\zeta)\frac{\partial \tilde{u}}{\partial t}(x,t,\zeta) - D_u r(t,\zeta)\Delta\tilde{u}(x,t,\zeta) = \zeta(x,t)r(t,\zeta)\tilde{u}(x,t,\zeta). \tag{2.5}$$

It then follows from $\int_\Omega \tilde{u}^2(x,t,\zeta)\mathrm{d}x = 1$ that

$$r_t(t,\zeta) = a(t,\zeta)r(t,\zeta), \tag{2.6}$$

where

$$a(t,\zeta) = -D_u \int\limits_\Omega |\nabla\tilde{u}(x,t,\zeta)|^2\mathrm{d}x + \int\limits_\Omega \zeta(x,t)|\tilde{u}(x,t,\zeta)|^2\mathrm{d}x, \tag{2.7}$$

and it is almost periodic because $\Delta\tilde{u}(x,t,\zeta)$, $\tilde{u}(x,t,\zeta)$ and $\zeta(x,t)$ are uniformly almost periodic in t ([16, Lemma 3.13 3)]). From (2.5) and (2.6), we see that $\tilde{u}(x,t,\zeta)$ satisfies

$$\begin{cases} \frac{\partial \tilde{u}}{\partial t} - D_u\Delta\tilde{u} = \zeta(x,t)\tilde{u} - a(t,\zeta)\tilde{u}, & x \in \Omega,\ t > 0, \\ \frac{\partial \tilde{u}}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{2.8}$$

It is easy to verify that $u(x,t,\zeta) = e^{\int_0^t a(\tau,\zeta)\mathrm{d}\tau}\tilde{u}(x,t,\zeta)$ is a solution of (2.3) with the initial value $u(\cdot,0,\zeta) = \tilde{u}(\cdot,0,\zeta) = \tilde{u}(\zeta)$, and hence,

$$\Phi(t,\zeta)\tilde{u}(\zeta) = e^{\int_0^t a(\tau,\zeta)\mathrm{d}\tau}\tilde{u}(\cdot,t,\zeta) = u(\cdot,t,\tilde{u}(\zeta),\zeta) = r(t,\zeta)\tilde{u}(\cdot,t,\zeta). \tag{2.9}$$

The existence of the continuous separation of the linear skew-product semiflow shows that

$$\lambda^* = \sup_{\zeta\in H(\xi)} \limsup_{t\to\infty} \frac{\ln\|\Phi(t,\zeta)\tilde{u}(\zeta)\|}{t}. \tag{2.10}$$

Since $a(t,\zeta)$ is almost periodic, $\lim_{t\to\infty}\frac{1}{t}\int_0^t a(\tau,\zeta)\mathrm{d}\tau$ exists and is independent of $\zeta \in H(\xi)$ (see, e.g., [16, Lemma 3.2]). Thus, $\lim_{t\to\infty}\frac{\ln\|\Phi(t,\zeta)\tilde{u}(\zeta)\|}{t} = \lim_{t\to\infty}\frac{1}{t}\int_0^t a(\tau,\zeta)\mathrm{d}\tau$ exists and is independent of $\zeta \in H(\xi)$. It then follows from (2.9) and (2.10) that

$$\lambda^* = \lim_{t\to\infty}\frac{\ln\|u(x,t,\zeta)\|}{t} = \lim_{t\to\infty}\frac{1}{t}\int\limits_0^t a(\tau,\zeta)\mathrm{d}\tau.$$

It remains to prove the monotonicity. In fact, it is obvious that λ^* is nonincreasing by inspection of (2.7). At the beginning, we differentiate both sides of (2.8) by D_u to obtain

$$(\tilde{u}_t)' - \Delta\tilde{u} - D_u\Delta(\tilde{u})' = \zeta(x,t)(\tilde{u})' - a'(t,\zeta)\tilde{u} - a(t,\zeta)(\tilde{u})', \tag{2.11}$$

where the notation $()'$ denotes differentiation by D_u. Next, we multiply (2.8) by $(\tilde{u})'$ and (2.11) by $\tilde{u}$, subtract the resulting equations and then integrate by parts over $(0,t)\times\Omega$ to obtain

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t a'(\tau,\zeta)\mathrm{d}\tau = -\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega |\nabla\tilde{u}(\tau,\zeta)|^2\mathrm{d}x\mathrm{d}\tau.$$

Since $\tilde{u}(t,\zeta)\in \mathrm{Int}X_+$, we conclude that $(\lambda^*)'\leqslant 0$. Finally, we prove that $(\lambda^*)'=0$ is impossible. On the contrary, if $(\lambda^*)'=0$, then the uniform almost periodicity of $\nabla\tilde{u}(t,\zeta)$ (see, e.g., [16, Lemma 3.13 2)]) implies that $\tilde{u}(t,\zeta)$ is a positive constant function with respect to x. Thus, we follows from (2.8) that $\lambda^* = \lim_{t\to\infty}\frac{1}{t}\int_0^t a(\tau,\zeta)\mathrm{d}\tau = \lim_{t\to\infty}\frac{1}{t}\int_0^t \zeta(x,\tau)\mathrm{d}\tau$. Since $\zeta(\cdot,t)$ changes sign on Ω, it is a contradiction. Hence, we have $(\lambda^*)'<0$. □

Let $\Psi_\zeta(t,s)$ $(t\geq s,\ s\in\mathbb{R})$ be the evolution operator of (2.3). Obviously, $\Psi_\zeta(t,0)=\Phi(t,\zeta)$. For each $\zeta\in H(\xi)$, we define the *exponential growth bound* of $\Psi_\zeta(t,s)$ to be

$$\omega(\Psi_\zeta) = \inf\{\tilde{\omega}\in\mathbb{R} : \exists K_0\geqslant 1, \|\Psi_\zeta(t+s,s)\|\leqslant K_0e^{\tilde{\omega}t},\ \forall s\in\mathbb{R},\ t\geqslant 0\}. \tag{2.12}$$

Lemma 2.2. *$\omega(\Psi_\zeta)=\omega(\Psi_\xi)=\lambda^*=\sup\Sigma(H(\xi))$ for all $\zeta\in H(\xi)$.*

Proof. The conclusion of $\lambda^*=\sup\Sigma(H(\xi))$ can be obtained straight by [16, Lemma 3.10] (see also [29, Proposition II. 4.1]).

By [27, Lemma 1], we see that the spectrum $\Sigma(\zeta)=\Sigma(H(\xi))$ for each $\zeta\in H(\xi)$. Since $\Psi_\zeta(t+s,s)=\Psi_{\zeta\cdot s}(t,0)=\Phi(t,\zeta\cdot s)$, the similar arguments to those in the proof of [32, Lemma 2.4] (see also [27, Lemma 4 (A)]) and (2.12) imply that $\omega(\Psi_\zeta)\geqslant\sup\Sigma(H(\xi))$, $\forall\zeta\in H(\xi)$. For any $\epsilon>0$, let $\lambda_0=\sup\Sigma(H(\xi))$ and $\lambda_*=\lambda_0+\epsilon$. It then follows that from the properties of exponential dichotomy that there exists $K_3\geqslant 1$ such that

$$\|e^{-\lambda_* t}\Phi(t,\zeta)\|\leqslant K_3,\quad \forall t\geqslant 0,\ \zeta\in H(\xi),$$

that is,

$$\|\Phi(t,\zeta)\|\leqslant K_3e^{\lambda_* t},\quad \forall t\geqslant 0,\ \zeta\in H(\xi).$$

Hence, we conclude that $\omega(\Psi_\zeta)\leqslant\lambda_*=\lambda_0+\epsilon$. Letting $\epsilon\to 0$, we have $\omega(\Psi_\zeta)\leqslant\lambda_0=\sup\Sigma(H(\xi))$. Thus, $\omega(\Psi_\zeta)=\sup\Sigma(H(\xi))$, $\forall\zeta\in H(\xi)$. In particular, $\omega(\Psi_\zeta)=\sup\Sigma(\zeta)$. In view of $\lambda^*=\sup\Sigma(H(\xi))$, we have $\omega(\Psi_\zeta)=\omega(\Psi_\xi)=\lambda^*=\sup\Sigma(H(\xi))$. □

3. Basic reproduction ratios

In epidemiology, the basic reproduction ratio R_0 (sometimes called basic reproductive rate, basic reproductive number) of an infection is defined as the expected number of secondary cases produced by a single (typical) infection in a completely susceptible population (see [8]). It is used to measure the transmission potential of a disease, i.e., it helps determine whether or not an infectious disease can spread through a population. Mathematically, R_0 is a crucial threshold parameter in population models. In this section, we will give the definition of the basic reproduction ratio of system (1.2) and analyze its properties.

Let $\Psi_{-\gamma}(t,s)$ $(t\geqslant s,\ s\in\mathbb{R})$ be the evolution operator of reaction–diffusion equation

$$\begin{cases} \frac{\partial I}{\partial t} - D_I\Delta I = -\gamma(x,t)I, & x\in\Omega,\ t>0,\\ \frac{\partial I}{\partial\nu}=0, & x\in\partial\Omega,\ t>0. \end{cases} \tag{3.1}$$

It then follows from the standard semigroup theory (see also [25]) that there exist $K_4 > 0$ and $\sigma_4 > 0$ such that

$$\|\Psi_{-\gamma}(t,s)\| \leqslant K_4 e^{-\sigma_4(t-s)}, \quad \forall t \geq s,\ s \in \mathbb{R}. \tag{3.2}$$

Let $\omega(\Psi_{-\gamma})$ be the exponential growth bound of $\Psi_{-\gamma}(t,s)$, then it is easy to see from the definition of the exponential growth bound that $\omega(\Psi_{-\gamma}) < 0$.

Assume that the uniformly almost periodic function $\phi(x,s)$ is the initial distribution of infectious individuals at the location $x \in \Omega$ and time s, then $\beta(x,s)\phi(x,s)$ is the distribution of new infections produced by the infected individuals who were introduced at time s. Given $t \geqslant s$, then $\Psi_{-\gamma}(t,s)\beta(x,s)\phi(x,s)$ is the distribution at location x of those infected individuals who were newly infected at time s and remains infected at time t. It then follows that

$$\int_{-\infty}^{t} \Psi_{-\gamma}(t,s)\beta(\cdot,s)\phi(\cdot,s)\mathrm{d}s = \int_{0}^{\infty} \Psi_{-\gamma}(t,t-a)\beta(\cdot,t-a)\phi(\cdot,t-a)\mathrm{d}a$$

represents the distribution of accumulative new infections at location x and time t produced by all those infected individuals $\phi(x,s)$ introduced at previous time to t.

Define $AP_{(\beta,\gamma)} := \{\phi : \phi \in AP, mod(\phi) \subset mod(\beta,\gamma)\}$. By [32, Lemma 2.1], $AP_{(\beta,\gamma)}$ is a Banach space with the supremum norm, and the positive cone $AP^+_{(\beta,\gamma)} = \{\phi \in AP_{(\beta,\gamma)} : \phi(t)(x) \geqslant 0,\ \forall x \in \bar{\Omega}, t \in \mathbb{R}\}$ has a nonempty interior $\mathrm{Int}(AP^+_{(\beta,\gamma)})$. Then we define a linear mapping L by

$$(L\phi)(t) = \int_{0}^{\infty} \Psi_{-\gamma}(t,t-a)\beta(\cdot,t-a)\phi(\cdot,t-a)\mathrm{d}a. \tag{3.3}$$

Since $\omega(\Psi_{-\gamma}) < 0$, applying the similar arguments to those in [32, Lemma 3.1], we can verify that L is continuous and positive operator from $AP_{(\beta,\gamma)}$ to $AP_{(\beta,\gamma)}$.

Using the ideas in [8,31] (see also [25,32,34]), we call L the next-generation operator and define the spectral radius of L as the basic reproduction ratio

$$R_0 := r(L)$$

for almost periodic reaction–diffusion epidemic model (1.2).

Consider the linear almost periodic parabolic equation

$$\begin{cases} \frac{\partial I}{\partial t} - D_I \Delta I = \beta(x,t)I - \gamma(x,t)I, & x \in \Omega,\ t > 0, \\ \frac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{3.4}$$

Let $\Psi_{(\beta,-\gamma)}(t,s)$ $(t \geqslant s, s \in \mathbb{R})$ be the evolution operator of (3.4). Similarly, we can define the exponential growth bound of $\Psi_{(\beta,-\gamma)}(t,s)$, denoted by $\omega(\Psi_{(\beta,-\gamma)})$.

Lemma 3.1. *$R_0 - 1$ and $\omega(\Psi_{(\beta,-\gamma)})$ have the same sign.*

Proof. For simplicity, Let $A := D_I\Delta + \beta - \gamma$, $B := D_I\Delta - \gamma$ and $C := \beta$. Here, we denote the evolution operators $\Psi_{(\beta,-\gamma)}(t,s)$ and $\Psi_{-\gamma}(t,s)$ by $\Psi_A(t,s)$ and $\Psi_B(t,s)$, respectively.

Following the approach similar to that in [30], we define the evolution semigroup Γ_A associated with $\Psi_A(t,s)$ as

$$[\Gamma_A(t)\phi](\cdot,s) = \Psi_A(s,s-t)\phi(\cdot,s-t),\ \phi \in AP(\beta,\gamma), \quad \forall t \geq 0, \quad s \in \mathbb{R}.$$

Then we define the exponential growth bound of Γ_A as

$$\omega(\Gamma_A) = \inf\{\tilde{\omega} \in \mathbb{R} : \exists K_0 \geq 1, \|\Gamma_A(t)\| \leq K_0 e^{\tilde{\omega}t},\ \forall t \geq 0\}.$$

Similarly, we can define the evolution semigroup Γ_B associated with $\Psi_B(t,s)$ and then define the exponential growth bound of Γ_B, denoted by $\omega(\Gamma_B)$. Let the generator $\mathcal{A}$ of Γ_A be the part of the operator $-\frac{\mathrm{d}}{\mathrm{d}t} + A$

on $AP_{(\beta,\gamma)}$, the generator $\mathcal{B}$ of Γ_B be the part of the generator $-\frac{\mathrm{d}}{\mathrm{d}t}+B$, and define $(\mathcal{C}\phi)(\cdot,t)=\beta(\cdot,t)\phi(\cdot,t)$. Then $\mathcal{A}$ is a positive perturbation of $\mathcal{B}$, and $\mathcal{A}=\mathcal{B}+\mathcal{C}$. Let $\sigma(\mathcal{A})$ be the spectrum of $\mathcal{A}$. Recall that the spectral bound of $\mathcal{A}$ (see also [30]) is defined as

$$s(\mathcal{A})=\sup\{Re(\lambda):\lambda\in\sigma(\mathcal{A})\}.$$

Similarly, we can define the spectral bound of $\mathcal{B}$, denoted by $s(\mathcal{B})$. Since $AP_{(\beta,\gamma)}$ is an abstract M-space, we follow from [32, Lemma 2.3] and [30, Theorem 3.14] that $\omega(\Psi_A)=\omega(\Gamma_A)=s(\mathcal{A})$ and $\omega(\Psi_B)=\omega(\Gamma_B)=s(\mathcal{B})$.

Note that the positive cone $AP^+_{(\beta,\gamma)}$ is normal and generating. It then follows from [30, Theorem 3.12] that $\mathcal{A}$ and $\mathcal{B}$ are resolvent-positive on $AP_{(\beta,\gamma)}$, and $\mathcal{B}^{-1}$ exists such that

$$(-\mathcal{B}^{-1}\phi)(\cdot,t)=\int_0^\infty\Psi_B(t,t-a)\phi(\cdot,t-a)\mathrm{d}a,\quad\forall\phi\in AP_{(\beta,\gamma)}.$$

Hence,

$$\begin{aligned}(-\mathcal{C}\mathcal{B}^{-1}\phi)(\cdot,t)&=\int_0^\infty\beta(\cdot,t)\Psi_B(t,t-a)\phi(\cdot,t-a)\mathrm{d}a\\&=\beta(\cdot,t)\int_0^\infty\Psi_{-\gamma}(t,t-a)\phi(\cdot,t-a)\mathrm{d}a\\&=:(\bar{L}\phi)(t).\end{aligned}$$

Let

$$\mathcal{I}(\phi)(t)=\int_0^\infty\Psi_{-\gamma}(t,t-a)\phi(\cdot,t-a)\mathrm{d}a,\quad\mathcal{J}(\phi)(t)=\beta(\cdot,t)\phi(\cdot,t).$$

By the arguments similar to the above, we see that $\bar{L}$, $\mathcal{I}$ and $\mathcal{J}$ are the linear operators from $AP_{(\beta,\gamma)}$ to $AP_{(\beta,\gamma)}$. The spectral radius of $\bar{L}$, $r(\bar{L})$, can be defined as the basic reproduction ratio of (1.2) (see [4,30]). Since $L=\mathcal{I}\mathcal{J}$ and $\bar{L}=\mathcal{J}\mathcal{I}$, it follows that $r(L)=r(\bar{L})$. Thus, $r(L)$ and $r(\bar{L})$ give rise to the same R_0. Since $\omega(\Psi_{-\gamma})<0$, that is $s(\mathcal{B})<0$, [30, Theorem 3.5] implies that $s(\mathcal{A})$ and $r(-\mathcal{C}\mathcal{B}^{-1})-1$ have the same sign. Based on the above all, we conclude that R_0-1 and $\omega(\Psi_{(\beta,-\gamma)})$ have the same sign. □

Let $\Psi_{(\beta/\rho,-\gamma)}(t,s)$ $(t\geqslant s,\ s\in\mathbb{R})$ be the evolution operator of the linear almost periodic parabolic equation

$$\begin{cases}\frac{\partial I}{\partial t}-D_I\Delta I=\frac{\beta(x,t)}{\rho}I-\gamma(x,t)I, & x\in\Omega,\ t>0,\\ \frac{\partial I}{\partial\nu}=0, & x\in\partial\Omega,\ t>0,\end{cases}\tag{3.5}$$

where $\rho\in(0,\infty)$ is a parameter. Let $\omega(\Psi_{(\beta/\rho,-\gamma)})$ be the exponential growth bound of $\Psi_{(\beta/\rho,-\gamma)}(t,s)$.

In order to computer R_0, we make the following assumption:

$(H3)$ Either $(\beta(\cdot,t),\gamma(\cdot,t))$ is quasi-periodic, or $\mathrm{mod}(\beta,\gamma)$ has no finite limit point.

Lemma 3.2. *Let (H3) hold. Then $\rho=R_0>0$ is the unique solution of $\omega(\Psi_{(\beta/\rho,-\gamma)})=0$.*

Proof. Since the exponential growth bound associated with (3.5) is nonincreasing in $\rho\in(0,\infty)$ and $\omega(\Psi_{-\gamma})<0$, without loss of generality, we assume that $\omega(\Psi_{(\beta/\rho,-\gamma)})=0$ for some $\rho=\rho_0>0$. Let (ρ_0,ψ_0) be the solution of (3.5). Since $\omega(\Psi_{(\beta/\rho_0,-\gamma)})=0$, it follows from the definition of the exponential growth bound that $\psi_0(x,t)$ is a bounded solution of (3.5) with $\rho=\rho_0$. By the constant variation formula, we have

$$\psi_0(x,t)=\Psi_{-\gamma}(t,\tau)\psi_0(x,\tau)+\int_{\tau}^{t}\Psi_{-\gamma}(t,s)\frac{\beta(x,s)}{\rho_0}\psi_0(x,s)\mathrm{d}s. \tag{3.6}$$

In view of (3.2) and the boundedness of $\psi_0(x,t)$ on $\mathbb{R}$, letting $\tau\to-\infty$ in (3.6), we obtain

$$\psi_0(x,t)=\int_{-\infty}^{t}\Psi_{-\gamma}(t,s)\frac{\beta(x,s)}{\rho_0}\psi_0(x,s)\mathrm{d}s,\ \forall t\in\mathbb{R}.$$

It then follows from (3.3) that $L\psi_0=\rho_0\psi_0$. Since $\rho_0\in\sigma(L)\setminus\{0\}$, we have $R_0:=r(L)>0$.

For any sufficiently small $\epsilon>0$, let $\Psi_{((\beta+\epsilon)/\rho,-\gamma)}$ be the evolution operator of the linear almost periodic parabolic equation

$$\begin{cases}\frac{\partial I}{\partial t}-D_I\Delta I=\frac{\beta(x,t)+\epsilon}{\rho}I-\gamma(x,t)I, & x\in\Omega,\ t>0,\\ \frac{\partial I}{\partial\nu}=0, & x\in\partial\Omega,\ t>0,\end{cases} \tag{3.7}$$

where $\rho\in(0,\infty)$ is a parameter. We use $\omega(\Psi_{((\beta+\epsilon)/\rho,-\gamma)})$ to denote the exponential growth bound of $\Psi_{((\beta+\epsilon)/\rho,-\gamma)}$. Define

$$L_\epsilon(\phi)(t):=\int_0^{\infty}\Psi_{-\gamma}(t,s)(\beta(\cdot,t-a)+\epsilon)\phi(\cdot,t-a)\mathrm{d}a,$$

and its spectral radius $R_0^\epsilon=r(L_\epsilon)$. By Theorem 2.1 and Lemma 2.2, we conclude that there exist almost periodic functions $\tilde I(\cdot,t,(\beta+\epsilon)/R_0^\epsilon,\gamma)\in\mathrm{Int}X_+$ and $a(t,(\beta+\epsilon)/R_0^\epsilon,\gamma)$ such that

$$I_\epsilon(\cdot,t):=e^{\int_0^t a(t,(\beta+\epsilon)/R_0^\epsilon,\gamma)\mathrm{d}\tau}\tilde I(\cdot,t,(\beta+\epsilon)/R_0^\epsilon,\gamma),\ \forall t\in\mathbb{R}_+$$

is a solution of (3.7) with $\rho=R_0^\epsilon$, and

$$\bar\omega:=\omega(\Psi_{((\beta+\epsilon)/R_0^\epsilon,-\gamma)})=\lim_{t\to\infty}\frac{\ln\|I_\epsilon(x,t)\|}{t}. \tag{3.8}$$

Since $\beta(x,t)+\epsilon>0$, $L_\epsilon:AP_{(\beta,\gamma)}\to AP_{(\beta,\gamma)}$ is strongly positive. Furthermore, the similar arguments to those in the proof of [32, Lemma 3.3] imply that L_ϵ is compact, and $\lim_{\epsilon\to0^+}\omega(\Psi_{(\beta+\epsilon)/\rho_0,-\gamma)})=\omega(\Psi_{(\beta/\rho_0,-\gamma)})$, and $\lim_{\epsilon\to0^+}R_0^\epsilon=R_0$. By the Krein–Rutman theorem (see, e.g., [13, Theorem 7.1]), we conclude that there exists $\psi_0^\epsilon\in\mathrm{Int}(AP^+_{(\beta,\gamma)})$ such that $L_\epsilon\psi_0^\epsilon=R_0^\epsilon\psi_0^\epsilon$. Hence, $\psi_0^\epsilon(x,t)$ is a uniformly almost periodic solution of (3.7) with $\rho=R_0^\epsilon$. In the following, we prove that $\bar\omega=0$.

First, by contradiction, suppose that $\bar\omega>0$. Since I_ϵ, $\psi_0^\epsilon\in\mathrm{Int}(AP^+_{(\beta,\gamma)})$, there exists a sufficiently small number $\nu_1>0$ such that $\psi_0^\epsilon(x,0)\geq\nu_1 I_\epsilon(x,0)$. The comparison principle implies that $\psi_0^\epsilon(x,t)\geq\nu_1 I_\epsilon(x,t)$, $\forall t\geq0$ and $x\in\Omega$. Hence,

$$\|\psi_0^\epsilon(x,t)\|\geq\nu_1\|I_\epsilon(x,t)\|,\quad\forall t\geq0,\ x\in\Omega. \tag{3.9}$$

In view of (3.8), we know that for any given $\varepsilon_*\in(0,\bar\omega)$, there exists $t_0>0$ such that

$$\frac{\ln\|I_\epsilon(x,t)\|}{t}\geq\bar\omega-\varepsilon_*,\quad\forall t>0,$$

which implies

$$\|I_\epsilon(x,t)\|\geq e^{(\bar\omega-\varepsilon_*)t},\quad\forall t>0. \tag{3.10}$$

Letting $t\to\infty$, we see from (3.9) and (3.10) that $\|\psi_0^\epsilon(x,t)\|\to\infty$, which contradicts the boundedness of the uniformly almost periodic function $\psi_0^\epsilon(x,t)$. Hence, $\bar\omega>0$ is impossible.

Next, suppose that $\bar{\omega} < 0$. For any given $\varepsilon^* \in (0, -\bar{\omega})$, (3.8) implies that there exists $\bar{t}_0 > 0$ such that

$$\frac{\ln \|I_\epsilon(x,t)\|}{t} \leq \bar{\omega} + \varepsilon^*, \quad \forall t > \bar{t}_0,$$

that is,

$$\|I_\epsilon(x,t)\| \leq e^{(\bar{\omega}+\varepsilon^*)t}, \quad \forall t > \bar{t}_0.$$

Similarly, there exists a sufficiently large number $\nu_2 > 0$ such that $\psi_0^\epsilon(x, \bar{t}_0) \leq \nu_2 I_\epsilon(x, \bar{t}_0)$. By the comparison principle, we then have $\psi_0^\epsilon(x,t) \leq \nu_2 I_\epsilon(x,t)$, and hence,

$$\|\psi_0^\epsilon(x,t)\| \leq \nu_2 \|I_\epsilon(x,t)\| \leq \nu_2 e^{(\bar{\omega}+\varepsilon^*)t}, \quad \forall t > \bar{t}_0.$$

Letting $t \to \infty$, we see that $\lim_{t\to\infty} \|\psi_0^\epsilon(x,t)\| = 0$. Let $A^\epsilon(\cdot,t) := D_I\Delta + (\beta(\cdot,t)/R_0^\epsilon - \gamma(\cdot,t))$. By [9, Theorem 5.7] and its proof, we further see that the uniformly almost periodic solution $\psi_0^\epsilon(x,t)$ of $\frac{\mathrm{d}I}{\mathrm{d}t} = A^\epsilon(x,t)I$ satisfies either $\inf_{t\in\mathbb{R}} \|\psi_0^\epsilon(\cdot,t)\| > 0$ or $\psi_0^\epsilon(\cdot,t) \equiv 0$. Thus, we obtain $\psi_0^\epsilon(\cdot,t) \equiv 0$, a contradiction. Hence, $\bar{\omega} < 0$ is impossible.

The arguments above imply that $\bar{\omega} = \omega(\Psi_{((\beta+\epsilon)/R_0^\epsilon, -\gamma)}) = 0$. Letting $\epsilon \to 0^+$, we get $\omega(\Psi_{(\beta/R_0,-\gamma)}) = 0$.

It remains to prove that $\omega(\Psi_{(\beta/\rho,-\gamma)}) = 0$ has at most one positive solution for ρ. By the standard comparison principle, since the exponential growth bound associated with (3.5) is nonincreasing in $\rho \in (0,\infty)$, Lemma 2.2 tells us that $\omega(\Psi_{(\beta/\rho,-\gamma)})$ admits the same property. On the contrary, we assume that $\omega(\Psi_{(\beta/\rho,-\gamma)}) = 0$ has two positive solutions $\rho_1 < \rho_2$. Then $\omega(\Psi_{(\beta/\rho,-\gamma)}) = 0$ for all $\rho \in [\rho_1, \rho_2]$. Hence, we see that any $\rho \in [\rho_1, \rho_2]$ is an eigenvalue of L, which is impossible since the compact linear operator L has countably many eigenvalues. □

Based on the above, we present the properties of the basic reproduction ratio of (1.2) for various rates of disease transmission and disease recovery. At the beginning, we assume that $\beta(x,t) - \gamma(x,t)$ or both $\beta(x,t)$ and $\gamma(x,t)$ are spatially homogeneous. For an almost periodic function $g(t)$, we denote the mean value of $g(t)$ by

$$[g] := \lim_{t\to\infty} \frac{1}{t} \int_0^t g(\tau)\mathrm{d}\tau.$$

Lemma 3.3. *The following statements are valid:*

(i) *If $\beta(x,t) \equiv \beta(t)$, $\gamma(x,t) \equiv \gamma(t)$ and (H3) holds, then $R_0 = [\beta]/[\gamma]$.*
(ii) *If $\beta(x,t) - \gamma(x,t) \equiv \mu(t)$, then $R_0 > 1$ if $[\mu] > 0$, $R_0 = 1$ if $[\mu] = 0$ and $R_0 < 1$ if $[\mu] < 0$.*

Proof. (i) By Theorem 2.1 and its proof, there exist almost periodic functions $\tilde{\varphi}^* := \tilde{\varphi}^*(\cdot, t, \beta/\rho, \gamma) \in \mathrm{Int}X_+$ and $a(t, \beta/\rho, \gamma))$ associated with (3.5) such that

$$\begin{cases} \frac{\partial \tilde{\varphi}^*}{\partial t} - D_I \Delta \tilde{\varphi}^* = \frac{\beta(x,t)}{\rho}\tilde{\varphi}^* - \gamma(x,t)\tilde{\varphi}^* - a(t,\beta/\rho,\gamma)\tilde{\varphi}^*, & x \in \Omega,\ t > 0, \\ \frac{\partial \tilde{\varphi}^*}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{3.11}$$

Multiply (3.11) by $\tilde{\varphi}^*$ and integrate by parts over $\Omega \times (0,t)$, then we get

$$-D_I \lim_{t\to\infty} \frac{1}{t} \int_0^t \int_\Omega |\nabla \tilde{\varphi}^*|^2 \mathrm{d}x\mathrm{d}\tau + \lim_{t\to\infty} \frac{1}{t} \int_0^t \int_\Omega \left(\frac{\beta(x,\tau)}{\rho} - \gamma(x,\tau) - a(\tau,\beta/\rho,\gamma) \right) (\tilde{\varphi}^*)^2 \mathrm{d}x\mathrm{d}\tau = 0. \tag{3.12}$$

Since $\beta(x,t)\equiv\beta(t)$, $\gamma(x,t)\equiv\gamma(t)$, it follows from Theorem 2.1 and its proof that $\tilde{\varphi}^*$ is independent on the spatial factor. Hence, we rewrite (3.12) as

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\left(\frac{\beta(\tau)}{\rho}-\gamma(\tau)-a(\tau,\beta/\rho,\gamma)\right)\mathrm{d}\tau=0.$$

Thus, Theorem 2.1 and Lemma 2.2 imply that

$$\omega(\Psi_{(\beta/\rho,-\gamma)})=\lim_{t\to\infty}\int_0^t a(\tau,\beta/\rho,\gamma)\mathrm{d}\tau=\lim_{t\to\infty}\frac{1}{t}\int_0^t\left(\frac{\beta(\tau)}{\rho}-\gamma(\tau)\right)\mathrm{d}\tau.$$

It then follows from Lemma 3.2 that $R_0=[\beta]/[\gamma]$.

(ii) Applying the completely similar arguments to the above, we conclude that

$$\omega(\Psi_{(\beta,-\gamma)})=\lim_{t\to\infty}\frac{1}{t}\int_0^t(\beta(x,\tau)-\gamma(x,\tau))\mathrm{d}\tau=\lim_{t\to\infty}\frac{1}{t}\int_0^t\mu(\tau)\mathrm{d}\tau=[\mu].$$

By Theorem 3.1, we deduce that R_0-1 and $\omega(\Psi_{(\beta,-\gamma)})$ have the same sign, that is, $R_0>1$ if $[\mu]>0$, $R_0=1$ if $[\mu]=0$ and $R_0<1$ if $[\mu]<0$. □

In the following, we investigate the property of R_0 as $\beta(x,t)-\gamma(x,t)$ or both $\beta(x,t)$ and $\gamma(x,t)$ are only dependent on the spatial factor.

Lemma 3.4. *Suppose that $\beta(x,t)-\gamma(x,t)\equiv\mu(x)$. Then we have the following statements:*

(i) *If $\int_\Omega\mu(x)dx\geqslant 0$ and $\mu(x)\not\equiv 0$ in Ω, then $R_0>1$ for all D_I.*
(ii) *If $\int_\Omega\mu(x)dx<0$ and $\mu(x)\leqslant 0$ on $\bar{\Omega}$, then $R_0<1$ for all D_I.*
(iii) *If $\int_\Omega\mu(x)dx<0$ and $\max_{x\in\bar{\Omega}}\mu(x)>0$, then there exists a threshold value $D_I^*>0$ such that $R_0>1$ for all $D_I<D_I^*$, $R_0<1$ for all $D_I>D_I^*$, and $R_0=1$ for all $D_I=D_I^*$.*

In particular, if $\beta(x,t)\equiv\beta(x)$, $\gamma(x,t)\equiv\gamma(x)$, then we have

$$R_0=\sup_{\phi\in H^1(\Omega),\phi\neq 0}\left\{\frac{\int_\Omega\beta\phi^2\,dx}{\int_\Omega(D_I|\nabla\phi|^2+\gamma\phi^2)\,dx}\right\},$$

and R_0 is a nonincreasing function of D_I with $\lim_{D_I\to 0}R_0=\max_{x\in\bar{\Omega}}\{\frac{\beta(x)}{\gamma(x)}\}$, and $\lim_{D_I\to\infty}R_0=\int_\Omega\beta\,dx/\int_\Omega\gamma\,dx$.

Proof. By Theorem 2.1 and Lemma 2.2, there exist almost periodic functions $\tilde{\varphi}_*:=\tilde{\varphi}_*(\cdot,t,\beta,\gamma)\in\mathrm{Int}X_+$ and $a(t,\beta,\gamma)$ associated with (3.4) such that

$$\begin{aligned}\omega(\Psi_{(\beta,-\gamma)})&=\lim_{t\to\infty}\frac{1}{t}\int_0^t a(\tau,\beta,\gamma))\mathrm{d}\tau\\&=-D_I\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega|\nabla\tilde{\varphi}_*|^2\mathrm{d}x\mathrm{d}\tau+\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega\mu(x)|\tilde{\varphi}_*|^2\mathrm{d}x\mathrm{d}\tau,\end{aligned}\tag{3.13}$$

and

$$\begin{cases}\frac{\partial\tilde{\varphi}_*}{\partial t}-D_I\Delta\tilde{\varphi}_*=\mu(x)\tilde{\varphi}_*-a(t,\beta,\gamma)\tilde{\varphi}_*, & x\in\Omega,\ t>0,\\ \frac{\partial\tilde{\varphi}_*}{\partial\nu}=0, & x\in\partial\Omega,\ t>0.\end{cases}\tag{3.14}$$

Since $\mu(x)$ is independent on t, Theorem 2.1 and its proof imply that $\tilde{\varphi}_*$ and $a(t,\beta,\gamma)$ are independent on the spatial factor. For simplicity, we denote $a(t,\beta,\gamma) \equiv a(\beta,\gamma)$. Then we rewrite (3.13) and (3.14) as

$$\omega(\Psi_{(\beta,-\gamma)}) = a(\beta,\gamma) = -D_I \int_\Omega |\nabla \tilde{\varphi}_*|^2 \mathrm{d}x + \int_\Omega \mu(x)|\tilde{\varphi}_*|^2 \mathrm{d}x, \tag{3.15}$$

and

$$\begin{cases} -D_I \Delta \tilde{\varphi}_* = \mu(x)\tilde{\varphi}_* - a(\beta,\gamma)\tilde{\varphi}_*, & x \in \Omega, \\ \frac{\partial \tilde{\varphi}_*}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{3.16}$$

respectively. Furthermore, we have the following two claims.

Claim 1: $\omega(\Psi_{(\beta,-\gamma)}) \to \max_{x\in\bar{\Omega}} \mu(x)$ as $D_I \to 0$.

It is easy to see from (3.15) that it suffices to prove $a(\beta,\gamma) \to \max_{x\in\bar{\Omega}} \mu(x)$ as $D_I \to 0$. Applying (3.15) again, we have

$$\omega(\Psi_{(\beta,-\gamma)}) \leqslant \int_\Omega \mu(x)|\tilde{\varphi}_*|^2 \mathrm{d}x\mathrm{d}\tau \leqslant \max_{x\in\bar{\Omega}} \mu(x) := \mu^*.$$

Hence, $\limsup_{D_I\to 0} a(\beta,\gamma) \leqslant \mu^*$. Thus, it is sufficient to show that $\liminf_{D_I\to 0} a(\beta,\gamma) \geqslant \mu^*$. Suppose not, then there exists $\epsilon^* > 0$ such that $\liminf_{D_I\to 0} a(\beta,\gamma) \leqslant \mu^* - \epsilon^*$. Passing to a sequence if necessary, we assume that there exists $D^* > 0$ such that if $D_I < D^*$, then $a(\beta,\gamma) \leqslant \mu^* - \epsilon^*/2$. By the continuity of $\mu(x)$, there exist $x^* \in \Omega$ and $\delta > 0$ such that $\mu^* \leqslant \mu(x) + \epsilon^*/4$ for every $x \in B_\delta(x^*) \subset \Omega$, where $B_\delta(x^*) = \{x : dist(x,x^*) \leqslant \delta\}$. Hence, $a(\beta,\gamma) \leqslant \mu(x) - \epsilon^*/4$ for $0 < D_I < D^*$ and $x \in B_\delta(x^*)$. It then follows from (3.16) that

$$-D_I \Delta \tilde{\varphi}_* = \mu(x)\tilde{\varphi}_* - a(\beta,\gamma)\tilde{\varphi}_* \geqslant \frac{\epsilon^*}{4}\tilde{\varphi}_*, \quad x \in B_\delta(x^*). \tag{3.17}$$

Since $\tilde{\varphi}_* \in \mathrm{Int}X_+$, let $D_I \to 0$ in (3.17), then we get a contradiction. Claim 1 is proved.

Claim 2: $\omega(\Psi_{(\beta,-\gamma)}) \to \frac{1}{|\Omega|}\int_\Omega \mu(x)\mathrm{d}x$ as $D_I \to \infty$.

Integrate (3.16) over Ω by parts, then we obtain

$$\int_\Omega (\mu(x) - a(\beta,\gamma))\tilde{\varphi}_* \mathrm{d}x = 0,$$

which is equivalent to

$$a(\beta,\gamma)\int_\Omega \tilde{\varphi}_* \mathrm{d}x = \int_\Omega \mu(x)\tilde{\varphi}_* \mathrm{d}x. \tag{3.18}$$

Hence, it is easy to see that $\omega(\Psi_{(\beta,-\gamma)}) = a(\beta,\gamma) \geqslant \min_{x\in\bar{\Omega}} \mu(x)$. Since Theorem 2.1 and Lemma 2.2 imply that $\omega(\Psi_{(\beta,-\gamma)})$ is nonincreasing of D_I, we conclude that it is uniformly bounded. Thus, it has a finite limit $\hat{\omega}$. It then follows from [10] (see also [2]) that the elliptic equation (3.16) admits the property that there exists some positive constant $\hat{\varphi}$ such that $\tilde{\varphi}_* \to \hat{\varphi}$ as $D_I \to \infty$. By Theorem 2.1 and Lemma 2.2 again, let $D_I \to \infty$ in (3.18), we conclude that $\hat{\omega} \to \frac{1}{|\Omega|}\int_\Omega \mu(x)\mathrm{d}x$, and $\omega(\Psi_{(\beta,-\gamma)})$ is strictly decreasing with respect to $D_I > 0$ if $\mu(x)$ is not a constant in Ω. Hence, the conclusions (i)–(iii) can be obtained from two claims above and Lemma 3.1.

In the case of $\beta(x,t) \equiv \beta(x)$ and $\gamma(x,t) \equiv \gamma(x)$, the conclusions can be obtained from [25]. □

For the general case of β and γ, motivated by [25, Theorem 2.5], we have the following assertions.

Theorem 3.5. *Suppose that (H3) holds, then the following statements are valid:*

(i) *For all $D_I > 0$, $R_0 \geqslant \lim_{t\to\infty} \frac{1}{t}\int_0^t \int_\Omega \beta(x,\tau)\mathrm{d}x\mathrm{d}\tau / \lim_{t\to\infty} \frac{1}{t}\int_0^t \int_\Omega \gamma(x,\tau)\mathrm{d}x\mathrm{d}\tau$, and the strict inequality holds if and only if $\beta(x,t) - \gamma(x,t)$ nontrivially depends on the spatial variable x.*

(ii) *If* $\lim_{t\to\infty}\frac{1}{t}\int_0^t \max_{x\in\bar\Omega}(\beta(x,\tau)-\gamma(x,\tau))\mathrm{d}\tau \leqslant 0$ *and* $\beta(x,t)-\gamma(x,t)$ *nontrivially depends on the spatial variable* x*, then* $R_0<1$ *for all* $D_I>0$.

(iii) $R_0 \to \lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega \beta(x,\tau)\mathrm{d}x\mathrm{d}\tau/\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega \gamma(x,\tau)\mathrm{d}x\mathrm{d}\tau$ *as* $D_I\to\infty$.

(iv) $R_0 \leqslant \max_{x\in\bar\Omega}\left\{\frac{\lim_{t\to\infty}\frac{1}{t}\int_0^t\beta(x,\tau)\mathrm{d}\tau}{\lim_{t\to\infty}\frac{1}{t}\int_0^t\gamma(x,\tau)\mathrm{d}\tau}\right\}$ *as* $D_I\to 0$.

(v) *In general,* $R_0(D_I):=R_0$ *is not a nonincreasing function of* D_I*; in particular, if* $\beta(x,t)=a(x)b(t)$ *and* $\gamma(x,t)=a(x)c(t)$ *with* $a>0$ *on* $\bar\Omega$*, a is not identically equal to a constant,* $b,c\in AP_{(\beta,\gamma)}$*, b,* $c>0$ *on* $[0,\infty)$ *and* $b-c$ *is not identically equal to a constant, then there exist* D_I^1 *and* D_I^2 *with* $0<D_I^1<D_I^2$ *such that* $R_0(D_I^1)=R_0(D_I^2)$.

Proof. (i) Since $\tilde\varphi^*\in \mathrm{Int}X_+$, we divide Eq. (3.11) by $\tilde\varphi^*$ and integrate the resulting equation over $\Omega\times(0,t)$ by parts to get

$$-D_I\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega\frac{|\nabla\tilde\varphi^*|^2}{(\tilde\varphi^*)^2}\mathrm{d}x\mathrm{d}\tau=\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega\left(\frac{\beta}{\rho}-\gamma\right)\mathrm{d}x\mathrm{d}\tau-\lim_{t\to\infty}\frac{1}{t}\int_0^t a(\tau,\beta/\rho,\gamma))\mathrm{d}\tau.$$

By Lemmas 2.2 and 3.2, we get

$$\omega(\Psi_{(\beta/R_0,-\gamma)})=D_I\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega\frac{|\nabla\tilde\varphi^*|^2}{(\tilde\varphi^*)^2}\mathrm{d}x\mathrm{d}\tau+\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega\left(\frac{\beta}{R_0}-\gamma\right)\mathrm{d}x\mathrm{d}\tau=0.$$

This implies that $R_0\geqslant \lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega\beta(x,\tau)\mathrm{d}x\mathrm{d}\tau/\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega\gamma(x,\tau)\mathrm{d}x\mathrm{d}\tau$. Furthermore, since the condition that $\beta(x,t)-\gamma(x,t)$ nontrivially depends on the spatial variable x is equivalent to

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega\frac{|\nabla\tilde\varphi^*|^2}{(\tilde\varphi^*)^2}\mathrm{d}x\mathrm{d}\tau>0,$$

we obtain the conclusion.

(ii) For Eq. (3.4), it follows from (3.13) that

$$\begin{aligned}\omega(\Psi_{(\beta,-\gamma)})&=-D_I\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega|\nabla\tilde\varphi_*|^2\mathrm{d}x\mathrm{d}\tau+\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega(\beta-\gamma)|\tilde\varphi_*|^2\mathrm{d}x\mathrm{d}\tau\\&\leqslant\lim_{t\to\infty}\frac{1}{t}\int_0^t\max_{x\in\bar\Omega}(\beta-\gamma)\mathrm{d}\tau.\end{aligned}$$

If $\beta(x,t)-\gamma(x,t)$ nontrivially depends on the spatial variable x, then the strict inequality holds. Hence, the conclusion can be obtained by Lemma 3.1.

(iii) To obtain this assertion, we use similar arguments to those in the proof of [25, Theorem 2.5 (c)] (see also [15, Lemma 2.4]). Since some necessary modifications are required, here we provide a detailed proof.

Let φ be the solution of (3.5) with $\rho=R_0$ and assume that $\gamma>0$ on $\bar\Omega\times\mathbb{R}$. Integrating Eq.(3.5) with $\rho=R_0$ that φ satisfies over $\Omega\times(0,t)$, then we conclude that

$$R_0=\frac{\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega\beta\varphi\mathrm{d}x\mathrm{d}\tau}{\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega\gamma\varphi\mathrm{d}x\mathrm{d}\tau}\leqslant\frac{\max_{\bar\Omega\times(0,\infty)}\beta}{\min_{\bar\Omega\times(0,\infty)}\gamma}. \tag{3.19}$$

Consequently, the above and (i) imply that R_0 is bounded independent of D_I. Furthermore, we assume that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega \varphi^2\mathrm{d}x\mathrm{d}\tau = 1. \tag{3.20}$$

Multiply (3.5) that φ satisfies by φ and then integrate to get

$$D_I\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega |\nabla\varphi|^2\mathrm{d}x\mathrm{d}\tau = \frac{1}{R_0}\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega \beta\varphi^2\mathrm{d}x\mathrm{d}\tau - \lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega \gamma\varphi^2\mathrm{d}x\mathrm{d}\tau.$$

Thus, we can find a positive constant C such that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega |\nabla\varphi|^2\mathrm{d}x\mathrm{d}\tau \leqslant \frac{C}{D_I}. \tag{3.21}$$

It is obvious to see that the constant C does not depend on D_I and may change from place to place (see also [25]).

Let

$$\bar{\varphi}(t) = \frac{1}{|\Omega|}\int_\Omega \varphi(x,t)\mathrm{d}x \qquad \text{and} \qquad \Gamma(x,t) = \varphi(x,t) - \bar{\varphi}(t).$$

It is easy to see that $\int_\Omega \Gamma(x,t)\mathrm{d}x = 0$. It then follows from Poincaré inequality that

$$\int_\Omega \Gamma^2\mathrm{d}x \leqslant C\int_\Omega |\nabla\Gamma|^2\mathrm{d}x \quad \text{for all } t.$$

Since $\nabla\Gamma = \nabla\varphi$, the inequality (3.21) implies that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega \Gamma^2\mathrm{d}x\mathrm{d}\tau \leqslant \frac{C}{D_I}, \qquad \text{and hence,} \quad \lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega |\Gamma|\mathrm{d}x\mathrm{d}\tau \leqslant \frac{C}{\sqrt{D_I}}. \tag{3.22}$$

In the following, integrate (3.5) that φ satisfies over Ω, we have

$$\frac{\mathrm{d}\bar{\varphi}}{\mathrm{d}t} = \int_\Omega \left(\frac{\beta}{R_0} - \gamma\right)\mathrm{d}x\bar{\varphi} + \int_\Omega \left(\frac{\beta}{R_0} - \gamma\right)\Gamma\mathrm{d}x. \tag{3.23}$$

The assertion (i) implies that the zero solution of the homogeneous equation associated with (3.23) is uniformly asymptotically stable (see also [32, Theorm 2.6]), and hence [9, Theorem 7.7 and Sect. XI.4] shows that system (3.23) admits a unique positive almost periodic solution (see also the proof of [32, Lemma 3.1]). Without loss of generality, we also denote the almost periodic solution by $\bar{\varphi}(t)$. Using the result in (i), we follow from (3.19) and (3.22) that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\left|\int_\Omega \left(\frac{\beta}{R_0} - \gamma\right)\Gamma\mathrm{d}x\right|\mathrm{d}\tau = O\left(\frac{1}{\sqrt{D_I}}\right). \tag{3.24}$$

□

By integrating (3.23) that $\bar{\varphi}(t)$ satisfies over $(0,t)$, then we obtain that

$$0 = \lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega \left(\frac{\beta}{R_0} - \gamma\right)\bar{\varphi}\mathrm{d}x\mathrm{d}\tau + \lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega \left(\frac{\beta}{R_0} - \gamma\right)\Gamma\mathrm{d}x\mathrm{d}\tau. \tag{3.25}$$

It then follows from (3.24) and (3.25) that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega\left(\frac{\beta}{R_0}-\gamma\right)\bar{\varphi}\mathrm{d}x\mathrm{d}\tau\to 0\quad\text{as } D_I\to\infty. \tag{3.26}$$

Furthermore, we claim that $\bar{\varphi}\to 0$ is impossible as $D_I\to\infty$. On the contrary, if $\bar{\varphi}\to 0$ as $D_I\to\infty$, then together with (3.22), we conclude that $\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega\varphi^2\mathrm{d}x\mathrm{d}\tau\to 0$. This contradicts (3.20). Hence, the boundedness of $\bar{\varphi}$ as an almost periodic function and (3.26) imply that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega\left(\frac{\beta}{R_0}-\gamma\right)\mathrm{d}x\mathrm{d}\tau\to 0\quad\text{as } D_I\to\infty.$$

This leads to the assertion (iii).

In the general case of $\gamma\geqslant,\not\equiv 0$ on $\bar{\Omega}\times\mathbb{R}$, replace γ above by $\gamma+\varepsilon$ for any give $\varepsilon>0$, then we have

$$R_0\to\frac{\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega\beta\mathrm{d}x\mathrm{d}\tau}{\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega(\gamma+\varepsilon)\mathrm{d}x\mathrm{d}\tau}. \tag{3.27}$$

Letting $\varepsilon\to 0$ in (3.27), we get the desired result.

(iv) Without loss of generality, we assume that $\beta>0$ and $\gamma>0$ on $\bar{\Omega}\times\mathbb{R}$. For the general case, as above, we replace $\beta>0$ and $\gamma>0$ by $\beta+\varepsilon$ and $\gamma+\varepsilon$, respectively, and then, the result can be obtained by letting $\varepsilon\to 0$. Let

$$\chi=\frac{\lim_{t\to\infty}\frac{1}{t}\int_0^t\beta(x_0,\tau)\mathrm{d}\tau}{\lim_{t\to\infty}\frac{1}{t}\int_0^t\gamma(x_0,\tau)\mathrm{d}\tau}=\max_{x\in\bar{\Omega}}\left\{\frac{\lim_{t\to\infty}\frac{1}{t}\int_0^t\beta(x,\tau)\mathrm{d}\tau}{\lim_{t\to\infty}\frac{1}{t}\int_0^t\gamma(x,\tau)\mathrm{d}\tau}\right\}\quad\text{for some } x_0\in\bar{\Omega}.$$

In the following, we prove $R_0\leqslant\chi$. On the contrary, we suppose that $R_0>\chi$. Consider Eq.(3.5) with $\rho=R_0$ and rewrite (3.11) as

$$\begin{cases}\frac{\partial\tilde{\varphi}^*}{\partial t}-D_I\Delta\tilde{\varphi}^*=\frac{\beta(x,t)}{R_0}\tilde{\varphi}^*-\gamma(x,t)\tilde{\varphi}^*-a(t,\beta/R_0,\gamma)\tilde{\varphi}^*, & x\in\Omega,\ t>0,\\ \frac{\partial\tilde{\varphi}^*}{\partial\nu}=0, & x\in\partial\Omega,\ t>0.\end{cases} \tag{3.28}$$

Multiply (3.28) by $1/\tilde{\varphi}^*$ and then integrate over $(0,t)$, it then follows from Theorem 2.1, Lemmas 2.2 and 3.2 that

$$\begin{aligned}0=\omega(\Psi_{(\beta/R_0,-\gamma)})&=\lim_{t\to\infty}\frac{1}{t}\int_0^t a(\tau,\beta/R_0,\gamma))\mathrm{d}\tau\\&=D_I\lim_{t\to\infty}\frac{1}{t}\int_0^t\frac{\Delta\tilde{\varphi}^*}{\tilde{\varphi}^*}\mathrm{d}\tau+\lim_{t\to\infty}\frac{1}{t}\int_0^t\left(\frac{\beta}{R_0}-\gamma\right)\mathrm{d}\tau\\&\leqslant D_I\lim_{t\to\infty}\frac{1}{t}\int_0^t\frac{\Delta\tilde{\varphi}^*}{\tilde{\varphi}^*}\mathrm{d}\tau+\left(\frac{\chi}{R_0}-1\right)\lim_{t\to\infty}\frac{1}{t}\int_0^t\gamma\mathrm{d}\tau\\&<0\end{aligned}$$

as $D_I\to 0$. It is a contradiction. Hence, we get $R_0\leqslant\chi$.

(v) It is known that the exponential growth bound $\omega(\Psi_{(\beta/\rho,-\gamma)})$ associated with (3.5) is nonincreasing in $\rho\in(0,\infty)$. Furthermore, Theorem 2.1 and Lemma 2.2 imply that $\omega(\Psi_{(\beta/\rho,-\gamma)})$ is not a nonincreasing function of D_I. Hence, by Lemma 3.2, we conclude that $R_0(D_I):=R_0$ is not a nonincreasing function of D_I.

By the forms of β and γ, it is easy from (i), (iii) and (iv) to see that

$$R_0 > \frac{\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega \beta(x,\tau)\mathrm{d}x\mathrm{d}\tau}{\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega \gamma(x,\tau)\mathrm{d}x\mathrm{d}\tau} \quad \text{for all } D_I,$$

and

$$\lim_{D_I\to 0} R_0(D_I) = \lim_{D_I\to\infty} R_0(D_I) = \frac{\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega \beta(x,\tau)\mathrm{d}x\mathrm{d}\tau}{\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega \gamma(x,\tau)\mathrm{d}x\mathrm{d}\tau}.$$

Obviously, there exist D_I^1 and D_I^2 with $0 < D_I^1 < D_I^2$ such that $R_0(D_I^1) = R_0(D_I^2)$.

4. Threshold dynamics

In this section, we consider the threshold dynamical behavior of (1.2) in terms of R_0. At the beginning, we introduce the result of the uniform boundedness of the solution of (1.2).

Lemma 4.1. [25, Lemma 3.2] *There exists a positive constant C_0 independent of the initial data (S_0, I_0) satisfying (H2) such that for the corresponding unique solution (S, I) of (1.2), we have*

$$\|S(x,t)\|_{L^\infty(\Omega)} + \|I(x,t)\|_{L^\infty(\Omega)} \leqslant C_0, \quad \forall t \in [0,\infty).$$

Theorem 4.2. *The following two statements are valid:*

(i) *If $R_0 < 1$, then $(S, I) \to (N/|\Omega|, 0)$ uniformly on $\bar\Omega$ as $t \to \infty$.*
(ii) *If $R_0 > 1$, then for any solution of (1.2) with the initial value (S_0, I_0) satisfying (H2), there exists $\varrho > 0$ such that*

$$\liminf_{t\to\infty} S(x,t) \geqslant \varrho \quad \text{and} \quad \liminf_{t\to\infty} I(x,t) \geqslant \varrho$$

uniformly on $\bar\Omega$.

Proof. (i) Let (S, I) be the solution of (1.2), then we have

$$\frac{\partial I}{\partial t} - D_I\Delta I \leqslant \beta(x,t)I - \gamma(x,t)I, \quad x\in\Omega,\ t>0. \tag{4.1}$$

Consider the comparison equation of (4.1):

$$\frac{\partial \bar I}{\partial t} - D_I\Delta \bar I = \beta(x,t)\bar I - \gamma(x,t)\bar I, \quad x\in\Omega,\ t>0. \tag{4.2}$$

which admits the same form with (3.4). Suppose that $R_0 < 1$, then Lemma 3.1 implies that $\omega(\Psi_{(\beta,-\gamma)})<0$. According to the arguments in the proof of Lemma 3.4, there exist almost periodic functions $\tilde\varphi_*(\cdot,t,\beta,\gamma) \in \mathrm{Int}X_+$ and $a(t,\beta,\gamma)$ such that $\bar I(x,t,\beta,\gamma) = e^{\int_0^t a(\tau,\beta,\gamma)\mathrm{d}\tau}\tilde\varphi_*(x,t,\beta,\gamma)$ is the solution of (4.2), and

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t a(\tau,\beta,\gamma)\mathrm{d}\tau = \omega(\Psi_{(\beta,-\gamma)}) < 0. \tag{4.3}$$

It then follows that $\bar I(x,t,\beta,\gamma) \to 0$ uniformly on $\bar\Omega$ as $t\to\infty$. The comparison principle shows that $I(x,t)\to 0$ as $t\to\infty$.

In the following, we prove $S(x,t) \to N/|\Omega|$ as $t\to\infty$. Some ideas come from the proof of [2, Lemma 2.5]. Observe from (1.2) that

$$\frac{\partial S}{\partial t} - D_S\Delta S = \left[\gamma(x,t) - \frac{\beta(x,t)S}{S+I}\right]I, \quad x\in\Omega,\ t>0.$$

The preceding arguments about $\bar{I}$ and I imply that

$$\left|\frac{\partial S}{\partial t} - D_S \Delta S\right| \leqslant C_1 e^{\int_0^t a(\tau,\beta,\gamma)\mathrm{d}\tau}, \quad x \in \Omega,\ t > 0, \tag{4.4}$$

for some positive constant C_1. It then follows from (4.3) that the right side of (4.4) tends to 0. Hence, $S(x,t)$ tends to a positive constant as $t \to \infty$. Let $S(x,t) = S_1(t)+S_2(x,t)$, where $S_1(t) = \frac{1}{|\Omega|}\int_\Omega S(x,t)\mathrm{d}x$. Thus, we have $|\partial S_1/\partial t| \leqslant C_2 e^{\int_0^t a(\tau,\beta,\gamma)\mathrm{d}\tau}$ for some positive constant C_2. Hence, S_2 satisfies

$$\begin{cases} \frac{\partial S_2}{\partial t} = D_S \Delta S_2 + h(x,t), & x \in \Omega,\ t > 0, \\ \frac{\partial S_2}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \end{cases}$$

where $h(x,t)$ satisfies $|h(x,t)| \leqslant C_3 e^{\int_0^t a(\tau,\beta,\gamma)\mathrm{d}\tau}$ for a constant $C_3 > 0$, and $\int_\Omega S_2(x,t)\mathrm{d}x \equiv 0$. It follows from (1.3) and $I \to 0$ as $t \to \infty$ that $S_1(t) \to N/|\Omega|$ as $t \to \infty$. Denote $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ be the eigenvalue of $-\Delta$ with zero Neumann boundary condition, and $\{\psi_k\}_{k=0}^\infty$ denote corresponding normalized eigenfunctions. Let $S_2(x,t) = \sum_{k=0}^\infty a_k(t)\psi_k(x)$ and $h(x,t) = \sum_{k=0}^\infty h_k(t)\psi_k(x)$. Since $\int_\Omega S_2(x,t)\mathrm{d}x \equiv 0$, we conclude that $a_0 = h_0 = 0$. Note that $|h_k(t)| \leqslant C_4 e^{\int_0^t a(\tau,\beta,\gamma)\mathrm{d}\tau}$ for every $k \geqslant 1$, we have $|a_k(t)| \leqslant C_5 e^{-\lambda_* t}$ for every $k \geqslant 1$, where C_4 and C_5 are positive constants and $\lambda_* = \min\{-\int_0^t a(\tau,\beta,\gamma)\mathrm{d}\tau, \lambda_1\} > 0$. Consequently, we deduce that $S_2(x,t) \to 0$ as $t \to \infty$ for every $x \in \Omega$. Thus, we follows that $S(x,t) \to N/|\Omega|$ as $t \to \infty$.

(ii) In the case of where $R_0 > 1$, we use the skew-product semiflows approach to prove the desired uniform persistence (see, e.g., [36]). Define the hull of $\beta(x,t)$ and $\gamma(x,t)$ as

$$H = \mathrm{cls}\{(\beta_s,\gamma_s) : s \in \mathbb{R}, \beta_s(\cdot,t) = \beta(\cdot,s+t), \gamma_s(\cdot,t) = \gamma(\cdot,s+t)\},$$

where the closure is taken in the compact open topology. It then follows that the translation $\sigma : \mathbb{R} \times H \to H$, $\sigma(t)(\theta,\vartheta) = (\theta_t,\vartheta_t)$, $\forall(\theta,\vartheta) \in H$, defines a continuous, compact, almost periodic minimal and distal flow (see [28, Lemma VI.C], which is denoted by $(H,\sigma,\mathbb{R})$. Let

$$X_0 = \left\{(w,z) \in L^p(\Omega) \times L^p(\Omega) : \int_\Omega (w+z)\mathrm{d}x = N\right\} \text{ and } U = (X_+ \times X_+) \cap X_0,$$

and

$$U_0 := \{(S_0,I_0) \in U : I_0 \not\equiv 0\},\ \partial U_0 := \{(S_0,I_0) \in U : I_0 \equiv 0\},$$
$$P := U \times H,\ P_0 := U_0 \times H,\ \partial P_0 := P \setminus P_0,\ \mathrm{Int}P := ((\mathrm{Int}X_+ \times \mathrm{Int}X_+) \cap X_0) \times H.$$

Then P_0 and ∂P_0 are relatively open and closed in P, respectively. By the standard regularity theory for parabolic equation and (1.3), we conclude that for every $(S_0,I_0,\theta,\vartheta) \in P$, the reaction–diffusion equation

$$\begin{cases} \frac{\partial S}{\partial t} - D_S\Delta S = -\frac{\theta(x,t)SI}{S+I} + \vartheta(x,t)I, & x \in \Omega,\ t > 0, \\ \frac{\partial I}{\partial t} - D_I\Delta I = \frac{\theta(x,t)SI}{S+I} - \vartheta(x,t)I, & x \in \Omega,\ t > 0, \\ \frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0 \end{cases} \tag{4.5}$$

admits a unique solution $y(\cdot,t,S_0,I_0,\theta,\vartheta) := (S(\cdot,t,S_0,I_0,\theta,\vartheta)), I(\cdot,t,S_0,I_0,\theta,\vartheta))$ which exists for any $t \geq 0$, and satisfies $(S(x,0,S_0,I_0,\theta,\vartheta), I(x,0,S_0,I_0,\theta,\vartheta)) = (S_0(x),I_0(x))$, $\forall(\theta,\vartheta) \in H$. Then the solution family of (4.5) generates a skew-product semiflow:

$$\pi_t : P \to P, \quad t \geqslant 0$$
$$(S_0,I_0,\theta,\vartheta) \mapsto (y(\cdot,t,S_0,I_0,\theta,\vartheta), \sigma(t)(\theta,\vartheta)).$$

It is easy to see that $\pi_t(P_0) \subset P_0$ and $\pi_t(\partial P_0) \subset \partial P_0$ for all $t \geqslant 0$. In view of Lemma 4.1, we conclude that π_t is continuous and compact for any $t > 0$. Furthermore, Lemma 4.1 and the similar arguments to those in [7, Theorem 23.3] imply that $\pi_t : P \to P$, $\forall t \geqslant 0$ is point dissipative. It then follows from [11, Theorem 2.4.7] that $\pi_t : P \to P$ has a global attractor. Let $\omega(S_0, I_0, \theta, \vartheta)$ be the omega limit set for π_t. It is easy to see that for $(S_0, I_0) \in \partial U_0$, $I(x, t, S_0, I_0, \theta, \vartheta) \equiv 0$ and $S(x, t, S_0, I_0, \theta, \vartheta)$ satisfies

$$\begin{cases} \frac{\partial S}{\partial t} - D_S \Delta S = 0, & x \in \Omega,\ t > 0, \\ \frac{\partial S}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ \int_\Omega S \mathrm{d}x = N, & t \geqslant 0, \\ S(x, 0, S_0, \theta, \vartheta) = S_0(x) \geqslant, \not\equiv 0, & x \in \Omega. \end{cases} \tag{4.6}$$

A similar argument to those in (i) implies that $S \to N/|\Omega|$ uniformly on $\bar{\Omega}$ as $t \to \infty$. It then follows that $\cup_{(S_0, I_0, \theta, \vartheta) \in \partial P_0} \omega(S_0, I_0, \theta, \vartheta) = \{(N/|\Omega|, 0, \theta, \vartheta) : (\theta, \vartheta) \in H\}$. For simplicity, we denote $\mathcal{M} = \{(N/|\Omega|, 0, \theta, \vartheta) : (\theta, \vartheta) \in H\}$. Let M_∂ be the maximal positively invariant set for π_t in ∂P_0. Then we conclude that $\tilde{M}_\partial = \cup_{(S_0, I_0, \theta, \vartheta) \in M_\partial} \omega(S_0, I_0, \theta, \vartheta) = \mathcal{M}$, $\mathcal{M}$ is a compact and isolated invariant set, and no subset of $\mathcal{M}$ forms a cycle for π_t in ∂P_0.

Since $R_0 > 1$, Lemma 3.1 implies that $\omega(\Psi_{(\beta, -\gamma)}) > 0$. Hence, we can choose sufficiently small $\delta > 0$ such that the exponential growth bound, $\omega(\Psi_{(\delta, \beta, -\gamma)})$, associated with the equation

$$\begin{cases} \frac{\partial \hat{I}}{\partial t} - D_I \Delta \hat{I} = \frac{\beta(x,t)(N/|\Omega| - \delta)}{N/|\Omega| + 2\delta} \hat{I} - \gamma(x, t)\hat{I}, & x \in \Omega,\ t > 0, \\ \frac{\partial \hat{I}}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0 \end{cases} \tag{4.7}$$

satisfies $\omega(\Psi_{(\delta, \beta, -\gamma)}) > 0$ (see e.g., [32, Lemma 3.3]), where $\Psi_{(\delta, \beta, -\gamma)}$ is the evolution operator of (4.7). Furthermore, we claim that

$$\limsup_{t \to \infty} d(\pi_t(S_0, I_0, \theta, \vartheta), \mathcal{M}) \geqslant \delta, \quad \forall (S_0, I_0, \theta, \vartheta) \in P_0.$$

On the contrary, we suppose that for some $(S_0, I_0, \theta, \vartheta) \in P_0$, there holds

$$\limsup_{t \to \infty} d(\pi_t(S_0, I_0, \theta, \vartheta), \mathcal{M}) < \delta.$$

Hence, there exists $t_1 > 0$ such that $d(\pi_t(S_0, I_0, \theta, \vartheta), \mathcal{M}) < \delta$, $\forall t \geqslant t_1$. Thus, we conclude that $\|I(x, t, S_0, I_0, \theta, \vartheta))\| < \delta$, $\forall t \geqslant t_1$. Applying the similar arguments to the proof of (i) above, we conclude that there exists t_2 such that $S \geqslant N/|\Omega| - \delta$ for all $t > t_2$ and sufficiently small δ. It then follows from (4.5) that

$$\begin{cases} \frac{\partial I}{\partial t} - D_I \Delta I \geqslant \frac{\theta(x,t)(N/|\Omega| - \delta)}{N/|\Omega| + 2\delta} I - \vartheta(x, t) I, & x \in \Omega,\ t > t_2, \\ \frac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega,\ t > t_2. \end{cases} \tag{4.8}$$

By Theorem 2.1 and Lemma 2.2, there exist an almost periodic function $\hat{a}(t, \delta, \theta, \vartheta)$ and a uniformly almost periodic function $\tilde{I}(t, \delta, \theta, \vartheta) \in \mathrm{Int}X_+$ such that $\hat{I}(t, \delta, \theta, \vartheta) = e^{\int_0^t \hat{a}(\tau, \delta, \theta, \vartheta)\mathrm{d}\tau} \tilde{I}(t, \delta, \theta, \vartheta)$ is a solution of (4.7). Furthermore,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \hat{a}(\tau, \delta, \theta, \vartheta) \mathrm{d}\tau = \omega(\Psi_{(\delta, \beta, -\gamma)}) > 0.$$

On the other hand, as $(S_0, I_0, \theta, \vartheta) \in P_0$, the strong maximum principle for parabolic equations shows $(S(t, S_0, I_0, \theta, \vartheta)), I(t, S_0, I_0, \theta, \vartheta)) \in (\mathrm{Int}X_+) \times (\mathrm{Int}X_+)$. Hence, there exist t_3 and $\varepsilon > 0$ to be small enough such that $I(t_3, S_0, I_0, \theta, \vartheta)) \geqslant \varepsilon \tilde{I}(t_3, \delta, \theta, \vartheta)$. By the comparison principle, as applied to (4.8), we obtain that

$$I(t, S_0, I_0, \theta, \vartheta) \geqslant \varepsilon \hat{I}(t, \delta, \theta, \vartheta) = \varepsilon e^{\int_0^t \hat{a}(\tau, \delta, \theta, \vartheta)\mathrm{d}\tau} \tilde{I}(t, \delta, \theta, \vartheta), \quad t \geqslant t_4,$$

where $t_4 = \max\{t_1, t_2, t_3\}$. Since $\tilde{I}(t, \delta, \theta, \vartheta)$ is almost periodic in t, and

$$\lim_{t \to \infty} e^{\int_0^t \hat{a}(\tau, \delta, \theta, \vartheta)\mathrm{d}\tau} = \lim_{t \to \infty} \left(e^{\frac{1}{t}\int_0^t \hat{a}(\tau, \delta, \theta, \vartheta)\mathrm{d}\tau} \right)^t = \infty,$$

we get $\lim_{t \to \infty} I(t, S_0, I_0, \theta, \vartheta) = \infty$, a contradiction to Lemma 4.1.

Since $\mathcal{M}$ is an isolated invariant set for π_t in ∂P_0, the claim above implies that $\mathcal{M}$ is also an isolated invariant for π_t in P. The claim above also implies that $W^s(\mathcal{M}) \cap P_0 = \emptyset$, where the set

$$W^s(\mathcal{M}) := \{(S_0, I_0, \theta, \vartheta) \in P : \omega(S_0, I_0, \theta, \vartheta) \neq \emptyset, \omega(S_0, I_0, \theta, \vartheta) \subset \mathcal{M}\}$$

is the stable set of $\mathcal{M}$ for π_t. By the continuous-time version of [38, Theorem 1.3.1 and Remark 1.3.1], the skew-product semiflow $\pi_t : P \to P$ is uniformly persistent with respect to $(P_0, \partial P_0)$, and hence, $\pi_t : P_0 \to P_0$ admits a global attractor A_0.

It remains to prove the practical uniform persistence. Since $A_0 \subset P_0$ and $\pi_t(A_0) = A_0$, $\forall t \geqslant 0$, we follows that for any $(S, I, \theta, \vartheta) \in A_0$, there exists $(\hat{S}, \hat{I}, \hat{\theta}, \hat{\vartheta}) \in A_0$ such that

$$(S, I, \theta, \vartheta) = \pi_1(\hat{S}, \hat{I}, \hat{\theta}, \hat{\vartheta}) \in \mathrm{Int}P.$$

Then we conclude that $A_0 \subset \mathrm{Int}P$. Define a continuous function $\mathcal{F} : P \to \mathbb{R}_+$ by

$$\mathcal{F}(S_0, I_0, \theta, \vartheta) = \inf\{\upsilon \in \mathbb{R}_+ : S_0(x) \geqslant \upsilon, I_0(x) \geqslant \upsilon, x \in \bar{\Omega}\}, \ \forall (S_0, I_0, \theta, \vartheta) \in P.$$

Obviously, $\mathcal{F}(S_0, I_0, \theta, \vartheta) > 0$ if and only if $(S_0, I_0, \theta, \vartheta) \in \mathrm{Int}P$. It then follows that $\mathcal{F} : P \to \mathbb{R}_+$ is lower semicontinuous in the sense that for any $(\check{S}_0, \check{I}_0, \check{\theta}, \check{\vartheta}) \in P$ and $\varepsilon^* > 0$, there exists $\varepsilon_* > 0$ such that $\mathcal{F}(S_0, I_0, \theta, \vartheta) > \mathcal{F}(\check{S}_0, \check{I}_0, \check{\theta}, \check{\vartheta}) - \varepsilon^*$, for all $(\check{S}_0, \check{I}_0, \check{\theta}, \check{\vartheta}) \in P$ with $\|(S_0, I_0) - (\check{S}_0, \check{I}_0)\| < \varepsilon_*$ and $d((\theta, \vartheta), (\check{\theta}, \check{\vartheta})) < \varepsilon_*$, where d is the metric on $C(\mathbb{R}, \mathbb{R}^2)$ equipped with the compact open topology. Since $\mathcal{F}(S_0, I_0, \theta, \vartheta) > 0$, $\forall (S_0, I_0, \theta, \vartheta) \in A_0$, the compactness of A_0 and the lower semicontinuity of $\mathcal{F}$ imply that there exist an open neighborhood O of A_0 in P and a number $\varrho > 0$ such that

$$\mathcal{F}(S_0, I_0, \theta, \vartheta) \geqslant \varrho, \quad \forall (S_0, I_0, \theta, \vartheta) \in O.$$

Thus, the global attractivity of A_0 for $\pi_t : P_0 \to P_0$ completes the proof. □

Based on the conclusions in Lemmas 3.3 and 3.4, Theorems 3.5 and 4.2, the following assertions are obtained straight.

Theorem 4.3. *The following assertions are valid:*

(i) *The disease-free constant solution* $(N/|\Omega|, 0)$ *is globally attractive for* (1.2) *if one of the following conditions holds:*
  - (i-a) $\beta(x,t) - \gamma(x,t) \equiv \mu(t)$ *and* $[\mu] < 0$*;*
  - (i-b) $\beta(x,t) - \gamma(x,t) \equiv \mu(x)$ *and either* $\mu \leqslant, \not\equiv 0$ *on* $\bar{\Omega}$ *or* $\max_{x \in \bar{\Omega}} \mu(x) > 0$ *and* $\int_\Omega \mu(x)dx < 0$ *but* $D_I > D_I^*$*, where* D_I^* *is given in Lemma* 3.4*;*
  - (i-c) *If* $\lim_{t\to\infty} \frac{1}{t}\int_0^t \max_{x \in \bar{\Omega}} (\beta(x,\tau) - \gamma(x,\tau))d\tau \leqslant 0$ *and* $\beta(x,t) - \gamma(x,t)$ *nontrivially depends on the spatial variable* x*;*
  - (i-d) $\lim_{t\to\infty} \frac{1}{t}\int_0^t \int_\Omega \beta(x,\tau)dx d\tau - \lim_{t\to\infty} \frac{1}{t}\int_0^t \int_\Omega \gamma(x,\tau)dx d\tau < 0$ *and* D_I *is sufficiently large.*

(ii) *The uniform persistence for* (1.2) *holds if one of the following conditions holds:*
  - (ii-a) $\beta(x,t) - \gamma(x,t) \equiv \mu(t)$ *and* $[\mu] > 0$*;*
  - (ii-b) $\beta(x,t) - \gamma(x,t) \equiv \mu(x)$ *and either* $\mu \not\equiv 0$ *on* $\bar{\Omega}$ *and* $\int_\Omega \mu(x)dx \geqslant 0$ *or* $\max_{x \in \bar{\Omega}} \mu(x) > 0$ *and* $\int_\Omega \mu(x)dx < 0$ *but* $0 < D_I < D_I^*$*;*
  - (ii-c) $\lim_{t\to\infty} \frac{1}{t}\int_0^t \int_\Omega (\beta(x,\tau) - \gamma(x,\tau))dx d\tau > 0$*;*

5. Global attractivity

The uniqueness and global attractivity of the endemic almost periodic solution to almost periodic reaction–diffusion system (1.2) is a hard work since the existence of an almost periodic positive solution is a challenging problem. Allen et al. [2] gave a conjecture about the global attractivity of endemic equilibrium of the autonomous system (1.2). Since [24] answered the problem partially, Peng and Zhao

[25] gave a complete description of the global attractivity of the disease-free constant solution and the endemic ω-periodic solution in the periodic environment for two special cases. For the more generalized case, i.e., almost periodic case, if $\beta(x,t) \equiv \beta(t)$ and $\gamma(x,t) \equiv \gamma(t)$ are almost periodic functions, the global attractivity of the ODE system associated with system (1.2) can be obtained from the conclusions developed in [32]. When it comes to the almost periodic reaction–diffusion system (1.2), we would present the global attractivity of the endemic almost periodic solution for two special cases. At first, we consider the case that the diffusion rate of the susceptible individuals and the infected individuals is equal, i.e., $D_S = D_I$. In order to obtain the existence of almost periodic positive solution, we consider

$$\begin{cases} \frac{\partial w}{\partial t} - D\Delta w = \bar{a}(x,t)w - \bar{b}(x,t)w^2, & x \in \Omega,\ t > 0, \\ \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{5.1}$$

where D is a positive constant, and $\bar{a}(x,t)$ is Hölder continuous and uniformly almost periodic in t, $\bar{b}(x,t)$ is uniformly almost periodic in t with $\bar{b}(x,t) > 0$ on $\bar{\Omega} \times \mathbb{R}$. We denote that $w(t,x)$ is the solution of (5.1) satisfying $w(0,x) = w_0(x)$. Let $\Psi_{(\bar{a},\bar{b})}(t,s)$ be the evolution operator of

$$\begin{cases} \frac{\partial w}{\partial t} - D\Delta w = \bar{a}(x,t)w, & x \in \Omega,\ t > 0, \\ \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0. \end{cases}$$

Hence, we can define the exponential growth bound of $\Psi_{(\bar{a},\bar{b})}(t,s)$, denoted by $\omega(\Psi_{(\bar{a},\bar{b})})$. Thanks to [37, Theorem 3.1](see also [35]), we have

Lemma 5.1. *The following two statements are valid:*

(i) *If* $\omega(\Psi_{(\bar{a},\bar{b})}) < 0$, *then* $\lim_{t\to\infty} \|w(t,x)\| = 0$ *for every* $w_0 \in X_+$;

(ii) *If* $\omega(\Psi_{(\bar{a},\bar{b})}) > 0$, *then* (5.1) *admits a unique positive almost periodic solution* $w^*(t,x)$ *and* $\lim_{t\to\infty} \|w(t,\cdot) - w^*(t,\cdot)\| = 0$ *for every* $w_0 \in X_+ \setminus \{0\}$.

By Lemmas 3.2 and 5.1, the similar arguments to those in [24, Theorem 1.1] imply the following result.

Theorem 5.2. *Suppose that* $D_S = D_I$ *and (H3) holds. If* $R_0 \leqslant 1$, *then* $(N/|\Omega|, 0)$ *is globally attractive; if* $R_0 > 1$ *and* $\beta(x,t) > 0$ *on* $\bar{\Omega} \times \mathbb{R}$, *then* (1.2) *admits an endemic almost periodic solution, which is globally attractive in* $X_+ \setminus \{0\}$.

Next, for the case of where $\beta(x,t) = r\gamma(x,t)$ for some positive constant $r \in (0,\infty)$, as noted in [25], we deduce that for $r > 1$,

$$(S_*, I_*) = \left(\frac{1}{r}\frac{N}{|\Omega|}, \frac{r-1}{r}\frac{N}{|\Omega|}\right)$$

is an endemic almost periodic solution of (1.2). In order to get the global attractivity of (S_*, I_*), we draw support from the following lemma, which can be found in [25, Lemma 4.3] (see also [22, Lemma 1]). For reader's convenience, we present the lemma below.

Lemma 5.3. *Let* $\hat{a}$ *and* $\hat{b}$ *be the positive constants. Assume that* ϕ, $\psi \in C^1([\hat{a},\infty))$, $\psi \geqslant 0$, *and* ϕ *is bounded from below in* $[\hat{a},\infty)$. *If* $\phi'(t) \leqslant -\hat{b}\psi(t)$ *and* $\psi'(t) \leqslant K$ *on* $[\hat{a},\infty)$ *for some positive constant* K, *then* $\lim_{t\to\infty} \psi(t) = 0$.

Theorem 5.4. *Let (H3) hold and* $\beta(x,t) = r\gamma(x,t)$ *on* $\bar{\Omega} \times \mathbb{R}$ *for some constant* $r \in (0,\infty)$ *and* $\gamma(x,t) > 0$ *in* $\bar{\Omega} \times \mathbb{R}_+$. *If* $r \leqslant 1$, *then* $(N/|\Omega|, 0)$ *is globally attractive; if* $r > 1$, *then* (S_*, I_*) *is globally attractive for* *(1.2)*.

Proof. Theorem 3.5 (i) tells us that $R_0 \geqslant r$. For Eq. (3.5), we follow from (3.12) that

$$\omega(\Psi_{(\beta/\rho,-\gamma)}) = -D_I \lim_{t\to\infty} \frac{1}{t} \int_0^t \int_\Omega |\nabla\tilde{\varphi}^*|^2 \mathrm{d}x\mathrm{d}\tau + \lim_{t\to\infty} \frac{1}{t} \int_0^t \int_\Omega \left(\frac{\beta}{\rho} - \gamma\right) |\tilde{\varphi}^*|^2 \mathrm{d}x\mathrm{d}\tau.$$

Hence, Lemma 3.2 implies that $R_0 \leqslant r$. Thus, we have $R_0 = r$. It then follows that $R_0 < 1$ if $0 < r < 1$, $R_0 = 1$ if $r = 1$ and $R_0 > 1$ if $r > 1$. As $r < 1$, the conclusion can be obtained from Theorem 4.2 (i). For the case $r = 1$, the global attractivity of $(N/|\Omega|, 0)$ can be gotten by the similar arguments to those in [24, Theorem 1.2].

In the case of where $r > 1$, we draw support from the idea of the Lyapunov functional which introduced in [24,25]. Let

$$F(t) := \int_\Omega \left(S(x,t) + \frac{S_*^2}{S(x,t)} + I(x,t) + \frac{I_*^2}{I(x,t)} \right) \mathrm{d}x.$$

By carrying out the same computations as in the proof of [24, Theorem 1.2], we conclude that

$$\frac{\mathrm{d}F}{\mathrm{d}t} = -\int_\Omega \left[D_S \frac{2S_*^2}{S^3}|\nabla S|^2 + D_I \frac{2I_*^2}{I^3}|\nabla I|^2 + \frac{\beta(x,t)SI^2}{(S_* + I_*)(S+I)} \left(\frac{S_*}{S} + \frac{I_*}{I} \right) \left(\frac{S_*}{S} - \frac{I_*}{I} \right) \right] \mathrm{d}x. \tag{5.2}$$

It is easy to see that

$$\frac{\mathrm{d}F}{\mathrm{d}t} \leqslant -\int_\Omega \frac{\beta(x,t)SI^2}{(S_* + I_*)(S+I)} \left(\frac{S_*}{S} + \frac{I_*}{I} \right) \left(\frac{S_*}{S} - \frac{I_*}{I} \right)^2 \mathrm{d}x.$$

We then follow from Lemma 4.1 and Theorem 4.2 (ii) that there exist positive constants C_6 and T_0 sufficiently large such that

$$\frac{\mathrm{d}F}{\mathrm{d}t} \leqslant -C_6 \int_\Omega \left(\frac{S_*}{S} - \frac{I_*}{I} \right)^2 \mathrm{d}x =: -\psi(t), \quad \forall t \geqslant T_0.$$

Using Lemma 4.1 and Theorem 4.2 (ii) again, the similar arguments to those in the proof of [25, Theorem 4.4] show that $\psi'(t)$ is bounded in $[T_0, \infty)$ and $F(t)$ is bounded from below in $[T_0, \infty)$. Hence, Lemma 5.3 implies that $\lim_{t\to\infty} \psi(t) = 0$. Thus, we have

$$\lim_{t\to\infty} \int_\Omega |(r-1)S(x,t) - I(x,t)| \mathrm{d}x = 0.$$

This, together with (1.3), gives rise to

$$\lim_{t\to\infty} \frac{1}{|\Omega|} \int_\Omega S(x,t)\mathrm{d}x = S_*, \quad \lim_{t\to\infty} \frac{1}{|\Omega|} \int_\Omega I(x,t)\mathrm{d}x = I_*. \tag{5.3}$$

On the other hand, we follow from (5.2) that

$$\frac{\mathrm{d}F}{\mathrm{d}t} \leqslant -\int_\Omega \left[D_S \frac{2S_*^2}{S^3}|\nabla S|^2 + D_I \frac{2I_*^2}{I^3}|\nabla I|^2 \right] \mathrm{d}x.$$

The similar arguments to those above imply that

$$\lim_{t\to\infty} \int_\Omega (|\nabla S|^2 + |\nabla I|^2)\mathrm{d}x = 0$$

It then follows from Poincaré inequality that

$$\lim_{t\to\infty} \int_\Omega |S(x,t) - S^*(t)|^2 \mathrm{d}x = 0, \quad \lim_{t\to\infty} \int_\Omega |I(x,t) - I^*(t)|^2 \mathrm{d}x = 0, \tag{5.4}$$

where $S^*(t) = \frac{1}{|\Omega|}\int_\Omega S(x,t)\mathrm{d}x$ and $I^*(t) = \frac{1}{|\Omega|}\int_\Omega I(x,t)\mathrm{d}x$. In view of (5.3) and (5.4), we get

$$\lim_{t\to\infty}\int_\Omega (|S(x,t) - S_*| + |I(x,t) - I_*|)\mathrm{d}x = 0. \tag{5.5}$$

In the following, we use the tool of skew-product semiflows and inherit the notations in Theorem 4.2. Recall that for any given $(\tilde{S}, \tilde{I}, \tilde{\theta}, \tilde{\vartheta}) \in P_0$, $\omega(\tilde{S}, \tilde{I}, \tilde{\theta}, \tilde{\vartheta})$ denotes its omega limit set. Then for any $(\hat{S}, \hat{I}, \hat{\theta}, \hat{\vartheta}) \in \omega(\tilde{S}, \tilde{I}, \tilde{\theta}, \tilde{\vartheta})$, there exists a sequence $t_k \to \infty$ such that $\lim_{t_k\to\infty}\pi_{t_k}(\tilde{S}, \tilde{I}, \tilde{\theta}, \tilde{\vartheta}) = (\hat{S}, \hat{I}, \hat{\theta}, \hat{\vartheta})$. Letting $(S, I, \theta, \vartheta) = \pi_t(\tilde{S}, \tilde{I}, \tilde{\theta}, \tilde{\vartheta})$ and $t = t_k$ in (5.5), we conclude that

$$\int_\Omega (|\hat{S}(x) - S_*| + |\hat{I}(x) - I_*|)\mathrm{d}x = 0,$$

and hence $(\hat{S}, \hat{I}) = (S_*, I_*)$. Thus, we have $\omega(\tilde{S}, \tilde{I}, \tilde{\theta}, \tilde{\vartheta}) = \{(S_*, I_*, \theta_*, \vartheta_*) : (\theta_*, \vartheta_*) \in H\}$. It then follows that $\lim_{t\to\infty}\pi_t(\tilde{S}, \tilde{I}, \tilde{\theta}, \tilde{\vartheta}) = (S_*, I_*, \theta_*, \vartheta_*)$. The global attractivity of (S_*, I_*) is obtained. □

6. Discussion

In this paper, we have considered the basic reproduction ratio of almost periodic reaction–diffusion system (1.2) and represent its properties. Applying the developed theory, we have obtained a threshold-type result for the uniform persistence, the global extinction and the global attractivity of (1.2) by means of the basic reproduction ratio R_0. Since almost periodic functions are a generalization of periodic functions, the study of almost periodic reaction–diffusion system (1.2) is more reasonable to reveal the mechanism of the disease transmission.

Using the terminology similar to those in [25] (see also [2]), we present the biological interpretations corresponding to our results. We call x is a *high-risk* site if $\lim_{t\to\infty}\frac{1}{t}\int_0^t \beta(x,\tau)\mathrm{d}\tau > \lim_{t\to\infty}\frac{1}{t}\int_0^t \gamma(x,\tau)\mathrm{d}\tau$. A *low-risk* site is defined in a reversed mode. The habitat Ω is called to be *high-risk* if $\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega \beta(x,\tau)\mathrm{d}x\mathrm{d}\tau > \lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega \gamma(x,\tau)\mathrm{d}x\mathrm{d}\tau$ and *low-risk* if $\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega \beta(x,\tau)\mathrm{d}x\mathrm{d}\tau < \lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega \gamma(x,\tau)\mathrm{d}x\mathrm{d}\tau$. A *moderate-risk* habitat is defined if $\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega \beta(x,\tau)\mathrm{d}x\mathrm{d}\tau = \lim_{t\to\infty}\frac{1}{t}\int_0^t\int_\Omega \gamma(x,\tau)\mathrm{d}x\mathrm{d}\tau$.

At the beginning, Theorem 4.3 (i-a) and (ii-a) shows that the extinction happens in a *low-risk* habitat, while the persistence happens in a *high-risk* habitat when the disease transmission and recovery depend on the temporal factor alone. For the case of where the disease transmission and recovery depend on the spatial factor, we follows from Theorem 4.3 (ii-b) that the disease will be persistent once a *high-risk* habitat exists. However, Theorem 4.3 (i-b) and (ii-b) also shows that the disease is not necessarily extinct or persistent though the individuals live in a *low-risk* habitat where a *high-risk* site is contained at least. In the case, the movement of the infected population by a quick way will be helpful for the disease extinction, and in other words, there exists a threshold value $D_I^* \in (0,\infty)$ such that the disease is extinct if $D_I > D_I^*$; otherwise, the disease will persist if $D_I < D_I^*$.

For the general case of where the disease transmission and recovery depend on the spatiotemporal variables, Theorem 4.3 (ii-c) implies that the disease will persist if the habitat is a *high-risk* type. Conversely, Theorem 4.3 (i-d) tells us that the disease will die out if the habitat is a *low-risk* one and the movement of the infected individuals is sufficiently quick.

Next, we investigate the affection of spatial heterogeneity and almost periodic environment for the disease transmission. Assume that

$$\beta(x,t) = a(x)b(t), \quad \gamma(x,t) = a(x)c(t),$$

where $a(x)$ is a positive Hölder continuous function on $\bar{\Omega}$ and b, c are positive almost periodic Hölder continuous functions on $\mathbb{R}$. If $b \equiv c$, then the habitat is a *moderate-risk* one and we follow from

Theorem 5.4 that the disease will eventually die out because of the diffusion rates. If a is not a constant, $b \not\equiv c$, and $[b] = [c]$, and hence the habitat is still a *moderate-risk* one, then we see from Theorem 3.5 that the basic reproduction ratio $R_0(D_I) = R_0 > 1$ for all $D_I > 0$ and $R_0(D_I) \to 1$ as $D_I \to 0$ or $D_I \to \infty$. Consequently, Theorem 4.2 tells us that the disease will persist in this *moderate-risk* habitat.

The above discussions show that the interaction of spatial heterogeneity and temporal almost periodicity tends to enhance the persistence of the disease for the SIS model (1.2). In other words, if we only consider the temporal almost periodicity or spatial heterogeneity in (1.2), then the infection risk would be underestimated. Furthermore, for the case of where a is not a constant, $b \not\equiv c$, and $[b] = [c]$, when the infected individuals move at the speed $D_I = \hat{D}_I$, where $\hat{D}_I > 0$ satisfying $R_0(\hat{D}_I) = \max_{D_I \in (0,\infty)} R_0(D_I) > 1$, the risk of the outbreak of the disease in the population will be enlarged; on the other hand, the small or large diffusion rate will weaken the persistence of the disease since the basic reproduction ratio is close to unity in the case.

Remark 6.1. *We only prove* $R_0 \leqslant \max_{x\in\bar{\Omega}} \left\{ \frac{\lim_{t\to\infty} \frac{1}{t}\int_0^t \beta(x,\tau)\mathrm{d}\tau}{\lim_{t\to\infty} \frac{1}{t}\int_0^t \gamma(x,\tau)\mathrm{d}\tau} \right\}$ *as* $D_I \to 0$ *in Theorem* 3.5 *(iv). Similar to the periodic case in* [25]*, we conjecture that* $R_0 \to \max_{x\in\bar{\Omega}} \left\{ \frac{\lim_{t\to\infty} \frac{1}{t}\int_0^t \beta(x,\tau)\mathrm{d}\tau}{\lim_{t\to\infty} \frac{1}{t}\int_0^t \gamma(x,\tau)\mathrm{d}\tau} \right\}$ *as* $D_I \to 0$*. Thus, if* $\max_{x\in\bar{\Omega}} \left\{ \frac{\lim_{t\to\infty} \frac{1}{t}\int_0^t \beta(x,\tau)\mathrm{d}\tau}{\lim_{t\to\infty} \frac{1}{t}\int_0^t \gamma(x,\tau)\mathrm{d}\tau} \right\} > 1$ *and* D_I *is sufficiently small, which imply that if there exists at least one high-risk site and the infected individuals are almost stationary, then the disease will persist.*

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