Z. Angew. Math. Phys. 66 (2015), 2305–2341

0044-2275/15/052305-37
published online April 1, 2015
DOI 10.1007/s00033-015-0522-1

Zeitschrift für angewandte Mathematik und Physik ZAMP

The Cauchy problem for compressible Navier–Stokes equations with shear viscosity and large data

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Abstract. In this paper, we establish the existence of global-in-time smooth solutions to the compressible Navier–Stokes system for a viscous and heat-conducting ideal polytropical gas with shear viscosity and large data. Here, the viscosity coefficients can be degenerate functions on density, and the heat-conductive coefficient can also be a degenerate function on both density and temperature.

Mathematics Subject Classification. 35D30 · 35Q35 · 35D35 · 76D05 · 76N10.

Keywords. Compressible Navier–Stokes equations, Shear viscosity, Viscous coefficients, Heat-conducting coefficient.

1. Introduction

In this paper, we are concerned with the Cauchy problem for the viscous compressible flow between two horizontal plates. The governing equations are derived from the general three-dimensional Navier–Stokes equations:

$$\begin{cases} \rho_t + \nabla_\xi \cdot (\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \nabla_\xi \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_\xi p = \nabla_\xi \cdot \left(\nu(\nabla_\xi \cdot \mathbf{u})\mathbf{I} + \mu\left(\nabla_\xi \mathbf{u} + (\nabla_\xi \mathbf{u})^{\mathrm{T}}\right)\right), \\ E_t + \nabla_\xi \cdot (\mathbf{u}(E+p)) = \nabla_\xi \cdot \left(\nu(\nabla_\xi \cdot \mathbf{u})\mathbf{u} + \mu \mathbf{u}\left(\nabla_\xi \mathbf{u} + (\nabla_\xi \mathbf{u})^{\mathrm{T}}\right) + \kappa \nabla_\xi \theta\right). \end{cases} \tag{1.1}$$

Here, $\xi \in \mathbf{R^3}$ is the spatial variable, and $t > 0$ is the time variable. $\rho > 0$, $u = (u^1, u^2, u^3)$, $\theta > 0$, and $p = p(\rho, \theta)$ denote the density, the velocity, the absolute temperature, and the pressure, respectively. $(\nabla u)^{\mathrm{T}}$ is the transpose of the matrix ∇u. The specific total energy $\mathbf{E} = \rho(\frac{1}{2}|u|^2 + e)$ with e being the specific internal energy, the viscous coefficients $\mu(\rho, \theta) > 0$, and $\nu(\rho, \theta) > 0$ is assumed to satisfy $\mu(\rho, \theta) + \frac{2}{3}\nu(\rho, \theta) > 0$, and $\kappa(\rho, \theta) > 0$ denotes the coefficient of heat conductivity. The thermodynamic variables p, ρ, and e are related through Gibbs' equation $\mathrm{d}e = \theta \mathrm{d}s - p\mathrm{d}\rho^{-1}$ with s being the specific entropy. The viscosity coefficients μ, ν and heat conductivity coefficient κ can be functions of density ρ and temperature θ. Such a dependence will have an obviously influence on the solutions of the field equations and the mathematical analysis.

Let us consider the three-dimensional flow (1.1) with spatial variable $\xi = (x, x_2, x_3)$, which is moving in the x direction and uniform in the transverse direction (x_2, x_3), with

$$\rho = \rho(t, x), \quad \theta = \theta(t, x), \quad \mathbf{u}(t, x) = (u, \mathbf{w})(t, x), \quad \mathbf{w}(t, x) = (u_2, u_3)(t, x); \tag{1.2}$$

here, $u \in \mathbf{R}$ is the longitudinal velocity and $\mathbf{w} \in R^2$ is the transverse velocity. With the structure (1.2), Eq. (1.1) become the following system in one space dimension with $\lambda = \nu + 2\mu > 0$,

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$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = (\lambda u_x)_x, \\ (\rho \mathbf{w})_t + (\rho u \mathbf{w})_x = (\mu \mathbf{w}_x)_x, \\ E_t + (u(E+p))_x = (\lambda u u_x + \mu \mathbf{w} \cdot \mathbf{w}_x + \kappa \theta_x)_x . \end{cases} \tag{1.3}$$

Here, just as in (1.1), $x \in \mathbf{R}$ is the spatial variable, $t > 0$ is the time variable, $u \in \mathbf{R}$ is the longitudinal velocity, $\mathbf{w}$ is the transverse velocity, $\lambda(\rho, \theta)$ and $\mu(\rho, \theta)$ are the viscosity coefficients, $\kappa(\rho, \theta)$ is the heat conductivity; $\mu(\rho, \theta)$ is particularly called the shear viscosity. They depend on both viscosity and temperature generally.

We begin with a rough review in this direction. When the viscosity coefficients are constant, the local classical solutions to the Navier–Stokes equations with heat-conducting fluid in Hölder spaces were obtained, respectively, by Itaya [1] for the Cauchy problem and by Tani [10] for initial-boundary-value problem with $\inf \rho_0 > 0$, where the spatial dimension $N = 3$. Matsumura and Nishida [6,7] showed that the global classical solutions exist provided that the initial data are small in some sense and away from vacuum with spatial dimension $N = 3$. For large initial data, as to polytropic perfect gas with constant viscosity, Kazhikhov and Shelukhi [4]; Kawohl [3]; Jenssen and Karper [5]; and Tan et al. [9] got global classical solutions in dimension $N = 1$, respectively. As to the vacuum case, recently, Wen and Zhu [12,13] get the global well-posedness of strong and classical vacuum solutions with large initial data in one dimension and the symmetric solutions in high dimensions.

As to the initial-boundary-value problem of (1.3) in a bounded spatial domain $\Omega = (0, 1)$, with the following initial condition and mixed Dirichlet–Neumann impermeable thermally insulated boundaries,

$$\begin{cases} (\rho, u, \mathbf{w}, \theta)|_{t=0} = (\rho_0, u_0, \mathbf{w}_0, \theta_0)(x), x \in \Omega, \\ (u, \mathbf{w})|_{\partial\Omega} = 0, \\ \theta_x|_{\partial\Omega} = 0, \end{cases} \tag{1.4}$$

Wang [11] deals with the real viscous heat-conducting flow with shear viscosity. They need to ask whether there are positive constant bounds for viscosity coefficients, that is, $\mu_1 \le \mu(v) \le \mu_2, \lambda_1 \le \lambda(v) \le \lambda_2$, with $\mu_i, \lambda_i, i = 1, 2$ as positive constants. They also assume the growth conditions with exponents $r \in [0, 1]$ and $q \ge 2 + 2r$ such that the following hold:

(1) There exists a constant $e_0 > 0$ such that, for $v > 0$ and $\theta \ge 0$, $p_v(v, \theta) \le 0, e(v, \theta) \ge e_0(1 + \theta^r); e_\theta(v, \theta) \ge 0$;
(2) For any given $v_1 > 0$, there exist positive constants $\kappa_0 = \kappa_0(v_1), p_0 = p_0(v_1), e_1 = e_1(v_1)$ such that, for $v \ge v_1, \theta \ge 0$, $0 \le vp(v, \theta) \le p_0(1 + \theta^{1+r}), \kappa(v, \theta) \ge \kappa_0(1 + \theta^q), e_\theta(v, \theta) \ge e_1(1 + \theta^r)$;
(3) For any given $v_2 > v_1 > 0$, there exist positive constants $p_i = p_i(v_1, v_2)$ $(i = 1, 2, 3)$, $e_j = e_j(v_1, v_2), (j = 2, 3)$ and $\kappa_1 = \kappa_1(v_1, v_2)$ such that, for any $v \in [v_1, v_2], \theta \ge 0$

$$\left.\begin{aligned} |vp_\theta(v, \theta)| &\le p_1\left(1 + \theta^r\right), \\ -p_3\left(1 + \theta^{1+r}\right) \le v^2 p_v(v, \theta) &\le -p_2\left(1 + \theta^{1+r}\right), \\ |e_v(v, \theta)| \le e_2(1 + \theta^{1+r}),\ e_\theta(v, \theta) &\le e_3(1 + \theta^r), \\ \kappa(v, \theta) + |\kappa_v(v, \theta)| + |\kappa_{vv}(v, \theta)| &\le \kappa_1\left(1 + \theta^q\right). \end{aligned}\right\} \tag{1.5}$$

With the above conditions, they get that the initial-boundary-value problems (1.3) and (1.4) have a unique global solution $(v, u, \mathbf{w}, \theta)(t, x)$ such that $v \in L^\infty(0, T); H^1 \bigcap W^{1,\infty}(\Omega)$, and $(u, \mathbf{w}, \theta) \in L^\infty(0, T; H^1(\Omega))$.

Our main purpose in this paper is devoted to the construction of globally smooth, non-vacuum solutions to the Cauchy problem for the one-dimensional non-isentropic compressible Navier–Stokes equations with density-dependent viscous coefficient and density- and temperature-dependent heat conductivity coefficient for arbitrary large smooth initial data. Also, the viscosity coefficient $\mu(\rho)$, $\lambda(\rho)$, and heat-conducting coefficient $\kappa(\rho,\theta)$ can be degenerate functions of density ρ and temperature θ. That means when ρ or θ goes to 0, the viscosity coefficients or the heat-conducting coefficient can be 0.

Here, let x be the Lagrangian space variable, t be the time variable, and $v=\frac{1}{\rho}$ denote the specific volume, then the one-dimensional compressible Navier–Stokes equations (1.3) can be rewritten as

$$\begin{cases} v_t-u_x=0,\\ u_t+p(v,\theta)_x=\left(\frac{\lambda(v)u_x}{v}\right)_x,\\ \mathbf{w}_t=\left(\frac{\mu(v)\mathbf{w}_x}{v}\right)_x,\\ \varepsilon_t+(up(v,\theta))_x=\left(\frac{\lambda(v)uu_x}{v}+\frac{\mu(v)\mathbf{w}\cdot\mathbf{w}_x}{v}+\frac{\kappa(v,\theta)\theta_x}{v}\right)_x. \end{cases} \tag{1.6}$$

Here, $\varepsilon=e+\frac{1}{2}(u^2+|\mathbf{w}|^2)$, with the initial data

$$\begin{aligned} &(v(0,x),u(0,x),\mathbf{w}(0,x),\theta(0,x))=(v_0(x),u_0(x),\mathbf{w}_0(x),\theta_0(x)),\\ &\lim_{x\to\pm\infty}(v_0(x),u_0(x),\mathbf{w}_0(x),\theta_0(x))=(v_\pm,u_\pm,\mathbf{w}_\pm,\theta_\pm). \end{aligned} \tag{1.7}$$

Throughout this paper, we will concentrate on the case of ideal, polytropic gases, that is,

$$p(v,\theta)=\frac{R\theta}{v}=Av^{-\gamma}\exp\left(\frac{\gamma-1}{R}s\right),\quad e=C_v\theta=\frac{R\theta}{\gamma-1}, \tag{1.8}$$

where the specific gas constants R and C_v are positive constants, and $\gamma>1$ is the adiabatic exponent of the gases.

Then, (1.6) can be rewritten as

$$\begin{cases} v_t-u_x=0,\\ u_t+p(v,\theta)_x=\left(\frac{\lambda(v)u_x}{v}\right)_x,\\ \mathbf{w}_t=\left(\frac{\mu(v)\mathbf{w}_x}{v}\right)_x,\\ C_v\theta_t+u_xp(v,\theta)=\frac{\lambda(v)u_x^2}{v}+\frac{\mu(v)|\mathbf{w}_x|^2}{v}+\left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_x. \end{cases} \tag{1.9}$$

For simplicity, without the loss of generality, we consider the case when the far fields of the initial data satisfy $(v_\pm,u_\pm,\mathbf{w}_\pm,\theta_\pm)=(1,0,\mathbf{0},1)$.

Our first result in this paper is concerned with the case $\lambda(v)\equiv\bar{\lambda}>0$, $\mu(v)$ depends on v, and the heat-conducting coefficient $\kappa(v,\theta)>0$ depends on density and temperature, which can be summarized as the following theorem:

Theorem 1.1. *Suppose that the following conditions hold*

- $(v_0(x)-1,u_0(x),\mathbf{w}_0(x),\theta_0(x)-1)\in H^1(\mathbf{R})$ *and there exist positive constants* $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ *such that*

$$\underline{V}\le v_0(x)\le\overline{V},\quad \underline{\Theta}\le\theta_0(x)\le\overline{\Theta}; \tag{1.10}$$

- $$\lambda(v)\equiv\bar{\lambda}>0,\ \inf_{v\ge\underline{V},\theta\ge\underline{\Theta}}\kappa(v,\theta)\ge C(\underline{V},\underline{\Theta})>0. \tag{1.11}$$

  $\mu(v)\ge0$ *is smooth function of* v.

Then, the Cauchy problem (1.7)–(1.9) with $\mu(v)$, $\lambda(v)$, and $\kappa(v,\theta)$ given above admits a unique global solution $(v(t,x), u(t,x), \mathbf{w}(t,x), \theta(t,x))$ which satisfies

$$\begin{aligned}
&(v(t,x)-1, u(t,x), \mathbf{w}(t,x), \theta(t,x)-1) \in C^0\left(0,T;H^1(R)\right),\\
&(u_x(t,x), \mathbf{w}_x(t,x), \theta_x(t,x)) \in L^2\left(0,T;H^1(R)\right),\\
&0 < V_0^{-1} \le v(t,x) \le V_0, \quad 0 < \Theta_0^{-1} \le \theta(t,x) \le \Theta_0, \quad \forall (t,x) \in [0,T]\times\mathbf{R}
\end{aligned} \tag{1.12}$$

Here, $T>0$ is any given positive constant, and V_0, Θ_0 are some positive constants which may depend on T.

The assumptions imposed on $\lambda(v) \equiv \bar{\lambda}$ do not cover the case when the viscosity coefficient $\lambda(v)$ depends on density. So, our next main concern is the solvability of (1.7)–(1.9) when the viscosity coefficient $\lambda(v)$ depends on density, and the heat conductivity coefficient $\kappa(v,\theta)$ depends on density and temperature. Here, for simplicity, we just consider about the polynomial case, i.e.,

$$\lambda(v) = v^{-a}, \quad \mu(v) = v^{-c}, \quad \kappa(v,\theta) = \theta^b,$$

Here, a, b, c are constants to be determined later. Then, our next result in this direction can be summarized in the following theorem.

Theorem 1.2. *Suppose that the following conditions hold*

- $(v_0(x)-1, u_0(x), \mathbf{w}_0(x), \theta_0(x)-1) \in H^1(\mathbf{R})$ *and there exist positive constants* $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$ *such that*
$$\underline{V} \le v_0(x) \le \overline{V}, \quad \underline{\Theta} \le \theta_0(x) \le \overline{\Theta}; \tag{1.13}$$
- $$\frac{1}{3} < a < \frac{1}{2}$$
- *b and c satisfy one of the following conditions:*
  (i.) $1 \le b < \frac{2a}{1-a} < 2$;
  (ii.) $0 < b < 1, \frac{2-b}{2} + \frac{(a^2-a+2)(1-b)}{(3a-1)(1-2a)} < 1, \frac{(1-b)(3+a-2a^2)}{(3a-1)(1-2a)} < 1, |\mathrm{sign}(1+c)|P_2(1-b) < 1$, *$P_2$ are positive constants that can be determined in* Sect. 3.

Then, the Cauchy problem (1.7)–(1.9) with $\mu(v)$, $\lambda(v)$, and $\kappa(v,\theta)$ given above admits a unique global solution $(v(t,x), u(t,x), \mathbf{w}(t,x), \theta(t,x))$, which satisfies (1.12).

Our next result is concerned with the case when $\mu(v)$, $\lambda(v)$, and $\kappa(v,\theta)$ are more general smooth functions of density and temperature, which contains the case when μ, λ, and κ are positive constants. The main idea is using the smallness of $\gamma-1$ to control the possible growth of the solutions caused by the nonlinearity of the systems to deduce an uniform lower and upper bound on the absolute temperature, which is based on the observation that when $(v_0(x)-1, u_0(x), \mathbf{w}_0(x), s_0(x)-\overline{s}) \in H^2(\mathbf{R})$ with its $H^2(\mathbf{R})-$norm being bounded by some constant independent of $\gamma-1$, and $\|\theta_0(x)-1\|_{L^\infty(\mathbf{R})}$ can be chosen as small as wanted provided that $\gamma-1$ is sufficiently small. Here, $\overline{s} = \frac{R}{\gamma-1}\ln\frac{R}{A}$ is the far field of the initial entropy $s_0(x)$, i.e.,

$$\lim_{|x|\to+\infty} s_0(x) = \lim_{|x|\to+\infty} \frac{R}{\gamma-1}\ln\frac{R\theta_0(x)v_0(x)^{\gamma-1}}{A} = \overline{s}.$$

It is easy to see that $\overline{s}$ depends on $\frac{1}{\gamma-1}$. It is more convenient to use $v, u, \mathbf{w}$, and s as independent variables in such a case. And our result in this direction can be summarized in the following theorem:

Theorem 1.3. *Suppose that the following conditions hold*

- $(v_0(x)-1, u_0(x), \mathbf{w}_0(x), s_0(x)-\overline{s}) \in H^2(\mathbf{R})$ *with* $\|(v_0(x)-1, u_0(x), \mathbf{w}_0(x), s_0(x)-\overline{s}\|_{H^2(\mathbf{R})}$ *being bounded by some positive constant independent of $\gamma-1$ and (1.8) holds true for some $\gamma-1$independent positive constants* $\underline{V}$, $\overline{V}$, $\underline{\Theta}$, $\overline{\Theta}$;

- $\lambda(v) > 0$ *for all* $v > 0$, *and*

$$\lim_{v\to 0^+} \Psi(v) = -\infty, \qquad \lim_{v\to +\infty} \Psi(v) = +\infty. \tag{1.14}$$

Here,

$$\Psi(v) = \int_1^v \frac{\sqrt{z - \ln z - 1}}{z} \lambda(z) \mathrm{d}z; \tag{1.15}$$

- $\mu(v) > 0$ *for all* $v > 0$, *and* $\mu(v)$ *is smooth function of* v;
- *We assume that* $\kappa(v,\theta)$ *is smooth function of* v, θ, *and* $\kappa(v,\theta) > 0$ *for all* $v > 0, \theta > 0$ *and if we set* $\kappa_1(v) = \min_{\underline{\Theta}\le\theta\le\overline{\Theta}} \kappa(v,\theta)$, *we may further assume that*

$$\overline{\lim_{v\to 0_+}} \frac{\frac{\lambda(v)}{\kappa_1(v)}}{|\Psi(v)|^2} = \overline{\lim_{v\to +\infty}} \frac{\frac{\lambda(v)}{\kappa_1(v)}}{|\Psi(v)|^2} = 0; \tag{1.16}$$

- $\gamma - 1$ *is sufficiently small.*

Then, the Cauchy problem (1.7), (1.9) *admits a unique global solution* $(v(t,x), u(t,x), \mathbf{w}(t,x), \theta(t,x))$, *which satisfies* (1.12) *and*

$$\lim_{t\to+\infty} \sup_{x\in\mathbf{R}} |(v(t,x) - 1, u(t,x), \mathbf{w}(t,x), \theta(t,x) - 1)| = 0. \tag{1.17}$$

This paper is organized as follows: after this introduction and the statement of our main results, which constitutes Sect. 1, the proofs of Theorems 1.1–1.3 will be given in Sects. 2–4, respectively.

Notations: Almost all the notations used in this manuscript are standard: O(1) or $C_i (i \in \mathbf{N})$ stands for a generic positive constant, which is independent of t and x, while $C_i(\cdot, \ldots, \cdot)$ $(i \in \mathbf{N})$ is used to denote some positive constant depending on the arguments listed in the parenthesis. Note that all these constants may vary from line to line. We denote $f^+ = \max\{f, 0\}, f^- = \max\{-f, 0\}$. $\|\cdot\|_s$ represents the norm in $H^s(\mathbf{R})$ with $\|\cdot\| = \|\cdot\|_0$ and for $1 \le p \le +\infty$, $L^p(\mathbf{R})$ denotes the standard Lebesgue space with the norm

$$\begin{cases} \|f(\tau)\|_{L^p} := \left(\int_{\mathbf{R}} |f(\tau,x)|^p \mathrm{d}x\right)^{\frac{1}{p}}, \quad 1 \le p < +\infty, \\ \|f(\tau)\|_{L^\infty_x} := \sup_{x\in\mathbf{R}} |f(\tau,x)|, \\ \|f\|_{L^\infty_{t,x}} := \sup_{(\tau,x)\in[0,t]\times\mathbf{R}} |f(\tau,x)| \end{cases} \tag{1.18}$$

for $f(\tau,x) \in C([0,t], L^p(\mathbf{R}))$.

2. The proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. To this end, we first define the following closed set for which we seek the solutions of the Cauchy problem (1.7), (1.9)

$$X(0,T; M_0, M_1; N_0, N_1) = \left\{ (v,u,\mathbf{w},\theta)(t,x) \left| \begin{array}{c} (v-1, u, \mathbf{w}, \theta-1)(t,x) \in C^0\left(0,T; H^k(\mathbf{R})\right) \\ (v_x, u_x, \mathbf{w}_x, \theta_x)(t,x) \in L^2\left(0,T; H^2(\mathbf{R})\right) \\ M_0 \le v(t,x) \le M_1, \quad N_0 \le \theta(t,x) \le N_1 \end{array} \right. \right\} \tag{2.1}$$

Here, $T > 0, M_1 \ge M_0 > 0, N_1 \ge N_0 > 0$ are some positive constants, and when $k = 1, 2$, the space $C^0(0,T; H^k(\mathbf{R})$ is different in different theorems. We can get the following local existence result.

Lemma 2.1. (Local existence) *Under the assumptions stated in* Theorems 1.1–1.3*, there exists a sufficiently small positive constant* t_1*, which depends only on* $\underline{V}, \overline{V}, \underline{\Theta}, \overline{\Theta}$*, and* $\|(v_0-1, u_0, \mathbf{w}_0, \theta_0-1)\|_1$*, such that the Cauchy problem* (1.7), (1.9) *admits a unique smooth solution* $(v(t,x), u(t,x), \mathbf{w}(t,x), \theta(t,x))$ *and* $(v(t,x), u(t,x), \mathbf{w}(t,x), \theta(t,x))$ *satisfies*

$$\begin{cases} 0 < \frac{\underline{V}}{2} \leq v(t,x) \leq 2\overline{V}, \\ 0 < \frac{\underline{\Theta}}{2} \leq \theta(t,x) \leq 2\overline{\Theta}, \end{cases} \tag{2.2}$$

$$\sup_{[0,t_1]} \left(\|(v-1, u, \mathbf{w}, \theta-1,)(t)\|_k \right) \leq 2\|(v_0-1, u_0, \mathbf{w}_0, \theta_0-1)\|_k, \tag{2.3}$$

and

$$\lim_{|x|\to\infty} (v(t,x)-1, u(t,x), \mathbf{w}(t,x), \theta(t,x)-1) = (0,0,0,0). \tag{2.4}$$

Here, to prove Theorems 1.1 *and* 1.2*, we need* $k=1$*, and for* Theorem 1.3*, we need* $k=2$*.*

Lemma 2.1 can be proved by the standard iteration argument as in [8] for the one-dimensional compressible Navier–Stokes system; we thus omit the details for brevity.

Suppose that the local solution $(v(t,x), u(t,x), \mathbf{w}(t,x), \theta(t,x))$ constructed in Lemma 2.1 has been extended to the time step $t = T \geq t_1$ and satisfies the a priori assumption

$$\overline{V}_0 \leq v(t,x) \leq \overline{V}_1, \quad \overline{\Theta}_0 \leq \theta(t,x) \leq \overline{\Theta}_1 \tag{H$_1$}$$

for all $x \in \mathbf{R}$, $0 \leq t \leq T$, and some positive constants $0 < \overline{\Theta}_0 \leq \overline{\Theta}_1, 0 < \overline{V}_0 \leq \overline{V}_1$, we now deduce certain a priori estimates on $(v(t,x), u(t,x), \mathbf{w}(t,x), \theta(t,x))$ which are independent of $\overline{\Theta}_0, \overline{\Theta}_1, \overline{V}_0, \overline{V}_1$ but may depend on T.

First, we will get the basic energy estimate. For this purpose, note that

$$\eta(v,u,\mathbf{w},\theta) = R\phi(v) + \frac{u^2+|\mathbf{w}|^2}{2} + \frac{R\phi(\theta)}{\gamma-1}, \quad \phi(x) = x - \ln x - 1$$

is a convex entropy to (1.9), which satisfies the following identity

$$\begin{aligned} &\eta(v,u,\mathbf{w},\theta)_t + \tfrac{\kappa\theta_x^2}{v\theta^2} + \tfrac{\lambda u_x^2}{v\theta} + \tfrac{\mu|\mathbf{w}|_x^2}{v\theta} = \left\{ \tfrac{\kappa\theta_x}{v} - \tfrac{\kappa\theta_x}{v\theta} + \tfrac{\lambda u u_x}{v} + \tfrac{\mu(v)\mathbf{w}\cdot\mathbf{w}_x}{v} \right\}_x \\ &\quad - \left\{ \left(\tfrac{R\theta}{v} - R \right) u \right\}_x \end{aligned} \tag{2.5}$$

we can deduce by integrating (2.5) with respect to t and x over $[0,T]\times\mathbf{R}$ and from (2.5) that

Lemma 2.2. (Basic energy estimates) *Let the conditions listed in* Lemma 2.1 *hold and suppose that the local solution* $(v(t,x), u(t,x), \mathbf{w}(t,x), \theta(t,x))$ *constructed in* Lemma 2.1 *has been extended to the time step* $t = T \geq t_1$ *and satisfies the* a priori *assumption* (H$_1$)*, then we have for* $0 \leq t \leq T$ *that*

$$\int_{\mathbf{R}} \eta(v,u,\mathbf{w},\theta)\mathrm{d}x + \int_0^t \int_{\mathbf{R}} \left(\frac{\kappa\theta_x^2}{v\theta^2} + \frac{\lambda u_x^2}{v\theta} + \frac{\mu|\mathbf{w}_x|^2}{v\theta} \right)(\tau,x)\mathrm{d}x\mathrm{d}\tau = \int_{\mathbf{R}} \eta(v_0,u_0,\mathbf{w}_0,\theta_0)(x)\mathrm{d}x. \tag{2.6}$$

From the argument used in [4], and the basic energy estimate (2.6), by making use of the Jenssen's inequality, we get for each $i \in \mathbf{Z}$, there are positive constants $A_0 > 0, A_1 > 0$ such that

$$A_0 \leq \int_i^{i+1} v(t,x)\mathrm{d}x, \quad \int_i^{i+1} \theta(t,x)\mathrm{d}x \leq A_1, \quad \forall t \in [0,T]. \tag{2.7}$$

And there exist $a_i(t) \in [i, i+1]$, $b_i(t) \in [i, i+1]$ such that

$$A_0 \le v(t, a_i(t)), \quad \theta(t, b_i(t)) \le A_1. \tag{2.8}$$

From $(1.7)_2$, we have

$$u_t = -R\left(\frac{\theta}{v}\right)_x + (\lambda \ln v)_{tx}. \tag{2.9}$$

Integrating (2.9) with respect to t over $[0, t]$,

$$u(t, y) - u_0(y) = -R\left(\int_0^t \frac{\theta(s, y)}{v(s, y)} \mathrm{d}s\right)_y + \lambda\left(\ln\frac{v(t, y)}{v_0(y)}\right)_y. \tag{2.10}$$

Integrating (2.10) with respect to y from $a_i(t)$ to any $x \in [i, i+1]$, we obtain that

$$\begin{aligned}\int_{a_i(t)}^x u(t, y) - u_0(y)\mathrm{d}y &= -R\int_0^t \frac{\theta(s, x)}{v(s, x)}\mathrm{d}s \\ &\quad + R\int_0^t \frac{\theta(s, a_i(t))}{v(s, a_i(t))}\mathrm{d}s + \lambda \ln\frac{v(t, x)}{v_0(x)} - \lambda\ln\frac{v(t, a_i(t))}{v_0(a_i(t))}.\end{aligned} \tag{2.11}$$

Multiply $\frac{1}{\lambda}$ and take the exponential on both sides of (2.11), we arrive at

$$\begin{aligned}&\exp\left(\frac{1}{\lambda}\int_{a_i(t)}^x u(t, y) - u_0(y)\mathrm{d}y\right) \\ &= \frac{v(t, x)}{v_0(x)} \cdot \frac{v_0(a_i(t))}{v(t, a_i(t))} \exp\left(-\frac{R}{\lambda}\int_0^t \frac{\theta(s, x)}{v(s, x)}\mathrm{d}s\right) \cdot \exp\left(\frac{R}{\lambda}\int_0^t \frac{\theta(s, a_i(t))}{v(s, a_i(t))}\mathrm{d}s\right)\mathrm{d}s.\end{aligned}$$

Consequently, we get

$$\frac{1}{v(t, x)} \exp\left(\frac{R}{\lambda}\int_0^t \frac{\theta(s, x)}{v(s, x)}\mathrm{d}s\right) = B_i(t, x) \cdot Y_i(t), \tag{2.12}$$

with

$$\begin{aligned}B_i(t, x) &= \frac{v_0(a_i(t))}{v_0(x)v(t, a_i(t))} \exp\left(\frac{1}{\lambda}\int_{a_i(t)}^x u(t, y) - u_0(y)\mathrm{d}y\right), \\ Y_i(t) &= \exp\left(\frac{R}{\lambda}\int_0^t \frac{\theta(s, a_i(t))}{v(s, a_i(t))}\right).\end{aligned} \tag{2.13}$$

Notice that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(\frac{R}{\lambda}\int_0^t \frac{\theta(s, x)}{v(s, x)}\mathrm{d}s\right)\right] = \frac{R}{\lambda}\frac{\theta(t, x)}{v(t, x)} \exp\left(\frac{R}{\lambda}\int_0^t \frac{\theta(s, x)}{v(s, x)}\mathrm{d}s\right). \tag{2.14}$$

Multiply $\frac{R\theta(t,x)}{\lambda}$ on both sides of (2.14), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(\frac{R}{\lambda}\int_0^t \frac{\theta(s,x)}{v(s,x)}\mathrm{d}s\right)\right] = \frac{R}{\lambda}B_i(t,x)Y_i(t)\theta(t,x). \tag{2.15}$$

Integrating (2.15) with respect to variable t over the interval $[0,t]$, it follows that

$$\exp\left(\frac{R}{\lambda}\int_0^t \frac{\theta(s,x)}{v(s,x)}\mathrm{d}s\right) = 1 + \frac{R}{\lambda}\int_0^t B_i(s,x)Y_i(s)\theta(s,x)\mathrm{d}s. \tag{2.16}$$

Then, plug (2.16) into (2.12), it yields

$$v(t,x) = \frac{1}{B_i(t,x)Y_i(t)}\exp\left(\frac{R}{\lambda}\int_0^t \frac{\theta(s,x)}{v(s,x)}\mathrm{d}s\right) = \frac{1+\frac{R}{\lambda}\int_0^t B_i(s,x)Y_i(s)\theta(s,x)\mathrm{d}s}{B_i(t,x)Y_i(t)}. \tag{2.17}$$

Integrating (2.17) with respect to x over $[i,i+1]$ yields

$$\int_i^{i+1} v(t,x)\mathrm{d}x = \int_i^{i+1} \frac{1+\frac{R}{\lambda}\int_0^t B_i(s,x)Y_i(s)\theta(s,x)\mathrm{d}s}{B_i(t,x)Y_i(t)}\mathrm{d}x. \tag{2.18}$$

From the definition of $B_i(t,x), Y_i(t)$, we can deduce there is $C>0$, such that

$$0 < C^{-1} \le B_i(t,x) \le C, \quad Y_i(t) \ge 1, \quad \forall x \in [i,i+1], \quad t \in [0,T]. \tag{2.19}$$

Applying (2.19) to (2.18), we can discover that

$$\int_i^{i+1} v(t,x)\mathrm{d}x \le \frac{C}{Y_i(t)}\left(1 + \frac{R}{\mu}\int_0^t Y_i(s)\int_i^{i+1}\theta(s,x)\mathrm{d}x\mathrm{d}s\right). \tag{2.20}$$

Consequently, from (2.7), (2.8)

$$Y_i(t) \le C\left(1 + \int_0^t Y_i(s)\mathrm{d}s\right). \tag{2.21}$$

Using Gronwall's inequality,

$$1 \le Y_i(t) \le C, \quad t \in [0,T], \quad i = 0, \pm 1, \pm 2, \ldots. \tag{2.22}$$

Combine (2.19), (2.22) together, we have $v(t,x) \ge C$, $\forall x \in [i,i+1]$. Since i is arbitrary, C is independent of i, we actually obtain

Lemma 2.3. *Under the same conditions listed in* Lemma 2.1*, there is* $\underline{V} > 0$*, such that for* $\forall t \in [0,T]$*,* $v(t,x)$ *can be bounded from below as*

$$v(t,x) \ge \underline{V}, \quad \forall x \in \mathbf{R}. \tag{2.23}$$

Our next result is concerned with the lower bound estimate of $\theta(t,x)$,

Lemma 2.4. *Under the assumptions listed in* Lemma 2.3*, we have*

$$\frac{1}{\theta(t,x)} \le \frac{1}{\underline{\Theta}}, \quad 0 \le t \le T. \tag{2.24}$$

Proof. It is easy to see from $(1.7)_4$ that

$$C_v\left(\frac{1}{\theta}\right)_t = \left(\frac{\kappa(v,\theta)}{v}\left(\frac{1}{\theta}\right)_x\right)_x - \left[\frac{2\kappa\theta}{v}\left(\left(\frac{1}{\theta}\right)_x\right)^2 + \frac{\lambda}{v\theta^2}\left(u_x - \frac{\theta R}{2\lambda}\right)^2\right] + \frac{R^2}{4\lambda v} - \frac{\mu \mathbf{w}_x^2}{v\theta^2}. \tag{2.25}$$

If we set $h(t,x) = \frac{1}{\theta} - \frac{R^2 t}{4C_v\lambda}\|\frac{1}{v}\|_{L^\infty_{T,x}}$, then

$$\begin{cases} C_v h_t \le \left(\frac{\kappa(v,\theta)}{v}h_x\right)_x, \\ h(0,x) = h_0(x) = \frac{1}{\theta_0(x)}. \end{cases} \tag{2.26}$$

From the maximal principle, we get

$$h(t,x) = \frac{1}{\theta} - \frac{R^2 t}{4C_v\lambda}\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}} \le \frac{1}{\underline{\Theta}}. \tag{2.27}$$

Thus, we have completed the proof. □

With the lower bounds of $v(t,x)$ and $\theta(t,x)$ at hand, our next job is getting the upper bound of $v(t,x)$. Since

$$\theta(t,x) \le C\int_i^{i+1}\frac{\kappa\theta_x^2}{v\theta^2}\mathrm{d}x \cdot \int_i^{i+1}\theta(t,x)\mathrm{d}x \max_x v(t,x)\max_x\frac{1}{\kappa(v,\theta)} + C. \tag{2.28}$$

and due to the assumptions imposed on $\kappa(v,\theta)$ in (1.11), and (2.7), we obtain that

$$\theta(t,x) \le C\int_i^{i+1}\frac{\kappa\theta_x^2}{v\theta^2}\mathrm{d}x \cdot \max_x v(t,x) + C. \tag{2.29}$$

Inserting (2.29) into (2.17), we finally get

$$\|v(t)\|_{L^\infty_x} \le C + \int_0^t\int_i^{i+1}\frac{\kappa\theta_x^2}{v\theta^2}\mathrm{d}x\|v(\tau)\|_{L^\infty_x}\mathrm{d}\tau. \tag{2.30}$$

From the Gronwall's inequality, we can get the following lemma:

Lemma 2.5. *Under the same conditions listed in* Lemma 2.1*, there is a positive constant* $\overline{V} > 0$*, such that*

$$v(t,x) \le \overline{V}. \tag{2.31}$$

With (2.23), (2.24), and (2.31) at hand, (2.6) can be rewritten as

$$\int_{\mathbf{R}}\eta(t,x)\mathrm{d}x + \int_0^t\int_{\mathbf{R}}\left(\frac{\kappa(v,\theta)\theta_x^2}{\theta^2} + \frac{u_x^2}{\theta} + \frac{|\mathbf{w}_x|^2}{\theta}\right)\mathrm{d}x\mathrm{d}\tau \le C. \tag{2.32}$$

Then, our next main job is to get the upper bound of $\theta(t,x)$, and for the purpose in this direction, we get the following lemma:

Lemma 2.6. *Under the same conditions listed in* Lemma 2.1*, we get*

$$\|\theta - 1\|_{L^\infty_{t,x}} \le \int_0^t\left(\|u_x\|^2_{L^\infty} + \|\mathbf{w}_x\|^2_{L^\infty} + \|\theta\|^2_{L^\infty}\right)\mathrm{d}\tau + C. \tag{2.33}$$

Proof. Multiply $(1.7)_4$ by $2p(\theta-1)^{2p-1}$, and integrating the result with respect to x,

$$
\begin{aligned}
&2pC_v\|\theta-1\|_{L^{2p}}^{2p-1}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta-1\|_{L^{2p}}+\int_{\mathbf{R}}2p(2p-1)(\theta-1)^{2p-2}\frac{\kappa(v,\theta)}{v}\left((\theta-1)_x\right)^2\mathrm{d}x\\
&=\int_{\mathbf{R}}(\theta-1)^{2p-1}\left(\frac{\lambda u_x^2}{v}+\frac{\mu w_x^2}{v}\right)\mathrm{d}x-\int_{\mathbf{R}}2p(\theta-1)^{2p-1}\frac{R\theta u_x}{v}\mathrm{d}x. \qquad (2.34)
\end{aligned}
$$

By making use of the Young's inequality, we obtain that

$$
\|\theta-1\|_{L^{2p}}\le C\int_0^t\left(\|u_x\|_{L^{4p}}^2+\|\mathbf{w}_x\|_{L^{4p}}^2+\|\theta\|_{L_x^\infty}^2\right)\mathrm{d}\tau. \qquad (2.35)
$$

Letting $p\to+\infty$, we get (2.33); thus, the proof of Lemma 2.6 is completed. □

From Lemma 2.6, we can see that to get the upper bound of θ, we need to get the estimate of the terms appeared in the right-hand side of (2.33). For the purpose in this direction, our next job is getting the estimates of $\|(u_x,\mathbf{w}_x)\|$, $\|(u_{xx},\mathbf{w}_{xx})\|$. First, we will get the estimates of $\|v_x\|$. From $(1.9)_2$, we obtain that

$$
\begin{aligned}
\frac{\lambda}{2}\left\|\frac{v_x}{v}\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{R\theta}{v^3}v_x^2\mathrm{d}x\mathrm{d}\tau&=\int_0^t\int_{\mathbf{R}}\left(\frac{R\theta_x v_x}{v^2}+\frac{u_x^2}{v}\right)\mathrm{d}x\mathrm{d}\tau\\
&\quad+\int_{\mathbf{R}}\left(\frac{uv_x}{v}(t,x)-\frac{u_0v_{0x}}{v_0}(x)\right)\mathrm{d}x. \qquad (2.36)
\end{aligned}
$$

By making use of the Cauchy's inequality, we obtain that

$$
\int_0^t\int_{\mathbf{R}}\frac{R\theta_x v_x}{v^2}\mathrm{d}x\mathrm{d}\tau\le\frac{1}{8}\int_0^t\int_{\mathbf{R}}\frac{R\theta v_x^2}{v^3}\mathrm{d}x\mathrm{d}\tau+C\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v\theta}\mathrm{d}x\mathrm{d}\tau. \qquad (2.37)
$$

$$
\int_{\mathbf{R}}\frac{uv_x}{v}(t,x)\mathrm{d}x\le\frac{1}{2}\int_{\mathbf{R}}\lambda\frac{v_x^2}{v^2}\mathrm{d}x+C\int_{\mathbf{R}}\frac{u^2}{\lambda}\mathrm{d}x. \qquad (2.38)
$$

Then, we can obtain that

$$
\left\|\frac{v_x}{v}\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{\theta v_x^2}{v^3}\mathrm{d}x\mathrm{d}\tau\le C+C\int_0^t\int_{\mathbf{R}}\left(\frac{\theta_x^2}{v\theta}+\frac{u_x^2}{v}\right)\mathrm{d}x\mathrm{d}\tau. \qquad (2.39)
$$

Next, we will get the estimates of the terms appeared in the right-hand side of (2.39). For this purpose, if we multiply $(1.9)_2$ by u, it follows that

$$\left(\frac{u^2}{2}\right)_t + \lambda\frac{u_x^2}{v} = u_x(p-R) + \left(u(R-p) + \frac{\lambda u u_x}{v}\right)_x. \tag{2.40}$$

With the estimates of (2.6) and from easily calculation, we can get that

$$\|u\|^2 + \int_0^t \int_{\mathbf{R}} \lambda \frac{u_x^2}{v} \mathrm{d}x\mathrm{d}\tau \leq C. \tag{2.41}$$

$$\frac{1}{2}\|w(t)\|^2 + \int_0^t \int_{\mathbf{R}} \frac{\mu w_x^2}{v} \mathrm{d}x\mathrm{d}\tau = \frac{1}{2}\|w_0\|^2. \tag{2.42}$$

Then, plug (2.41) into (2.39), we get

$$\left\|\frac{v_x}{v}\right\|^2 + \int_0^t \int_{\mathbf{R}} \frac{\theta v_x^2}{v^3} \mathrm{d}x\mathrm{d}\tau \leq C + C\int_0^t \int_{\mathbf{R}} \frac{\theta_x^2}{v\theta} \mathrm{d}x\mathrm{d}\tau. \tag{2.43}$$

If we denote $f = C_v(\theta - 1) + \frac{1}{2}u^2 + \frac{|\mathbf{w}|^2}{2}$, then from (1.9),

$$f_t + (up)_x = \left(\frac{\kappa(v,\theta)}{v}\theta_x + \frac{\lambda}{v}uu_x + \frac{\mu(v)}{v}\mathbf{w}\cdot\mathbf{w}_x\right)_x. \tag{2.44}$$

Multiply (2.44) by f, we obtain

$$\begin{aligned}
&\left(\frac{1}{2}f^2\right)_t - \left(\left(\frac{\kappa(v,\theta)}{v}\theta_x + \frac{\lambda}{v}uu_x + \frac{\mu(v)}{v}\mathbf{w}\cdot\mathbf{w}_x - up\right)f\right)_x \\
&= -\left(\frac{\kappa(v,\theta)}{v}\theta_x + \frac{\lambda}{v}uu_x + \frac{\mu(v)}{v}\mathbf{w}\cdot\mathbf{w}_x\right)f_x + upf_x.
\end{aligned} \tag{2.45}$$

Since $f_x = C_v\theta_x + uu_x + \mathbf{w}\cdot\mathbf{w}_x$, integrating (2.45) with respect to t and x over $[0,t]\times\mathbf{R}$, and combining (2.23), (2.31) together, by making use of Cauchy's inequality, we assume further that $\lim\limits_{\theta\to+\infty}\kappa(v,\theta) = \overline{K} < +\infty$, and then, we get

$$\|f(t)\|^2 + \int_0^t \int_{\mathbf{R}} \kappa(v,\theta)\theta_x^2 \mathrm{d}x\mathrm{d}\tau \leq C + C\int_0^t \int_{\mathbf{R}} \left(u^2u_x^2 + (\mathbf{w}\cdot\mathbf{w}_x)^2 + \theta^2u^2\right)\mathrm{d}x\mathrm{d}\tau. \tag{2.46}$$

To get the estimates of the terms appeared on the right-hand side of (2.46), multiply $(1.9)_2$ by u^3,

$$\|u\|_{L^4}^4 + \int_0^t \int_{\mathbf{R}} \frac{\lambda}{v}u^2u_x^2 \mathrm{d}x\mathrm{d}\tau \leq C + \int_0^t \int_{\mathbf{R}} (\theta-1)^2u^2 \mathrm{d}x\mathrm{d}\tau. \tag{2.47}$$

Similarly, we have

$$\|\mathbf{w}\|_{L^4}^4 + \int_0^t \int_{\mathbf{R}} \mathbf{w}^2\mathbf{w}_x^2 \mathrm{d}x\mathrm{d}\tau \leq C. \tag{2.48}$$

If we plug (2.47), (2.48) into (2.46), with (2.41) in hand, it yields that

$$\|\theta - 1\|^2 + \int_0^t \int_{\mathbf{R}} \kappa(v,\theta)\theta_x^2 \mathrm{d}x\mathrm{d}\tau \leq C + \int_0^t \|(\theta-1)\|_{L_x^\infty}^2 \mathrm{d}\tau. \tag{2.49}$$

By making use of the Cauchy's inequality and the Gronwall's inequality, we actually get

$$\|\theta-1\|^2+\int_0^t\int_{\mathbf{R}}\kappa(v,\theta)\theta_x^2\mathrm{d}x\mathrm{d}\tau\le C. \tag{2.50}$$

Thus, inserting (2.50) into (2.43) and combining (2.24), we actually get

$$\|v_x\|^2+\int_0^t\int_{\mathbf{R}}\theta v_x^2\mathrm{d}x\mathrm{d}\tau\le C. \tag{2.51}$$

Next, we will get the estimates of $\mathbf{w}_{xx}, u_{xx}$. As to the estimates of $\mathbf{w}_{xx}$, we have

$$\frac{1}{2}(w_x^2)_t=\left(\left(\frac{\mu}{v}w_x\right)_x w_x\right)_x-\left(\frac{\mu}{v}w_x\right)_x w_{xx}, \tag{2.52}$$

From integrating parts and by making use of the Cauchy's inequality, it follows that

$$\|w_x(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{\mu}{v}w_{xx}^2\mathrm{d}x\mathrm{d}\tau\le C+\int_0^t\int_{\mathbf{R}}\left(\frac{\mu}{v}\right)_v^2 v_x^2w_x^2\mathrm{d}x\mathrm{d}\tau. \tag{2.53}$$

Since

$$\int_0^t\int_{\mathbf{R}}v_x^2w_x^2\mathrm{d}x\mathrm{d}\tau\le\int_0^t\|w_x\|\|v_x\|^2\|w_{xx}\|\mathrm{d}\tau\le\epsilon\int_0^t\|w_{xx}\|^2\mathrm{d}\tau+C\int_0^t\|w_x\|^2\|v_x\|^4\mathrm{d}\tau. \tag{2.54}$$

Plug (2.51), (2.54) into (2.53), it follows that

$$\|w_x(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{\mu}{v}w_{xx}^2\mathrm{d}x\mathrm{d}\tau\le C. \tag{2.55}$$

Similarly, we can get that

$$\|u_x(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{\mu}{v}u_{xx}^2\mathrm{d}x\mathrm{d}\tau\le C\int_0^t\int_{\mathbf{R}}\left(\theta_x^2+\theta^2v_x^2+u_x^2v_x^2\right)\mathrm{d}x\mathrm{d}\tau\le C+\int_0^t\|\theta\|_{L_x^\infty}^2\mathrm{d}\tau. \tag{2.56}$$

If we plug (2.50) into (2.56), it follows that

$$\|u_x(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{\mu}{v}u_{xx}^2\mathrm{d}x\mathrm{d}\tau\le C. \tag{2.57}$$

Inserting (2.55), (2.57) into (2.33), and using (2.50), we have

$$\|\theta-1\|_{L_{t,x}^\infty}\le\int_0^t\|\theta-1\|_{L^\infty}^2\mathrm{d}\tau+C\le\int_0^t\|\theta-1\|\|\theta_x\|\mathrm{d}\tau\le C. \tag{2.58}$$

Thus, we have

Corollary 2.1. *Under the conditions of* Lemma 2.1, *if we further assume that* $\lim_{\theta\to+\infty}\inf_{\underline{V}\le v}\kappa(v,\theta)=\overline{K}<+\infty$, *then there is* $\overline{\Theta}>0$, *such that*

$$\theta\le\overline{\Theta}. \tag{2.59}$$

Otherwise, if $\lim_{\theta\to+\infty} \inf_{\underline{V}\leq v} \kappa(v,\theta) = +\infty$, we have to get the upper bound of θ in a quite different way. From (2.43), it follows that

$$\left\|\frac{v_x}{v}\right\|^2 + \int_0^t\int_{\mathbf{R}} \frac{\theta v_x^2}{v^3}\mathrm{d}x\mathrm{d}\tau \leq C + C\left\|\frac{\theta}{\kappa(v,\theta)}\right\|_{L^\infty_{t,x}}. \tag{2.60}$$

Then, we have

$$\begin{aligned} &\|u_x(t)\|^2 + \|\mathbf{w}_x(t)\|^2 + \int_0^t\int_{\mathbf{R}} (u_{xx}^2 + |\mathbf{w}_{xx}|^2)\mathrm{d}x\mathrm{d}\tau \\ &\leq C + \left\|\frac{\theta^2}{\kappa}\right\|_{L^\infty_{t,x}} + \left\|\frac{\theta}{\kappa}\right\|_{L^\infty_{t,x}} \int_0^t \|\theta\|^2_{L^\infty_x}\mathrm{d}\tau + \left\|\frac{\theta}{\kappa}\right\|^2_{L^\infty_{t,x}}. \end{aligned} \tag{2.61}$$

Then, plug (2.61) into (2.33), it can be easily obtain that

$$\|\theta - 1\|_{L^\infty_{t,x}} \leq C + \left\|\frac{\theta^2}{\kappa}\right\|^{\frac{1}{2}}_{L^\infty_{t,x}} + \int_0^t \|\theta - 1\|^2_{L^\infty_x}\mathrm{d}\tau + \left\|\frac{\theta}{\kappa}\right\|_{L^\infty_{t,x}}. \tag{2.62}$$

from easily calculation, we get

Corollary 2.2. *Under the assumptions listed in* Lemma 2.1, *if we further assume that*

$$\lim_{\theta\to+\infty} \inf_{\underline{V}\leq v} \kappa(v,\theta) = +\infty,$$

then there is $\overline{\Theta} > 0$*, such that*

$$\theta \leq \overline{\Theta}. \tag{2.63}$$

With Corollaries 2.1 and 2.2, we can get the proof of Theorem 1.1.

3. Proof of Theorem 1.2

In this section, we are devoted to consider the case when $\mu(v) = v^{-c}$, $\lambda(v) = v^{-a}$, and $\kappa(v,\theta) = \theta^b$.

Let $(v(t,x), u(t,x), \mathbf{w}(t,x), \theta(t,x)) \in X(0,T; M_0, M_1; N_0, N_1)$ be a solution of the Cauchy problem (1.9), (1.7), which is defined in the time strip $[0,T]$ for some $T > 0$, and to extend such a solution globally, as pointed out in the proofs of Theorem 1.1, we only need to deduce positive lower and upper bounds on $v(t,x)$ and $\theta(t,x)$, which are independent of M_0, M_1, N_0, and N_1 but may depend on T.

Suppose that the local solution $(v(t,x), u(t,x), \mathbf{w}(t,x), \theta(t,x))$ constructed in Lemma 2.1 has been extended to the time step $t = T \geq t_1$ and satisfies the a priori assumption

$$\overline{V}_0 \leq v(t,x) \leq \overline{V}_1, \quad \overline{\Theta}_0 \leq \theta(t,x) \leq \overline{\Theta}_1 \tag{H$_2$}$$

for all $x \in \mathbf{R}$, $0 \leq t \leq T$, and some positive constants $0 < \overline{\Theta}_0 \leq \overline{\Theta}_1, 0 < \overline{V}_0 \leq \overline{V}_1$, we now deduce certain a priori estimates on $(v(t,x), u(t,x), \mathbf{w}(t,x), \theta(t,x))$, which are independent of $\overline{\Theta}_0, \overline{\Theta}_1, \overline{V}_0, \overline{V}_1$ but may depend on T.

First, just as the proof of Theorem 1.1, we get the following basic energy estimates.

Lemma 3.1. *Let the conditions listed in* Theorem 1.2 *hold and suppose that the local solution* $(v(t,x), u(t,x), \mathbf{w}(t,x), \theta(t,x))$ *constructed in* Lemma 2.1 *has been extended to the time step* $t = T \geq t_1$ *and satisfies the* a priori *assumption* (H_2)*, then we have for* $0 \leq t \leq T$ *that*

$$\int_{\mathbf{R}} \eta(v,u,\mathbf{w},\theta)\mathrm{d}x + \int_0^t\int_{\mathbf{R}} \left(\frac{\kappa\theta_x^2}{v\theta^2} + \frac{\lambda u_x^2}{v\theta} + \frac{\mu|\mathbf{w}_x|^2}{v\theta}\right)(\tau,x)\mathrm{d}x\mathrm{d}\tau = \int_{\mathbf{R}} \eta(v_0,u_0,\mathbf{w}_0,\theta_0)\mathrm{d}x. \tag{3.1}$$

Here, as in Sect. 2*,* $\eta(t,x) = C_v\phi(\theta) + R\phi(v) + \frac{u^2+|\mathbf{w}|^2}{2}$*, with* $\phi(x) = x - \ln x - 1$*.*

Lemma 3.2. *Under the assumptions in* Lemma 2.1*, when* $a<1$*, we have*

$$\frac{1}{\theta(t,x)} \leq O(1) + O(1)\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{1-a}, \quad x\in\mathbf{R}, \quad 0\leq t\leq T. \tag{3.2}$$

Proof. First of all, $(1.7)_4$ implies

$$\begin{aligned} C_v\left(\frac{1}{\theta}\right)_t &= \left(\frac{\kappa(v,\theta)}{v}\left(\frac{1}{\theta}\right)_x\right)_x - \left[\frac{2\kappa\theta}{v}\left(\left(\frac{1}{\theta}\right)_x\right)^2 + \frac{\lambda}{v\theta^2}\left(u_x - \frac{\theta R}{2\lambda}\right)^2\right] \\ &\quad + \frac{R^2}{4\lambda v} - \frac{\mu\mathbf{w}_x^2}{v\theta^2}. \end{aligned} \tag{3.3}$$

Set

$$h(t,x) = \frac{1}{\theta} - \frac{R^2 t}{4C_v}\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{1-a},$$

we can deduce that $h(t,x)$ satisfies

$$\begin{cases} C_v h_t \leq \left(\frac{\theta^b}{v}h_x\right)_x, & x\in\mathbf{R}, \quad 0\leq t\leq T, \\ h(0,x) = \frac{1}{\theta_0(x)} \leq \frac{1}{\underline{\Theta}}, \end{cases} \tag{3.4}$$

and the standard maximum principle for parabolic equation implies that $h(t,x) \leq \frac{1}{\underline{\Theta}}$ holds for all $(t,x) \in [0,T]\times\mathbf{R}$, that is, for $x\in\mathbf{R}, 0\leq t\leq T$

$$\frac{1}{\theta} - \frac{R^2 t}{4C_v}\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{1-a} \leq \frac{1}{\underline{\Theta}}. \tag{3.5}$$

Thus, the proof of Lemma 3.2 is completed. □

To use Kanel's method to deduce a lower bound and an upper bound on $v(t,x)$, we need to deduce an estimate on $\|\frac{v_x}{v^{1+a}}\|$, which is the main concern of our next lemma. It is worth to pointing out that it is in this step that we ask the viscous coefficient μ depends only on v.

Lemma 3.3. *Under the assumptions listed in* Lemma 3.1*, we have*

$$\begin{aligned} &\left\|\frac{v_x}{v^{1+a}}\right\|^2 + \int_0^t\int_{\mathbf{R}} \frac{\theta v_x^2}{v^{3+a}}\mathrm{d}x\mathrm{d}\tau \\ &\quad \leq C\left(\|v_{0x}\|^2 + \|(v_0-1,u_0,\theta_0-1)\|^2\right) + \int_0^t\int_{\mathbf{R}} \frac{u_x^2}{v^{1+a}}\mathrm{d}x\mathrm{d}s + C\int_0^t\int_{\mathbf{R}} \frac{\theta_x^2}{v^{1+a}\theta}\mathrm{d}x\mathrm{d}s \end{aligned} \tag{3.6}$$

Proof. Notice that

$$\left(\frac{v_x}{v^{1+a}}\right)_t = \left(\frac{v_t}{v^{1+a}}\right)_x = \left(\frac{u_x}{v^{1+a}}\right)_x = u_t + p(v,\theta)_x,$$

we have by multiplying the above identity by $\frac{v_x}{v^{1+a}}$ and integrating the resulting equation with respect to t and x over $[0,T] \times \mathbf{R}$ that

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{v_x}{v^{1+a}}\right\|^2 + \int_0^t \int_{\mathbf{R}} \frac{R\theta v_x^2}{v^{3+a}} \mathrm{d}x\mathrm{d}s \\
&\quad \le O(1)\|v_{0x}\|^2 + \underbrace{\int_0^t \int_{\mathbf{R}} \frac{R\theta_x v_x}{v^{2+a}} \mathrm{d}x\mathrm{d}s}_{I_1} + \underbrace{\int_0^t \int_{\mathbf{R}} \frac{u_t v_x}{v^{1+a}} \mathrm{d}x\mathrm{d}s}_{I_2},
\end{aligned} \tag{3.7}$$

As to I_1, we have

$$I_1 \le \frac{1}{2}\int_0^t \int_{\mathbf{R}} \frac{R\theta v_x^2}{v^{3+a}} \mathrm{d}x\mathrm{d}s + O(1)\int_0^t \int_{\mathbf{R}} \frac{\theta_x^2}{v^{1+a}\theta} \mathrm{d}x\mathrm{d}s. \tag{3.8}$$

From easily calculation, we have

$$\begin{aligned}
I_2 &= \int_{\mathbf{R}} \frac{uv_x}{v^{1+a}} \mathrm{dx}\Big|_0^t - \int_0^t \int_{\mathbf{R}} u\left(\frac{v_x}{v^{1+a}}\right)_t \mathrm{d}x\mathrm{d}s \\
&\le \int_{\mathbf{R}} \frac{uv_x}{v^{1+a}} \mathrm{d}x + O(1)\left\|(u_0, v_{0x})\right\|^2 - \int_0^t \int_{\mathbf{R}} u\left(\frac{u_x}{v^{1+a}}\right)_x \mathrm{d}x\mathrm{d}s \\
&\le \frac{1}{2}\left\|\frac{v_x}{v^{1+a}}\right\|^2 + O(1)\left\|(v_0 - 1, v_{0x}, u_0, \theta_0 - 1)\right\|^2 + \int_0^t \int_{\mathbf{R}} \frac{u_x^2}{v^{1+a}} \mathrm{d}x\mathrm{d}s.
\end{aligned} \tag{3.9}$$

Inserting the estimates of (3.8) and (3.9) into (3.7), we can get (3.6). This completes the proof of Lemma 2.5. □

To bound the two terms on the right-hand side of (3.6), we now estimate $\int_0^t \int_{\mathbf{R}} \frac{u_x^2}{v^{1+a}} \mathrm{d}x\mathrm{d}s$ in the following lemma.

Lemma 3.4. *Under the assumptions in* Lemma 3.1*, we have*

$$\|u(t)\|^2 + \int_0^t \int_{\mathbf{R}} \frac{u_x^2}{v^{1+a}} \mathrm{d}x\mathrm{d}s \le O(1)\|(v_0 - 1, u_0, \theta_0 - 1)\|^2 + O(1)\int_0^t \int_{\mathbf{R}} \frac{(\theta - 1)^2}{v^{1-a}} \mathrm{d}x\mathrm{d}s. \tag{3.10}$$

Proof. Multiplying $(1.9)_2$ by u, we have by integrating the resulting equation with respect to t and x over $[0,T]\times\mathbf{R}$ that

$$\begin{aligned}
&\frac{1}{2}\|u(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v^{1+a}}\mathrm{d}x\mathrm{d}s\\
&\quad\le O(1)\|u_0\|^2+\underbrace{\int_0^t\int_{\mathbf{R}}\frac{R(\theta-1)u_x}{v}\mathrm{d}x\mathrm{d}s}_{I_3}+\underbrace{\int_0^t\int_{\mathbf{R}}R\left(1-\frac{1}{v}\right)u_x\mathrm{d}x\mathrm{d}s}_{I_4}
\end{aligned}\tag{3.11}$$

From the basic energy estimate (3.1) and the Cauchy–Schwarz inequality, we can bound I_j $(j=3,4)$ as follows:

$$\begin{aligned}
I_4&=\int_0^t\int_{\mathbf{R}}R\left(1-\frac{1}{v}\right)v_t dxds=R\int_{\mathbf{R}}\Phi(v)\mathrm{d}x\Big|_0^t\\
&=R\left(\int_{\mathbf{R}}\Phi(v)\mathrm{d}x-\int_{\mathbf{R}}\Phi(v_0)\mathrm{d}x\right)\\
&\le O(1)\|(u_0,v_0-1,\theta_0-1,\Phi_{0x})\|^2,\\
I_3&\le\frac{1}{2}\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v^{1+a}}\mathrm{d}x\mathrm{d}s+O(1)\int_0^t\int_{\mathbf{R}}\frac{(\theta-1)^2}{v^{1-a}}\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Substituting the above estimates into (3.11), we can deduce (3.10) and complete the proof of the lemma. □

To bound the two terms on the right-hand side of (3.6) and (3.10), we need the following.

Lemma 3.5. *Under the assumptions in* Lemma 2.3, *we have for* $b\neq 0,-1$ *that*

$$\int_0^t\max_{x\in\mathbf{R}}|\theta(s,x)|^b\mathrm{d}s\le C(T),\tag{3.12}$$

$$\int_0^t\max_{x\in\mathbf{R}}|\theta(s,x)|^{b+1}\mathrm{d}s\le C(T)\left(1+\|\theta\|_{L^\infty_{T,x}}\right),\tag{3.13}$$

and

$$\int_0^t\max_{x\in\mathbf{R}}|\theta(s,x)|^{b+1}\mathrm{d}s\le C(T)\left(1+\|v\|_{L^\infty_{T,x}}\right).\tag{3.14}$$

Proof. We only prove (3.13) because (3.12) and (3.14) can be proved similarly.

From the argument used in [4], we have from the basic energy estimate (3.1), the Jenssen inequality that from each $i\in\mathbf{Z}$, there are positive constants $A_0>0, A_1>0$ such that

$$A_0\le\int_i^{i+1}v(t,x)\mathrm{d}x,\quad\int_i^{i+1}\theta(t,x)\mathrm{d}x\le A_1,\quad\forall t\in[0,T].\tag{3.15}$$

Hence, there exist $a_i(t)\in[i,i+1]$, $b_i(t)\in[i,i+1]$ such that

$$A_0\le v\left(t,a_i(t)\right),\quad\theta\left(t,b_i(t)\right)\le A_1.\tag{3.16}$$

Define

$$g_1(\theta)=\int_1^{\theta} s^{\frac{b-1}{2}}\mathrm{d}s=\frac{2}{b+1}\left(\theta^{\frac{b+1}{2}}-1\right),$$

for each $x\in\mathbf{R}$, there exists an integer $i\in\mathbf{Z}$ such that $x\in[i,i+1]$, and we can assume without the loss of generality that $x\geq b_i(t)$. Thus,

$$\begin{aligned}
g_1(\theta(t,x)) &= g_1\left(\theta(t,b_i(t))\right)+\int_{b_i(t)}^{x} g_1\left(\theta(t,y)\right)_y\mathrm{d}y\\
&\leq O(1)+\int_i^{i+1}\left|\theta^{\frac{b-1}{2}}\theta_x\right|\mathrm{d}x\\
&\leq O(1)+\left(\int_{\mathbf{R}}\frac{\theta_x^2}{v\theta^{2-b}}\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_i^{i+1}v\theta\mathrm{d}x\right)^{\frac{1}{2}}\\
&\leq O(1)+\|\theta\|_{L^\infty_{T,x}}^{\frac{1}{2}}\left(\int_{\mathbf{R}}\frac{\theta_x^2}{v\theta^{2-b}}\mathrm{d}x\right)^{\frac{1}{2}}.
\end{aligned}$$

The above estimate and (3.1) give (3.13) and then complete the proof of the lemma. □

As a direct corollary of (3.12)–(3.14), we have

Corollary 3.1. *Under the conditions listed in* Lemma 3.5, *we have*

$$\int_0^t\int_{\mathbf{R}}(\theta-1)^2\mathrm{d}x\mathrm{d}s\leq O(1)\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}. \tag{3.17}$$

Proof. In fact, (3.19) together with (3.12) implies that

$$\begin{aligned}
\int_0^t\int_{\mathbf{R}}(\theta-1)^2\mathrm{d}x\mathrm{d}\tau &\leq O(1)\int_0^t\int_{\mathbf{R}}(\theta+1)\phi(\theta)\mathrm{d}x\mathrm{d}\tau\\
&\leq O(1)\int_0^t\max_{x\in\mathbf{R}}\theta(\tau,x)\mathrm{d}\tau+O(1)\\
&= O(1)\int_0^t\max_{x\in\mathbf{R}}\left(\theta^{1-b}\theta^b\right)\mathrm{d}\tau+O(1)\\
&\leq O(1)\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}\int_0^t\max_{x\in\mathbf{R}}\theta^b(\tau,x)\mathrm{d}\tau+O(1)\\
&\leq O(1)\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}+O(1),
\end{aligned}$$

and this completes the proof of corollary. □

Having obtained (3.17), we can deduce that

$$\int_0^t\int_{\mathbf{R}}\frac{(\theta-1)^2}{v^{1-a}}\mathrm{d}x\mathrm{d}\tau\leq\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{1-a}\int_0^t\int_{\mathbf{R}}(\theta-1)^2\mathrm{d}x\mathrm{d}\tau\leq O(1)\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{1-a}. \tag{3.18}$$

On the other hand, from (3.1), we have

$$\begin{aligned}\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{\theta v^{1+a}}\mathrm{d}x\mathrm{d}\tau&=\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v\theta^{2-b}}\frac{1}{v^a\theta^{b-1}}\mathrm{d}x\mathrm{d}\tau\\&\leq\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{a}\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v\theta^{2-b}}dxd\tau\\&\leq O(1)\|(v_0-1,u_0,\mathbf{w}_0,\theta_0-1)\|^2\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{a}\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}.\end{aligned} \tag{3.19}$$

Substituting (3.18) and (3.19) into (3.10) and (3.6), we have

Corollary 3.2. *Under the assumptions in* Lemma 3.1*, we have*

$$\|u(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v^{1+a}}\mathrm{d}x\mathrm{d}\tau\leq O(1)\left\|(v_0-1,u_0,\mathbf{w}_0,\theta_0-1)\right\|^2+O(1)\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{a}\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}, \tag{3.20}$$

$$\begin{aligned}&\left\|\frac{v_x}{v^{1+a}}\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{\theta v_x^2}{v^{3+a}}\mathrm{d}x\mathrm{d}\tau\leq O(1)\left\|(v_0-1,u_0,\mathbf{w}_0,\theta_0-1)\right\|^2\\&\quad+O(1)\left(\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{a}+\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{1-a}\right)\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}.\end{aligned} \tag{3.21}$$

Now, we apply Kanel's approach to deduce a lower bound and an upper bound on $v(t,x)$ in terms of $\|\theta^{1-b}\|_{L^\infty_{T,x}}$. To this end, set

$$\Psi(v)=\int_1^v\frac{\sqrt{\phi(z)}}{z^{1+a}}\mathrm{d}z. \tag{3.22}$$

Note that there exist positive constants A_2, A_3 such that

$$|\Psi(v)|\geq A_2\left(v^{-a}+v^{\frac{1}{2}-a}\right)-A_3. \tag{3.23}$$

Since

$$\begin{aligned}|\Psi(v)|&=\left|\int_{-\infty}^x\Psi(v(t,y))_y\mathrm{d}y\right|\\&\leq\int_{\mathbf{R}}\left|\frac{\sqrt{\phi(v)}}{v^{1+a}}v_x\right|\mathrm{d}x\\&\leq\left\|\sqrt{\phi(v)}\right\|\left\|\frac{v_x}{v^{1+a}}\right\|\\&\leq O(1)\left(1+\left(\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{\frac{a}{2}}+\left\|\frac{1}{v}\right\|_{L^\infty_{T,x}}^{\frac{1-a}{2}}\right)\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{1}{2}}\right),\end{aligned} \tag{3.24}$$

we have from (3.23) and (3.24) that

$$\left\|\frac{1}{v}\right\|^{a}_{L^\infty_{T,x}}+\|v\|^{\frac{1}{2}-a}_{L^\infty_{T,x}}\le O(1)\left(1+\left(\left\|\frac{1}{v}\right\|^{\frac{a}{2}}_{L^\infty_{T,x}}+\left\|\frac{1}{v}\right\|^{\frac{1-a}{2}}_{L^\infty_{T,x}}\right)\|\theta^{1-b}\|^{\frac{1}{2}}_{L^\infty_{T,x}}\right). \tag{3.25}$$

Thus, if $\frac{1}{3}<a<\frac{1}{2}$, we can deduce from (3.25)

Corollary 3.3. *Under the conditions in* Lemma 3.1*, if we assume further that* $\frac{1}{3}<a<\frac{1}{2}$*, then we have*

$$\frac{1}{v(t,x)}\le O(1)\left(1+\|\theta^{1-b}\|^{\frac{1}{3a-1}}_{L^\infty_{T,x}}\right), \tag{3.26}$$

and

$$v(t,x)\le O(1)\left(1+\|\theta^{1-b}\|^{\frac{2a}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right) \tag{3.27}$$

hold for any $(t,x)\in[0,T]\times\mathbf{R}$*.*

Consequently, (3.20) *and* (3.21) *can be rewritten as*

$$\|u(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{u_x^2}{v^{1+a}}\,dx\,d\tau\le O(1)\left(1+\|\theta^{1-b}\|^{\frac{2a}{3a-1}}_{L^\infty_{T,x}}\right), \tag{3.28}$$

$$\left\|\frac{v_x}{v^{1+a}}\right\|^2+\int_0^t\int_{\mathbf{R}}\left(\frac{\theta v_x^2}{v^{3+a}}\right)dx\,d\tau\le O(1)\left(1+\|\theta^{1-b}\|^{\frac{2a}{3a-1}}_{L^\infty_{T,x}}\right). \tag{3.29}$$

To get the upper bound of $\theta(t,x)$, we need to get the estimates of u_x and $\mathbf{w}_x$. Thus, we have the following lemma:

Lemma 3.6. *Under the same conditions of* Theorem 1.2*, we have*

$$\frac{1}{2}\|\mathbf{w}(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{\mu(v)}{v}\mathbf{w}_x^2\mathrm{d}x\mathrm{d}\tau=\frac{1}{2}\|\mathbf{w}_0\|, \tag{3.30}$$

$$\|\mathbf{w}_x(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{\mu(v)}{v}\mathbf{w}_{xx}^2\mathrm{d}x\mathrm{d}\tau\le\|\mathbf{w}_{0x}\|^2+(1+c)^4\|\theta^{1-b}\|^{P_1}_{L^\infty_{t,x}}. \tag{3.31}$$

Here, we denote P_1 *as*

$$P_1=\frac{4a[2a-1-c]^++4a[1+c]^+}{(3a-1)(1-2a)}+\frac{2[2a-c-1]^-+2[1+c]^-+4a}{3a-1}.$$

Proof. From $(1.9)_3$, we get

$$\frac{1}{2}\|\mathbf{w}(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{\mu(v)}{v}\mathbf{w}_x^2\mathrm{d}x\mathrm{d}\tau=\frac{1}{2}\|\mathbf{w}_0\|. \tag{3.32}$$

$$\frac{1}{2}\|\mathbf{w}_x(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{\mu(v)}{v}\mathbf{w}_{xx}^2\mathrm{d}x\mathrm{d}\tau=\frac{1}{2}\|\mathbf{w}_{0x}\|-\int_0^t\int_{\mathbf{R}}\left(\frac{\mu(v)}{v}\right)_v v_x\mathbf{w}_x\mathbf{w}_{xx}\mathrm{d}x\mathrm{d}\tau. \tag{3.33}$$

From easily calculation with the Cauchy's inequality, it follows that

$$\|\mathbf{w}_x(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{\mu(v)}{v}\mathbf{w}_{xx}^2\mathrm{d}x\mathrm{d}\tau\le\|\mathbf{w}_{0x}\|^2+\underbrace{\int_0^t\int_{\mathbf{R}}\frac{v}{\mu(v)}\left(\left(\frac{\mu(v)}{v}\right)_v v_x\mathbf{w}_x\right)^2\mathrm{d}x\mathrm{d}\tau}_{I_5}. \tag{3.34}$$

As for I_5, since (3.32), and $\mu(v)=v^{-c}$, $\lambda(v)=v^{-a}$, we get

$$\begin{aligned}\int_0^t\|\mathbf{w}_x^2\|_{L_x^\infty}\mathrm{d}\tau&\le\int_0^t\|\mathbf{w}_x\|\|\|\mathbf{w}_{xx}\|\mathrm{d}\tau\\&\le\|v^{1+c}\|_{L^\infty_{t,x}}\left(\int_0^t\int_{\mathbf{R}}\frac{\mu(v)\mathbf{w}_x^2}{v}dx\mathrm{d}\tau\right)^{\frac12}\left(\int_0^t\int_{\mathbf{R}}\frac{\mu(v)\mathbf{w}_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau\right)^{\frac12}\\&\le O(1)\|\theta^{1-b}\|_{L^\infty_{t,x}}^{\frac{2a[1+c]^+}{(1-2a)(3a-1)}+\frac{[1+c]^-}{3a-1}}\left(\int_0^t\int_{\mathbf{R}}\frac{\mu(v)\mathbf{w}_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau\right)^{\frac12}.\end{aligned} \tag{3.35}$$

Thus, we have

$$I_5\le\int_0^t\int_{\mathbf{R}}\frac{v}{\mu(v)}\left(\left(\frac{\mu(v)}{v}\right)_v v_x\mathbf{w}_x\right)^2\mathrm{d}x\mathrm{d}\tau$$

$$\le\left\|\frac{v}{\mu(v)}\left(\frac{v}{\lambda(v)}\left(\frac{\mu(v)}{v}\right)_v\right)^2\right\|_{L^\infty_{t,x}}\int_0^t\|\mathbf{w}_x(\tau)\|^2_{L_x^\infty}\left\|\frac{\lambda(v)v_x}{v}\right\|^2\mathrm{d}\tau \tag{3.36}$$

$$\le O(1)(1+c)^2\left\|v^{2a-1-c}\right\|_{L^\infty_{t,x}}\left(1+\|\theta^{1-b}\|^{\frac{2a}{3a-1}}\right)\int_0^t\|\mathbf{w}_x\|^2_{L_x^\infty}\mathrm{d}\tau \tag{3.37}$$

$$\le O(1)(1+c)^4\|\theta^{1-b}\|^{\frac{4a[2a-1-c]^++4a[1+c]^+}{(3a-1)(1-2a)}+\frac{2[2a-c-1]^-+2[1+c]^-+4a}{3a-1}}+\varepsilon\int_0^t\int_{\mathbf{R}}\frac{\mu(v)\mathbf{w}_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau.$$

Plug the estimates of (3.36) into (3.34) and combine (3.26) and (3.27) together, we can get the proof of Lemma 3.6.

Lemma 3.7. *Under the conditions listed in* Lemma 3.1*, we have for* $0\le t\le T$ *that*

$$\begin{aligned}&\|u_x(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}\mathrm{d}x\mathrm{d}\tau\\&\le O(1)\left\|(v_0-1,u_0,\mathbf{w}_0,\theta_0-1)\right\|^2\\&\quad+O(1)\left\|\theta^{2-b}\right\|_{L^\infty_{T,x}}\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2a^2}{(3a-1)(1-2a)}}\right)\\&\quad+O(1)\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2(2a-2a^2+1)}{(3a-1)(1-2a)}}\right).\end{aligned} \tag{3.38}$$

Proof. By differentiating $(1.9)_2$ with respect to x, multiplying the resulting identity by u_x, and integrating the result with respect to t and x over $[0,T]\times\mathbf{R}$, we have

$$\begin{aligned}&\|u_x(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}\mathrm{d}x\mathrm{d}\tau\\&\le O(1)\|u_{0x}\|^2+\underbrace{2\int_0^t\int_{\mathbf{R}}u_{xx}p(v,\theta)_x\mathrm{d}x\mathrm{d}\tau}_{I_6}+\underbrace{2(1+a)\int_0^t\int_{\mathbf{R}}\frac{u_xv_xu_{xx}}{v^{2+a}}\mathrm{d}x\mathrm{d}\tau}_{I_7}.\end{aligned}\tag{3.39}$$

For I_6, we have from (3.1) that

$$\begin{aligned}I_6&=2R\int_0^t\int_{\mathbf{R}}u_{xx}\left(\frac{\theta_x}{v}-\frac{\theta v_x}{v^2}\right)\mathrm{d}x\mathrm{d}\tau\\&\le\frac14\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}\mathrm{d}x\mathrm{d}\tau+O(1)\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v^{1-a}}\mathrm{d}x\mathrm{d}\tau+O(1)\int_0^t\int_{\mathbf{R}}\frac{\theta^2v_x^2}{v^{3-a}}\mathrm{d}x\mathrm{d}\tau\\&\le\frac14\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}\mathrm{d}x\mathrm{d}\tau+O(1)\left\|\theta^{2-b}\right\|_{L^\infty_{T,x}}\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2a^2}{(3a-1)(1-2a)}}\right)\\&\quad+O(1)\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{7a-4a^2-1}{(3a-1)(1-2a)}}\right).\end{aligned}\tag{3.40}$$

Here, we have used the fact that

$$\begin{aligned}\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v^{1-a}}\mathrm{d}x\mathrm{d}\tau&=\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v\theta^{2-b}}v^a\theta^{2-b}\mathrm{d}x\mathrm{d}\tau\\&\le O(1)\|v\|^a_{L^\infty_{T,x}}\left\|\theta^{2-b}\right\|_{L^\infty_{T,x}}\int_0^t\int_{\mathbf{R}}\frac{\theta_x^2}{v\theta^{2-b}}\mathrm{d}x\mathrm{d}\tau\\&\le O(1)\|v\|^a_{L^\infty_{T,x}}\left\|\theta^{2-b}\right\|_{L^\infty_{T,x}}\\&\le O(1)\left\|\theta^{2-b}\right\|_{L^\infty_{T,x}}\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2a^2}{(3a-1)(1-2a)}}\right),\end{aligned}$$

and

$$\begin{aligned}\int_0^t\int_{\mathbf{R}}\frac{\theta^2v_x^2}{v^{3-a}}\mathrm{d}x\mathrm{d}\tau&=\int_0^t\int_{\mathbf{R}}\frac{v_x^2}{v^{2+2a}}\frac{\theta^2}{v^{1-3a}}\mathrm{d}x\mathrm{d}\tau\\&\le\int_0^t\int_{\mathbf{R}}\left(\max_{x\in\mathbf{R}}\theta^2(s,x)\mathrm{d}s\right)\|v\|^{3a-1}_{L^\infty_{T,x}}\left(\int_{\mathbf{R}}\frac{v_x^2}{v^{2+2a}}\mathrm{d}x\right)\mathrm{d}\tau\\&\le O(1)\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2a}{3a-1}}\right)\|v\|^{3a-1}_{L^\infty_{T,x}}\int_0^t\left(\max_{x\in\mathbf{R}}\theta^2(s,x)\mathrm{d}s\right)\end{aligned}$$

$$
\begin{aligned}
&\leq O(1)\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2a}{3a-1}}\right)\|v\|_{L^\infty_{T,x}}^{3a-1}\int_0^t\left(\max_{x\in\mathbf{R}}\theta^{1-b}\theta^{1+b}(s,x)\mathrm{d}s\right)\\
&\leq O(1)\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{5a-1}{3a-1}}\right)\|v\|_{L^\infty_{T,x}}^{3a-1}\int_0^t\left(\max_{x\in\mathbf{R}}\theta^{1+b}(s,x)\mathrm{d}s\right)\\
&\leq O(1)\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{5a-1}{3a-1}}\right)\|v\|_{L^\infty_{T,x}}^{3a-1}\left(1+\|v\|_{L^\infty_{T,x}}\right)\\
&\leq O(1)\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{7a-4a^2-1}{(3a-1)(1-2a)}}\right),
\end{aligned}
$$

where (3.1), (3.12)–(3.14), and (3.29) are used.

As for I_7, since (3.28), (3.29) together with the Sobolev inequality imply

$$
\begin{aligned}
\int_0^t\|u_x(\tau)\|_{L^\infty_x}^2\mathrm{d}\tau &\leq \int_0^t\|u_x(\tau)\|\|u_{xx}(\tau)\|\mathrm{d}\tau\\
&\leq\left(\int_0^t\|u_x(\tau)\|^2\mathrm{d}\tau\right)^{\frac12}\left(\int_0^t\|u_{xx}(\tau)\|^2\mathrm{d}\tau\right)^{\frac12}\\
&\leq\|v\|_{L^\infty_{T,x}}^{1+a}\left(\int_0^t\left\|\frac{u_x}{v^{\frac{1+a}{2}}}(\tau)\right\|^2\mathrm{d}\tau\right)^{\frac12}\left(\int_0^t\left\|\frac{u_{xx}}{v^{\frac{1+a}{2}}}(\tau)\right\|^2\mathrm{d}\tau\right)^{\frac12}\\
&\leq O(1)\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{3a}{(3a-1)(1-2a)}}\right)\left(\int_0^t\left\|\frac{u_{xx}}{v^{\frac{1+a}{2}}}(\tau)\right\|^2\mathrm{d}\tau\right)^{\frac12},
\end{aligned}
\tag{3.41}
$$

we can deduce from (3.26) and (3.27) that

$$
\begin{aligned}
I_7&\leq\frac14\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}\mathrm{d}x\mathrm{d}\tau+O(1)\int_0^t\int_{\mathbf{R}}\frac{u_x^2v_x^2}{v^{3+a}}\mathrm{d}x\mathrm{d}\tau\\
&\leq\frac14\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}\mathrm{d}x\mathrm{d}\tau+O(1)\int_0^t\left\|\frac{u_x^2}{v^{1-a}}\right\|_{L^\infty_x}\int_{\mathbf{R}}\frac{v_x^2}{v^{2+2a}}\mathrm{d}x\mathrm{d}\tau\\
&\leq\frac14\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}\mathrm{d}x\mathrm{d}\tau+O(1)\left\|\frac1v\right\|_{L^\infty_{T,x}}^{1-a}\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2a}{3a-1}}\right)\int_0^t\|u_x(\tau)\|_{L^\infty_x}^2\mathrm{d}\tau\\
&\leq\frac14\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}\mathrm{d}x\mathrm{d}\tau+O(1)\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2a-2a^2+1}{(3a-1)(1-2a)}}\right)\left(\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}\mathrm{d}x\mathrm{d}\tau\right)^{\frac12}\\
&\leq\frac12\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}\mathrm{d}x\mathrm{d}\tau+O(1)\left(1+\left\|\theta^{1-b}\right\|_{L^\infty_{T,x}}^{\frac{2\left(2a-2a^2+1\right)}{(3a-1)(1-2a)}}\right).
\end{aligned}
\tag{3.42}
$$

Putting (3.39), (3.40), and (3.42) together and noticing that $2(2a-2a^2+1)>7a-4a^2-1$ imply (3.38), this completes the proof of Lemma 3.7. □

Now, we turn to deduce the upper bound on $\theta(t,x)$.

Lemma 3.8. *Under the conditions in* Lemma 3.1, *we have*

$$\|\theta\|_{L^\infty_{T,x}} \le O(1)\left\{1+\int_0^t\left(\left\|\frac{u_x^2}{v^{1+a}}\right\|_{L^\infty_x}+\left\|\frac{\mathbf{w}_x^2}{v^{1+c}}\right\|_{L^\infty_x}+\left\|\frac{u_x^2}{v^2}\right\|_{L^\infty_x}+\|\theta\|^2_{L^\infty_x}\right)d\tau\right\}. \tag{3.43}$$

Proof. From $(1.9)_4$, it is easy to see that for each $p>1$,

$$\begin{aligned}&C_v\left[(\theta-1)^{2p}\right]_t+2p(2p-1)(\theta-1)^{2(p-1)}\frac{\theta^b\theta_x^2}{v}\\&=\left\{\frac{2p(\theta-1)^{2p-1}\theta^b\theta_x}{v}\right\}_x+\frac{2p(\theta-1)^{2p-1}}{v^{1+c}}\mathbf{w}_x^2+\frac{2p(\theta-1)^{2p-1}}{v^{1+a}}u_x^2-\frac{2pR\theta}{v}u_x(\theta-1)^{2p-1}.\end{aligned} \tag{3.44}$$

Integrating (3.44) with respect to x over $\mathbf{R}$ and using the Young's inequality, we get

$$\|\theta-1\|_{L^{2p}} \le O(1)+O(1)\int_0^t\left(\left\|\frac{u_x^2}{v^{1+a}}\right\|_{L^{2p}}+\left\|\frac{\mathbf{w}_x^2}{v^{1+c}}\right\|_{L^{2p}}+\left\|\frac{\theta u_x}{v}\right\|_{L^{2p}}\right)\mathrm{d}\tau. \tag{3.45}$$

Letting $p\to+\infty$, we get (3.43); thus, the proof of Lemma 3.8 is completed. □

We are now ready to use (3.26), (3.27), and (3.43) to deduce a lower bound and an upper bound on $\theta(t,x)$. Firstly, from (3.38) and (3.41), we have

$$\begin{aligned}&\int_0^t\|u_x(s)\|^2_{L^\infty_x}\mathrm{d}s\\&\le O(1)\left(1+\|\theta^{1-b}\|^{\frac{3a}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right)\\&\quad\times\left[\|\theta^{2-b}\|^{\frac12}_{L^\infty_{T,x}}\left(1+\|\theta^{1-b}\|^{\frac{a^2}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right)+1+\|\theta^{1-b}\|^{\frac{2a-2a^2+1}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right]\\&\le O(1)\|\theta^{2-b}\|^{\frac12}_{L^\infty_{T,x}}\left(1+\|\theta^{1-b}\|^{\frac{3a+a^2}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right)+O(1)\|\theta^{1-b}\|^{\frac{5a-2a^2+1}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}+O(1).\end{aligned} \tag{3.46}$$

Thus, we have from (3.26) and (3.27), and (3.41) that

$$\begin{aligned}&\int_0^t\left(\left\|\frac{u_x^2}{v^{1+a}}\right\|_{L^\infty_x}+\left\|\frac{u_x^2}{v^2}\right\|_{L^\infty_x}\right)\mathrm{d}\tau\\&\le O(1)\left(\left\|\frac1v\right\|^{1+a}_{L^\infty_{T,x}}+\left\|\frac1v\right\|^2_{L^\infty_{T,x}}\right)\int_0^t\|u_x(\tau)\|^2_{L^\infty_x}\mathrm{d}\tau\\&\le O(1)\left(1+\|\theta^{1-b}\|^{\frac{2}{3a-1}}_{L^\infty_{T,x}}\right)\int_0^t\|u_x(\tau)\|^2_{L^\infty_x}\mathrm{d}\tau\\&\le O(1)\|\theta^{2-b}\|^{\frac12}_{L^\infty_{T,x}}\left(1+\|\theta^{1-b}\|^{\frac{a^2-a+2}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right)+O(1)\|\theta^{1-b}\|^{\frac{3+a-2a^2}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}+O(1),\end{aligned} \tag{3.47}$$

and similarly, from (3.30) and (3.31), we get

$$
\begin{aligned}
&\int_0^t \left\| \frac{\mathbf{w}_x^2}{v^{1+c}} \right\|_{L_x^\infty} \mathrm{d}\tau \\
&\quad \le O(1) \left\| \left(\frac{1}{v}\right)^{1+c} \right\|_{L_{T,x}^\infty} \int_0^t \|\mathbf{w}_x(\tau)\|_{L_x^\infty}^2 \mathrm{d}\tau \\
&\quad \le O(1) \left\|\theta^{1-b}\right\|_{L_{T,x}^\infty}^{\frac{[1+c]^+}{3a-1}+\frac{2a[1+c]^-}{(1-2a)(3a-1)}} \int_0^t \|\mathbf{w}_x(\tau)\|_{L_x^\infty}^2 \mathrm{d}\tau \\
&\quad \le O(1) \left\|\theta^{1-b}\right\|_{L_{T,x}^\infty}^{\frac{[1+c]^+}{3a-1}+\frac{2a[1+c]^-}{(1-2a)(3a-1)}} \left\|v^{1+c}\right\|_{L_{t,x}^\infty} \left(\int_0^t \int_{\mathbf{R}} \frac{\mu(v)\mathbf{w}_x^2}{v} \mathrm{d}x\mathrm{d}\tau\right)^{\frac{1}{2}} \left(\int_0^t \int_{\mathbf{R}} \frac{\mu(v)\mathbf{w}_{xx}^2}{v} \mathrm{d}x\mathrm{d}\tau\right)^{\frac{1}{2}} \\
&\quad \le O(1) \left\|\theta^{1-b}\right\|_{L_{T,x}^\infty}^{\frac{[1+c]^+}{3a-1}+\frac{2a[1+c]^-}{(1-2a)(3a-1)}} \left\|\theta^{1-b}\right\|_{L_{T,x}^\infty}^{\frac{[1+c]^-}{3a-1}+\frac{2a[1+c]^-}{(1-2a)(3a-1)}} \left(\int_0^t \int_{\mathbf{R}} \frac{\mu(v)\mathbf{w}_{xx}^2}{v} \mathrm{d}x\mathrm{d}\tau\right)^{\frac{1}{2}} \\
&\quad \le O(1) \left\|\theta^{1-b}\right\|_{L_{T,x}^\infty}^{\frac{[1+c]^+ + [1+c]^-}{(1-2a)(3a-1)}} \left(\int_0^t \int_{\mathbf{R}} \frac{\mu(v)\mathbf{w}_{xx}^2}{v} \mathrm{d}x\mathrm{d}\tau\right)^{\frac{1}{2}} \\
&\quad \le O(1)(1+c)^2 \left\|\theta^{1-b}\right\|^{\frac{2a}{3a-1}+\frac{(1+2a)[1+c]^+ + (2-2a)[1+c]^- + 2a[2a-1-c]^+ + (1-2a)[2a-1-c]^-}{(1-2a)(3a-1)}}
\end{aligned}
\tag{3.48}
$$

Here, for brevity, if we denote P_2 as

$$
P_2 = \frac{2a}{3a-1} + \frac{(1+2a)[1+c]^+ + (2-2a)[1+c]^- + 2a[2a-1-c]^+ + (1-2a)[2a-1-c]^-}{(1-2a)(3a-1)},
$$

then (3.48) can be rewritten as

$$
\int_0^t \left\| \frac{\mathbf{w}_x^2}{v^{1+c}} \right\|_{L_x^\infty} \mathrm{d}\tau \le O(1)(1+c)^2 \|\theta^{1-b}\|_{L_{t,x}^\infty}^{P_2}. \tag{3.49}
$$

Also, from direct calculation, we have

$$
\begin{aligned}
\int_0^t \max_{x\in\mathbf{R}} \theta^2(s,x)\mathrm{d}s &\le \int_0^t \max_{x\in\mathbf{R}} \left(\theta^{1-b}(s,x)\theta^{b+1}(s,x)\right) \mathrm{d}s \\
&\le \left\|\theta^{1-b}\right\|_{L_{T,x}^\infty} \int_0^t \max_{x\in\mathbf{R}} \theta^{1+b}(s,x)\mathrm{d}s \\
&\le O(1) \left\|\theta^{1-b}\right\|_{L_{T,x}^\infty} \left(1 + \|v\|_{L_{T,x}^\infty}\right) \\
&\le O(1) \left(1 + \left\|\theta^{1-b}\right\|_{L_{T,x}^\infty}^{\frac{7a-6a^2-1}{(3a-1)(1-2a)}}\right).
\end{aligned}
\tag{3.50}
$$

Inserting (3.47), (3.49), and (3.50) into (3.43) yields

$$\begin{aligned}\|\theta\|_{L^\infty_{T,x}} \leq\ & O(1) + O(1)\left\|\theta^{2-b}\right\|^{\frac{1}{2}}_{L^\infty_{T,x}}\left(1+\left\|\theta^{1-b}\right\|^{\frac{a^2-a+2}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right)\\ & + O(1)\left\|\theta^{1-b}\right\|^{\frac{3+a-2a^2}{(3a-1)(1-2a)}}_{L^\infty_{T,x}} + O(1)\left\|\theta^{1-b}\right\|^{\frac{7a-6a^2-1}{(3a-1)(1-2a)}}_{L^\infty_{T,x}} + O(1)(1+c)^2\|\theta^{1-b}\|^{P_2}_{L^\infty_{t,x}}\\ \leq\ & O(1) + O(1)\left\|\theta^{2-b}\right\|^{\frac{1}{2}}_{L^\infty_{T,x}}\left(1+\left\|\theta^{1-b}\right\|^{\frac{a^2-a+2}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right)\\ & + O(1)\left\|\theta^{1-b}\right\|^{\frac{3+a-2a^2}{(3a-1)(1-2a)}}_{L^\infty_{T,x}} + O(1)(1+c)^2\|\theta^{1-b}\|^{P_2}_{L^\infty_{t,x}}. \end{aligned} \tag{3.51}$$

Then, based on (3.2), (3.26), and (3.27), we finally get

Corollary 3.4. *Under the assumptions in* Lemma 3.1, *we further assume that* $\frac{1}{3} < a < \frac{1}{2}$, *and one of the following conditions holds*

(i.) $1 \leq b < \frac{2a}{1-a} < 2$;

(ii.) $0 < b < 1, \frac{2-b}{2} + \frac{(a^2-a+2)(1-b)}{(3a-1)(1-2a)} < 1, \frac{(1-b)(3+a-2a^2)}{(3a-1)(1-2a)} < 1, |sign(1+c)|(1-b)(P_2) < 1.$

Then, there exists positive constants $V_1 > 0, \Theta_1 > 0$, *such that*

$$\begin{cases} V_1^{-1} \leq v(t,x) \leq V_1, \\ \Theta_1^{-1} \leq \theta(t,x) \leq \Theta_1. \end{cases} \tag{3.52}$$

Proof. We first consider the case $b \geq 1$. In this case, we have from (3.2), (3.26), and (3.27) that

$$\left\|\frac{1}{\theta}\right\|_{L^\infty_{T,x}} \leq O(1) + O(1)\left\|\theta^{1-b}\right\|^{\frac{1-a}{3a-1}}_{L^\infty_{T,x}} \leq O(1) + O(1)\left\|\frac{1}{\theta}\right\|^{\frac{(1-a)(b-1)}{3a-1}}_{L^\infty_{T,x}},$$

which, together with the assumption $b < \frac{2a}{1-a}$, implies that there exists a positive constant $\Theta_1 > 0$ such that

$$\theta(t,x) \geq \Theta_1^{-1} > 0, \quad \forall (t,x) \in [0,T] \times \mathbf{R}. \tag{3.53}$$

Moreover, (3.26) and (3.27), and (3.53) together with the fact that $b \geq 1$ imply that there exists a positive constant $V_1 > 0$, which may depends on T, such that

$$V_1^{-1} \leq v(t,x) \leq V_1, \quad \forall (t,x) \in [0,T] \times \mathbf{R}. \tag{3.54}$$

Thus, to prove (3.52), we only need to deduce the upper bound on $\theta(t,x)$. For this purpose, we have from the fact $1 \leq b < \frac{2a}{1-a} < 2$, that

$$\begin{aligned}\|\theta\|_{L^\infty_{T,x}} \leq\ & O(1) + O(1)\,\|\theta\|^{\frac{2-b}{2}}_{L^\infty_{T,x}}\left(1+\left\|\frac{1}{\theta}\right\|^{\frac{(a^2-a+2)(b-1)}{(3a-1)(1-2a)}}_{L^\infty_{T,x}}\right)\\ & + O(1)\left\|\frac{1}{\theta}\right\|^{\frac{(3+a-2a^2)(b-1)}{(3a-1)(1-2a)}}_{L^\infty_{T,x}} + O(1)(1+c)^2\left\|\frac{1}{\theta}\right\|^{P_2}_{L^\infty_{T,x}}\\ \leq\ & O(1)\left(1+\|\theta\|^{\frac{2-b}{2}}_{L^\infty_{T,x}}\right). \end{aligned} \tag{3.55}$$

From (3.55) and the fact that $0 < \frac{2-b}{2} < 1$, one can easily deduce an upper bound on $\theta(t,x)$. This completes the proof of (3.52) for the case $1 \leq b < \frac{2a}{1-a}$. □

When $b < 1$,

$$\|\theta\|_{L^\infty_{T,x}} \leq O(1) + O(1)\|\theta\|_{L^\infty_{T,x}}^{\frac{2-b}{2}+\frac{(a^2-a+2)(1-b)}{(3a-1)(1-2a)}} + O(1)\|\theta\|_{L^\infty_{T,x}}^{\frac{(3+a-2a^2)(1-b)}{(3a-1)(1-2a)}} + O(1)(1+c)^2\|\theta\|^{P_2}_{L^\infty_{T,x}}. \tag{3.56}$$

From (3.56) and the assumption (ii) of Corollary 3.4, we can deduce an upper bound on $\theta(t,x)$. With this, the lower and upper bound on $v(t,x)$ can be deduced from (3.26) and (3.27). And then, (3.2) implies the lower bound on $\theta(t,x)$. This completes the proof of the corollary.

From corollary 3.4, we can get Theorem 1.2.

4. The proof of Theorem 1.3

First of all, the local solvability of the Cauchy problem (1.7), (1.9) in the function space $X(0,t_1;\frac{1}{2}\underline{V},2\overline{V};\frac{1}{2}\underline{\Theta},2\overline{\Theta})$ with t_1 depending on $\underline{V},\overline{V},\underline{\Theta},\overline{\Theta},\|(v_0-1,v_0,\theta_0-1,\Phi_{0x})\|_1$ can be proved as in Lemma 3.1. Suppose this solution $(v(t,x),u(t,x),\theta(t,x),\Phi(t,x))$ is extended to $t=T\geq t_1$. To apply the continuation argument for global existence, we first set the following a priori assumption:

$$\|\theta(t,x)-1\|_2 \leq \epsilon, \quad (t,x)\in[0,T]\times\mathbf{R}. \tag{H$_3$}$$

Here, ϵ is small positive constant, and without the loss of generality, we can assume that $\epsilon < \frac{1}{2}$.

Note that the smallness of $\gamma-1$ is needed to close the a priori assumption, the generic constants used later are independent of $\gamma-1$, and whenever the dependence on this factor will be clearly stated in the estimates. First, just as the proof of Theorem 1.1, we get the following basic energy estimates:

Lemma 4.1. *Let the conditions listed in* Lemma 2.1 *hold and suppose that the local solution* $(v(t,x),u(t,x),\mathbf{w}(t,x),\theta(t,x))$ *constructed in* Lemma 2.1 *has been extended to the time step* $t=T\geq t_1$ *and satisfies the* a priori *assumption* (H$_3$)*, then we have for* $0\leq t\leq T$ *that*

$$\int_{\mathbf{R}}\eta(v,u,\mathbf{w},\theta)\mathrm{d}x + \int_0^t\int_{\mathbf{R}}\left(\frac{\kappa\theta_x^2}{v\theta^2}+\frac{\lambda u_x^2}{v\theta}+\frac{\mu|\mathbf{w}_x|^2}{v\theta}\right)(\tau,x)\mathrm{d}x\mathrm{d}\tau = \int_{\mathbf{R}}\eta(v_0,u_0,\mathbf{w}_0,\theta_0)(x)\mathrm{d}x. \tag{4.1}$$

Here, as in Sect. 2*,* $\eta(t,x)=C_v\phi(\theta)+R\phi(v)+\frac{u^2+|\mathbf{w}|^2}{2}$*, with* $\phi(x)=x-\ln x-1$.

Now, we turn to deduce an estimate on $\|\frac{\lambda(v)v_x}{v}\|$. For this, similar to Lemma 3.3, we can deduce

$$\begin{aligned}&\left\|\frac{\lambda(v)v_x}{v}\right\|^2 + \int_0^t\int_{\mathbf{R}}\frac{\lambda(v)\theta v_x^2}{v^3}\mathrm{d}x\mathrm{d}\tau\\ &\quad\leq O(1)\|v_{0x}\|^2 + O(1)\underbrace{\int_0^t\int_{\mathbf{R}}\frac{\lambda(v)u_x^2}{v}\mathrm{d}x\mathrm{d}\tau}_{J_1} + O(1)\underbrace{\int_0^t\int_{\mathbf{R}}\frac{\lambda(v)\theta_x^2}{v\theta}\mathrm{d}x\mathrm{d}\tau}_{J_2}.\end{aligned} \tag{4.2}$$

If the a priori estimate (H$_3$) holds, we have from (4.1) and the assumptions imposed on $\kappa(v,\theta)$ in Theorem 1.3 that

$$J_1 \leq O(1)\int_0^t\int_{\mathbf{R}}\frac{\lambda(v)u_x^2}{v\theta}\mathrm{d}x\mathrm{d}\tau \leq O(1)\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|^2, \tag{4.3}$$

and

$$\begin{aligned} J_2 &\le \int_0^t \int_{\mathbf{R}} \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2} \frac{\theta\lambda(v)}{\kappa(v,\theta)} \mathrm{d}x\mathrm{d}\tau \\ &\le O(1) \left\| \frac{\lambda(v)}{\kappa_1(v)} \right\|_{L^\infty} \int_0^t \int_{\mathbf{R}} \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2} \mathrm{d}x\mathrm{d}\tau \\ &\le O(1) \left\| \left(v_0 - 1, u_0, \mathbf{w}_0, \frac{\theta_0 - 1}{\sqrt{\gamma - 1}} \right) \right\|^2 \left\| \frac{\lambda(v)}{\kappa_1(v)} \right\|_{L^\infty}. \end{aligned} \tag{4.4}$$

Putting (4.2)–(4.4) together, we obtain

Lemma 4.2. *Under the assumptions in* Lemma 4.1 *and the* a priori *assumption* (H_3)*, we have*

$$\left\| \frac{\lambda(v)v_x}{v} \right\|^2 + \int_0^t \int_{\mathbf{R}} \frac{\lambda(v)v_x^2}{v^3} \mathrm{d}x\mathrm{d}\tau \le O(1) \left\| \left(v_0 - 1, u_0, \mathbf{w}_0, \frac{\theta_0 - 1}{\sqrt{\gamma - 1}}, v_{0x} \right) \right\|^2 \left(1 + \left\| \frac{\lambda(v)}{\kappa_1(v)} \right\|_{L^\infty_{T,x}} \right). \tag{4.5}$$

Having obtained (4.1) and (4.5), we can use Kanel's argument, cf. [2], to deduce the lower and upper bounds on $v(t,x)$ as follows.

Lemma 4.3. *Under the assumptions in* Theorem 1.3 *and* Lemma 4.2*, there exists a positive constant* $V_2 \ge 1$*, which depends only on* $\|(v_0 - 1, u_0, \frac{\theta_0 - 1}{\sqrt{\gamma-1}}, v_{0x})\|$, $\underline{V}, \overline{V}, \underline{\Theta}$*, and* $\overline{\Theta}$*, but is independent of* T*, such that*

$$V_2^{-1} \le v(t,x) \le V_2, \ \ (t,x) \in [0,T] \times \mathbf{R}. \tag{4.6}$$

Proof. Define

$$\Psi(v) = \int_1^v \frac{\sqrt{\phi(z)}}{z} \lambda(z) \mathrm{d}z, \quad \phi(z) = z - \ln z - 1,$$

and notice that

$$\begin{aligned} |\Psi(v)| &= \left| \int_{-\infty}^x \Psi(v(t,y))_y \mathrm{d}y \right| \\ &\le \int_{\mathbf{R}} \left| \sqrt{\phi(v)} \frac{\lambda(v)v_x}{v} \right| \mathrm{d}x \\ &\le \|\phi(v)\|_{L^1}^{\frac{1}{2}} \left\| \frac{\lambda(v)v_x}{v} \right\| \\ &\le O(1) \left\| \left(v_0 - 1, u_0, \mathbf{w}_0, \frac{\theta_0 - 1}{\sqrt{\gamma - 1}}, v_{0x} \right) \right\|^2 \left(1 + \left\| \frac{\lambda(v)}{\kappa_1(v)} \right\|_{L^\infty_{T,x}} \right)^{\frac{1}{2}}. \end{aligned}$$

It is straightforward to deduce (4.6) from the assumptions in Theorem 1.3. This completes the proof of the lemma. □

Thus, combining (4.5) and (4.6), we actually get

$$\left\| \frac{\lambda(v)v_x}{v} \right\|^2 + \int_0^t \int_{\mathbf{R}} \frac{\lambda(v)v_x^2}{v^3} \mathrm{d}x\mathrm{d}\tau \le O(1) \left\| \left(v_0 - 1, u_0, \mathbf{w}_0, \frac{\theta_0 - 1}{\sqrt{\gamma - 1}}, v_{0x} \right) \right\|^2. \tag{4.7}$$

From easily calculation, we can deduce

Lemma 4.4. *Under the same conditions of* Theorem 1.3, *we can get*

$$\|(u,\mathbf{w})(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{\lambda(v)}{v}u_x^2+\frac{\mu(v)}{v}\mathbf{w}_x^2\,dx\,d\tau\le O(1)\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|^2. \tag{4.8}$$

Similar to Lemmas 3.6 and 3.7, we can get the estimates of u_x and $\mathbf{w}_x$.

Lemma 4.5. *Under the assumptions in* Lemma 4.3, *we have for each* $0\le t\le T$ *that*

$$\|u_x(t)\|^2+\|\mathbf{w}_x\|^2+\int_0^t\int_{\mathbf{R}}\frac{u_{xx}^2}{v^{1+a}}+\frac{|\mathbf{w}_{xx}|^2}{v^{1+c}}\,dx\,d\tau\le O(1)\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^6. \tag{4.9}$$

Here, for brevity, we omit the proof of Lemma 4.5. To close the a priori estimate (H$_3$), we need to deduce an estimate on $\|\theta_x(t)\|$.

Now, we turn to the case when $\kappa(v,\theta)$ depends on both v and θ. For this , we have

Lemma 4.6. *Under the assumptions in* Lemma 4.1, *if the assumption* H$_3$ *holds true, then we have*

$$\left\|\frac{\theta_x(t)}{\sqrt{\gamma-1}}\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{\kappa(v,\theta)}{v}\theta_{xx}^2\mathrm{d}x\mathrm{d}\tau\le O(1)\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^{10}. \tag{4.10}$$

Proof. Differentiating (1.9) $_4$ with respect to x and multiplying the resulting equation by θ_x, we have by integrating it over $[0,t]\times\mathbf{R}$ that

$$\begin{aligned}
&\frac{C_v}{2}\|\theta_x(t)\|^2+\int_0^t\int_{\mathbf{R}}\frac{\kappa(v,\theta)}{v}\theta_{xx}^2\mathrm{d}x\mathrm{d}\tau\\
&=\frac{C_v}{2}\|\theta_{0x}\|^2+\underbrace{\int_0^t\int_{\mathbf{R}}\theta_x\left(\frac{\lambda(v)}{v}u_x^2\right)_x\mathrm{d}x\mathrm{d}\tau}_{J_3}+\underbrace{\int_0^t\int_{\mathbf{R}}\theta_x\left(\frac{\mu(v)}{v}\mathbf{w}_x^2\right)_x\mathrm{d}x\mathrm{d}\tau}_{J_4}\\
&\quad-\underbrace{\int_0^t\int_{\mathbf{R}}\theta_x\left(\frac{\kappa(v,\theta)}{v}\right)_x\theta_{xx}\mathrm{d}x\mathrm{d}\tau}_{J_5}+\underbrace{\int_0^t\int_{\mathbf{R}}\theta_{xx}p(v,\theta)u_x\mathrm{d}x\mathrm{d}\tau}_{J_6}.
\end{aligned} \tag{4.11}$$

For J_3,J_4,J_5, and J_6, we have from Lemmas 4.1–4.6 and the a priori estimate (H$_3$) that

$$\begin{aligned}
J_3&=-\int_0^t\int_{\mathbf{R}}\frac{\lambda(v)}{v}u_x^2\theta_{xx}\mathrm{d}x\mathrm{d}\tau\\
&\le\frac{1}{6}\int_0^t\int_{\mathbf{R}}\frac{\kappa(v,\theta)}{v}\theta_{xx}^2\mathrm{d}x\mathrm{d}\tau+O(1)\int_0^t\int_{\mathbf{R}}u_x^4\mathrm{d}x\mathrm{d}\tau\\
&\le\frac{1}{6}\int_0^t\int_{\mathbf{R}}\frac{\kappa(v,\theta)}{v}\theta_{xx}^2\mathrm{d}x\mathrm{d}\tau+O(1)\int_0^t\int_{\mathbf{R}}\|u_x(\tau)\|^3\|u_{xx}(\tau)\|\mathrm{d}x\mathrm{d}\tau\\
&\le\frac{1}{6}\int_0^t\int_{\mathbf{R}}\frac{\kappa(v,\theta)}{v}\theta_{xx}^2\mathrm{d}x\mathrm{d}\tau+O(1)\left\|\left((v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^{10},
\end{aligned} \tag{4.12}$$

In the same way, we can get

$$J_4 = -\int_0^t\int_{\mathbf{R}} \frac{\mu(v)}{v}\mathbf{w}_x^2\theta_{xx}\mathrm{d}x\mathrm{d}\tau \le \frac{1}{6}\int_0^t\int_{\mathbf{R}} \frac{\kappa(v,\theta)}{v}\theta_{xx}^2\mathrm{d}x\mathrm{d}\tau + O(1)\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^{10}. \tag{4.13}$$

$$\begin{aligned} J_6 &\le \frac{1}{6}\int_0^t\int_{\mathbf{R}} \frac{\kappa(v,\theta)}{v}\theta_{xx}^2\mathrm{d}x\mathrm{d}\tau + O(1)\int_0^t\int_{\mathbf{R}} u_x^2\mathrm{d}x\mathrm{d}\tau \\ &\le \frac{1}{6}\int_0^t\int_{\mathbf{R}} \frac{\kappa(v,\theta)}{v}\theta_{xx}^2\mathrm{d}x\mathrm{d}\tau + O(1)\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^{2}, \end{aligned} \tag{4.14}$$

As to J_5, from the assumption H_3, there is positive constant $C>0$, such that $\|\theta_x(t)\|_{L^\infty}\le C$ and also from (4.7), we can deduce

$$\begin{aligned} J_9 &\le O(1)\int_0^t\int_{\mathbf{R}} \theta_x^4\mathrm{d}x\mathrm{d}\tau + \frac{1}{6}\int_0^t\int_{\mathbf{R}} \frac{\kappa(v,\theta)}{v}\theta_{xx}^2\mathrm{d}x\mathrm{d}\tau + O(1)\int_0^t\int_{\mathbf{R}} \theta_x^2v_x^2\mathrm{d}x\mathrm{d}\tau \\ &\le \frac{1}{6}\int_0^t\int_{\mathbf{R}} \frac{\kappa(v,\theta)}{v}\theta_{xx}^2\mathrm{d}x\mathrm{d}\tau + O(1)\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^{6}. \end{aligned} \tag{4.15}$$

□

Inserting (4.12)–(4.15) into (4.11), we can deduce (4.10), and then, we complete the proof of Lemma 4.7. Lemmas 4.1–4.7 imply that under the a priori estimate (H_3), there exist two positive constants $V_2\ge 1$ and $C_1\ge 1$ with V_2 depending only on $\|(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},v_{0x})\|$, $\underline{V},\overline{V},\underline{\Theta}$, and $\overline{\Theta}$ but independent of T and $\gamma-1$, and C_1 depending only on V_2 but independent of $T>0$ and $\gamma-1$, such that

$$\begin{aligned} &V_2^{-1}\le v(t,x)\le V_2, \quad (t,x)\in[0,T]\times\mathbf{R}, \\ &\left\|\left(v-1,u,\mathbf{w},\frac{\theta-1}{\sqrt{\gamma-1}}\right)(t)\right\|^2 + \int_0^t\int_{\mathbf{R}}\left(u_x^2+\theta_x^2\right)(\tau,x)\mathrm{d}x\mathrm{d}\tau \le C_1\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|^2, \\ &\|v_x(t)\|^2 + \int_0^t\int_{\mathbf{R}} v_x^2(\tau,x)\mathrm{d}x\mathrm{d}\tau \le C_1\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^2, \\ &\|u_x(t)\|^2 + \int_0^t\int_{\mathbf{R}} u_{xx}^2(\tau,x)\mathrm{d}x\mathrm{d}\tau \le C_1\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^6, \\ &\|\mathbf{w}_x(t)\|^2 + \int_0^t\int_{\mathbf{R}} \mathbf{w}_{xx}^2(\tau,x)\mathrm{d}x\mathrm{d}\tau \le C_1\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^6, \\ &\left\|\frac{\theta_x(\tau)}{\sqrt{\gamma-1}}\right\|^2 + \int_0^t\int_{\mathbf{R}} \theta_{xx}^2(\tau,x)\mathrm{d}x\mathrm{d}\tau \le C_1\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^{10} \end{aligned} \tag{4.16}$$

hold for $0\le t\le T$.

To obtain the global existence of solutions, we only need to close the a priori estimate (H_3); thus, our main job is getting the higher estimate of $\theta(t,x)$.

Differentiate $(1.9)_3$ twice with respect to x and multiply the result with θ_{xx}, then it follows that

$$
\begin{aligned}
&\frac{1}{2}\left\|\frac{\theta_{xx}}{\sqrt{\gamma-1}}\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xxx}^2}{v}\mathrm{d}x\mathrm{d}\tau \\
&=\frac{1}{2}\left\|\frac{\theta_{0xx}}{\sqrt{\gamma-1}}\right\|^2+\underbrace{\int_0^t\int_{\mathbf{R}}\left(\frac{\lambda(v)u_x^2}{v}\right)_x\theta_{xxx}\mathrm{d}x\tau}_{J_7}+\underbrace{\int_0^t\int_{\mathbf{R}}\left(\frac{\mu(v)\mathbf{w}_x{}_x^2}{v}\right)_x\theta_{xxx}\mathrm{d}x\tau}_{J_8} \\
&\quad+\underbrace{\int_0^t\int_{\mathbf{R}}\left(\frac{R\theta u_x}{v}\right)_x\theta_{xxx}\mathrm{d}x\mathrm{d}\tau}_{J_9}-\underbrace{\int_0^t\int_{\mathbf{R}}\left(\left(\frac{\kappa\theta_x}{v}\right)_{xx}-\frac{\kappa\theta_{xxx}}{v}\right)\theta_{xxx}\mathrm{d}x\tau}_{J_{10}}. \qquad (4.17)
\end{aligned}
$$

Notice that, from (4.16), we have

$$
\begin{aligned}
J_7&=\int_0^t\int_{\mathbf{R}}\left(\frac{\lambda(v)u_x^2}{v}\right)_x\theta_{xxx}\mathrm{d}x\tau \\
&\le\frac{1}{6}\int_0^t\int_{\mathbf{R}}(\frac{\kappa}{v}\theta_{xxx}^2)\mathrm{d}x\mathrm{d}\tau+C\int_0^t\int_{\mathbf{R}}(v_x^2u_x^4+u_{xx}^2u_x^2)\mathrm{d}x\mathrm{d}\tau \\
&\le\frac{1}{6}\int_0^t\int_{\mathbf{R}}\frac{\kappa}{v}\theta_{xxx}^2\mathrm{d}x\mathrm{d}\tau+C\int_0^t\int_{\mathbf{R}}u_x^2u_{xx}^2\mathrm{d}x\mathrm{d}\tau+C\int_0^t\|v_x\|^2\|u_x\|^2\|u_{xx}\|^2\mathrm{d}\tau \\
&\le\frac{1}{6}\int_0^t\int_{\mathbf{R}}\frac{\kappa}{v}\theta_{xxx}^2\mathrm{d}x\mathrm{d}\tau+C\int_0^t\int_{\mathbf{R}}u_x^2u_{xx}^2\mathrm{d}x\mathrm{d}\tau+C\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^{14}. \qquad (4.18)
\end{aligned}
$$

Also, we can obtain

$$
\begin{aligned}
J_8&=\int_0^t\int_{\mathbf{R}}\left(\frac{\mu(v)\mathbf{w}_x{}_x^2}{v}\right)_x\theta_{xxx}\mathrm{d}x\mathrm{d}\tau \\
&\le\frac{1}{6}\int_0^t\int_{\mathbf{R}}\frac{\kappa}{v}\theta_{xxx}^2\mathrm{d}x\mathrm{d}\tau+C\int_0^t\int_{\mathbf{R}}(v_x^2|\mathbf{w}_x|^4+|\mathbf{w}_{xx}^2||\mathbf{w}_x|^2)\mathrm{d}x\mathrm{d}\tau \\
&\le\frac{1}{6}\int_0^t\int_{\mathbf{R}}\frac{\kappa}{v}\theta_{xxx}^2\mathrm{d}x\mathrm{d}\tau+C\int_0^t\int_{\mathbf{R}}|\mathbf{w}_{xx}^2||\mathbf{w}_x|^2\mathrm{d}x\mathrm{d}\tau+C\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^{14}. \qquad (4.19)
\end{aligned}
$$

In the same method as above, we can get

$$
\begin{aligned}
J_9 &= \int_0^t \int_{\mathbf{R}} \left(\frac{R\theta u_x}{v}\right)_x \theta_{xxx} \mathrm{d}x\mathrm{d}\tau \\
&\le \frac{1}{6}\int_0^t \int_{\mathbf{R}} \frac{\kappa}{v}\theta_{xxx}^2 \mathrm{d}x\mathrm{d}\tau + C\int_0^t \int_{\mathbf{R}} (u_x^2(\theta_x^2+v_x^2)+u_{xx}^2)\mathrm{d}x\mathrm{d}\tau \\
&\le \frac{1}{6}\int_0^t \int_{\mathbf{R}} \frac{\kappa}{v}\theta_{xxx}^2 \mathrm{d}x\mathrm{d}\tau + C\left\|\left(v_0-1, u_0, \mathbf{w}_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^{30}.
\end{aligned} \tag{4.20}
$$

Our next job is getting the estimate of J_{10}, here

$$
\begin{aligned}
J_{10} &= \int_0^t \int_{\mathbf{R}} \left(\left(\frac{\kappa\theta_x}{v}\right)_{xx} - \frac{\kappa\theta_{xxx}}{v}\right)\theta_{xxx}\mathrm{d}x\tau \\
&\le \frac{1}{6}\int_0^t \int_{\mathbf{R}} \frac{\kappa}{v}\theta_{xxx}^2 \mathrm{d}x\mathrm{d}\tau + C\int_0^t \int_{\mathbf{R}} (\theta_x^6 + \theta_x^2 v_x^4 + \theta_x^2\theta_{xx}^2 + \theta_x^2 v_{xx}^2 + v_x^2\theta_{xx}^2)\mathrm{d}x\mathrm{d}\tau \\
&\le \frac{1}{6}\int_0^t \int_{\mathbf{R}} \frac{\kappa}{v}\theta_{xxx}^2 \mathrm{d}x\mathrm{d}\tau + C\int_0^t \int_{\mathbf{R}} ((v_x^2+\theta_x^2)\theta_{xx}^2 + \theta_x^2 v_{xx}^2)\mathrm{d}x\mathrm{d}\tau \\
&\quad + C\int_0^t (\|\theta_x\|^4\|\theta_{xx}\|^2 + \|v_x\|^2\|\theta_x\|^2\|v_{xx}\|^2)\mathrm{d}\tau \\
&\le \frac{1}{6}\int_0^t \int_{\mathbf{R}} \frac{\kappa}{v}\theta_{xxx}^2 \mathrm{d}x\mathrm{d}\tau + C\int_0^t \int_{\mathbf{R}} ((v_x^2+\theta_x^2)\theta_{xx}^2 + \theta_x^2 v_{xx}^2)\mathrm{d}x\mathrm{d}\tau \\
&\quad + C\left\|\left(v_0-1, u_0, \mathbf{w}_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^{30}.
\end{aligned} \tag{4.21}
$$

Thus, input the estimate of J_7, J_8, J_9, J_{10} into (4.17), we get the following lemma:

Lemma 4.7. *Under the condition listed in* Lemma 4.1*, if the* a priori *assumption* H_3 *holds, then we get*

$$
\begin{aligned}
\left\|\frac{\theta_{xx}}{\sqrt{\gamma-1}}\right\|^2 + \int_0^t \int_{\mathbf{R}} \frac{\kappa(\theta)\theta_{xxx}^2}{v}\mathrm{d}x\mathrm{d}\tau &\le C\left\|\left(v_0-1, u_0, \mathbf{w}_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^{30} \\
&\quad + C\int_0^t \int_{\mathbf{R}} \left(\left(u_x^2+v_x^2+\theta_x^2+\mathbf{w}_x^2\right)\left(\theta_{xx}^2+u_{xx}^2+\mathbf{w}_{xx}^2\right)+\theta_x^2 v_{xx}^2\right)\mathrm{d}x\mathrm{d}\tau.
\end{aligned} \tag{4.22}
$$

As the second-order estimate of $u_{xx}, \mathbf{w}_{xx}$, we have the following lemma:

Lemma 4.8. *Under the same condition listed in* Lemma 4.7, *we have*

$$\|u_{xx}\|^2 + \int_0^t \int_{\mathbf{R}} \frac{\lambda}{v} u_{xxx}^2 \le C \left\| \left(v_0 - 1, u_0, \mathbf{w}_0, \frac{\theta_0 - 1}{\sqrt{\gamma - 1}} \right) \right\|_2^{10} + \int_0^t \int_{\mathbf{R}} (u_x^2 (v_{xx}^2 + v_x^4) + u_{xx}^2 v_x^2) \mathrm{d}x \mathrm{d}\tau + \int_0^t \|v_{xx}\|^2 \mathrm{d}\tau, \tag{4.23}$$

$$\|\mathbf{w}_{xx}\|^2 + \int_0^t \int_{\mathbf{R}} \frac{\mu}{v} u_{xxx}^2 \le C \left\| \left(v_0 - 1, u_0, \mathbf{w}_0, \frac{\theta_0 - 1}{\sqrt{\gamma - 1}} \right) \right\|_1^{10} + \int_0^t \int_{\mathbf{R}} \mathbf{w}_x^2 \left(v_{xx}^2 + v_x^4 \right) \mathrm{d}x \mathrm{d}\tau. \tag{4.24}$$

Just perform the same calculation as Lemma 4.7, we can get Lemma 4.8. Here, for brevity, we omit the proof.

We will give the estimate for v_{xx} in the following. First, since

$$\left(\frac{\mu(v) v_x}{v} \right)_t = \left(\frac{\mu(v) v_t}{v} \right)_x ;$$

thus, there is

$$\left(\frac{\mu(v) v_t}{v} \right)_x = u_t + \left(\frac{R\theta}{v} \right)_x . \tag{4.25}$$

Differentiate (4.25) with respect to x and then multiply the result by $(\frac{\mu(v) v_x}{v})_x$, we can get

$$\begin{aligned} \frac{1}{2} \left[\left(\frac{\mu v_x}{v} \right)_x \right]_t^2 &= \left(\frac{\mu(v) v_x}{v} \right)_x u_{tx} + \left(\frac{R\theta}{v} \right)_{xx} \left(\frac{\mu(v) v_x}{v} \right)_x \\ &= \left[\left(\frac{\mu(v) v_x}{v} \right)_x u_x \right]_t - \left(\frac{\mu(v) v_x}{v} \right)_{tx} u_x + \left(\frac{R\theta}{v} \right)_{xx} \left(\frac{\mu(v) v_x}{v} \right)_x \\ &= \left[\left(\frac{\mu(v) v_x}{v} \right)_x u_x \right]_t - \left[\left(\frac{\mu(v) u_x}{v} \right)_x u_x \right]_x + \left(\frac{\mu(v) u_x}{v} \right)_x u_{xx} \\ &\quad + \left(\frac{\mu(v) v_x}{v} \right)_x \left(\frac{R\theta_{xx}}{v} + \frac{2R\theta_x v_x - R\theta v_{xx}}{v^2} + \frac{2Rv_x^2}{v^3} \right). \end{aligned} \tag{4.26}$$

Integrating the above equation over $[0, t] \times \mathbf{R}$, we have

$$\frac{1}{2} \left\| \left(\frac{\mu v_x}{v} \right)_x (t) \right\|^2 + \int_0^t \int_{\mathbf{R}} \frac{\mu v_{xx}^2}{v} \mathrm{d}x \mathrm{d}\tau = \underbrace{\int_{\mathbf{R}} \left(\frac{\mu(v) v_x}{v} \right)_x u_x(t) - \left(\frac{\mu(v) v_x}{v} \right)_x u_x(0) \mathrm{d}x}_{J_{11}} + \frac{1}{2} \left\| \left(\frac{\mu v_x}{v} \right)_x (0) \right\|^2 + \underbrace{\int_0^t \int_{\mathbf{R}} \left(\frac{\mu(v) u_x}{v} \right)_x u_{xx} \mathrm{d}x \mathrm{d}\tau}_{J_{12}}$$

$$
\underbrace{+\int_0^t\int_{\mathbf{R}}\left(\frac{\mu(v)}{v}\right)' v_x^2\left(\frac{R\theta_{xx}}{v}+\frac{2R\theta_x v_x-R\theta v_{xx}}{v^2}+\frac{2Rv_x^2}{v^3}\right)\mathrm{d}x\mathrm{d}\tau}_{J_{13}}
$$

$$
\underbrace{+\int_0^t\int_{\mathbf{R}}\frac{\mu v_{xx}}{v}\left(\frac{R\theta_{xx}}{v}+\frac{2R\theta_x v_x}{v^2}+\frac{2Rv_x^2}{v^3}\right)\mathrm{d}x\mathrm{d}\tau}_{J_{14}}. \tag{4.27}
$$

From easily calculation, with (4.16), we have

$$
\begin{aligned}
J_{12}&=\int_0^t\int_{\mathbf{R}}\left(\frac{\mu(v)u_x}{v}\right)_x u_{xx}\mathrm{d}x\mathrm{d}\tau\le O(1)\int_0^t\int_{\mathbf{R}}(u_{xx}^2+v_x^2)\mathrm{d}x\mathrm{d}\tau\\
&\le C\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^6. &(4.28)\\
J_{11}&=\int_{\mathbf{R}}\left(\left(\frac{\mu(v)v_x}{v}\right)_x u_x(t)-\left(\frac{\mu(v)v_x}{v}\right)_x u_x(0)\right)\mathrm{d}x\\
&\le\frac14\left\|\left(\frac{\mu(v)v_x}{v}\right)_x(t)\right\|^2+C\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_2^2, &(4.29)\\
J_{13}+J_{14}&\le O(1)\int_0^t\int_{\mathbf{R}}(v_x^4+\theta_x^4+\theta_{xx}^2)\mathrm{d}x\mathrm{d}\tau+\frac14\left\|\left(\frac{\mu(v)v_x}{v}\right)_x(t)\right\|^2\mathrm{d}x\mathrm{d}\tau\\
&\le O(1)\int_0^t\int_{\mathbf{R}}(v_x^4+\theta_x^4)\mathrm{d}x\mathrm{d}\tau+\frac14\left\|\left(\frac{\mu(v)v_x}{v}\right)_x(t)\right\|^2\mathrm{d}x\mathrm{d}\tau\\
&\quad+C\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^{10}, &(4.30)
\end{aligned}
$$

Thus, plug the estimate of (4.28)–(4.30) into (4.27), it follows that

$$
\begin{aligned}
&\left\|\left(\frac{\mu v_x}{v}\right)_x(t)\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{\mu v_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau\\
&\le O(1)\int_0^t\int_{\mathbf{R}}(v_x^4+\theta_x^4)\mathrm{d}x\mathrm{d}\tau+C\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_2^{10}\\
&\le O(1)\int_0^t(\|v_x\|^3\|v_{xx}\|+\|\theta_x\|^3\|\theta_{xx}\|)\mathrm{d}\tau+C\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_2^{10}\\
&\le C\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_2^{16}+\frac12\int_0^t\int_{\mathbf{R}}\frac{\mu v_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau; &(4.31)
\end{aligned}
$$

thus, we have the following lemma:

Lemma 4.9. *Under the condition of* Theorem 1.3*, there is positive constant* $C > 0$*, such that*

$$\left\|\left(\frac{\mu v_x}{v}\right)_x(t)\right\|^2 + \int_0^t\int_{\mathbf{R}} \frac{\mu v_{xx}^2}{v}\mathrm{d}x\mathrm{d}\tau \le C\left\|\left(v_0-1, u_0, \mathbf{w}_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_2^{16}. \tag{4.32}$$

Since $(\frac{\mu v_x}{v})_x = (\frac{\mu}{v})' v_x^2 + \frac{\mu}{v}v_{xx}$ and v is bounded from above and below, thus combining the estimate of (4.31), (4.32), we actually obtain

$$\|v_{xx}(t)\|^2 + \int_0^t \|v_{xx}\|^2\mathrm{d}\tau \le C\left\|\left(v_0-1, u_0, \mathbf{w}_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_2^{16}. \tag{4.33}$$

Then, if we plug (4.33) into (4.24), we can get

$$\begin{aligned}
\|\mathbf{w}_{xx}\|^2 + \int_0^t\int_{\mathbf{R}} \frac{\mu}{v}\mathbf{w}_{xxx}^2 &\le C\left\|\left(v_0-1, u_0, \mathbf{w}_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^{10} + \int_0^t\int_{\mathbf{R}} \mathbf{w}_x^2(v_{xx}^2+v_x^4)\mathrm{d}x\mathrm{d}\tau \\
&\le C\left\|\left(v_0-1, u_0, \mathbf{w}_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^{10} \\
&\quad + \int_0^t \left(\|v_{xx}\|^2\|\mathbf{w}_x\|\|\mathbf{w}_{xx}\| + \|v_x\|^2\|\|v_{xx}\|^2\|\mathbf{w}_x\|^2\right)\mathrm{d}\tau \\
&\le C\left\|\left(v_0-1, u_0, \mathbf{w}_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_2^{20}.
\end{aligned} \tag{4.34}$$

Also, from (4.16), (4.23), and (4.33), we have

$$\begin{aligned}
\|u_{xx}\|^2 + \int_0^t\int_{\mathbf{R}} \frac{\lambda}{v}u_{xxx}^2 &\le C\left\|\left(v_0-1, u_0, \mathbf{w}_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_2^{10} \\
&\quad + \int_0^t\int_{\mathbf{R}} \left(u_x^2(v_{xx}^2+v_x^4) + u_{xx}^2v_x^2\right)\mathrm{d}x\mathrm{d}\tau + \int_0^t \|v_{xx}\|^2\mathrm{d}\tau \\
&\le C\left\|\left(v_0-1, u_0, \mathbf{w}_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_2^{16} \\
&\quad + \int_0^t\int_{\mathbf{R}} \left(u_x^2(v_{xx}^2+v_x^4) + u_{xx}^2v_x^2\right)\mathrm{d}x\mathrm{d}\tau \\
&\le C\left\|\left(v_0-1, u_0, \mathbf{w}_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_2^{16} \\
&\quad + \int_0^t \left(\|u_x\|\|u_{xx}\|\|v_{xx}\|^2 + \|v_x\|\|v_{xx}\|\|u_{xx}\|^2 + \|u_x\|^2\|v_x\|^2\|v_{xx}\|^2\right)\mathrm{d}\tau \\
&\le C\left\|\left(v_0-1, u_0, \mathbf{w}_0, \frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_2^{20}.
\end{aligned} \tag{4.35}$$

Finally, plug (4.33)–(4.35) into (4.12), we have

$$\begin{aligned}
&\left\|\frac{\theta_{xx}}{\sqrt{\gamma-1}}\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xxx}^2}{v}\mathrm{d}x\mathrm{d}\tau\\
&\le C\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^{30}\\
&\quad+C\int_0^t\int_{\mathbf{R}}\left(\left(u_x^2+v_x^2+\theta_x^2+\mathbf{w}_x^2\right)\left(\theta_{xx}^2+u_{xx}^2+\mathbf{w}_{xx}^2\right)+\theta_x^2v_{xx}^2\right)\mathrm{d}x\mathrm{d}\tau\\
&\le C\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^{30}\\
&\quad+C\int_0^t\|(u_x,v_x,\theta_x,\mathbf{w}_x)\|\left(\|\theta_{xx}\|\|\theta_{xxx}\|+\|\theta_x\|\|\theta_{xx}\|\|v_{xx}\|^2+\|u_{xx}\|\|u_{xxx}\|+\|\mathbf{w}_{xx}\|\|\mathbf{w}_{xxx}\|\right)\mathrm{d}\tau\\
&\le C\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_2^{30}
\end{aligned}\tag{4.36}$$

Consequently, we obtain the following lemma:

Lemma 4.10. *Under the conditions listed in* Theorem 1.3*, there is positive constant* $C>0$*, such that*

$$\left\|\frac{\theta_{xx}}{\sqrt{\gamma-1}}\right\|^2+\int_0^t\int_{\mathbf{R}}\frac{\kappa(\theta)\theta_{xxx}^2}{v}\mathrm{d}x\mathrm{d}\tau\le C\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_2^{30}.\tag{4.37}$$

Lemmas 4.7–4.10 tell us that if the local solution $(v(t,x),u(t,x),\mathbf{w}(t,x),\theta(t,x))$ to the Cauchy problem (1.9) constructed in Lemma 2.1 has been extended to the time step $t=T\ge t_1$, and the a priori assumption (H_3) holds true, then there exist positive constants $C_2\ge 1$ depending only on $\|(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}},v_{0x})\|$, $\underline{V},\overline{V},\underline{\Theta}$, and $\overline{\Theta}$ but independent of T and $\gamma-1$, and C_2 depending only on V_2 but independent of $T>0$ and $\gamma-1$, such that (4.16) and

$$\begin{aligned}
\|v_{xx}(t)\|^2+\int_0^t\|v_{xx}\|^2\mathrm{d}\tau&\le C_2\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_2^{16},\\
\|u_{xx}(t)\|^2+\int_0^t\|u_{xxx}\|^2\mathrm{d}\tau&\le C_2\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_2^{20},\\
\|\mathbf{w}_{xx}(t)\|^2+\int_0^t\|\mathbf{w}_{xxx}\|^2\mathrm{d}\tau&\le C_2\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_2^{20},\\
\left\|\frac{\theta_{xx}(t)}{\sqrt{\gamma-1}}\right\|^2+\int_0^t\|\theta_{xxx}\|^2\mathrm{d}\tau&\le C_2\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_2^{30}.
\end{aligned}\tag{4.38}$$

holds true. □

To use the continue method to get the global solution step by step, we only need to show that the a priori assumption H_3 can be closed. For this purpose, here we need $\gamma-1>0$ to be sufficiently small. In fact, from $(4.16)_2$, $(4.16)_6$, and $(4.38)_4$, we know that

$$\begin{aligned}\left\|\frac{\theta-1}{\sqrt{\gamma-1}}\right\|^2 &\le C_1\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|^2,\\ \left\|\frac{\theta_x(\tau)}{\sqrt{\gamma-1}}\right\|^2 &\le C_1\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_1^{10},\\ \left\|\frac{\theta_{xx}(t)}{\sqrt{\gamma-1}}\right\|^2 &\le C_2\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_2^{30}. \end{aligned}\tag{4.39}$$

Due to $\theta=\frac{A}{r}v^{1-\gamma}\exp(\frac{\gamma-1}{R}s)$, if we set $\overline{s}=\frac{R}{\gamma-1}\ln\frac{R}{A}$, we have

$$\begin{aligned}\theta-1&=\frac{A}{R}v^{1-\gamma}\exp\left(\frac{\gamma-1}{R}s\right)-1\\ &=\frac{A}{R}v^{1-\gamma}\exp\left(\frac{\gamma-1}{R}s\right)-\frac{A}{R}\exp\left(\frac{\gamma-1}{R}\overline{s}\right)\\ &=\frac{A}{R}\left(v^{1-\gamma}-1\right)\exp\left(\frac{\gamma-1}{R}s\right)+\frac{A}{R}\left(\exp\left(\frac{\gamma-1}{R}s\right)-\exp\left(\frac{\gamma-1}{R}\overline{s}\right)\right).\end{aligned}$$

Consequently,

$$\|\theta_0-1\|\le O(1)\frac{A(\gamma-1)}{R}\exp\left(\frac{\gamma-1}{R}\|s_0\|_{L_x^\infty}\right)\left[\left\|v_0^{-\gamma}\right\|_{L_x^\infty}\|v_0-1\|+\frac{1}{R}\|s_0(x)-\overline{s}\|\right],\tag{4.40}$$

$$\begin{aligned}\|\theta_{0x}\|&\le O(1)\frac{A(\gamma-1)}{R}\exp\left(\frac{\gamma-1}{R}\|s_0\|_{L_x^\infty}\right)\left[(\inf v_0(x))^{-\gamma}\|v_{0x}\|+\frac{1}{R}(\inf v_0(x))^{1-\gamma}\|s_{0x}\|\right],\\ \|\theta_{0xx}\|&\le O(1)\frac{A(\gamma-1)}{R}\exp\left(\frac{\gamma-1}{R}\|s_0\|_{L_x^\infty}\right)\left[(\inf v_0(x))^{-\gamma}+(\inf v_0(x))^{1-\gamma}+(\inf v_0(x))^{-\gamma-1}\right]\\ &\quad\times\left(\|v_{0x}\|^2+\|v_{0xx}\|^2+\|s_{0x}\|^2+\|s_{0xx}\|^2\right).\end{aligned}\tag{4.41}$$

Since $\inf v_0(x), \|v_0\|_2, \|s_0\|_2$ is assumed to be independent of $\gamma-1$, then from (4.39), we know that

$$\|\theta_0-1\|_2\le C(\inf v_0(x),\|v_0\|_2,\|s_0\|_2)(\gamma-1).$$

Thus, we have

$$\begin{aligned}\|\theta(t)-1\|_2^2&\le C(\gamma-1)\left\|\left(v_0-1,u_0,\mathbf{w}_0,\frac{\theta_0-1}{\sqrt{\gamma-1}}\right)\right\|_2^{30}\\ &\le C(\gamma-1)\left\|(v_0-1,u_0,\mathbf{w}_0)\right\|_2^{30}+C(\gamma-1)^{-14}\|\theta_0-1\|_2^{30}\\ &\le C(\gamma-1)\left\|(v_0-1,u_0,\mathbf{w}_0)\right\|_2^{30}+C(\inf v_0(x),\|v_0\|_2,\|s_0\|_2)(\gamma-1)^{16},\end{aligned}\tag{4.42}$$

holds for $0\le t\le T$ and some positive constants independent of T and the quantity $(\gamma-1)^{-1}$.

Thus, if $\gamma-1$ is chosen sufficiently small, then the a priori assumption H_3 can be hold; thus, we can use the continuity method to get the global solution.

Acknowledgements

The first author is supported in part by the grant from the National Natural Science Foundation of China under Contract 11401449, Natural Science Foundation of Hubei Province under Contract 2013CFA131, and the Open Foundation of Hubei Province Key Laboratory of Systems Science in Metallurgical Process (Wuhan University of Science and Technology) under Grant Y201414. This work is also supported in part by the grant from the National Natural Science Foundation of China under Contract 11201355. The authors express much gratitude to Professor Huijiang Zhao for his support and advice.

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(Received: March 2, 2014; revised: March 16, 2015)