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Rigidity of planar central configurations

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Abstract. In this article, we study the rigidity of planar central configurations in the non-collinear n-body problem relative to the change of masses. More precisely, we study central configurations for which it is possible to change the values of k masses keeping fixed all the positions and the values of the masses of the other $n-k$ bodies and still have central configurations. Here, we consider the cases $k=1$ and $k=2$. The central configurations that have such properties are closely related to the so-called stacked central configurations and super central configurations.

Mathematics Subject Classification. 70F10 · 70F15 · 37N05.

Keywords. Central configuration · n-body problem · Planar central configuration · Rigidity · Celestial mechanics.

1. Introduction and statement of the main results

The Newtonian n-body problem consists in the study of a system formed by n punctual bodies described by the position vectors $r_1, \ldots, r_n$ in $\mathbb{R}^d$, $d = 2, 3$ and positive masses $m_1, \ldots, m_n$ interacting between themselves by their mutual gravitational attraction according to Newton's gravitational law [19]. The equations of the motion are given by

$$\ddot{r}_i = -\sum_{j=1, j\neq i}^{n} \frac{m_j}{r_{ij}^3}(r_i - r_j), \tag{1.1}$$

for $i = 1, \ldots, n$, where $r_{ij} = |r_i - r_j|$ is the Euclidean distance between the bodies at r_i and r_j. In (1.1), we consider the gravitational constant $G = 1$. In order to avoid the singular case, the space of configuration is $\{(r_1, r_2, \ldots, r_n) \in \mathbb{R}^{dn} : r_i \neq r_j, i \neq j\}$.

Some particular solutions of (1.1) appear in the literature, such as the *homographic* solutions in which the initial shape of the configuration remains the same as time varies. The first homographic solutions are due to Euler [5] and Lagrange [12].

At a given instant $t = t_0$, the n bodies are in a *central configuration* if there exists $\lambda \neq 0$ such that $\ddot{r}_i = \lambda(r_i - \mathcal{C}_n)$, for all $i = 1, \ldots, n$, where $\mathcal{C}_n$ is the center of mass of the system which is given by $\mathcal{C}_n = \sum_{j=1}^{n} m_j r_j / \mathcal{M}_n$ and $\mathcal{M}_n = m_1 + \ldots + m_n$ is the total mass of the system. Such configurations are closely related to homographic solutions. See [18, 20, 21] and [24].

From the definition of central configuration and using Eq. (1.1), in order to compute the planar, that is $d = 2$, central configurations, it is necessary to solve the following set of algebraic equations

$$\lambda(r_i - \mathcal{C}_n) = -\sum_{j=1, j\neq i}^{n} \frac{m_j}{r_{ij}^3}(r_i - r_j), \tag{1.2}$$

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for $i = 1, 2, \ldots, n$. For the planar case, simple computations allow us to write Eq. (1.2) in the following form

$$f_{i,j} = \sum_{k=1, k\neq i,j}^{n} m_k \left(R_{i,k} - R_{j,k}\right) \Delta_{i,j,k} = 0, \tag{1.3}$$

for $1 \leq i < j \leq n$, where $R_{i,j} = 1/r_{ij}^3$ and $\Delta_{i,j,k} = (r_i - r_j) \wedge (r_i - r_k)$. In fact, $\Delta_{i,j,k}$ is twice the oriented area of the triangle formed by the bodies at r_i, r_j and r_k (see [8]). These $n(n-1)/2$ equations are called Dziobek–Laura–Andoyer equations or simply Andoyer equations.

A central configuration is invariant by mass scale, that is, if we have a central configuration with position vectors $r_1, r_2, \ldots, r_n$ and positive masses $m_1, m_2, \ldots, m_n$, then we still have a central configuration with the same position vectors $r_1, r_2, \ldots, r_n$ and masses $\alpha m_1, \alpha m_2, \ldots, \alpha m_n$, for any positive real constant α. In fact, the Andoyer equations for the new masses can be written as $\alpha f_{i,j} = 0$.

Two central configurations $(r_1, \ldots, r_n)$ and $(\bar{r}_1, \ldots, \bar{r}_n)$ of the n bodies are *related* if we can pass from one to the other through a dilation or a rotation (centered at the center of mass). So we can study the classes of central configurations defined by the above equivalence relation.

Taking into account this equivalence we have exactly five classes of central configurations in the 3-body problem. The finiteness of the number of central configurations performed by n bodies with positive masses is a question posed by Chazy in [2], Wintner in [24] and reformulated to the planar case by Smale in [23]. For $n = 4$, this problem has an affirmative answer [10]. Alternatively, see [1] for a proof of the finiteness when $n = 4$ and a partial answer when $n = 5$. This question is open when $n > 5$.

A *stacked central configuration* is a central configuration in which a proper subset of the n bodies is already in a central configuration. This class of central configurations was introduced by Hampton in [9]. A central configuration is called (n, k)–*stacked* when the n bodies perform a central configuration, and we can remove $0 < k < n$ bodies (the k with greater indices) such that the remaining $n - k$ bodies are already in a central configuration.

Some cases of $(5, 2)$–stacked central configurations were studied by Llibre and Mello in [13] and by Llibre, Mello and Perez–Chavela in [14]. Stacked central configurations for the spatial 7–body problem can be found in [11] and [17].

Fernandes and Mello in [6] studied $(5, 1)$–stacked planar central configurations. The authors conclude that the only non-collinear $(5, 1)$–stacked planar central configuration is formed by four bodies in a co-circular central configuration and one body of arbitrary mass at the center of the circle.

Recently, Fernandes and Mello [7] provided an extension of the above result to $(n, 1)$–stacked non-collinear central configurations.

Theorem 1.1. *[7] Consider the planar non-collinear n–body problem with $n \geq 4$. Then, the only $(n, 1)$–stacked central configurations are formed by $n - 1$ bodies in a co-circular central configuration and one body (to be removed) of arbitrary mass at the center of the circle.*

As a consequence of the above theorem, in an $(n, 1)$–stacked central configuration, it is possible to change the value of the mass of the body at the center of the circle, keeping fixed all the positions and the values of the other $n - 1$ masses, and still have a central configuration.

Another kind of central configuration related to the change of the values of some masses is a *super central configuration*: a central configuration in which it is possible to make a non-trivial permutation of the values of two or more different masses keeping fixed the position vectors and still have a central configuration. See [25] for a more precise definition.

We have the following question concerning central configurations in the planar n–body problem.

Question 1.2. *Consider n bodies with position vectors $r_1, r_2, \ldots, r_n$ and positive masses $m_1, m_2, \ldots, m_n$ in a planar central configuration. Consider also $0 < k < n$. Is it possible to change the values of the masses*

$$\bar{m}_{n-k} \neq m_{n-k}, \bar{m}_{n-k+1} \neq m_{n-k+1}, \ldots, \bar{m}_n \neq m_n$$

keeping fixed all the position vectors $r_1, r_2, \ldots, r_n$ and the values of the other masses $m_1, m_2, \ldots, m_{n-k-1}$ such that the n bodies are still in a central configuration?

An answer for Question 1.2 is given by the super central configurations [25] where the change of the values of the masses is obtained in the original set of masses. Central configurations considered here can be seen as generalizations of super central configurations.

Question 1.2 concerns on a type of rigidity of planar central configurations in the n–body problem relative to the change of masses. This question is motivated by the instructive example of the central configuration due to Lagrange: three bodies of arbitrary positive masses at the vertices of an equilateral triangle. In this central configuration, the positive value of the mass of each body can be changed by any other positive value or even can vary continuously assuming any positive value.

In Question 1.2, we do not consider the case in which some masses are changed to zero because this can be seen like a stacked central configuration. In fact, there exist several examples of (n, k)–stacked central configurations, see for example [9,13,14].

For the case $k = 1$, we have the following result.

Theorem 1.3. *Consider the planar non-collinear n–body problem with $n \geq 4$. Then, the central configurations for which it is possible to change the value of one mass keeping fixed all the positions and the values of the other $n-1$ masses and still have a central configuration are the central configurations with n bodies such that $n-1$ bodies are in a co-circular central configuration, and one body of arbitrary mass is at the center of the circle. Moreover, the center of mass of the $n-1$ co-circular bodies must be at the center of the circle.*

It is important to mention that Theorem 1.3 for $n = 4$ is contained in the following theorem due to MacMillan and Bartky [15], page 872.

Theorem 1.4. [15] *Associated with each admissible quadrilateral, there is one and only one set of mass ratios, with the single exception of three equal masses at the vertices of an equilateral triangle and a fourth arbitrary mass at the center of gravity of the other three.*

We also study the case $k = 2$. We have the following result which can be seen as a type of rigidity of central configurations.

Theorem 1.5. *Consider the planar non-collinear n–body problem with $n \geq 4$. There is not a central configuration for which it is possible to change the values of the masses of two bodies keeping fixed all the positions and the values of the masses of the other $n-2$ bodies and still have a central configuration.*

Theorem 1.3 is proved in Sect. 2. In Sect. 3, we prove Theorem 1.5. Concluding comments are presented in Sect. 4.

2. The case $k = 1$

Without loss of generality, suppose that the mass to be change is m_n. The proof of Theorem 1.3 is divided into three lemmas.

Lemma 2.1. *In order to have a central configuration in which the mass m_n can be changed, it is necessary that the other $n-1$ bodies must be in a co-circular configuration with center at r_n.*

Proof. The planar Andoyer Eq. (1.3) must be satisfied for the n bodies. Consider the Andoyer equations with $i \neq n$ and $j \neq n$. These equations can be written as

$$f_{i,j} = \sum_{k \neq i,j,n} m_k \left(R_{i,k} - R_{j,k}\right) \Delta_{i,j,k} + m_n \left(R_{i,n} - R_{j,n}\right) \Delta_{i,j,n} = 0, \tag{2.1}$$

for all indices i and j, such that $0 < i < j < n$. Note that in Eq. (2.1), the part under summation does not depend on the mass m_n. So, the variation of the mass m_n implies that the part under summation and the coefficient of m_n must vanish. Thus, we have

$$(R_{i,n} - R_{j,n})\,\Delta_{i,j,n} = 0, \tag{2.2}$$

for all indices i and j, such that $0 < i < j < n$.

By assumption, the configuration is non-collinear, so in Eq. (2.2), at least one $\Delta_{i,j,n} \neq 0$. Without loss of generality, suppose $\Delta_{1,2,n} \neq 0$. Thus, from

$$(R_{1,n} - R_{2,n})\,\Delta_{1,2,n} = 0$$

we have

$$R_{1,n} - R_{2,n} = 0,$$

which implies that $r_{1n} = r_{2n} = d > 0$. Therefore, r_1 and r_2 belong to the circle of radius d and center at r_n.

We can classify the other indices into two sets

$$\mathfrak{C}_1 = \{j : \Delta_{1,j,n} = 0\}\,, \quad \mathfrak{C}_2 = \{j : \Delta_{1,j,n} \neq 0\}\,,$$

that is, $\mathfrak{C}_1$ contains the indices of the bodies whose position vectors are collinear with r_1 and r_n, while $\mathfrak{C}_2$ contains the indices of the bodies whose position vectors are not collinear with r_1 and r_n. For $j \in \mathfrak{C}_2$ and from

$$(R_{1,n} - R_{j,n})\,\Delta_{1,j,n} = 0$$

we have

$$R_{1,n} - R_{j,n} = 0.$$

Thus, $r_{jn} = r_{1n} = d > 0$, for all $j \in \mathfrak{C}_2$. Then, r_1, r_2 and r_j belong to the circle of radius d and center at r_n, for all $j \in \mathfrak{C}_2$.

To complete the proof of the lemma, we need to show that $\mathfrak{C}_1$ has at most one element. Suppose, by contradiction, that there exist two indices $b, c \in \mathfrak{C}_1$. So $\Delta_{1,b,n} = 0$, which implies that $\Delta_{2,b,n} \neq 0$. From

$$(R_{2,n} - R_{b,n})\,\Delta_{2,b,n} = 0,$$

we have

$$R_{2,n} - R_{b,n} = 0,$$

which implies that $r_{bn} = r_{2n} = d > 0$. Thus, r_b belongs to the circle of radius d and center at r_n. As the central configurations are out of the collision set, r_b must be diametrically opposite to r_1. Now consider the index $c \in \mathfrak{C}_1$. So $\Delta_{1,c,n} = 0$, which implies that $\Delta_{2,c,n} \neq 0$. From

$$(R_{2,n} - R_{c,n})\,\Delta_{2,c,n} = 0$$

we have

$$R_{2,n} - R_{c,n} = 0,$$

which implies that $r_{cn} = r_{2n} = d > 0$. Here, we have a contradiction, since in this case r_c coincides with either r_1 or r_b. See Fig. 1. The lemma is proved. □

Note that with the assumptions of Lemma 2.1, the central configuration is independent of the value of the mass m_n, thus such a configuration will be a central configuration for all positive values of m_n.

The next lemma says that the $n-1$ bodies must perform a central configuration too.

Lemma 2.2. *In order to have a planar central configuration in which the mass m_n can be changed, it is necessary that the other $n-1$ co-circular bodies must be in a central configuration.*

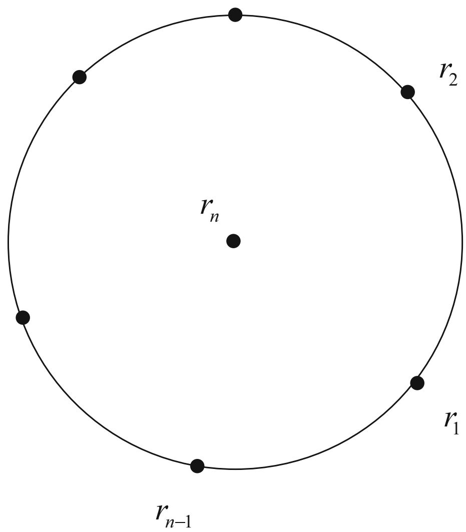

FIG. 1. Only possible configuration with the continuous variation of m_n

Proof. Note that the part under summation in Eq. (2.1) is exactly the Andoyer equation for the bodies $1, 2, \ldots, n-1$. So, taking into account Eq. (2.2) for $1 \leq i < j \leq n-1$, Eq. (2.1) is already satisfied for $1 \leq i < j \leq n-1$. Thus, the $n-1$ bodies are in a central configuration. □

Now we give information about the position of the center of mass of the $n-1$ co-circular bodies.

Lemma 2.3. *Consider a central configuration with n bodies, $n \geq 4$, formed by $n-1$ bodies on a circle C and one body of arbitrary mass located at the center of the circle C. Then, the center of mass of the co-circular bodies coincides with the center of the circle C.*

Proof. Without loss of generality, assume that the position vectors r_i of the co-circular bodies form a polygon inscribed in a circle C of radius d and center at r_n. Assume also that the indices $1, \ldots, n-1$ are disposed cyclically. As above denote by $\mathcal{M}_n$ the total mass of the n bodies and by $\mathcal{C}_n$ the center of mass of the n bodies. Denote by $\mathcal{M}_{n-1}$ the total mass of the $n-1$ co-circular bodies and by $\mathcal{C}_{n-1}$ the center of mass of these bodies.
Its easy to see that

$$\lambda = \frac{U}{2I},$$

where

$$U = \sum_{i<j} \frac{m_i m_j}{r_{ij}}, \quad I = \frac{1}{2\mathcal{M}_n} \sum_{i<j} m_i m_j r_{ij}^2$$

which are, respectively, the Newtonian potential function and the moment of inertia.
Consider the equation of central configuration for the body n. Under our hypotheses, we have

$$\lambda(r_n - \mathcal{C}_n) = -\sum_{j=1}^{n-1} \frac{m_j}{r_{jn}^3} (r_n - r_j) = -\sum_{j=1}^{n-1} \frac{m_j}{d^3} (r_n - r_j). \tag{2.3}$$

Equation (2.3) is equivalent to

$$d^3 \lambda(r_n - \mathcal{C}_n) = \sum_{j=1}^{n-1} m_j r_j - \mathcal{M}_{n-1} r_n = \mathcal{M}_{n-1}\mathcal{C}_{n-1} - \mathcal{M}_{n-1} r_n, \tag{2.4}$$

which can be written as

$$d^3\lambda(r_n - \mathcal{C}_n) = \mathcal{M}_n\mathcal{C}_n - \mathcal{M}_n r_n.$$

Thus, we have

$$(d^3\lambda + \mathcal{M}_n)(r_n - \mathcal{C}_n) = 0, \tag{2.5}$$

which must be satisfied for all positive values of m_n. Suppose, by contradiction, that $r_n - \mathcal{C}_n \neq 0$. So, from Eq. (2.5), $d^3\lambda + \mathcal{M}_n = 0$, for all positive values of m_n. Thus,

$$\mathcal{M}_n + d^3\lambda = \mathcal{M}_n - d^3\left(\frac{U}{I}\right) = \mathcal{M}_n - d^3\left[\frac{\sum_{i<j}\frac{m_i m_j}{r_{ij}}}{\frac{1}{2\mathcal{M}_n}\sum_{i<j} m_i m_j r_{ij}^2}\right] = 0, \tag{2.6}$$

which must be satisfied for all positive values of m_n. Equation (2.6) is equivalent to

$$\mathcal{M}_n\frac{1}{2\mathcal{M}_n}\sum_{i<j} m_i m_j r_{ij}^2 - d^3\sum_{i<j}\frac{m_i m_j}{r_{ij}} = 0, \tag{2.7}$$

which must be satisfied for all positive values of m_n. On the other hand, Eq. (2.7) is equivalent to

$$\frac{1}{2}\sum_{i<j} m_i m_j r_{ij}^2 - d^3\sum_{i<j}\frac{m_i m_j}{r_{ij}} = 0. \tag{2.8}$$

Note that Eq. (2.8) is linear (or constant) in the mass m_n, and this implies that it is satisfied for at most one value of m_n. It is easy to see that there exist positive values of m_n such that Eq. (2.8) is not satisfied. Hence, Eq. (2.5) is satisfied for all positive values of m_n if and only if $r_n - \mathcal{C}_n = 0$. In this case, we have $\mathcal{C}_n = r_n$, which implies that $r_n = \mathcal{C}_{n-1}$. The proof of the lemma is complete. □

Now the proof of Theorem 1.3 follows from Lemmas 2.1, 2.2 and 2.3. Theorem 1.3 gives a partial answer to Question 1.2 for the case $k = 1$. A complete answer remains open and requires the study of the co-circular central configurations, with center of mass at the center of the circle, which is a problem proposed by Chenciner in [3]. See also [4] and [16].

3. The case $k = 2$

Without loss of generality, suppose that the masses to be changed are m_{n-1} and m_n. The planar Andoyer Eq. (1.3) must be satisfied for the n bodies. Consider the Andoyer equations for $i = 1, 2, \ldots, n-2$ and $j = n-1$. These equations can be written as

$$f_{i,n-1} = \sum_{k\neq i,n-1,n} m_k\left(R_{i,k} - R_{n-1,k}\right)\Delta_{i,n-1,k} + m_n\left(R_{i,n} - R_{n-1,n}\right)\Delta_{i,n-1,n} = 0.$$

In Eq. (3.1), the part under summation does not depend on m_n. So, the change of the mass m_n implies that the coefficient of m_n must vanish, that is

$$\left(R_{i,n} - R_{n-1,n}\right)\Delta_{i,n-1,n} = 0, \tag{3.1}$$

for all $0 < i < n-1$. With the same arguments for the mass m_{n-1}, we have

$$\left(R_{i,n-1} - R_{n,n-1}\right)\Delta_{i,n,n-1} = 0, \tag{3.2}$$

for all $0 < i < n-1$. From Eqs. (3.1) and (3.2), the position vectors $r_1, \ldots, r_{n-2}$ must be either collinear with r_{n-1} and r_n or belong to the intersection of C_1 and C_2, where C_1 is the circle with center at r_{n-1} and radius $|r_n - r_{n-1}|$ and C_2 is the circle with center at r_n and radius $|r_n - r_{n-1}|$. Note that $C_1 \cap C_2$ determines two points in the plane. Since we consider non-collinear central configurations, these two points must be position vectors of two bodies of the configuration; otherwise by the Perpendicular Bisector Theorem (see [18]), there is no such a central configuration. Without loss of generality, suppose that $C_1 \cap C_2 = \{r_1, r_2\}$. See Fig. 2. We have proved the following lemma.

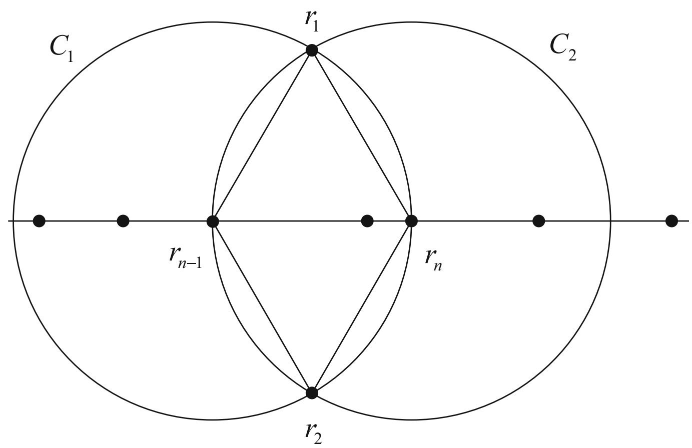

FIG. 2. Positions of the bodies with fixed masses must be either collinear or belong to the intersection of *two circles* with centers at r_{n-1} and r_n and radius $|r_{n-1} - r_n|$

Lemma 3.1. *Consider the planar non-collinear n-body problem with $n > 4$. The following conditions are necessary in order to have a central configuration for which it is possible to change the values of two masses keeping fixed all the positions and the values of the other $n-2$ masses and still have a central configuration (see Fig. 2):*

(a) *r_1, r_2, r_{n-1} and r_n are at the vertices of a rhombus with $|r_1 - r_{n-1}| = |r_1 - r_n| = |r_2 - r_{n-1}| = |r_2 - r_n| = |r_{n-1} - r_n| \neq 0$.*

(b) *The other $n-4$ bodies belong to the straight line containing r_{n-1} and r_n.*

In the following lemma is proved that the masses of the bodies at the intersections of C_1 and C_2 must be equal, that is $m_1 = m_2$.

Lemma 3.2. *Consider the planar n-body problem, $n > 4$. Suppose that r_1, r_2, r_{n-1} and r_n are at the vertices of a rhombus with $|r_1 - r_{n-1}| = |r_1 - r_n| = |r_2 - r_{n-1}| = |r_2 - r_n| = |r_{n-1} - r_n| \neq 0$ and the other $n-4$ bodies belong to the straight line containing r_{n-1} and r_n according to Fig. 2. Then, in order to have a central configuration, a necessary condition is $m_1 = m_2$.*

Proof. Consider the Andoyer equations $f_{i,n} = 0$, for $i = 3, \dots, n-2$, which can be written as

$$\begin{aligned} f_{i,n} &= m_1 (R_{i,1} - R_{n,1}) \Delta_{i,n,1} + m_2 (R_{i,2} - R_{n,2}) \Delta_{i,n,2} \\ &\quad + \sum_{k \neq 1,2,i,n} m_k (R_{i,k} - R_{n,k}) \Delta_{i,n,k} = 0. \end{aligned} \tag{3.3}$$

In Eq. (3.3), the part under summation is zero, since $\Delta_{i,n,k} = 0$. On the other hand, $\Delta_{i,n,1} = -\Delta_{i,n,2} \neq 0$, $R_{i,1} = R_{i,2}$ and $R_{n,1} = R_{n,2}$. Thus, Eq. (3.3) has the form

$$f_{i,n} = (m_1 - m_2)(R_{i,1} - R_{n,1}) \Delta_{i,n,1} = 0. \tag{3.4}$$

As $R_{i,1} - R_{n,1} \neq 0$, Eq. (3.4) is satisfied if and only if $m_1 = m_2$. This completes the proof of the lemma. □

The next lemma gives some information about the relation between the masses to be changed. Fix the following nomenclature: the masses to be changed are m_n and m_{n-1}; after the change, these masses will be denoted by $M_n = m_n - \mu_n$ and $M_{n-1} = m_{n-1} - \mu_{n-1}$.

Lemma 3.3. *Consider the planar n-body problem with $n > 4$. Suppose that r_1, r_2, r_{n-1} and r_n form a rhombus with $|r_1 - r_{n-1}| = |r_1 - r_n| = |r_2 - r_{n-1}| = |r_2 - r_n| = |r_{n-1} - r_n| \neq 0$ and the other $n-4$ bodies belong to the straight line containing r_{n-1} and r_n according to Fig. 2. Then, in order to have a central configuration in which it is possible to change the values of the masses m_n and m_{n-1} keeping fixed all the positions and other $n-2$ masses, the following equation must be satisfied*

$$\frac{\mu_{n-1}}{\mu_n} = -\frac{(R_{1,n} - R_{i,n})\,\Delta_{1,i,n}}{(R_{1,n-1} - R_{i,n-1})\,\Delta_{1,i,n-1}}, \tag{3.5}$$

for $2 < i < n-1$. Moreover, the quotient μ_{n-1}/μ_n must be positive.

Proof. Consider the Andoyer equations $f_{1,i} = 0$, with $2 < i < n-1$. These equations can be written as

$$f_{1,i} = \sum_{k\neq 1,i,n-1,n} m_k\,(R_{1,k} - R_{i,k})\,\Delta_{1,i,k} \tag{3.6}$$
$$+m_{n-1}\,(R_{1,n-1} - R_{i,n-1})\,\Delta_{1,i,n-1} + m_n\,(R_{1,n} - R_{i,n})\,\Delta_{1,i,n} = 0,$$

for $2 < i < n-1$. Consider the same equation after the change of the values of m_{n-1} and m_n

$$f_{1,i} = \sum_{k\neq 1,i,n-1,n} m_k\,(R_{1,k} - R_{i,k})\,\Delta_{1,i,k} + \tag{3.7}$$
$$+M_{n-1}\,(R_{1,n-1} - R_{i,n-1})\,\Delta_{1,i,n-1} + M_n\,(R_{1,n} - R_{i,n})\,\Delta_{1,i,n} = 0.$$

Note that the parts under summation in Eqs. (3.6) and (3.7) are equal. Taking the difference of (3.6) and (3.7), we have

$$m_{n-1}\,(R_{1,n-1} - R_{i,n-1})\,\Delta_{1,i,n-1} + m_n\,(R_{1,n} - R_{i,n})\,\Delta_{1,i,n}$$
$$-M_{n-1}\,(R_{1,n-1} - R_{i,n-1})\,\Delta_{1,i,n-1} - M_n\,(R_{1,n} - R_{i,n})\,\Delta_{1,i,n} = 0,$$

which implies Eq. (3.5). We need to prove that the quotient in Eq. (3.5) is positive. Consider Eq. (3.5) written in the following form

$$\mu_{n-1}\,(R_{1,n-1} - R_{i,n-1})\,\Delta_{1,i,n-1} + \mu_n\,(R_{1,n} - R_{i,n})\,\Delta_{1,i,n} = 0. \tag{3.8}$$

Suppose, by contradiction, that $\mu_{n-1}\mu_n < 0$. So, in order to satisfy (3.8), the coefficients of μ_{n-1} and μ_n must have the same sign. Suppose that there exists a body of index i out of the rhombus and located to the right of r_n. This implies that the terms $\Delta_{1,i,n-1}$, $\Delta_{1,i,n}$ are negative and the term $R_{1,n-1} - R_{i,n-1}$ is positive. So, the term $R_{1,n} - R_{i,n}$ must be positive. But this implies that $|r_1 - r_n| < |r_i - r_n|$. The same assertion holds for all bodies out of the rhombus and located to the right of r_n. All these cases lead a contradiction with the Perpendicular Bisector Theorem [18]. The same argument can be used for the bodies out of the rhombus and located to the left of r_{n-1}. Thus, all collinear bodies must be in the interior of the rhombus, but this also leads to a contradiction with the Perpendicular Bisector Theorem. This part of the proof implies that all bodies in the configuration must belong to the interior of the union of the sets bounded by the circles C_1 and C_2. □

An important consequence of this lemma is that in a central configuration in which it is possible to change the values of two masses (m_n and m_{n-1}) keeping fixed all the positions and other $n-2$ masses, the position vectors of the collinear bodies must satisfy

$$\frac{(R_{1,n} - R_{i,n})\,\Delta_{1,i,n}}{(R_{1,n-1} - R_{i,n-1})\,\Delta_{1,i,n-1}} = \frac{(R_{1,n} - R_{k,n})\,\Delta_{1,k,n}}{(R_{1,n-1} - R_{k,n-1})\,\Delta_{1,k,n-1}} < 0, \tag{3.9}$$

for all i and k such that $2 < i, k < n-1$.

Consider a system of coordinates formed by two axes: one passing through r_1 and r_2 and the other passing through the line containing the collinear bodies. See Fig. 3. Without loss of generality, we assume

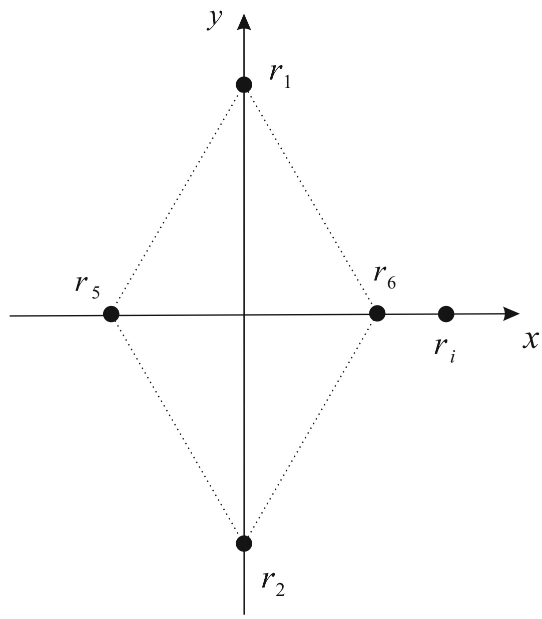

FIG. 3. System of coordinates formed by two axes: one passing through r_1 and r_2 and the other passing through r_{n-1}, r_n and the other bodies r_i, $i = 3, \dots, n-2$. See Figure 2

the coordinates $r_1 = (0, \sqrt{3})$, $r_2 = (0, -\sqrt{3})$, $r_{n-1} = (-1, 0)$, $r_n = (1, 0)$ and $r_i = (r_i, 0)$ (using r_i as a scalar variable) for $i = 3, \dots, n-2$. We study the equation

$$\frac{(R_{1,n} - R_{i,n})\,\Delta_{1,i,n}}{(R_{1,n-1} - R_{i,n-1})\,\Delta_{1,i,n-1}} = -a,$$

with $a > 0$ or equivalently the equation

$$(R_{1,n} - R_{i,n})\,\Delta_{1,i,n} + a\,(R_{1,n-1} - R_{i,n-1})\,\Delta_{1,i,n-1} = 0. \tag{3.10}$$

Since r_1, r_2, r_{n-1} and r_n are fixed in our system of coordinates, Eq. (3.10) can be written as a polynomial equation of degree five in the variable r_i.

By the construction of the coordinates, the terms $(R_{1,n} - R_{i,n})\,\Delta_{1,i,n}$ and $(R_{1,n-1} - R_{i,n-1})\,\Delta_{1,i,n-1}$ always have the pure imaginary roots $\imath\sqrt{3}|r_1 - r_n|/2 = \imath\sqrt{3}$ and $-\imath\sqrt{3}|r_1 - r_n|/2 = -\imath\sqrt{3}$, where $\imath = \sqrt{-1}$. Therefore, the polynomial Eq. (3.10) has at most three real roots, for every $a > 0$. An straightforward computation shows that a as function of r_i is strictly increasing in $(-3, -1) \cup (1, 3)$ and strictly decreasing in $(-1, 1)$, which are the intervals of interest in our problem, see Lemma 3.3.

Note that for each value of a, Eq. (3.9) is satisfied by an index i when r_i is a root of (3.10). Since (3.10) has at most three real roots, there are at most three possible positions to the collinear bodies. Moreover, for each positive value of a, we have exactly one root in $(-3, 1)$, one root in $(-1, 1)$ and one root in $(1, 3)$. Thus, n must be less than 8, because in the cases $n \geq 8$, collisions are always required in order to satisfy Eq. (3.9), for all indices. We have proved the following lemma.

Lemma 3.4. *Consider the planar non-collinear n-body problem with $n \geq 8$. There are no central configurations for which it is possible to change the values of two masses keeping fixed all the positions and the values of the other $n-2$ masses and still have a central configuration.*

Now we prove that there are no such kind of central configurations for the remaining cases: $n = 4$, $n = 5$, $n = 6$ and $n = 7$. We divide the proof into four lemmas.

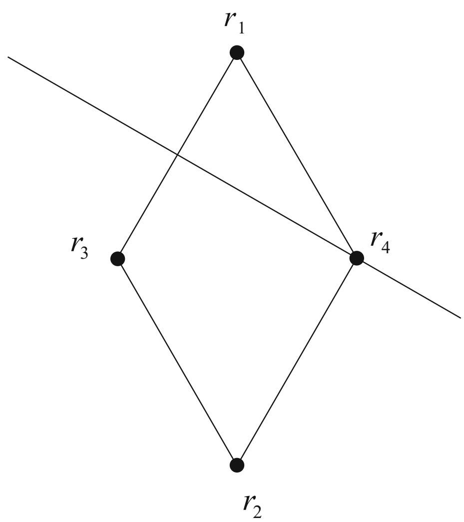

FIG. 4. This configuration cannot be a central configuration

Lemma 3.5. *Consider the planar 4-body problem. Suppose that r_1, r_2, r_3 and r_4 form a rhombus with $r_{13} = r_{14} = r_{23} = r_{24} = r_{34}$ according to Fig. 4. Then, there are no positive masses for which this configuration is a central configuration.*

The proof is a direct corollary of the Perpendicular Bisector Theorem [18].

Lemma 3.6. *Consider the planar 5-body problem. Suppose that r_1, r_2, r_4 and r_5 form a rhombus with $r_{14} = r_{15} = r_{24} = r_{25} = r_{45}$ and r_3 belongs to the straight line containing r_4 and r_5 according to Fig. 5. Then, there are no positive masses for which this configuration is a central configuration.*

Proof. The position vector r_3 cannot belong to the interior of the rhombus. This is a direct consequence of the Perpendicular Bisector Theorem [18]. With r_3 out of the convex hull of the rhombus, consider the Andoyer equation $f_{1,4} = 0$. Taking into account the symmetries, it can be written as

$$f_{1,4} = m_3 \left(R_{1,3} - R_{4,3}\right) \Delta_{1,4,3} = 0. \tag{3.11}$$

Equation (3.11) is satisfied if and only if r_3 coincides with either r_4 or r_5, but this is a contradiction. □

Lemma 3.7. *Consider the planar 6-body problem. Suppose that r_1, r_2, r_5 and r_6 form a rhombus with $r_{16} = r_{15} = r_{26} = r_{25} = r_{56}$, the position vectors r_3 and r_4 belong to the straight line containing r_5 and r_6 according to Fig. 6. Suppose also $m_1 = m_2$. Then, there are no positive masses for which this configuration form a central configuration satisfying: it is possible to change the values of m_5 and m_6 keeping fixed all the positions and other four masses and still have a central configuration.*

Proof. From Lemma 3.3, we consider just the case when the position vectors r_3 and r_4 are in the interior of the union of the sets bounded by the circles C_1 and C_2.
For six bodies, we have fifteen Andoyer Eq. (1.3). By our assumptions of symmetries, the following equations are already verified

$$f_{1,2} = 0, \quad f_{3,4} = 0, \quad f_{3,5} = 0, \quad f_{3,6} = 0, \quad f_{4,5} = 0, \quad f_{4,6} = 0, \quad f_{5,6} = 0.$$

The remaining equations are $f_{1,j} = 0$ and $f_{2,j} = 0$, with $3 \le j \le 6$. The assumption $m_1 = m_2$ and the symmetries imply that $f_{1,j} = 0$ if and only if $f_{2,j} = 0$. So we just study the equations $f_{1,j} = 0$. More

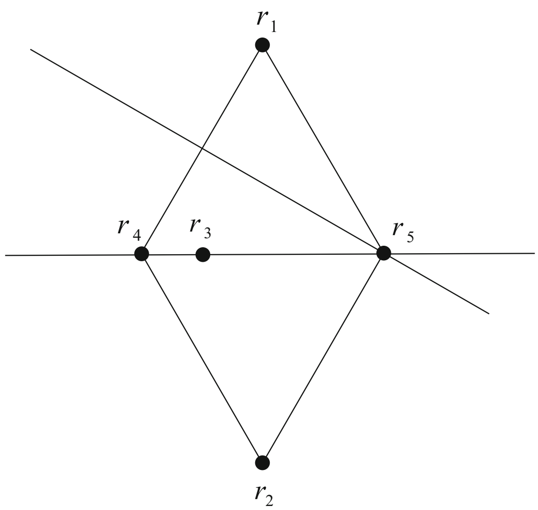

Fig. 5. This configuration cannot be a central configuration

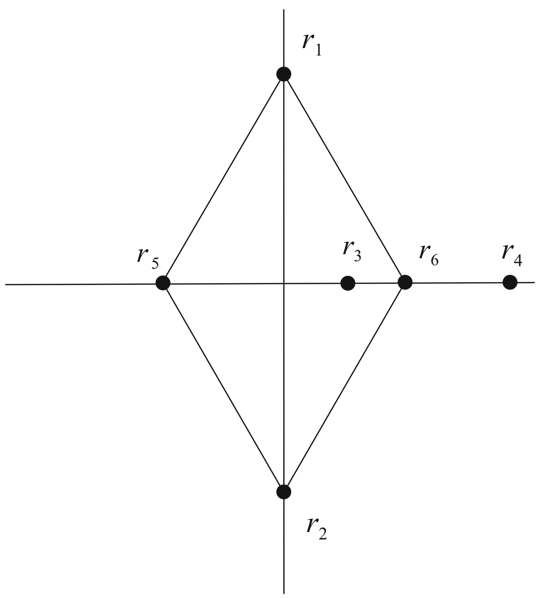

Fig. 6. This configuration cannot be a central configuration

explicitly, we study the following equations

$$\begin{aligned} f_{1,3} &= m_2 \left(R_{1,2} - R_{3,2}\right) \Delta_{1,3,2} + m_4 \left(R_{1,4} - R_{3,4}\right) \Delta_{1,3,4} \\ &\quad + m_5 \left(R_{1,5} - R_{3,5}\right) \Delta_{1,3,5} + m_6 \left(R_{1,6} - R_{3,6}\right) \Delta_{1,3,6} = 0, \\ f_{1,4} &= m_2 \left(R_{1,2} - R_{4,2}\right) \Delta_{1,4,2} + m_3 \left(R_{1,3} - R_{4,3}\right) \Delta_{1,4,3} \\ &\quad + m_5 \left(R_{1,5} - R_{4,5}\right) \Delta_{1,4,5} + m_6 \left(R_{1,6} - R_{4,6}\right) \Delta_{1,4,6} = 0, \end{aligned}$$

$$f_{1,5} = m_2 (R_{1,2} - R_{5,2}) \Delta_{1,5,2} + m_3 (R_{1,3} - R_{5,3}) \Delta_{1,5,3} + m_4 (R_{1,4} - R_{5,4}) \Delta_{1,5,4} + m_6 (R_{1,6} - R_{5,6}) \Delta_{1,5,6} = 0,$$

$$f_{1,6} = m_2 (R_{1,2} - R_{6,2}) \Delta_{1,6,2} + m_3 (R_{1,3} - R_{6,3}) \Delta_{1,6,3} + m_4 (R_{1,4} - R_{6,4}) \Delta_{1,6,4} + m_5 (R_{1,5} - R_{6,5}) \Delta_{1,6,5} = 0.$$

Equivalently, the above equations can be written as

$$H\vec{M} = \vec{0}, \tag{3.12}$$

where

$$\vec{M} = (m_2, m_3, m_4, m_5, m_6)^t, \qquad \vec{0} = (0, 0, 0, 0, 0)^t$$

and

$$H = \begin{bmatrix} h_{11} & 0 & h_{13} & h_{14} & h_{15} \\ h_{21} & h_{22} & 0 & h_{24} & h_{25} \\ h_{31} & h_{32} & h_{33} & 0 & 0 \\ h_{41} & h_{42} & h_{43} & 0 & 0 \end{bmatrix},$$

with

$$\begin{aligned} h_{11} &= (R_{1,2} - R_{3,2}) \Delta_{1,3,2}, & h_{13} &= (R_{1,4} - R_{3,4}) \Delta_{1,3,4}, \\ h_{14} &= (R_{1,5} - R_{3,5}) \Delta_{1,3,5}, & h_{15} &= (R_{1,6} - R_{3,6}) \Delta_{1,3,6}, \\ h_{21} &= (R_{1,2} - R_{4,2}) \Delta_{1,4,2}, & h_{22} &= (R_{1,3} - R_{4,3}) \Delta_{1,4,3}, \\ h_{24} &= (R_{1,5} - R_{4,5}) \Delta_{1,4,5}, & h_{25} &= (R_{1,6} - R_{4,6}) \Delta_{1,4,6}, \\ h_{31} &= (R_{1,2} - R_{5,2}) \Delta_{1,5,2}, & h_{32} &= (R_{1,3} - R_{5,3}) \Delta_{1,5,3}, \\ h_{33} &= (R_{1,4} - R_{5,4}) \Delta_{1,5,4}, & h_{41} &= (R_{1,2} - R_{6,2}) \Delta_{1,6,2}, \\ h_{42} &= (R_{1,3} - R_{6,3}) \Delta_{1,6,3}, & h_{43} &= (R_{1,4} - R_{6,4}) \Delta_{1,6,4}. \end{aligned}$$

Note that $h_{41} = -h_{31}$.

Equation (3.12) represents the intersection of four hyperplanes through the origin in the space $(m_2, m_3, m_4, m_5, m_6)$ and always admits non-trivial solutions, since we have more variables than equations. In the case that the matrix H has maximum rank, the solutions of (3.12) are parallel to the vector $\vec{T} = (T_1, -T_2, T_3, -T_4, T_5)$, where T_k is the determinant of the matrix obtained from H deleting the column k. In the problem of computing central configurations, only the positive solutions are considered. Even more, for the problem of this section, the position vectors must also satisfy other relations such as Eq. (3.10).

Consider again Eq. (3.10) in the context of six bodies

$$\frac{(R_{1,6} - R_{3,6}) \Delta_{1,3,6}}{(R_{1,5} - R_{3,5}) \Delta_{1,3,5}} = \frac{(R_{1,6} - R_{4,6}) \Delta_{1,4,6}}{(R_{1,5} - R_{4,5}) \Delta_{1,4,5}} = -a < 0.$$

Using this relation in the matrix H, we have

$$H = \begin{bmatrix} h_{11} & 0 & h_{13} & -ah_{15} & h_{15} \\ h_{21} & h_{22} & 0 & -ah_{25} & h_{25} \\ h_{31} & h_{32} & h_{33} & 0 & 0 \\ -h_{31} & h_{42} & h_{43} & 0 & 0 \end{bmatrix},$$

which implies that $\vec{T} = (T_1, -T_2, T_3, -T_4, T_5) = (0, 0, 0, aT_5, T_5)$. But in this case, it is necessary that the masses m_2, m_3 and m_4 vanish, assuming that the rank of H is equal to four.

Now we show that the rank of matrix H is in fact four. If the first and second rows are linearly dependent, the following equations must be satisfied

$$h_{13} = (R_{1,4} - R_{3,4})\Delta_{1,3,4} = 0 \tag{3.13}$$

and

$$h_{22} = (R_{1,3} - R_{4,3})\Delta_{1,4,3} = 0. \tag{3.14}$$

But without collisions, Eqs. (3.13) and (3.14) are not satisfied simultaneously.
If the third and fourth rows are linearly dependent, since $h_{41} = -h_{31}$, we must have $h_{32} = -h_{42}$ and $h_{33} = -h_{43}$, or more explicitly

$$(R_{1,3} - R_{5,3})\Delta_{1,5,3} = -(R_{1,4} - R_{5,4})\Delta_{1,5,4} \tag{3.15}$$

and

$$(R_{1,3} - R_{6,3})\Delta_{1,6,3} = -(R_{1,4} - R_{6,4})\Delta_{1,6,4} \tag{3.16}$$

But without collisions, neither Eqs. (3.15) nor (3.16) are satisfied.
Thus, if the matrix H has rank two, the third and fourth rows can be combined linearly to write the first and second rows. But it is not possible, because in this case, we must have $h_{14} = 0$, $h_{15} = 0$, $h_{24} = 0$ and $h_{25} = 0$. Such equations are never satisfied simultaneously. See the above definitions of h_{ij}.
Now suppose that the matrix H has rank three, so the dimension of the kernel of H is equal to one. Assume that the vector

$$\vec{M}_1 = (m_2, m_3, m_4, m_5, m_6)^t$$

is a solution of (3.12). Thus, after the change of m_5 to M_5 and m_6 to M_6 the vector

$$\vec{M}_2 = (m_2, m_3, m_4, M_5, M_6)^t$$

also is a solution of (3.12). Since the dimension of the kernel of H is one, we have $\vec{M}_1 = \alpha\vec{M}_2$. But in this case with the same m_2, m_3 and m_4, we have $\alpha = 1$, which implies that the masses m_5 and m_6 cannot change. So, in order to have a central configuration with six bodies satisfying our assumptions, the rank of H must be four. This completes the proof of the lemma. □

Our last case is $n = 7$. We have the following lemma.

Lemma 3.8. *Consider the planar 7-body problem. Suppose that r_1, r_2, r_6 and r_7 form a rhombus with $r_{16} = r_{17} = r_{26} = r_{27} = r_{67}$, the position vectors r_3, r_4 and r_5 belong to the straight line containing r_6 and r_7 and $m_1 = m_2$. Then, there are no positive masses for which this configuration form a central configuration satisfying: it is possible to change the values of m_6 and m_7 keeping fixed all the positions and other five masses and still have a central configuration.*

Proof. From Lemma 3.3, we consider just the case when the position vectors r_3 and r_4 are in the interior of the union of the sets bounded by the circles C_1 and C_2.
For seven bodies, we have 21 Andoyer Eq. (1.3). By our assumptions of symmetries, the following equations are already verified

$$f_{1,2} = 0, \quad f_{3,4} = 0, \quad f_{3,5} = 0, \quad f_{3,6} = 0,$$
$$f_{3,7} = 0, \quad f_{4,5} = 0, \quad f_{4,6} = 0, \quad f_{4,7} = 0.$$

The remaining equations are $f_{1,j} = 0$ and $f_{2,j} = 0$, with $3 \leq j \leq 7$. The assumption $m_1 = m_2$ and the symmetries imply that $f_{1,j} = 0$ if and only if $f_{2,j} = 0$. So we just study the equations $f_{1,j} = 0$. Similar to the proof of the previous lemma, equations can be written as

$$G\vec{M} = \vec{0}, \tag{3.17}$$

where,

$$\vec{M} = (m_2, m_3, m_4, m_5, m_6, m_7)^t, \qquad \vec{0} = (0, 0, 0, 0, 0, 0)^t$$

and

$$G = \begin{bmatrix} g_{11} & 0 & g_{13} & g_{14} & g_{15} & g_{16} \\ g_{21} & g_{22} & 0 & g_{24} & g_{25} & g_{26} \\ g_{31} & g_{32} & g_{33} & 0 & g_{35} & g_{36} \\ g_{41} & g_{42} & g_{43} & g_{44} & 0 & 0 \\ g_{51} & g_{52} & g_{53} & g_{54} & 0 & 0 \end{bmatrix},$$

with

$$\begin{aligned} g_{11} &= (R_{1,2} - R_{3,2})\,\Delta_{1,3,2}, & g_{13} &= (R_{1,4} - R_{3,4})\,\Delta_{1,3,4}, \\ g_{14} &= (R_{1,5} - R_{3,5})\,\Delta_{1,3,5}, & g_{15} &= (R_{1,6} - R_{3,6})\,\Delta_{1,3,6}, \\ g_{16} &= (R_{1,7} - R_{3,7})\,\Delta_{1,3,7}, & g_{21} &= (R_{1,2} - R_{4,2})\,\Delta_{1,4,2}, \\ g_{22} &= (R_{1,3} - R_{4,3})\,\Delta_{1,4,3}, & g_{24} &= (R_{1,5} - R_{4,5})\,\Delta_{1,4,5}, \\ g_{25} &= (R_{1,6} - R_{4,6})\,\Delta_{1,4,6}, & g_{26} &= (R_{1,7} - R_{4,7})\,\Delta_{1,4,7}, \\ g_{31} &= (R_{1,2} - R_{5,2})\,\Delta_{1,5,2}, & g_{32} &= (R_{1,3} - R_{5,3})\,\Delta_{1,5,3}, \\ g_{33} &= (R_{1,4} - R_{5,4})\,\Delta_{1,5,4}, & g_{35} &= (R_{1,6} - R_{5,6})\,\Delta_{1,5,6}, \\ g_{36} &= (R_{1,7} - R_{5,7})\,\Delta_{1,5,7}, & g_{41} &= (R_{1,2} - R_{6,2})\,\Delta_{1,6,2}, \\ g_{42} &= (R_{1,3} - R_{6,3})\,\Delta_{1,6,3}, & g_{43} &= (R_{1,4} - R_{6,4})\,\Delta_{1,6,4}, \\ g_{44} &= (R_{1,5} - R_{6,5})\,\Delta_{1,6,5}, & g_{51} &= (R_{1,2} - R_{7,2})\,\Delta_{1,7,2}, \\ g_{52} &= (R_{1,3} - R_{7,3})\,\Delta_{1,7,3}, & g_{53} &= (R_{1,4} - R_{7,4})\,\Delta_{1,7,4}, \\ g_{54} &= (R_{1,5} - R_{7,5})\,\Delta_{1,7,5}. \end{aligned}$$

Equation (3.17) represents the intersection of five hyperplanes through the origin in the space $(m_2, m_3, m_4, m_5, m_6, m_7)$ and always admits non-trivial solutions, since we have more variables than equations. In the case that the matrix G has maximum rank, the solutions of (3.17) are parallel to the vector $\vec{T} = (T_1, -T_2, T_3, -T_4, T_5, -T_6)$, where T_k is the determinant of the matrix obtained from G deleting the column k. Again using the Eq. (3.10), we get

$$G = \begin{bmatrix} g_{11} & 0 & g_{13} & g_{14} & -ag_{16} & g_{16} \\ g_{21} & g_{23} & 0 & g_{24} & -ag_{26} & g_{26} \\ g_{31} & g_{32} & g_{34} & 0 & -ag_{36} & g_{36} \\ g_{41} & g_{42} & g_{43} & g_{44} & 0 & 0 \\ g_{51} & g_{52} & g_{53} & g_{54} & 0 & 0 \end{bmatrix},$$

which implies that $\vec{T} = (T_1, -T_2, T_3, -T_4, T_5, -T_6) = (0, 0, 0, 0, aT_6, -T_6)$. But in this case, it is necessary that the masses m_2, m_3, m_4 and m_5 vanish, assuming that the rank of G is equal to five.
The proof of G has rank five which comes from the fact of H to be a sub-matrix of G, so the rank of G is at least four. But in this case, the same argument at the end of the proof of Lemma 3.7 can be used; thus, G has rank five. □

4. Concluding remarks

From the proofs of Theorems 1.3 and 1.5, it is possible to see that the variation of the values of some masses implies several restrictions to the geometry of a central configuration.

The only central configuration of n bodies that is not rigid with respect to the changing of the value of one mass is formed by $n - 1$ bodies in a co-circular central configuration and one body of arbitrary mass at the center of the circle. See Theorem 1.3.

The main result of Sect. 3 implies that all central configurations are rigid for the change of the values of two masses. In particular, we have the following corollary associated with super central configurations.

Corollary 4.1. *Consider the planar non-collinear n-body problem, $n \geq 4$. Then, there are no super central configurations that can be obtained permuting two different values of masses.*

The study of the change of the values of three or more masses can be applied for a possible classification and knowledge of super central configurations, for example. However, the extension of the results obtained here to the case of the change of the values of three or more masses is still an open question and requires a different kind of analysis. The approaches in the proof of Theorem 1.5 are more complicated than the ones in the proof of Theorem 1.3.

Acknowledgments

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