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Asymptotic limits of Navier–Stokes equations with quantum effects

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Abstract. This paper is concerned with the combined incompressible limit and semiclassical limit of the weak solutions to the barotropic quantum Navier–Stokes equations of compressible flows. By using the relative entropy method, we show that for well-prepared initial data, the weak solutions of the compressible quantum Navier–Stokes model converge to the strong solution of the incompressible Navier–Stokes equations as long as the latter exists. Furthermore, the convergence rates are also obtained.

Mathematics Subject Classification. Primary 35B25 · Secondary 76Y05.

Keywords. Incompressible limit · Semiclassical limit · Relative entropy method · Quantum Navier–Stokes equations · Incompressible Navier–Stokes equations.

1. Introduction

This paper is concerned with the following non-dimensional quantum Navier–Stokes equations in a three-dimensional torus $\mathbb{T}^3$:

$$\partial_t n + \mathrm{div}(nu) = 0, \quad x \in \mathbb{T}^3, \quad t > 0, \tag{1.1}$$

$$\partial_t(nu) + \mathrm{div}(nu \otimes u) + \nabla p(n) - 2\hbar^2 n \nabla \left(\frac{\Delta\sqrt{n}}{\sqrt{n}} \right) = 2\mathrm{div}(\mu(n)D(u)) - \alpha u, \tag{1.2}$$

$$n(x,0) = n_0, \quad u(x,0) = u_0, \tag{1.3}$$

where n and $u = (u^1, u^2, u^3)$ stand for the density and velocity, respectively. $p(n)$ is the pressure, and in this paper, we consider the case of isentropic flows with $p(n) = n^\gamma/\gamma$ for $\gamma > 1$. In (1.2), $D(u) = (\nabla u + \nabla u^T)/2$. And $\mu(n)$ denotes the density-dependent viscosity. In this paper, we assume that $\mu(n) = \mu n$ for some constant $\mu > 0$. $\kappa > 0$ is the scaled Planck constant. $2\kappa^2 n\nabla\left(\Delta\sqrt{n}/\sqrt{n}\right)$ can be interpreted as the quantum Bohm potential term or as a quantum correction to the pressure. Moreover, the following relation holds

$$2n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}} \right) = \nabla\Delta n - 4\mathrm{div}(\nabla\sqrt{n} \otimes \nabla\sqrt{n}). \tag{1.4}$$

which can avoid using too high regularities of the density n^λ. The constant $\alpha > 0$ is the damping coefficient which can guarantee to prove the existence of global weak solutions of the systems (1.1)–(1.3) as in [3]. Brull and Méhats [5] utilized a moment method and a Chapman–Enskog expansion around the quantum equilibrium to derive (1.1)–(1.2) from a Wigner equation.

Our main aim in the present paper is to rigorously prove a combined incompressible and semiclassical limit in the framework of the global weak solutions to (1.1)–(1.3). To begin with, we introduce the scaling

$$t \mapsto \epsilon t, \quad u \mapsto \epsilon u, \quad \mu(n) \mapsto \epsilon\mu(n), \quad \alpha \mapsto \epsilon\alpha,$$

and set $\hbar = \epsilon^\theta$ for $0 < \theta, \epsilon < 1$. With such scalings, the quantum Navier–Stokes equations (1.1)–(1.3) read as

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$$\partial_t n^\epsilon + \mathrm{div}(n^\epsilon u^\epsilon) = 0, \tag{1.5}$$

$$\partial_t(n^\epsilon u^\epsilon) + \mathrm{div}(n^\epsilon u^\epsilon \otimes u^\epsilon) + \frac{1}{\epsilon^2\gamma}\nabla(n^\epsilon)^\gamma - \frac{2}{\epsilon^{2(1-\theta)}} n^\epsilon \nabla\left(\frac{\Delta\sqrt{n^\epsilon}}{\sqrt{n^\epsilon}}\right) = 2\mu\mathrm{div}(n^\epsilon D(u^\epsilon)) - \alpha u^\epsilon, \tag{1.6}$$

with the initial conditions

$$n^\epsilon(\cdot,0) = n_0^\epsilon(x), \quad u^\epsilon(\cdot,0) = u_0^\epsilon(x). \tag{1.7}$$

Here, we use the superscript to emphasize the dependence of ϵ for each variables in (1.5) and (1.6).

When letting $\epsilon \to 0$, we formally obtain from the momentum equation (1.6) that n^ϵ converges to some function $n(t) > 0$. If we further assume that the initial datum n_0^ϵ is of order $1 + o(\epsilon)$ [see (2.2) below], then one can expect that $n(t) \equiv 1$. Thus, the continuity equation (1.5) yields to the limit $\mathrm{div}u = 0$, which is the incompressible condition of a fluid. Hence, we formally obtain the following incompressible Navier–Stokes equations with damping

$$\partial_t u + (u\cdot\nabla)u + \nabla\pi = 2\mu\mathrm{div}(D(u)) - \alpha u, \quad \mathrm{div}u = 0, \tag{1.8}$$

$$u(\cdot,0) = u_0. \tag{1.9}$$

In the present paper, we shall apply the method of relative entropy (or the modulated energy) to study the combined incompressible and semiclassical limit ($\epsilon \to 0$) of weak solutions to the system (1.5)–(1.6). Therefore, we shall prove the limit on any time intervals on which the incompressible Navier–Stokes equations possess a regular solution.

Define the energy of (1.5)–(1.7) by the sum of the kinetic, internal and quantum energy:

$$E^\epsilon(t) = \int_{\mathbb{T}^3} \left\{\frac{1}{2}n^\epsilon|u^\epsilon|^2 + \frac{1}{\epsilon^2\gamma(\gamma-1)}(n^\epsilon)^\gamma + \frac{2}{\epsilon^{2(1-\theta)}}|\nabla\sqrt{n^\epsilon}|^2\right\}\mathrm{d}x. \tag{1.10}$$

A formal computation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}E^\epsilon(t) + \int_{\mathbb{T}^3}(2\mu n^\epsilon|D(u^\epsilon)|^2 + \alpha|u^\epsilon|^2)\mathrm{d}x \le 0. \tag{1.11}$$

The initial data are taken in such a way as

$$n_0^\epsilon \in L^\gamma(\mathbb{T}^3),\ \frac{|m_0^\epsilon|^2}{n_0^\epsilon} \in L^1(\mathbb{T}^3),\ \nabla\sqrt{n_0^\epsilon} \in L^2(\mathbb{T}^3),\ \nabla\Phi_0^\epsilon \in L^2(\mathbb{T}^3),\ -\ln_- n_0^\epsilon \in L^1(\mathbb{T}^3), \tag{1.12}$$

where $m_0^\epsilon = 0$ when $n_0^\epsilon = 0$ and $\ln_- f = \ln\min\{f,1\}$.

The quantum Navier–Stokes system is hyperbolic–parabolic coupled; the presence of the third-order derivative and the viscosity depending on density in the momentum equation give rise to more difficulties than the common Navier–Stokes equations in the mathematical analysis. The existence of global-in-time weak solutions to the one-dimensional viscous quantum hydrodynamic equations was first proved by Gamba and Jüngel in [6]. The global existence of weak solutions for to the multidimensional quantum Navier–Stokes equations (1.1)–(1.3) was obtained in the non-classical sense of weak solutions (multiplying the momentum equation by the density)(see [9] for details). When the damping α is a positive constant, the velocity u^ϵ makes sense by itself independently of the density n^ϵ since u^ϵ belongs to $L^2([0,T];L^2(\mathbb{T}^3))$ so that existence of global weak solutions can be obtained in the classical sense of weak solutions (see [3] for details). The main idea of proof was based on the new entropy estimates which have been used in [2,3] by means of the so-called effective velocity $u^\epsilon + \mu\nabla\ln n^\epsilon$ for viscous Korteweg-type and shallow-water equations. More precisely, multiplying the momentum equations (1.6) by $\nabla\ln n^\epsilon$ provides an additional energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{E}^{\epsilon}(t) + \mu\int\limits_{\mathbb{T}^3}\Big((n^{\epsilon})^{\gamma-2}|\nabla n^{\epsilon}|^2 + \frac{2}{\epsilon^{2(1-\theta)}}n^{\epsilon}|\nabla^2 \ln n^{\epsilon}|^2 + \alpha|u^{\lambda}|^2\Big)\mathrm{d}x \le 0, \tag{1.13}$$

with

$$\overline{E}^{\epsilon}(t) = \int\limits_{\mathbb{T}^3}\left\{\frac{1}{2}n^{\epsilon}|u^{\epsilon}+\mu\nabla\ln n^{\epsilon}|^2 + \frac{1}{\epsilon^2\gamma(\gamma-1)}(n^{\epsilon})^{\gamma} + \frac{2}{\epsilon^{2(1-\theta)}}|\nabla\sqrt{n^{\epsilon}}|^2 - \alpha\mu\ln n^{\epsilon}\right\}\mathrm{d}x. \tag{1.14}$$

which can guarantee the global existence of weak solutions to the quantum Navier–Stokes model (1.5)–(1.6) as in [3]. We state the existence result of global weak solutions to the problem (1.5)–(1.6) in the following. The details of the proof are omitted here.

Theorem 1.1. *Let $T > 0$ and $\gamma > 1$. Assume that the initial data $(n_0^{\epsilon}, u_0^{\epsilon})$ are taken in such a way that (1.12) holds. Then there exists a weak solution $(n^{\epsilon}, u^{\epsilon})$ to quantum Navier–Stokes equations (1.5)–(1.7) with the regularity*

$$\begin{aligned}
&\sqrt{n^{\epsilon}} \in L^{\infty}([0,T]; H^1(\mathbb{T}^3)) \cap L^2([0,T]; H^2(\mathbb{T}^3)),\\
&n^{\epsilon} \in L^{\infty}([0,T]; L^{\gamma}(\mathbb{T}^3)),\\
&\sqrt{n^{\epsilon}}u^{\epsilon} \in L^{\infty}([0,T]; L^2(\mathbb{T}^3)),\\
&u^{\epsilon} \in L^2([0,T]; L^2(\mathbb{T}^3)), \sqrt{n^{\epsilon}}|D(u^{\epsilon})| \in L^2([0,T]; L^2(\mathbb{T}^3)),
\end{aligned}$$

and (1.5) holds in the sense of distributions, and for all smooth test functions satisfying v, compactly supported in $\mathbb{T}^3 \times [0,T)$, one has

$$\begin{aligned}
&\int\limits_{\mathbb{T}^3}(n^{\epsilon}u^{\epsilon}\cdot v)(t=0)\mathrm{d}x + \int\limits_0^T\int\limits_{\mathbb{T}^3}\Big(n^{\epsilon}u^{\epsilon}\cdot\partial_s v + n^{\epsilon}u^{\epsilon}\otimes u^{\epsilon} : \nabla v + p(n^{\epsilon})\mathrm{div}v + \frac{1}{\epsilon^{2(1-\alpha)}}n^{\epsilon}\Delta\mathrm{div}v\\
&\quad + \frac{4}{\epsilon^{2(1-\alpha)}}\hbar^2(\nabla\sqrt{n^{\epsilon}}\otimes\nabla\sqrt{n^{\epsilon}}) : \nabla v - 2\mu n^{\epsilon}D(u^{\epsilon}) : \nabla v - \alpha u^{\epsilon}\cdot v\Big)\mathrm{d}x\mathrm{d}s = 0.
\end{aligned} \tag{1.15}$$

There are a lot of studies on the rigorous verification of the incompressible limit of the compressible fluids without quantum effects, such as Klainerman and Majda [11], Beirao da Veiga [4], Isozaki [7], Schochet [16], Ukai [17], Lions and Masmoudi [13], Masmoudi [14], Alazard [1] and Jiang and Ou [8]. In particular, for the isentropic Navier–Stokes equations, Masmoudi [14] proved the inviscid and incompressible limit for the weak solutions by using the relative entropy method for the general initial data. The present paper will extend the results in [14] to the multidimensional barotropic quantum Navier–Stokes equations. However, being different from the system analyzed in [14], the third-order derivative term appears in the momentum equations. In addition, the viscosity coefficient we consider here is dependent on the density. Therefore, new techniques and ideas are introduced to treat them. To our knowledge, it is the first work on the combined incompressible limit and the semiclassical limit of weak solutions to the quantum Navier–Stokes equations.

In this present paper, we denote by χ the characteristics function and C the generic positive constants independent of ϵ.

The rest of this paper is organized as follows. In the next section, we state some useful known results, our main result and the main idea of the proof. Finally, Sect. 3 is devoted to the proof of our main result.

2. Main Result

Before stating our main results, we first recall the following classical results on the existence of regular solutions of the incompressible Navier–Stokes equations.

Proposition 2.1. (Ref.[10,15]) *Assume that* $u_0 \in H^s$, $s > 3/2 + 3$ *and* $\mathrm{div} u_0 = 0$. *Then there exist* $0 < T_* < \infty$, *the maximal existence time and a unique smooth solution* (u, π) *of the incompressible Navier–Stokes equations* (1.8)–(1.9) *on* $[0, T_*)$ *satisfying that* $\int_{\mathbb{T}^3} \pi(x,t)\mathrm{d}x = 0$, *and for any* $T_0 < T_*$,

$$\sup_{0\le t\le T_0} \Big(\|u\|_{H^s(\mathbb{T}^3)} + \|\partial_t u\|_{H^{s-2}(\mathbb{T}^3)} + \|\nabla \pi\|_{H^{s-2}(\mathbb{T}^3)} + \|\partial_t \pi\|_{H^{s-3}(\mathbb{T}^3)} \Big) \le C(T) \tag{2.1}$$

for some positive constant $C(T)$.

The main result of this paper can be stated as follows.

Theorem 2.2. *Let* $\theta \in (0,1), \gamma > 1$. *Assume that* $\{(n^\epsilon, u^\epsilon)\}_{\epsilon>0}$ *is a sequence of weak solutions to the compressible quantum Navier–Stokes equations* (1.5)–(1.7) *obtained in Theorem 1.1, satisfying the conditions:*

$$\frac{1}{\epsilon^2} \int_{\mathbb{T}^3} \left(|n_0^\epsilon - 1|^2 \chi_{(|n_0^\epsilon - 1| \le \delta)} + |n_0^\epsilon - 1|^\gamma \chi_{(|n_0^\epsilon - 1| > \delta)} \right) \mathrm{d}x \le C\epsilon, \tag{2.2}$$

$$\|\sqrt{n_0^\epsilon} u_0^\epsilon - u_0\|_{L^2(\mathbb{T}^3)}^2 \le C\epsilon, \tag{2.3}$$

$$\frac{1}{\epsilon^{2(1-\theta)}} \|\nabla \sqrt{n_0^\epsilon}\|_{L^2(\mathbb{T}^3)} \le C\epsilon \tag{2.4}$$

for any $\delta \in (0,1)$. *Also assume that* (u, π) *is the smooth solution to incompressible Navier–Stokes equations* (1.8)–(1.9) *satisfying the condition* (2.1). *For any* $T < T_*$, *we have that*

$$\frac{1}{\epsilon^2} \int_{\mathbb{T}^3} \left(|n^\epsilon - 1|^2 \chi_{(|n^\epsilon - 1| \le \delta)} + |n^\epsilon - 1|^\gamma \chi_{(|n^\epsilon - 1| > \delta)} \right) \mathrm{d}x \le C\epsilon^\beta, \tag{2.5}$$

$$\|\sqrt{n^\epsilon} u^\epsilon - u\|_{L^\infty([0,T];L^2(\mathbb{T}^3))}^2 \le C\epsilon^\beta, \tag{2.6}$$

$$\|n^\epsilon u^\epsilon - u\|_{L^\infty([0,T];L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3))}^2 \le C\epsilon^\beta, \tag{2.7}$$

where $\beta = \min\{1-\theta, \frac{2}{\gamma-1}\}$.

Remark 2.3. Theorem 2.2 describes the asymptotic limit of the quantum Navier–Stokes system (1.5)–(1.7) with well-prepared initial data. For the general initial data, the fast singular oscillation appears. It is more difficult to prove the asymptotic limit in this situation, which will be studied in a forthcoming paper.

Remark 2.4. We remark that the estimate in Theorem 2.2 is uniform with respect to μ. Therefore, Theorem 2.2 is a stability result not only with respect to ϵ but also with respect to μ. In fact, we can show that the combined incompressible limit, semiclassical limit and vanishing viscosity limit of the quantum Navier–Stokes system (1.5)–(1.6) are the incompressible Euler equation.

In the next section, we are going to prove Theorem 2.2. To this end, we introduce the following form of the modulated energy:

$$\mathcal{H}^\epsilon(t) = \int_{\mathbb{T}^3} \left\{ \frac{1}{2} n^\epsilon |u^\epsilon - u|^2 + \frac{1}{\epsilon^2} h(n^\epsilon) + \frac{2}{\epsilon^{2(1-\alpha)}} |\nabla \sqrt{n^\epsilon}|^2 \right\} \mathrm{d}x. \tag{2.8}$$

where u is the smooth solution of the incompressible Navier–Stokes equations (1.8)–(1.9) and

$$h(n^\epsilon) = \frac{1}{\gamma(\gamma-1)} \left((n^\epsilon)^\gamma - 1 - \gamma(n^\epsilon - 1) \right).$$

In fact, $h(n^\epsilon)$ stands for the *free energy per unit volume*. These terms in the right-hand side of equation (2.8) express the differences of the kinetic, internal and quantum energies. We shall employ the evolution equations and elaborated computations to prove the inequality

$$\mathcal{H}^\epsilon(t) \le C\int_0^t \mathcal{H}^\epsilon(s)\mathrm{d}s + \epsilon^\theta \tag{2.9}$$

for some positive constant $\theta > 0$. The Gronwall lemma then implies the result.

3. Proof of Theorem 2.2

From the energy inequality (1.11) and the conservation of mass, we have for almost all $t \in [0,T]$,

$$\begin{aligned}
&\int_{\mathbb{T}^3} \left\{ \frac{1}{2} n^\epsilon |u^\epsilon|^2 + \frac{1}{\epsilon^2} h(n^\epsilon) + \frac{2}{\epsilon^{2(1-\alpha)}} |\nabla \sqrt{n^\epsilon}|^2 \right\} \mathrm{d}x \\
&\quad + \int_0^t \int_{\mathbb{T}^3} (2\mu n^\epsilon |D(u^\epsilon)|^2 + \alpha |u^\epsilon|^2) \mathrm{d}x\mathrm{d}s \\
&\le \int_{\mathbb{T}^3} \left\{ \frac{1}{2} n_0^\epsilon |u_0^\epsilon|^2 + \frac{1}{\epsilon^2} h(n_0^\epsilon) + \frac{2}{\epsilon^{2(1-\alpha)}} |\nabla \sqrt{n_0^\epsilon}|^2 \right\} \mathrm{d}x \\
&\le C.
\end{aligned} \tag{3.1}$$

Therefore, we have the following properties:

$$\sqrt{n^\epsilon} u^\epsilon \quad \text{is bounded in } L^\infty([0,T]; L^2(\mathbb{T}^3)), \tag{3.2}$$

$$\frac{1}{\epsilon^2} h(n^\epsilon) \quad \text{is bounded in } L^\infty([0,T]; L^1(\mathbb{T}^3)), \tag{3.3}$$

$$\frac{2}{\epsilon^{2(1-\alpha)}} |\nabla \sqrt{n^\epsilon}|^2 \quad \text{is bounded in } L^\infty([0,T]; L^1(\mathbb{T}^3)). \tag{3.4}$$

Lemma 3.1. *Let (n^ϵ, u^ϵ) be the weak solution to quantum Navier–Stokes equations (1.5)–(1.7) on $[0,T]$. Then there exists a constant $C > 0$ such that for all $\epsilon \in (0,1)$ and $\gamma > 1$,*

$$\|n^\epsilon - 1\|_{L^\infty([0,T];L^\gamma(\mathbb{T}^3))} \le C\epsilon^{\frac{\lambda}{\gamma}} \quad \text{and} \quad \|n^\epsilon - 1\|_{L^\infty([0,T];L^\lambda(\mathbb{T}^3))} \le C\epsilon, \tag{3.5}$$

where $\lambda = \min\{2, \gamma\}$.

Proof. In view of Lemma 5.3 in [12], there exist two positive constants $c_1 \in (0,1)$ and $c_2 \in (1,+\infty)$ independent of n^ϵ such that the following inequality

$$\begin{aligned}
&c_1 \int_{\mathbb{T}^3} \left(|n^\epsilon - 1|^2 \chi_{(|n^\epsilon - 1| \le \delta)} + |n^\epsilon - 1|^\gamma \chi_{(|n^\epsilon - 1| > \delta)} \right) \mathrm{d}x \\
&\quad \le \int_{\mathbb{T}^3} h(n^\epsilon) \mathrm{d}x \le c_2 \int_{\mathbb{T}^3} \left(|n^\epsilon - 1|^2 \chi_{(|n^\epsilon - 1| \le \delta)} + |n^\epsilon - 1|^\gamma \chi_{(|n^\epsilon - 1| > \delta)} \right) \mathrm{d}x,
\end{aligned} \tag{3.6}$$

for any $\delta \in (0,1)$, which implies Lemma 3.1 holds due to (3.3). □

Lemma 3.2. *Let $T > 0, \gamma > 1$, and $0 < \alpha < 1$. Then*

$$\mathcal{H}^\epsilon(t) \le C\epsilon^\beta \tag{3.7}$$

uniformly in $[0,T]$, where $\beta = \min\{1-\theta, \frac{2}{\gamma-1}\}$.

Proof. To derive the integration inequality for $\mathcal{H}^\epsilon(t)$, we use u as a test function in the weak formulation of momentum equation (1.6) to yield the following equality for almost all t:

$$\begin{aligned}\int_{\mathbb{T}^3} n^\epsilon u^\epsilon \cdot u\mathrm{d}x &= \int_{\mathbb{T}^3} (n^\epsilon u^\epsilon \cdot u)(t=0)\mathrm{d}x + \int_0^t \int_{\mathbb{T}^3} n^\epsilon u^\epsilon \cdot \partial_s u\mathrm{d}x\mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{T}^3} (n^\epsilon u^\epsilon \otimes u^\epsilon) : \nabla u\mathrm{d}x\mathrm{d}s \\ &+ \frac{4}{\epsilon^{2(1-\theta)}} \int_0^t \int_{\mathbb{T}^3} (\nabla\sqrt{n^\epsilon} \otimes \nabla\sqrt{n^\epsilon}) : \nabla u\mathrm{d}x\mathrm{d}s \\ &- 2\mu \int_0^t \int_{\mathbb{T}^3} n^\epsilon D(u^\epsilon) : \nabla u\mathrm{d}x\mathrm{d}s - \alpha \int_0^t \int_{\mathbb{T}^3} u^\epsilon \cdot u\mathrm{d}x\mathrm{d}s, \end{aligned} \tag{3.8}$$

where we have used the fact that $\mathrm{div}u = 0$ and

$$n^\epsilon \nabla \left(\frac{\Delta\sqrt{n^\epsilon}}{\sqrt{n^\epsilon}} \right) = \frac{1}{2}\nabla\Delta n^\epsilon - 2\mathrm{div}(\nabla\sqrt{n^\epsilon} \otimes \nabla\sqrt{n^\epsilon}).$$

From (1.8)–(1.9), we have that the energy identity of the incompressible Navier–Stokes equations:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^3} |u|^2\mathrm{d}x + 2\mu \int_{\mathbb{T}^3} |D(u)|^2\mathrm{d}x = 0,$$

which implies that

$$\frac{1}{2}\int_{\mathbb{T}^3} |u_0|^2\mathrm{d}x = \frac{1}{2}\int_{\mathbb{T}^3} |u|^2\mathrm{d}x + 2\mu \int_0^t \int_{\mathbb{T}^3} |D(u)|^2\mathrm{d}x\mathrm{d}s. \tag{3.9}$$

Using (3.8)–(3.9) and the energy inequality (3.1), by integration by parts, we calculate $\mathcal{H}^\epsilon(t)$ as follows:

$$\begin{aligned} &\mathcal{H}^\epsilon(t) + 2\mu \int_0^t \int_{\mathbb{T}^3} n^\epsilon |D(u^\epsilon) - D(u)|^2\mathrm{d}x\mathrm{d}s + \alpha \int_0^t \int_{\mathbb{T}^3} |u^\epsilon - u|^2\mathrm{d}x\mathrm{d}s \\ &= \int_{\mathbb{T}^3} \left\{ \frac{1}{2} n^\epsilon |u^\epsilon|^2 + \frac{1}{\epsilon^2} h(n^\epsilon) + \frac{2}{\epsilon^{2(1-\theta)}} |\nabla\sqrt{n^\epsilon}|^2 \right\} \mathrm{d}x \\ &+ \int_0^t \int_{\mathbb{T}^3} (2\mu n^\epsilon |D(u)|^2 + \alpha |u^\epsilon|^2)\mathrm{d}x\mathrm{d}s - \int_{\mathbb{T}^3} (n^\epsilon u^\epsilon \cdot u)(t=0)\mathrm{d}x \\ &+ \frac{1}{2}\int_{\mathbb{T}^3} n^\epsilon |u|^2\mathrm{d}x - \alpha \int_0^t \int_{\mathbb{T}^3} u^\epsilon \cdot u\mathrm{d}x\mathrm{d}s + \alpha \int_0^t \int_{\mathbb{T}^3} |u|^2\mathrm{d}x\mathrm{d}s \\ &+ 2\mu \int_0^t \int_{\mathbb{T}^3} n^\epsilon |D(u)|^2\mathrm{d}x\mathrm{d}s - 2\mu \int_0^t \int_{\mathbb{T}^3} n^\epsilon D(u^\epsilon) : D(u)\mathrm{d}x\mathrm{d}s \\ &- \int_0^t \int_{\mathbb{T}^3} n^\epsilon u^\epsilon \cdot \partial_s u\mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} (n^\epsilon u^\epsilon \otimes u^\epsilon) : \nabla u\mathrm{d}x\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&- \frac{4}{\epsilon^{2(1-\alpha)}} \int_0^t \int_{\mathbb{T}^3} (\nabla\sqrt{n^\epsilon} \otimes \nabla\sqrt{n^\epsilon}) : \nabla u \mathrm{d}x\mathrm{d}s \\
\leq &\int_{\mathbb{T}^3} \left\{ \frac{1}{2} n_0^\epsilon |u_0^\epsilon|^2 + \frac{1}{\epsilon^2} h(n_0^\epsilon) + \frac{2}{\epsilon^{2(1-\theta)}} |\nabla\sqrt{n_0^\epsilon}|^2 \right\} \mathrm{d}x \\
&+ \frac{1}{2}\int_{\mathbb{T}^3} n_0^\epsilon |u_0|^2 \mathrm{d}x - \int_{\mathbb{T}^3} (n_0^\epsilon u_0^\epsilon \cdot u_0) \mathrm{d}x \\
&\underbrace{- \frac{1}{2}\int_{\mathbb{T}^3} |u_0|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3} |u|^2 \mathrm{d}x + 2\mu \int_0^t \int_{\mathbb{T}^3} |D(u)|^2 \mathrm{d}x\mathrm{d}s + \alpha \int_0^t \int_{\mathbb{T}^3} |u|^2 \mathrm{d}x\mathrm{d}s}_{=0 \text{ from } (3.9)} \\
&- \frac{1}{2}\int_{\mathbb{T}^3} (n_0^\epsilon - 1)|u_0|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3} (n^\epsilon - 1)|u|^2 \mathrm{d}x \\
&+ 2\mu \int_0^t \int_{\mathbb{T}^3} (n^\epsilon - 1)|D(u)|^2 \mathrm{d}x\mathrm{d}s - 2\mu \int_0^t \int_{\mathbb{T}^3} n^\epsilon D(u^\epsilon) : D(u) \mathrm{d}x\mathrm{d}s \\
&- \int_0^t \int_{\mathbb{T}^3} n^\epsilon u^\epsilon \cdot \partial_s u \mathrm{d}x\mathrm{d}s - \alpha \int_0^t \int_{\mathbb{T}^3} u^\epsilon \cdot u \mathrm{d}x\mathrm{d}s \\
&- \int_0^t \int_{\mathbb{T}^3} (n^\epsilon u^\epsilon \otimes u^\epsilon) : \nabla u \mathrm{d}x\mathrm{d}s - \frac{4}{\epsilon^{2(1-\theta)}} \int_0^t \int_{\mathbb{T}^3} (\nabla\sqrt{n^\epsilon} \otimes \nabla\sqrt{n^\epsilon}) : \nabla u \mathrm{d}x\mathrm{d}s \\
= &\mathcal{H}^\epsilon(0) - \frac{1}{2}\int_{\mathbb{T}^3} (n_0^\epsilon - 1)|u_0|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3} (n^\epsilon - 1)|u|^2 \mathrm{d}x + \sum_{k=1}^5 \mathcal{I}_k, \qquad (3.10)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{I}_1 &= 2\mu \int_0^t \int_{\mathbb{T}^3} (n^\epsilon - 1)|D(u)|^2 \mathrm{d}x\mathrm{d}s, \\
\mathcal{I}_2 &= - \int_0^t \int_{\mathbb{T}^3} n^\epsilon u^\epsilon \cdot \partial_s u \mathrm{d}x\mathrm{d}s - \alpha \int_0^t \int_{\mathbb{T}^3} u^\epsilon \cdot u \mathrm{d}x\mathrm{d}s, \\
\mathcal{I}_3 &= - \int_0^t \int_{\mathbb{T}^3} (n^\epsilon u^\epsilon \otimes u^\epsilon) : \nabla u \mathrm{d}x\mathrm{d}s, \\
\mathcal{I}_4 &= - \frac{4}{\epsilon^{2(1-\theta)}} \int_0^t \int_{\mathbb{T}^3} (\nabla\sqrt{n^\epsilon} \otimes \nabla\sqrt{n^\epsilon}) : \nabla u \mathrm{d}x\mathrm{d}s, \\
\mathcal{I}_5 &= -2\mu \int_0^t \int_{\mathbb{T}^3} n^\epsilon D(u^\epsilon) : D(u) \mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

Now, we begin to treat the integrals $\mathcal{I}_k(k=1,2,3,4,5)$ term by term. Using (2.1), Lemma 3.1 and Hölder inequality, we have that

$$\mathcal{I}_1 \leq C\|n^\epsilon - 1\|_{L^\infty([0,T];L^\lambda(\mathbb{T}^3))}\|D(u)\|^2_{L^\infty([0,T];L^{\frac{2\lambda}{\lambda-1}}(\mathbb{T}^3))} \leq C\epsilon. \tag{3.11}$$

From (1.8), we have

$$\begin{aligned}\mathcal{I}_2 = &\int_0^t\int_{\mathbb{T}^3} n^\epsilon u^\epsilon\cdot((u\cdot\nabla)u)\mathrm{d}x\mathrm{d}s - 2\mu\int_0^t\int_{\mathbb{T}^3} n^\epsilon u^\epsilon\cdot \mathrm{div}(D(u))\mathrm{d}x\mathrm{d}s\\ &+\int_0^t\int_{\mathbb{T}^3} n^\epsilon u^\epsilon\cdot\nabla\pi\mathrm{d}x\mathrm{d}s - \alpha\int_0^t\int_{\mathbb{T}^3}(n^\epsilon-1)u^\epsilon\cdot u\mathrm{d}x\mathrm{d}s.\end{aligned}$$

In fact, we only need to treat the last two terms on the right-hand side of (3.12) since the first two terms will be canceled later. Using Lemma 3.1, Young inequality, continuity equation (1.5) and integrating by parts, we get that

$$\begin{aligned}\int_0^t\int_{\mathbb{T}^3} n^\epsilon u^\epsilon\cdot\nabla\pi\mathrm{d}x\mathrm{d}s &= -\int_0^t\int_{\mathbb{T}^3}\mathrm{div}(n^\epsilon u^\epsilon)\pi\mathrm{d}x\mathrm{d}s = \int_0^t\int_{\mathbb{T}^3}\partial_s n^\epsilon\pi\mathrm{d}x\mathrm{d}s\\ &= \int_{\mathbb{T}^3}(n^\epsilon-1)\pi\mathrm{d}x - \int_{\mathbb{T}^3}(n_0^\epsilon-1)\pi_0\mathrm{d}x - \int_0^t\int_{\mathbb{T}^3}(n^\epsilon-1)\partial_s\pi\mathrm{d}x\mathrm{d}s\\ &\leq C\|n^\epsilon-1\|_{L^\infty([0,T];L^\lambda(\mathbb{T}^3))} - \int_{\mathbb{T}^3}(n_0^\epsilon-1)\pi_0\mathrm{d}x\\ &\leq C\epsilon + \frac{1}{\epsilon^2}\int_{\mathbb{T}^3}|n_0^\epsilon-1|^2\chi_{(|n_0^\epsilon-1|\leq\delta)}\mathrm{d}x + c\epsilon^2\\ &\quad+\frac{1}{\epsilon^2}\int_{\mathbb{T}^3}|n_0^\epsilon-1|^\gamma\chi_{(|n_0^\epsilon-1|>\delta)}\mathrm{d}x + c\epsilon^{\frac{2}{\gamma-1}}\\ &\leq C\epsilon^{\min\{1,\frac{2}{\gamma-1}\}},\end{aligned}$$

and

$$\begin{aligned}&-\alpha\int_0^t\int_{\mathbb{T}^3}(n^\epsilon-1)u^\epsilon\cdot u\mathrm{d}x\mathrm{d}s\\ &\leq C\|n^\epsilon-1\|_{L^\infty([0,T];L^2(\mathbb{T}^3))}\|u^\epsilon\|_{L^\infty([0,T];L^2(\mathbb{T}^3))}\\ &\leq C\epsilon.\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}\mathcal{I}_2 \leq &\int_0^t\int_{\mathbb{T}^3} n^\epsilon u^\epsilon\cdot((u\cdot\nabla)u)\mathrm{d}x\mathrm{d}s + C\epsilon^{\min\{1,\frac{2}{\gamma-1}\}}\\ &- 2\mu\int_0^t\int_{\mathbb{T}^3} n^\epsilon u^\epsilon\cdot\mathrm{div}(D(u))\mathrm{d}x\mathrm{d}s.\end{aligned}\tag{3.12}$$

In order to estimate $\mathcal{I}_3$, we express it as follows:

$$\mathcal{I}_3 = -\int_0^t \int_{\mathbb{T}^3} n^\epsilon (u^\epsilon - u) \otimes (u^\epsilon - u) : \nabla u \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} (n^\epsilon u \otimes u^\epsilon) : \nabla u \mathrm{d}x\mathrm{d}s$$
$$+ \int_0^t \int_{\mathbb{T}^3} (n^\epsilon u \otimes u) : \nabla u \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} (n^\epsilon u^\epsilon \otimes u) : \nabla u \mathrm{d}x\mathrm{d}s$$
$$\leq C \int_0^t \int_{\mathbb{T}^3} n^\epsilon |u^\epsilon - u|^2 \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{T}^3} n^\epsilon u^\epsilon \cdot ((u \cdot \nabla) u) \mathrm{d}x\mathrm{d}s + \mathcal{I}_{31} + \mathcal{I}_{32},$$

where

$$\mathcal{I}_{31} = \int_0^t \int_{\mathbb{T}^3} (n^\epsilon u \otimes u) : \nabla u \mathrm{d}x\mathrm{d}s$$

and

$$\mathcal{I}_{32} = -\int_0^t \int_{\mathbb{T}^3} (n^\epsilon u^\epsilon \otimes u) : \nabla u \mathrm{d}x\mathrm{d}s.$$

Notice that the second term on the right-hand side of the above inequality will be canceled by the first term on the right-hand side of (3.12). Similar to the estimate of $\mathcal{I}_1$, using Lemma 3.1 and the inequality (2.1), we get that

$$\mathcal{I}_{31} = \int_0^t \int_{\mathbb{T}^3} n^\epsilon (u \cdot \nabla) u \cdot u \mathrm{d}x\mathrm{d}s$$
$$= \int_0^t \int_{\mathbb{T}^3} (n^\epsilon - 1)(u \cdot \nabla) u \cdot u \mathrm{d}x\mathrm{d}s$$
$$\leq C\epsilon,$$

since

$$\int_0^t \int_{\mathbb{T}^3} (u \cdot \nabla) u \cdot u \mathrm{d}x = \frac{1}{2} \int_0^t \int_{\mathbb{T}^3} u \cdot \nabla |u|^2 \mathrm{d}x\mathrm{d}s = 0.$$

From the continuity equation (1.5) and the inequality (2.1), we get, by integration by parts, that

$$\mathcal{I}_{32} = -\frac{1}{2} \int_{\mathbb{T}^3} (n^\epsilon - 1)|u|^2 \mathrm{d}x + \frac{1}{2} \int_{\mathbb{T}^3} (n_0^\epsilon - 1)|u_0|^2 \mathrm{d}x + \frac{1}{2} \int_0^t \int_{\mathbb{T}^3} (n^\epsilon - 1) \partial_s |u|^2 \mathrm{d}x\mathrm{d}s$$
$$\leq -\frac{1}{2} \int_{\mathbb{T}^3} (n^\epsilon - 1)|u|^2 \mathrm{d}x + \frac{1}{2} \int_{\mathbb{T}^3} (n_0^\epsilon - 1)|u_0|^2 \mathrm{d}x$$

$$+ C\|n^\epsilon - 1\|_{L^\infty([0,T];L^\lambda(\mathbb{T}^3))}\|\partial_s|u|^2\|_{L^\infty([0,T];L^{\frac{\lambda}{\lambda-1}}(\mathbb{T}^3))}$$

$$\leq -\frac{1}{2}\int_{\mathbb{T}^3}(n^\epsilon - 1)|u|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3}(n_0^\epsilon - 1)|u_0|^2\mathrm{d}x + C\epsilon.$$

To justify the calculation in the above inequality, we need to use the definition of the weak solution to the compressible quantum Navier–Stokes system (1.5)–(1.7) as in Proposition 1.1. Therefore, we obtain that

$$\mathcal{I}_3 \leq C\int_0^t\int_{\mathbb{T}^3} n^\epsilon|u^\epsilon - u|^2\mathrm{d}x\mathrm{d}s - \int_0^t\int_{\mathbb{T}^3} n^\epsilon u^\epsilon\cdot((u\cdot\nabla)u)\mathrm{d}x\mathrm{d}s$$
$$-\frac{1}{2}\int_{\mathbb{T}^3}(n^\epsilon - 1)|u|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3}(n_0^\epsilon - 1)|u_0|^2\mathrm{d}x + C\epsilon. \tag{3.13}$$

For $\mathcal{I}_4$, one gets that

$$\mathcal{I}_4 \leq \frac{C\|u\|_{L^\infty((0,T)\times(\mathbb{T}^3)))}}{\epsilon^{2(1-\theta)}}\int_0^t\int_{\mathbb{T}^3}|\nabla\sqrt{n^\epsilon}|^2\mathrm{d}x\mathrm{d}s \leq C\int_0^t\mathcal{H}^\epsilon(s)\mathrm{d}s. \tag{3.14}$$

Now, we deal with the last term $\mathcal{I}_5$. To estimate it, we rewrite it as follows:

$$\mathcal{I}_5 = \mu\int_0^t\int_{\mathbb{T}^3} n^\epsilon u^\epsilon\cdot(\Delta u + \nabla\mathrm{div}u)\mathrm{d}x\mathrm{d}s$$
$$+ \mu\int_0^t\int_{\mathbb{T}^3}(\nabla n^\epsilon\otimes u^\epsilon + u^\epsilon\otimes\nabla n^\epsilon) : \nabla u\mathrm{d}x\mathrm{d}s$$
$$= 2\mu\int_0^t\int_{\mathbb{T}^3} n^\epsilon u^\epsilon\cdot\mathrm{div}(D(u))\mathrm{d}x\mathrm{d}s$$
$$+ 2\mu\int_0^t\int_{\mathbb{T}^3}(\nabla\sqrt{n^\epsilon}\otimes\sqrt{n^\epsilon}u^\epsilon + \sqrt{n^\epsilon}u^\epsilon\otimes\nabla\sqrt{n^\epsilon}) : \nabla u\mathrm{d}x\mathrm{d}s,$$

where we have used the equality $2\mathrm{div}(D(u)) = \Delta u$ with $\mathrm{div}u = 0$. Applying the Cauchy–Schwarz inequality and the properties (3.2) and (3.4), the last integral is bounded by

$$C\|\nabla\sqrt{n^\epsilon}\|_{L^\infty([0,T];L^2(\mathbb{T}^3))}\|\sqrt{n^\epsilon}u^\epsilon\|_{L^\infty([0,T];L^2(\mathbb{T}^3))} \leq C\epsilon^{1-\theta}.$$

Therefore, we obtain that

$$\mathcal{I}_5 \leq 2\mu\int_0^t\int_{\mathbb{T}^3} n^\epsilon u^\epsilon\cdot\mathrm{div}(D(u))\mathrm{d}x\mathrm{d}s + C\epsilon^{1-\theta}. \tag{3.15}$$

Inserting (3.11)–(3.15) into (3.10), we get

$$\mathcal{H}^\epsilon(t) \leq \mathcal{H}^\epsilon(t=0) + C\int_0^t\mathcal{H}^\epsilon(s)\mathrm{d}s + C\epsilon^\beta.$$

Using the initial conditions (2.2)–(2.4) and the inequality (3.6), we have that $\mathcal{H}^\epsilon(0) \le C\epsilon$ since

$$\begin{aligned}\int_{\mathbb{T}^3} n_0^\epsilon |u_0^\epsilon - u_0|^2 \mathrm{d}x &\le 2\int_{\mathbb{T}^3} |\sqrt{n_0^\epsilon} u_0^\epsilon - u_0|^2 \mathrm{d}x + 2\int_{\mathbb{T}^3} |(1-\sqrt{n_0^\epsilon})u_0|^2 \mathrm{d}x \\ &\le 2\int_{\mathbb{T}^3} |\sqrt{n_0^\epsilon} u_0^\epsilon - u_0|^2_{L^2(\mathbb{T}^3)} + C\int_{\mathbb{T}^3} |(1-\sqrt{n_0^\epsilon})|^2 \mathrm{d}x \\ &\le 2\int_{\mathbb{T}^3} |\sqrt{n_0^\epsilon} u_0^\epsilon - u_0|^2_{L^2(\mathbb{T}^3)} \\ &\quad + C\int_{\mathbb{T}^3} \left(|n_0^\epsilon - 1|^2 \chi_{(|n_0^\epsilon - 1| \le \delta)} + |n_0^\epsilon - 1|^\gamma \chi_{(|n_0^\epsilon - 1| > \delta)}\right) \mathrm{d}x \\ &\le C\epsilon.\end{aligned}$$

Here, we have used the following elementary inequality:

$$|1-\sqrt{x}|^2 \le C|1-x|^k, \quad \forall k \ge 1, \tag{3.16}$$

for some positive constant C and any $x \ge 0$. Thus the proof of Lemma 3.2 is completed. □

We are now in the position to prove Theorem 2.2. From Lemma 3.2 and inequality (3.6), we claim that the estimate (2.5) holds. Using Lemma 3.2, the inequality (3.16) and the Hölder inequality, we have that

$$\begin{aligned}\|\sqrt{n^\epsilon} u^\epsilon - u\|^2_{L^2(\mathbb{T}^3)} &\le 2\|\sqrt{n^\epsilon}(u^\epsilon - u)\|^2_{L^2(\mathbb{T}^3)} + 2\|(1-\sqrt{n^\epsilon})u\|^2_{L^2(\mathbb{T}^3)} \\ &\le C\epsilon^\beta + C\|1-\sqrt{n^\epsilon}\|^2_{L^2(\mathbb{T}^3)} \\ &\le C\epsilon^\beta + C\int_{\mathbb{T}^3} \left(|n^\epsilon - 1|^2 \chi_{(|n^\epsilon - 1| \le \delta)} + |n^\epsilon - 1|^\gamma \chi_{(|n^\epsilon - 1| > \delta)}\right) \mathrm{d}x \\ &\le C\epsilon^\beta + C\epsilon^2 \int_{\mathbb{T}^3} h(n^\epsilon) \mathrm{d}x \\ &\le C\epsilon^\beta,\end{aligned}$$

for any $t \in [0, T]$. Therefore, we conclude that (2.6) holds. Using the Hölder inequality and the fact that $1 < \frac{2\gamma}{\gamma+1} < \gamma$, as $\epsilon \to 0$,

$$\begin{aligned}\|n^\epsilon u^\epsilon - u\|^2_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)} &\le 2\|n^\epsilon(u^\epsilon - u)\|^2_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)} + 2\|(n^\epsilon - 1)u\|^2_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)} \\ &\le 2\|\sqrt{n^\epsilon}\|^2_{L^{2\gamma}(\mathbb{T}^3)} \|\sqrt{n^\epsilon}(u^\epsilon - u)\|^2_{L^2(\mathbb{T}^3)} \\ &\quad + 2\|n^\epsilon - 1\|^2_{L^\lambda(\mathbb{T}^3)} \|u\|^2_{L^{\frac{2\lambda\gamma}{\lambda\gamma+\lambda-2\gamma}}(\mathbb{T}^3)} \\ &\le C\epsilon^\beta + C\epsilon^2 \\ &\le C\epsilon^\beta.\end{aligned}$$

We remark that $0 < \lambda\gamma + \lambda - 2\gamma < 2\lambda\gamma$ for the definition of λ in Lemma 3.1. So we conclude that (2.7) holds.

Thus the proof of Theorem 2.2 is finished. □

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