Z. Angew. Math. Phys. 66 (2015), 1649–1669

0044-2275/15/041649-21
published online January 25, 2015
DOI 10.1007/s00033-015-0494-1

Zeitschrift für angewandte Mathematik und Physik ZAMP

On the nonlinear Schrödinger–Poisson systems with sign-changing potential

Juntao Sun and Tsung-fang Wu

Abstract. In this paper, we study a nonlinear Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V_\lambda(x)\, u + \mu K(x)\, \phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)\, u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\mu > 0$ is a parameter, V_λ is allowed to be sign-changing and f is an indefinite function. We require that $V_\lambda := \lambda V^+ - V^-$ with V^+ having a bounded potential well Ω whose depth is controlled by λ and $V^- \geq 0$ for all $x \in \mathbb{R}^3$. Under some suitable assumptions on K and f, the existence and the nonexistence of nontrivial solutions are obtained by using variational methods. Furthermore, the phenomenon of concentration of solutions is explored as well.

Mathematics Subject Classification. 35J50 · 35J60.

Keywords. Nonlinear Schrödinger–Poisson systems · Variational methods · Concentration of solutions.

1. Introduction

In this paper, we are concerned with the following nonlinear Schrödinger–Poisson system:

$$\begin{cases} -\Delta u + V_\lambda(x)\, u + \mu K(x)\, \phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)\, u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{$SP_{\lambda,\mu}$}$$

where $\mu > 0$ is a parameter, the potential $V_\lambda(x) = \lambda V^+(x) - V^-(x), K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$ and $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. We assume that the functions $V^\pm$ satisfy the following conditions:

- $(V1)$ $V \in C\left(\mathbb{R}^3, \mathbb{R}\right)$ with $V^\pm = \max\{\pm V, 0\}$ and V is bounded from below;
- $(V2)$ There exists $b > 0$ such that the set $\{V^+ < b\} := \left\{x \in \mathbb{R}^3 \mid V^+(x) < b\right\}$ is nonempty and has finite measure;
- $(V3)$ $\Omega = int\left\{x \in \mathbb{R}^3 \mid V^+(x) = 0\right\}$ is nonempty and has smooth boundary with $\overline{\Omega} = \left\{x \in \mathbb{R}^3 \mid V^+(x) = 0\right\}$;
- $(V4)$ There exists a constant $\mu_0 > 1$ such that

$$\mu_1(\lambda) := \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} \left[|\nabla u|^2 + \lambda V^+ u^2\right] \mathrm{d}x}{\int_{\mathbb{R}^3} V^- u^2 \mathrm{d}x} \geq \mu_0 \text{ for all } \lambda > 0.$$

J. Sun was supported by the National Natural Science Foundation of China (Grant Nos. 11201270 and 11271372), Shandong Natural Science Foundation (Grant No. ZR2012AQ010), Young Teacher Support Program of Shandong University of Technology, and China Postdoctoral Science Foundation (Grant No. 2014M551494). T.F. Wu was supported in part by the National Science Council and the National Center for Theoretical Sciences, Taiwan.

Birkhäuser

Remark 1.1. *It is easy to verify that the condition* $(V4)$ *holds. Indeed, if we choose a function* $V^- \in L^{\frac{3}{2}}(\mathbb{R}^3)$ *with* $\|V^-\|_{L^{3/2}} < \overline{S}^2$, *then by the conditions* $(V1)-(V3)$ *and the Hölder and Sobolev inequalities,*

$$\begin{aligned}\frac{\int_{\mathbb{R}^3}\left[|\nabla u|^2+\lambda V^+u^2\right]dx}{\int_{\mathbb{R}^3}V^-u^2\,dx} &\geq \frac{\int_{\mathbb{R}^3}|\nabla u|^2\,dx}{\|V^-\|_{L^{3/2}}\left(\int_{\mathbb{R}^3}|u|^6\,dx\right)^{\frac{1}{3}}}\\ &\geq \frac{\int_{\mathbb{R}^3}|\nabla u|^2\,dx}{\|V^-\|_{L^{3/2}}\overline{S}^{-2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx}\\ &= \frac{\overline{S}^2}{\|V^-\|_{L^{3/2}}}\quad \text{for all } \lambda\geq 0,\end{aligned}$$

which implies that

$$\mu_1(\lambda) \geq \frac{\overline{S}^2}{\|V^-\|_{L^{3/2}}} > 1 \text{ for all } \lambda > 0,$$

where $\overline{S}$ *is the best Sobolev constant for the imbedding of* $D^{1,2}(\mathbb{R}^3)$ *in* $L^6(\mathbb{R}^3)$.

The hypotheses $(V1)-(V3)$, first introduced by Bartsch and Wang [7] in the study of the nonlinear Schrödinger equations, imply that λV^+ represents a potential well whose depth is controlled by λ. For $\lambda > 0$, large one expects to find solutions which localize near its bottom Ω. We refer the reader to the papers [6,25,28,31] for recent results.

Schrödinger–Poisson systems, also known as the nonlinear Schrödinger–Maxwell equations, have a strong physical meaning. It was first introduced in [8] as a model describing solitary waves for the nonlinear stationary Schrödinger equations interacting with the electrostatic field and also in semiconductor theory, in nonlinear optics and in plasma physics. Indeed, in Eq. $(SP_{\lambda,\mu})$ the first equation is a nonlinear stationary Schrödinger equation (where, as usual, the nonlinear term simulates the interaction between many particles) that is coupled with a Poisson equation, to be satisfied by ϕ, meaning that the potential is determined by the charge of the wave function.

In recent years, Eq. $(SP_{\lambda,\mu})$ has been studied widely via variational methods under the various hypotheses on V_λ, K and f, see [2–4,10–12,16–21,23,24,27,29,30,32,33] and the references therein. For example, in [23], when $V_\lambda \equiv 1$ and $K \equiv 1$, the existence and nonexistence results on positive radial solutions for Eq. $(SP_{\lambda,\mu})$ with $f(x,u) = |u|^{p-2}u$ are obtained, depending on the parameters p and μ. It turns out that $p = 3$ is a critical value for the existence of solution. When $V_\lambda \equiv 1$ and K is a nonnegative L^2-function, in [11] the authors use the Nehari manifold method to find a positive ground-state solution and a bound-state solution for Eq. $(SP_{\lambda,\mu})$ with $f(x,u) = a(x)|u|^{p-2}u$ and $4 < p < 6$ under some suitable assumptions on K and a, but not requiring any symmetry property, respectively.

In [19], the steep potential well is first applied into Schrödinger–Poisson systems. The authors use variational methods to study the following problem:

$$\begin{cases} -\Delta u + (\lambda V(x)+1)u + \mu\phi u = |u|^{p-2}u & \text{in } \mathbb{R}^3,\\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3,\end{cases}$$

where λ and μ are positive constants, $V \geq 0$ and satisfies $(V1)-(V3)$ with V instead of V^+. They obtain the existence results of nontrivial solution for the case $p \in (2,3) \cup [4,6)$ by combining domains approximation with priori estimates. It is worth noting that the positivity of the infimum of the potential $V_\lambda(x) := \lambda V(x) + 1$ is the key in the arguments of [19].

In [32], under the assumptions $(V1)-(V3)$ with V instead of V^+, the authors consider a similar problem:

$$\begin{cases} -\Delta u + \lambda V(x)u + K(x)\phi u = |u|^{p-2}u & \text{in } \mathbb{R}^3,\\ -\Delta\phi = K(x)u^2 & \text{in } \mathbb{R}^3,\end{cases} \tag{1}$$

where $K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3), K \geq 0$ for all $x \in \mathbb{R}^3$. When $V \geq 0$, they find a nontrivial solution for Eq. (1) with $3 < p < 6$ and explore the phenomenon of concentration of solutions. When the potential V is allowed to be sign-changing, the existence of nontrivial solution for the case $p \in (4, 6)$ is obtained, but not studying the concentration of nontrivial solutions.

Based on the study of [32], very recently, Ye and Tang [29] consider the Eq. (1) with a general nonlinear term f, that is,

$$\begin{cases} -\Delta u + \lambda V(x)u + K(x)\phi u = f(x, u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{2}$$

where $\lambda > 0, V$ satisfies $(V1)-(V3)$ with V instead of $V^+, K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$ and $K \geq 0$ for all $x \in \mathbb{R}^3$. The authors mainly deal with two cases as follows:

(I) V is allowed to be sign-changing. When f satisfies the 4-superlinear conditions, like $f(x, u) \equiv |u|^{p-2}u(4 < p < 6)$, and the norm of K is small enough, they obtain a nontrivial solution for Eq. (2), which generalizes Theorem 1.1 in [32];

(II) $V \geq 0$. The existence and multiplicity of nontrivial solutions for Eq. (2) are obtained when f satisfies the 4-superlinear conditions, but without the restriction on the norm of K. In addition, using the symmetric mountain pass lemma, infinitely many solutions are found when f satisfies the sublinear conditions, which improves Theorem 1.1 in [26].

Motivated by the above works, in the present paper we consider Eq. $(SP_{\lambda,\mu})$ with a sign-changing potential V_λ satisfying $(V1)-(V4)$, which are different from those in the previous papers [19,29,32]. By using the mountain pass theorem, and combining some new inequalities, we mainly study the following three problems:

(i) The existence result when f is indefinite and satisfies the asymptotically linear conditions;
(ii) The nonexistence result;
(iii) The phenomenon of concentration of nontrivial solutions.

It is worth emphasizing that cases (i) and (ii) are not concerned in the previous papers. Moreover, we point out that we establish some new estimation, such as the inequality (9) below which will play an important role in our proof. These estimations are totally different from those in the literature.

Before stating our results, we need to introduce some notations and definitions.

Notation 1.1. *Throughout this paper, we denote by* $|\cdot|_r$ *the* L^r*-norm,* $1 \leq r \leq \infty$*, and we have to use the notation* $p^\pm = \max\{\pm p, 0\}$*. The letter* C *will denote various positive constants whose value may change from line to line but are not essential to the analysis of the problem. Also if we take a subsequence of a sequence* $\{u_n\}$*, we shall denote it again* $\{u_n\}$*. We use* $o(1)$ *to denote any quantity which tends to zero when* $n \to \infty$*.*

We need the following minimum problems:

$$\lambda_1(q) = \inf\left\{\int_\Omega |\nabla u|^2 \mathrm{d}x \mid u \in H_0^1(\Omega), \int_\Omega qu^2 \mathrm{d}x = 1\right\} \tag{3}$$

and

$$\mu_1(q) = \inf\left\{\int_\Omega\int_\Omega \frac{K(x)K(y)u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \mid u \in H_0^1(\Omega), \int_\Omega qu^4 \mathrm{d}x = 1\right\} \geq 0, \tag{4}$$

where $q \in L^\infty(\mathbb{R}^3)$ with $q > 0$ on $\overline{\Omega}$ and $K(x) > 0$ for $x \in \mathbb{R}^3$, $K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$. Then $\lambda_1(q) > 0$, which is achieved by some $\phi_1 \in H_0^1(\Omega)$ which $\int_\Omega q|\phi_1|^2 \mathrm{d}x = 1$ and $\phi_1 > 0$ a.e. in Ω by the compactness of Sobolev embedding from $H_0^1(\Omega)$ from $L^2(\Omega)$ and Fatou's lemma (see Figueiredo [15]). In particular,

$$\lambda_1(q)\int_\Omega q|u|^2 \mathrm{d}x \leq \int_\Omega |\nabla u|^2 \mathrm{d}x \text{ for all } u \in H_0^1(\Omega). \tag{5}$$

Now, we give our main results.

Theorem 1.1. *Suppose that conditions* $(V1)$–$(V4)$ *hold and* $K(x) > 0$ *for* $x \in \mathbb{R}^3, K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$. *In addition, for any* $k \in \{1, 3, 4\}$, *we assume that the function* f *satisfies the following conditions:*

$(D1)$ $f(x,s)$ *is a continuous function on* $\mathbb{R}^3 \times \mathbb{R}$ *such that* $f(x,s) \equiv 0$ *for all* $s < 0$ *and* $x \in \mathbb{R}^3$. *Moreover, there exists* $p \in L^\infty(\mathbb{R}^3)$ *with*

$$\left|p^+\right|_\infty < \Theta_0 := \min\left\{b, \frac{(\mu_0 - 1)\,\overline{S}^2}{\mu_0 \left|\{V^+ < b\}\right|^{\frac{2}{3}}}\right\}$$

such that

$$\lim_{s \to 0^+} \frac{f(x,s)}{s^k} = p(x) \ \text{ uniformly in } x \in \mathbb{R}^3$$

and

$$\frac{f(x,s)}{s^k} \geq p(x) \ \text{ for all } s > 0 \text{ and } x \in \overline{\Omega},$$

where $|\cdot|$ *is the Lebesgue measure;*

$(D2)$ *there exists a function* $q \in L^\infty(\mathbb{R}^3)$ *with* $q > 0$ *on* Ω *such that*

$$\lim_{s \to \infty} \frac{f(x,s)}{s^k} = q(x) \ \text{ uniformly in } x \in \mathbb{R}^3;$$

$(D3)$ *there exists a constant* d_0 *satisfying*

$$0 \leq d_0 < \frac{(\mu_0 - 1)\,\overline{S}^2}{4\mu_0 \left|\{V^+ < b\}\right|^{\frac{2}{3}}}$$

such that

$$F(x,s) - \frac{1}{4} f(x,s)s \leq d_0 s^2 \quad \text{for all } s > 0 \text{ and } x \in \mathbb{R}^3.$$

Then we have the following results.

(i) *If* $k = 1$ *and* $\lambda_1(q) < 1$, *then there exists a positive number* μ^* *such that for every* $\mu \in (0, \mu^*)$, *there exists* $\Lambda^* > 0$ *such that Eq.* $(SP_{\lambda,\mu})$ *has at least a nontrivial solution for all* $\lambda > \Lambda^*$.

(ii) *If* $k = 3$, *then for each* $\mu \in (0, 1/\mu_1(q))$ *(if* $\mu_1(q) = 0$, *then* $\mu > 0$*) there exists* $\Lambda^* > 0$ *such that Eq.* $(SP_{\lambda,\mu})$ *has at least a nontrivial solution for all* $\lambda > \Lambda^*$.

(iii) *If* $k = 4$, *then for each* $\mu > 0$ *there exists* $\Lambda^* > 0$ *such that Eq.* $(SP_{\lambda,\mu})$ *has at least a nontrivial solution for all* $\lambda > \Lambda^*$.

Remark 1.2. *In* [29], *Ye and Tang study the existence of nontrivial solutions for Eq. (2) under the 4-superlinear condition of* f *as follows:*

$$\lim_{|u| \to \infty} \frac{F(x,u)}{u^4} = +\infty \quad \text{uniformly in } x, \tag{6}$$

where $F(x,u) = \int_0^u f(x,s)ds$. *However, in our Theorem 1.1, for the cases of* $k = 1, 3$, *the nonlinearity* f *does not satisfies the condition (6). Therefore, we extend the corresponding results in* [29].

We need the following minimum problem:

$$\widehat{\mu}_1(q) = \inf\left\{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(x)K(y)u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \mid u \in H^1(\mathbb{R}^3), \int_{\mathbb{R}^3} qu^4 \mathrm{d}x = 1\right\} \geq 0.$$

Clearly, $\widehat{\mu}_1(q) \leq \mu_1(q)$. Then we have the following results.

Theorem 1.2. *Suppose that the conditions* $(V1)-(V4)$ *hold and* $K(x)>0$ *for* $x\in\mathbb{R}^3, K\in L^2(\mathbb{R}^3)\cup L^\infty(\mathbb{R}^3)$. *In addition, for each positive integer* $k=1,3$, *we assume that the function* f *satisfies the conditions* $(D2),(D3)$ *and the following condition:*

$(D4)$ $f\in C^1\left(\mathbb{R}^3\times\mathbb{R},\mathbb{R}\right)$ *and* $s\longmapsto\frac{f(x,s)}{s^k}$ *is nondecreasing function for any fixed* $x\in\mathbb{R}$.

Then we have the following results.

(i) *If* $k=1$ *and* $|q|_\infty<\frac{(\mu_0-1)}{\mu_0}\overline{S}^2|\Omega|^{-\frac{2}{3}}$, *then there exists positive number* Λ_* *such that for every* $\mu>0$ *and* $\lambda>\Lambda_*$, *Eq.* $(SP_{\lambda,\mu})$ *does not admit any nontrivial solution.*

(ii) *If* $k=3$ *and* $\widehat{\mu}_1(q)>0$, *then for every* $\mu\geq 1/\widehat{\mu}_1(q)$ *and* $\lambda>0$, *Eq.* $(SP_{\lambda,\mu})$ *does not admit any nontrivial solution.*

Remark 1.3. *Suppose that* q *is a bounded positive continuous function on* $\mathbb{R}^3$ *and* $K(x)>0$ *for* $x\in\mathbb{R}^3$, $K\in L^2(\mathbb{R}^3)\cup L^\infty(\mathbb{R}^3)$ *with*

$$\lim_{|x|\to\infty}q(x)=1\ \text{and}\ \lim_{|x|\to\infty}K(x)=0.$$

Let $\widehat{w}_0$ *be the unique positive solution with* $\widehat{w}_0(0)=\max_{x\in\mathbb{R}^3}\widehat{w}_0(x)$ *for the following nonlinear Schrödinger equation:*

$$-\Delta u+u=u^3\quad\text{in }\mathbb{R}^3, \tag{E_0^∞}$$

$u_n=\widehat{w}_0(x-ne)$ *and* $v_n=u_n\left(\int_{\mathbb{R}^3}qu_n^4dx\right)^{-1/4}$ *for* $n\in\mathbb{N}$, *where* $e=(1,0,0)$. *Then* $\int_{\mathbb{R}^3}qv_n^4dx=1$ *for all* $n\in\mathbb{N}$ *and*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(x)K(y)v_n^2(x)v_n^2(y)}{|x-y|}dxdy\to 0\ \text{as}\ n\to\infty,$$

which implies that $\widehat{\mu}_1(q)=0$. *Therefore, if we would like to obtain the inequality* $\widehat{\mu}_1(q)>0$, *then the condition* $\lim_{|x|\to\infty}K(x)=K_\infty>0$ *is necessary.*

On the concentration of solutions, we have the following result.

Theorem 1.3. *Let* u_λ *be the solution obtained by Theorem 1.1. Then* $u_\lambda\to u_0$ *in* $H^1(\mathbb{R}^3)$ *as* $\lambda\to\infty$, *where* $u_0\in H_0^1(\Omega)$ *is the nontrivial solution of*

$$\begin{cases}-\Delta u-V^-(x)u+\frac{\mu}{4\pi}\left((K(x)u^2)*\frac{1}{|x|}\right)K(x)u=f(x,u) & \text{in }\Omega,\\ u=0, & \text{on }\partial\Omega.\end{cases} \tag{SP_∞}$$

The remainder of this paper is organized as follows. In Sect. 2, some preliminary results are presented. In Sects. 3–5, we give the proofs of our main results.

2. Variational setting and preliminaries

In this section, we give the variational setting for Eq. $(SP_{\lambda,\mu})$ following [13] and establish compactness conditions. Let

$$X=\left\{u\in H^1\left(\mathbb{R}^3\right)\ \middle|\ \int_{\mathbb{R}^3}V^+(x)u^2\mathrm{d}x<\infty\right\}$$

be equipped with the inner product and norm

$$\langle u,v\rangle=\int_{\mathbb{R}^3}\left[\nabla u\nabla v+V^+(x)uv\right]\mathrm{d}x,\ \ \|u\|=\langle u,u\rangle^{1/2}.$$

For $\lambda > 0$, we also need the following inner product and norm

$$\langle u, v\rangle_\lambda = \int\limits_{\mathbb{R}^3} \left[\nabla u \nabla v + \lambda V^+(x) uv\right] \mathrm{d}x,\ \ \|u\|_\lambda = \langle u, u\rangle_\lambda^{1/2}.$$

It is clear that $\|u\| \le \|u\|_\lambda$ for $\lambda \ge 1$. Furthermore, it follows from the condition $(V4)$ that

$$\int\limits_{\mathbb{R}^N} |\nabla u|^2 + V_\lambda u^2 \mathrm{d}x \ge \frac{\mu_0 - 1}{\mu_0} \|u\|_\lambda^2 \ \text{ for all } \lambda \ge 0. \tag{7}$$

Set $X_\lambda = (X, \|u\|_\lambda)$. By the conditions $(V1)-(V2)$ and the Hölder and Sobolev inequalities, we have

$$\begin{aligned}
&\int\limits_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) \mathrm{d}x \\
&\quad = \int\limits_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \int\limits_{\{V^+ < b\}} u^2 \mathrm{d}x + \int\limits_{\{V^+ \ge b\}} u^2 \mathrm{d}x \\
&\quad \le \int\limits_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \left(\int\limits_{\{V^+ < b\}} 1 \mathrm{d}x\right)^{\frac{2}{3}} \left(\int\limits_{\{V^+ < b\}} |u|^6 \mathrm{d}x\right)^{\frac{1}{3}} + \frac{1}{b} \int\limits_{\{V^+ \ge b\}} V^+(x) u^2 \mathrm{d}x \\
&\quad \le \left(1 + \left|\{V^+ < b\}\right|^{\frac{2}{3}} \overline{S}^{-2}\right) \int\limits_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{1}{b} \int\limits_{\mathbb{R}^3} V^+(x) u^2 \mathrm{d}x \\
&\quad \le \max\left\{1 + \left|\{V^+ < b\}\right|^{\frac{2}{3}} \overline{S}^{-2}, \frac{1}{b}\right\} \left(\int\limits_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \int\limits_{\mathbb{R}^3} V^+(x) u^2 \mathrm{d}x\right),
\end{aligned}$$

which implies that the imbedding $X \hookrightarrow H^1(\mathbb{R}^3)$ is continuous. Moreover, using the conditions $(V1)$–$(V2)$ and the Hölder and Sobolev inequalities again, we have for any $r \in [2, 6]$,

$$\begin{aligned}
&\int\limits_{\mathbb{R}^3} |u|^r \mathrm{d}x \\
&\quad \le \left(\int\limits_{\{V^+ \ge b\}} u^2 \mathrm{d}x + \int\limits_{\{V^+ < b\}} u^2 \mathrm{d}x\right)^{\frac{6-r}{4}} \left(\overline{S}^{-6} \left(\int\limits_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^3\right)^{\frac{r-2}{4}} \\
&\quad \le \left[\frac{1}{\lambda b} \int\limits_{\{V^+ \ge b\}} \lambda V^+(x) u^2 \mathrm{d}x + \left(\int\limits_{\{V^+ < b\}} 1 \mathrm{d}x\right)^{\frac{2}{3}} \left(\int\limits_{\{V^+ < b\}} |u|^{2^*} \mathrm{d}x\right)^{\frac{2}{2^*}}\right]^{\frac{6-r}{4}} \\
&\quad \cdot \left[\overline{S}^{-6} \left(\int\limits_{\mathbb{R}^3} |\nabla u|^2 + \lambda V^+(x) u^2 \mathrm{d}x\right)^3\right]^{\frac{r-2}{4}}
\end{aligned}$$

$$\leq \left(\frac{1}{\lambda b}\int_{\mathbb{R}^3}\lambda V^+(x)\,u^2\mathrm{d}x + \left|\left\{V^+ < b\right\}\right|^{\frac{2^*-2}{2^*}}\overline{S}^{-2}\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\right)^{\frac{6-r}{4}}\left(\overline{S}^{-2^*}\|u\|_\lambda^{2^*}\right)^{\frac{r-2}{4}}$$

$$= \left(\max\left\{\frac{\overline{S}^2}{\lambda b}, \left|\left\{V^+ < b\right\}\right|^{\frac{2}{3}}\right\}\right)^{\frac{6-r}{4}}\overline{S}^{-r}\|u\|_\lambda^r \quad \text{for all } \lambda > 0, \tag{8}$$

which implies that

$$\int_{\mathbb{R}^3}|u|^r\,\mathrm{d}x \leq \left|\left\{V^+ < b\right\}\right|^{\frac{6-r}{6}}\overline{S}^{-r}\|u\|_\lambda^r \quad \text{for all } \lambda \geq \frac{\overline{S}^2}{b}\left|\left\{V^+ < b\right\}\right|^{\frac{2}{3}}. \tag{9}$$

It is well known that Eq. $(SP_{\lambda,\mu})$ can be easily transformed in a nonlinear Schrödinger equation with a nonlocal term (see [3,23] etc.). Briefly, the Poisson equation is solved by using the Lax–Milgram theorem, so, for all $u \in H^1(\mathbb{R}^3)$, a unique $\phi_{K,u} \in D^{1,2}(\mathbb{R}^3)$ given by

$$\phi_{K,u}(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{K(y)\,u^2(y)}{|x-y|}\mathrm{d}y,$$

such that $-\Delta\phi = Ku^2$ and that, inserted into the first equation, gives

$$-\Delta u + u + \lambda K(x)\,\phi_{K,u}u = a(x)\,|u|^{p-2}\,u \quad \text{in } \mathbb{R}^3.$$

Eq. $(SP_{\lambda,\mu})$ is variational and its solutions are the critical points of the functional defined in X_λ by

$$\begin{aligned} J_{\lambda,\mu}(u) &= \frac{1}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x + \int_{\mathbb{R}^3}V_\lambda u^2\mathrm{d}x\right) + \frac{\mu}{4}\int_{\mathbb{R}^3}K\phi_{K,u}u^2\mathrm{d}x - \int_{\mathbb{R}^3}F(x,u)\mathrm{d}x \\ &= \frac{1}{2}\|u\|_\lambda^2 - \int_{\mathbb{R}^3}V^-u^2\mathrm{d}x + \frac{\mu}{4}\int_{\mathbb{R}^3}K\phi_{K,u}u^2\mathrm{d}x - \int_{\mathbb{R}^3}F(x,u)\mathrm{d}x, \end{aligned}$$

where $F(x,u) = \int_0^u f(x,s)ds$. Furthermore, it is easy to prove that the functional $J_{\lambda,\mu}$ is of class C^1 in X_λ and that

$$\langle J'_{\lambda,\mu}(u), v\rangle = \int_{\mathbb{R}^3}\nabla u\cdot\nabla v\mathrm{d}x + \int_{\mathbb{R}^3}V_\lambda(x)\,uv\mathrm{d}x + \mu\int_{\mathbb{R}^3}K\phi_{K,u}uv\mathrm{d}x - \int_{\mathbb{R}^3}f(x,u)v\mathrm{d}x.$$

Hence, if $u \in X_\lambda$ is a critical point of $J_{\lambda,\mu}$, then $(u, \phi_{K,u})$ is a solution of Eq. $(SP_{\lambda,\mu})$. Furthermore, we have the following result.

Lemma 2.1. *Suppose that the conditions $(V1)$–$(V4)$ and $(D1)-(D3)$ hold. For every $\lambda \geq \frac{\overline{S}^2}{b}\left|\{V < b\}\right|^{-\frac{2}{3}}$ and u_λ a nontrivial solution of Eq. $(SP_{\lambda,\mu})$, we have $J_{\lambda,\mu}(u_\lambda) > 0$.*

Proof. If u_λ is a nontrivial solution of Eq. $(SP_{\lambda,\mu})$, then

$$\int_{\mathbb{R}^3}|\nabla u_\lambda|^2\,\mathrm{d}x + \int_{\mathbb{R}^3}V_\lambda u_\lambda^2\mathrm{d}x + \mu\int_{\mathbb{R}^3}K\phi_{K,u_\lambda}u_\lambda^2\mathrm{d}x = \int_{\mathbb{R}^3}f(x,u_\lambda)u_\lambda\mathrm{d}x.$$

Combining this with the condition $(D3)$, (7) and (9), we have

$$\begin{aligned} J_{\lambda,\mu}(u_\lambda) &= \frac{1}{2}\left(\int_{\mathbb{R}^3} |\nabla u_\lambda|^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} V_\lambda u_\lambda^2 \mathrm{d}x\right) + \frac{\mu}{4}\int_{\mathbb{R}^3} K\phi_{K,u_\lambda} u_\lambda^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(x,u_\lambda)\mathrm{d}x \\ &= \frac{1}{4}\left(\int_{\mathbb{R}^3} |\nabla u_\lambda|^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} V_\lambda u_\lambda^2 \mathrm{d}x\right) - \int_{\mathbb{R}^3} \left[F(x,u_\lambda) - \frac{1}{4} f(x,u_\lambda)u_\lambda\right] \mathrm{d}x \\ &\geq \frac{\mu_0 - 1}{4\mu_0}\|u_\lambda\|_\lambda^2 - \int_{\mathbb{R}^3} \left[F(x,u_\lambda) - \frac{1}{4} f(x,u_\lambda)u_\lambda\right] \mathrm{d}x \\ &\geq \frac{\mu_0 - 1}{4\mu_0}\|u_\lambda\|_\lambda^2 - d_0 \int_{\mathbb{R}^3} u_\lambda^2 \mathrm{d}x \\ &\geq \left(\frac{\mu_0 - 1}{4\mu_0} - d_0 \left|\left\{V^+ < b\right\}\right|^{\frac{2}{3}} \overline{S}^{-2}\right) \|u_\lambda\|_\lambda^2 > 0. \end{aligned} \tag{10}$$

This completes the proof. □

Set

$$N(u) = \int_{\mathbb{R}^3} K(x)\,\phi_{K,u} u^2 \mathrm{d}x = \frac{1}{4\pi} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{K(x)\,K(y)}{|x-y|} u^2(x)\,u^2(y)\,\mathrm{d}x\mathrm{d}y.$$

In [32], it was shown that the functional N and its derivative N' possess BL-splitting property, which is similar to Brezis–Lieb Lemma [9]. Now we recall them.

Lemma 2.2. *([32], Lemma 2.2). Let $K \in L^\infty(\mathbb{R}^3) \cup L^2(\mathbb{R}^3)$. If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$, then we have*

(i) $N(u_n - u) = N(u_n) - N(u) + o(1)$;
(ii) $N'(u_n - u) = N'(u_n) - N'(u) + o(1)$ in $H^{-1}(\mathbb{R}^3)$.

Next, we give a useful theorem. It is the variant version of the mountain pass theorem, which allows us to find a so-called Cerami type (PS) sequence.

Lemma 2.3. ([14], Mountain Pass Theorem). *Let E be a real Banach space with its dual space E^*, and suppose that $I \in C^1(E,\mathbb{R})$ satisfies*

$$\max\{I(0), I(e)\} \leq \mu < \eta \leq \inf_{\|u\|=\rho} I(u),$$

for some $\mu < \eta, \rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $c \geq \eta$ be characterized by

$$c = \inf_{\gamma\in\Gamma} \max_{0\leq\tau\leq 1} I(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e, then there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to c \geq \eta \quad \textit{and} \quad (1 + \|u_n\|)\|I'(u_n)\|_{E^*} \to 0 \quad \textit{as } n \to \infty.$$

In what follows, we give two lemmas which ensure that the functional $J_{\lambda,\mu}$ has the mountain pass geometry.

Lemma 2.4. *For any $k \in \{1,3,4\}$, assume that the conditions $(V1)$–$(V2)$ and $(D1)$–$(D2)$ hold. Then for each $\lambda \geq \frac{\overline{S}^2}{b}\left|\{V^+ < b\}\right|^{-\frac{2}{3}}$, there exist $\rho > 0$ and $\eta > 0$ such that $\inf\{J_{\lambda,\mu}(u) : u \in X_\lambda$ with $\|u\| = \rho\} > \eta$.*

Proof. For any $\epsilon > 0$, it follows from the conditions $(D1)$ and $(D2)$ that there exist $\max\{k, 2\} < r < 6$ and $C_\epsilon > 0$ such that

$$F(x,s) \le \frac{|p^+|_\infty + \epsilon}{2} s^2 + \frac{C_\epsilon}{r}|s|^r \quad \text{for all } s \in \mathbb{R}. \tag{11}$$

So that, from (9), (11) and the Sobolev inequality, for all $u \in X_\lambda$ and $\lambda \ge \frac{\overline{S}^2}{b}\left|\{V^+ < b\}\right|^{-\frac{2}{3}}$,

$$\begin{aligned}\int_{\mathbb{R}^3} F(x,u)\mathrm{d}x &\le \frac{|p^+|_\infty + \epsilon}{2}\int_{\mathbb{R}^3} u^2\mathrm{d}x + \frac{C_\epsilon}{r}\int_{\mathbb{R}^3}|u|^r\mathrm{d}x \\ &\le \frac{(|p^+|_\infty + \epsilon)\left|\{V^+ < b\}\right|^{\frac{2}{3}}}{2\overline{S}^2}\|u\|_\lambda^2 + \frac{C_\epsilon\left|\{V^+ < b\}\right|^{\frac{6-r}{6}}}{r\overline{S}^r}\|u\|_\lambda^r,\end{aligned}$$

which implies that

$$\begin{aligned}J_{\lambda,\mu}(u) &= \frac{1}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x + \int_{\mathbb{R}^3}V_\lambda u^2\mathrm{d}x\right) + \frac{\mu}{4}\int_{\mathbb{R}^3}K\phi_{K,u}u^2\mathrm{d}x - \int_{\mathbb{R}^3}F(x,u)\mathrm{d}x \\ &\ge \frac{\mu_0 - 1}{2\mu_0}\|u_\lambda\|_\lambda^2 - \frac{(|p^+|_\infty + \epsilon)\left|\{V^+ < b\}\right|^{\frac{2}{3}}}{2\overline{S}^2}\|u\|_\lambda^2 \\ &\quad - \frac{C_\epsilon\left|\{V^+ < b\}\right|^{\frac{6-r}{6}}}{r\overline{S}^r}\|u\|_\lambda^r \\ &\ge \frac{1}{2}\left(\frac{\mu_0 - 1}{\mu_0} - \frac{(|p^+|_\infty + \epsilon)\left|\{V^+ < b\}\right|^{\frac{2}{3}}}{\overline{S}^2}\right)\|u\|_\lambda^2 - \frac{C_\epsilon\left|\{V^+ < b\}\right|^{\frac{6-r}{6}}}{r\overline{S}^r}\|u\|_\lambda^r.\end{aligned} \tag{12}$$

So, by the condition $(D1)$ and fixing $\epsilon \in (0, \Theta_0 - |p^+|_\infty)$ and letting $\|u\| = \rho > 0$ small enough, it is easy to see that there is $\eta > 0$ such that this lemma holds. ∎

Lemma 2.5. *For any $k \in \{1, 3, 4\}$, assume that the conditions $(V1)-(V2)$ and $(D1)-(D2)$ hold. Let $\rho > 0$ be as in Lemma 2.4. Then we have the following results:*

(i) If $k = 1$ and $\lambda_1(q) < 1$, then there exist $\mu^* > 0$ and $e \in H^1(\mathbb{R}^3)$ with $\|e\|_\lambda > \rho$ such that $J_{\lambda,\mu}(e) < 0$ for all $\mu \in (0, \mu^*)$ and $\lambda > 0$.

(ii) If $k = 3$, then for each $0 < \mu < 1/\mu_1(q)$ (if $\mu_1(q) = 0$, then $\mu > 0$) and $\lambda > 0$, there exists $e \in H^1(\mathbb{R}^3)$ with $\|e\|_\lambda > \rho$ such that $J_{\lambda,\mu}(e) < 0$.

(iii) If $k = 4$, then for each $\mu > 0$ and $\lambda > 0$, there exists $e \in H^1(\mathbb{R}^3)$ with $\|e\|_\lambda > \rho$ such that $J_{\lambda,\mu}(e) < 0$.

Proof. Case $(I): k = 1$. By $\lambda_1(q) < 1$, the condition $(D2)$ and Fatou's lemma, we have

$$\begin{aligned}\lim_{t\to+\infty}\frac{J_{\lambda,0}(t\phi_1)}{t^2} &= \frac{1}{2}\left(\int_{\mathbb{R}^3}|\nabla\phi_1|^2\mathrm{d}x + \int_{\mathbb{R}^3}V_\lambda\phi_1^2\mathrm{d}x\right) - \lim_{t\to+\infty}\int_{\mathbb{R}^3}\frac{F(x,t\phi_1)}{t^2\phi_1^2}\phi_1^2\mathrm{d}x \\ &\le \frac{1}{2}\int_\Omega|\nabla\phi_1|^2\mathrm{d}x - \int_\Omega V^-\phi_1^2\mathrm{d}x - \int_\Omega\lim_{t\to+\infty}\frac{F(x,t\phi_1)}{t^2\phi_1^2}\phi_1^2\mathrm{d}x \\ &\le \frac{1}{2}\int_\Omega|\nabla\phi_1|^2\mathrm{d}x - \frac{1}{2}\int_\Omega q\phi_1^2\mathrm{d}x \\ &\le \frac{1}{2}\left(1 - \frac{1}{\lambda_1(q)}\right)\int_\Omega|\nabla\phi_1|^2\mathrm{d}x \\ &< 0,\end{aligned}$$

where $J_{\lambda,0}(u) = J_{\lambda,\mu}(u)$ with $\mu = 0$. So, if $J_{\lambda,0}(t\phi_1) \to -\infty$ as $t \to +\infty$, then there exists $e \in H^1(\mathbb{R}^3)$ with $\|e\|_\lambda > \rho$ such that $J_{\lambda,0}(e) < 0$. Since $J_{\lambda,\mu}(e) \to J_{\lambda,0}(e)$ as $\mu \to 0^+$, we see that there exists $\mu^* > 0$ such that $J_{\lambda,\mu}(e) < 0$ for all $\mu \in (0, \mu^*)$.

Case (II) : $k = 3, 4$. For $k = 3$, since $0 < \mu < 1/\mu_1(q)$, we can choose $\Phi \in H_0^1(\Omega)$ such that $\mu \int_\Omega K\phi_{K,\Phi}^2 \Phi^2 \mathrm{d}x < 1$ and $\int_\Omega q\Phi^4 \mathrm{d}x = 1$. Moreover, since $q > 0$ on Ω, we can choose a $\overline{\varphi} \in H_0^1(\Omega)$ such that $\int_\Omega q\overline{\varphi}^5 \mathrm{d}x > 0$. Define

$$\psi_k = \begin{cases} \Phi, & \text{if } k = 3, \\ \overline{\varphi}, & \text{if } k = 4. \end{cases}$$

Then, by the conditions $(D1)$, $(D2)$ and Fatou's lemma, one has

$$\begin{aligned} \lim_{t\to+\infty} \frac{J_{\lambda,\mu}(t\psi_k)}{t^{k+1}} &= \begin{cases} \frac{\mu}{4}\int_{\mathbb{R}^3} K\phi_{K,\Phi}^2\Phi^2\mathrm{d}x - \lim_{t\to+\infty}\int_{\mathbb{R}^3} \frac{F(x,t\Phi)}{t^4\Phi^4}\Phi^4\mathrm{d}x, & \text{if } k=3 \\ -\lim_{t\to+\infty}\int_{\mathbb{R}^3} \frac{F(x,t\overline{\varphi})}{t^5\overline{\varphi}^5}\overline{\varphi}^5\mathrm{d}x, & \text{if } k=4 \end{cases} \\ &\le \begin{cases} \frac{\mu}{4}\int_{\Omega} K\phi_{K,\Phi}^2\Phi^2\mathrm{d}x - \int_{\mathbb{R}^3}\lim_{t\to+\infty} \frac{F(x,t\Phi)}{t^4\Phi^4}\Phi^4\mathrm{d}x, & \text{if } k=3 \\ -\int_{\mathbb{R}^3}\lim_{t\to+\infty} \frac{F(x,t\overline{\varphi})}{t^5\overline{\varphi}^5}\overline{\varphi}^5\mathrm{d}x, & \text{if } k=4 \end{cases} \\ &= \begin{cases} \frac{1}{4}\left(\mu\int_\Omega K\phi_{K,\Phi}^2\Phi^2\mathrm{d}x - \int_\Omega q\Phi^4\mathrm{d}x\right), & \text{if } k=3 \\ -\frac{1}{5}\int_\Omega q\overline{\varphi}^5\mathrm{d}x, & \text{if } k=4 \end{cases} \\ &= \begin{cases} \frac{1}{4}\left(\mu\int_\Omega K\phi_{K,\Phi}^2\Phi^2\mathrm{d}x - 1\right), & \text{if } k=3 \\ -\frac{1}{5}\int_\Omega q\overline{\varphi}^5\mathrm{d}x, & \text{if } k=4 \end{cases} \\ &< 0. \end{aligned}$$

So, if $J_{\lambda,\mu}(t\psi_k) \to -\infty$ as $t \to +\infty$, then there exists $e \in H^1(\mathbb{R}^3)$ with $\|e\|_\lambda > \rho$ such that $J_{\lambda,\mu}(e) < 0$ and the lemma is proved. □

3. Proof of Theorem 1.1

First we define

$$c_{\lambda,\mu} = \inf_{\gamma\in\Gamma_\lambda} \max_{0\le t\le 1} J_{\lambda,\mu}(\gamma(t))$$

and

$$c_{0,\mu}(\Omega) = \inf_{\gamma\in\overline{\Gamma}_\lambda(\Omega)} \max_{0\le t\le 1} J_{\lambda,\mu}|_{H_0^1(\Omega)}(\gamma(t)),$$

where $J_{\lambda,\mu}|_{H_0^1(\Omega)}$ is a restriction of $J_{\lambda,\mu}$ on $H_0^1(\Omega)$,

$$\Gamma_\lambda = \{\gamma \in C([0,1], X_\lambda) : \gamma(0) = 0, \gamma(1) = e\}$$

and

$$\overline{\Gamma}_\lambda(\Omega) = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = e\}.$$

Note that for $u \in H_0^1(\Omega)$,

$$J_{\lambda,\mu}|_{H_0^1(\Omega)}(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} V^- u^2\mathrm{d}x + \frac{\mu}{4}\int_{\mathbb{R}^3} K\phi_{K,u}u^2\mathrm{d}x - \int_{\mathbb{R}^3} F(x,u)\mathrm{d}x$$

and $c_{0,\mu}(\Omega)$ independent of λ. Moreover, if the conditions $(D1)$–$(D3)$ hold, then by the proofs of Lemmas 2.4 and 2.5, we can conclude that $J_{\lambda,\mu}|_{H_0^1(\Omega)}$ satisfies the mountain pass hypothesis as in Theorem 2.3. Since $H_0^1(\Omega) \subset X_\lambda$ for all $\lambda > 0$. Then $0 < \eta \le c_{\lambda,\mu} \le c_{0,\mu}(\Omega)$ for all $\lambda \ge \frac{\overline{S}^2}{b}|\{V < b\}|^{-\frac{2}{3}}$. Define

$$m(k) = \begin{cases} \mu^*, & \text{if } k = 1, \\ 1/\mu_1(q), & \text{if } k = 3, \\ \infty, & \text{if } k = 4. \end{cases}$$

Then for each $k \in \{1,3,4\}$ and $\mu \in (0, m(k))$, take $D_\mu > c_{0,\mu}(\Omega)$. Thus,

$$0 < \eta \le c_{\lambda,\mu} \le c_{0,\mu}(\Omega) < D_\mu \text{ for all } \lambda \ge \frac{\overline{S}^2}{b}\left|\{V^+ < b\}\right|^{-\frac{2}{3}}.$$

Then by Lemmas 2.4 and 2.5 and Theorem 2.3, we obtain that for each $\mu \in (0, m(k))$ and $\lambda \ge \frac{\overline{S}^2}{b}\left|\{V^+ < b\}\right|^{-\frac{2}{3}}$ there exists $\{u_n\} \subset X_\lambda$ such that

$$J_{\lambda,\mu}(u_n) \to c_{\lambda,\mu} > 0 \quad \text{and} \quad (1 + \|u_n\|_\lambda)\|J'_{\lambda,\mu}(u_n)\|_{X_\lambda^{-1}} \to 0 \quad \text{as } n \to \infty, \tag{13}$$

where $0 < \eta \le c_{\lambda,\mu} \le c_{0,\mu}(\Omega) < D_\mu$. Furthermore, we have the following results.

Lemma 3.1. *For any $k \in \{1,3,4\}$, assume that the conditions $(V1)$–$(V4)$ and $(D1)$–$(D3)$ hold. Let $\{u_n\}$ defined in (13). Then $\{u_n\}$ is bounded in X_λ for each $\lambda \ge \frac{\overline{S}^2}{b}\left|\{V^+ < b\}\right|^{-\frac{2}{3}}$.*

Proof. For n large enough, by the condition $(D3)$, (7) and (9), one has

$$\begin{aligned}
c_{\lambda,\mu} + 1 &\ge J_{\lambda,\mu}(u_n) - \frac{1}{4}\langle J'_{\lambda,\mu}(u_n), u_n\rangle \\
&= \frac{1}{4}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} V_\lambda u_n^2 \mathrm{d}x\right) + \int_{\mathbb{R}^3}\left[\frac{1}{4}f(x,u_n) - F(x,u_n)\right]\mathrm{d}x \\
&\ge \frac{\mu_0 - 1}{4\mu_0}\|u_n\|_\lambda^2 - \int_{\mathbb{R}^3}\left[F(x,u_n) - \frac{1}{4}f(x,u_n)u_n\right]\mathrm{d}x \\
&\ge \frac{\mu_0 - 1}{4\mu_0}\|u_n\|_\lambda^2 - d_0\int_{\mathbb{R}^3} u_n^2 \mathrm{d}x \\
&\ge \frac{(\mu_0 - 1)\overline{S}^2 - 4\mu_0 d_0\left|\{V^+ < b\}\right|^{\frac{2}{3}}}{4\mu_0\overline{S}^2}\|u_n\|_\lambda^2,
\end{aligned}$$

which implies that

$$\|u_n\|_\lambda \le \left(\frac{4\mu_0\overline{S}^2(c_{\lambda,\mu}+1)}{(\mu_0 - 1)\overline{S}^2 - 4\mu_0 d_0\left|\{V^+ < b\}\right|^{\frac{2}{3}}}\right)^{1/2}.$$

Therefore, $\{u_n\}$ is bounded in X_λ. □

Lemma 3.2. *Suppose that the conditions $(V1) - (V2)$ and $(D1) - (D2)$ hold. In addition, assume that $K(x) > 0$ for $x \in \mathbb{R}^3$, $K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$. If $u_n \rightharpoonup u$ in X_λ and $u_n \to u$ a.e. in $\mathbb{R}^3$, then for each $\lambda \ge \frac{\overline{S}^2}{b}\left|\{V^+ < b\}\right|^{\frac{2}{3}}$, we have*

$$J_{\lambda,\mu}(u_n - u) = J_{\lambda,\mu}(u_n) - J_{\lambda,\mu}(u) + o(1) \tag{14}$$

and

$$J'_{\lambda,\mu}(u_n - u) = J'_{\lambda,\mu}(u_n) - J'_{\lambda,\mu}(u) + o(1). \tag{15}$$

Proof. By the definition of weak convergence in X_λ,

$$\int_{\mathbb{R}^3}\left[\nabla u_n \nabla u + \lambda V^+(x) u_n u\right]\mathrm{d}x = \int_{\mathbb{R}^3}\left[|\nabla u|^2 + \lambda V^+(x) u^2\right]\mathrm{d}x + o(1).$$

Thus,

$$\begin{aligned}\|u_n-u\|_\lambda^2 &= \int_{\mathbb{R}^3} |\nabla u_n-\nabla u|^2\,\mathrm{d}x+\lambda\int_{\mathbb{R}^3} V^+(x)\,|u_n-u|^2\,\mathrm{d}x\\ &= \int_{\mathbb{R}^3}\left(|\nabla u_n|^2+\lambda V^+(x)\,u_n^2\right)\mathrm{d}x+\int_{\mathbb{R}^3}\left(|\nabla u|^2+\lambda V^+(x)\,u^2\right)\mathrm{d}x\\ &\quad -2\int_{\mathbb{R}^3}\left[\nabla u_n\nabla u+\lambda V^+(x)\,u_nu\right]\mathrm{d}x\\ &= \|u_n\|_\lambda^2-\|u\|_\lambda^2+o(1)\,. \end{aligned}\tag{16}$$

Noticing that the conditions $(V1)$ and $(V2)$ imply that $V^-\geq 0$ for all $x\in\mathbb{R}^3$ and $V^-\in L^\infty(\mathbb{R}^3)$. Moreover, from the condition $(V2)$ it follows that $\{V^+=0\}$ has finite measure, which implies that $\{V^->0\}$ has finite measure. Therefore, using the facts that $u_n\rightharpoonup u$ in X_λ and $u_n\to u$ in $L^2_{loc}(\mathbb{R}^3)$, one has

$$\begin{aligned}\left|\int_{\mathbb{R}^3} V^-(u_n-u)^2\,\mathrm{d}x\right| &= \left|\int_{\mathrm{supp}\ V^-} V^-(u_n-u)^2\,\mathrm{d}x\right|\\ &\leq \left|V^-\right|_\infty \int_{\mathrm{supp}\ V^-} |u_n-u_0|^2\,\mathrm{d}x\longrightarrow 0\end{aligned}\tag{17}$$

and

$$\begin{aligned}\left|\int_{\mathbb{R}^3} V^-\left(u_n^2-u^2\right)\mathrm{d}x\right| &= \left|\int_{\mathrm{supp}\ V^-} V^-(u_n-u)(u_n+u)\,\mathrm{d}x\right|\\ &\leq \left|V^-\right|_\infty\int_{\mathrm{supp}\ V^-}|u_n-u|\,|u_n+u|\,\mathrm{d}x\\ &\leq \left|V^-\right|_\infty\left(\int_{\mathrm{supp}\ V^-}|u_n-u|^2\,\mathrm{d}x\right)^{1/2}\left(\int_{\mathrm{supp}\ V^-}|u_n+u|^2\,\mathrm{d}x\right)^{1/2}\\ &\longrightarrow 0,\end{aligned}$$

which implies that

$$\int_{\mathbb{R}^3} V^-(u_n-u)^2\,\mathrm{d}x=\int_{\mathbb{R}^3} V^-u_n^2\mathrm{d}x-\int_{\mathbb{R}^3} V^-u^2\mathrm{d}x+o(1)\,.\tag{18}$$

Similarly, for any $h\in X_\lambda$ we also have

$$\int_{\mathbb{R}^3} V^-(u_n-u)\,h\mathrm{d}x=\int_{\mathbb{R}^3} V^-u_nh\mathrm{d}x-\int_{\mathbb{R}^3} V^-uh\mathrm{d}x+o(1)\,.\tag{19}$$

Therefore, it follows from (16), (18)–(19) and Lemma 2.2 that, to obtain (14) and (15), it suffices to check that

$$\int_{\mathbb{R}^3}\left[F(x,u_n-u)-F(x,u_n)+F(x,u)\right]\mathrm{d}x=o(1)\tag{20}$$

and

$$\sup_{\|h\|_\lambda=1}\int_{\mathbb{R}^3}\left[f(x,u_n-u)-f(x,u_n)+f(x,u)\right]h\mathrm{d}x=o\left(1\right). \tag{21}$$

First, we verify (20). Inspired by [1], we observe that

$$F(x,u_n-u)-F(x,u_n)=\int_0^1\left(\frac{\mathrm{d}}{\mathrm{d}t}F(x,u_n-tu)\right)\mathrm{d}t.$$

Then

$$F(x,u_n-u)-F(x,u_n)=-\int_0^1 f(x,u_n-tu)u\mathrm{d}t,$$

and hence, by the conditions $(D1)$ and $(D2)$, we have

$$|F(x,u_n-u)-F(x,u_n)|\le\int_0^1\left[\delta\left|u_n-tu\right|\left|u\right|+C_\delta\left|u_n-tu\right|^{r-1}\left|u\right|\right]\mathrm{d}t,$$

where $\delta,C_\delta>0$ and $\max\{k,2\}<r<6$. This shows that

$$|F(x,u_n-u)-F(x,u_n)|\le\delta_1\left|u_n\right|\left|u\right|+\delta_1\left|u\right|^2+C_{\delta_1}\left|u_n\right|^{r-1}\left|u\right|+C_{\delta_1}\left|u\right|^r.$$

For each $\epsilon>0$, we use the Young's inequality to obtain that

$$|F(x,u_n-u)-F(x,u_n)+F(x,u)|\le C\left[\left(\epsilon\left|u_n\right|^2+C_\epsilon\left|u\right|^2\right)+\left(\epsilon\left|u_n\right|^r+C_\epsilon\left|u\right|^r\right)\right].$$

Next, we consider the function g_n given by

$$g_n\left(x\right):=\max\left\{|F(x,u_n-u)-F(x,u_n)+F(x,u)|-C\epsilon\left(\left|u_n\right|^2+\left|u_n\right|^r\right),0\right\}.$$

Then

$$0\le g_n\left(x\right)\le CC_\epsilon\left(\left|u\right|^2+\left|u\right|^r\right)\in L^1\left(\mathbb{R}^3\right).$$

Moreover, by the Lebesgue dominated convergence theorem,

$$\int_{\mathbb{R}^3}g_n\left(x\right)\mathrm{d}x\to 0\qquad\text{as } n\to\infty, \tag{22}$$

since $u_n\to u$ a.e. in $\mathbb{R}^3$. From the definition of g_n, it follows that

$$|F(x,u_n-u)-F(x,u_n)+F(x,u)|\le g_n\left(x\right)+C\epsilon\left(\left|u_n\right|^2+\left|u_n\right|^r\right),$$

which together with (22) and (9), shows that for n large enough,

$$\left|\int_{\mathbb{R}^3}\left[F(x,u_n-u)-F(x,u_n)+F(x,u)\right]\mathrm{d}x\right|\le C\epsilon,$$

which implies that

$$\int_{\mathbb{R}^3}\left[F(x,u_n-u)-F(x,u_n)+F(x,u)\right]\mathrm{d}x=o\left(1\right).$$

Similarly, we can verify (21) as well, and we omit it here. This completes the proof. □

Next, we investigate the compactness conditions for the functional $J_{\lambda,\mu}$. Recall that a C^1-functional $J_{\lambda,\mu}$ satisfies Cerami condition at level c ($(C)_c$ condition for short) if any sequence $\{u_n\} \subset X_\lambda$ such that $J_{\lambda,\mu}(u_n) \to c$ and $(1+\|u_n\|_\lambda)\|J'_{\lambda,\mu}(u_n)\|_{X_\lambda^{-1}} \to 0$ has a convergent subsequence, and such sequence is called a $(C)_c$-sequence.

Proposition 3.3. *For any $k \in \{1,3,4\}$, assume that the conditions $(V1)$–$(V4)$ and $(D1)$–$(D3)$ hold. Then for each $D>0$ there exists $\Lambda_0 = \Lambda(D) \geq \frac{4d_0}{b}$ such that $J_{\lambda,\mu}$ satisfies the $(C)_c$-condition in X_λ for all $c<D$ and $\lambda > \Lambda_0$.*

Proof. Let $\{u_n\}$ be a $(C)_c$-sequence with $c<D$. By Lemma 3.1, $\{u_n\}$ is bounded in X_λ, and there exists C_λ such that $\|u_n\|_\lambda \leq C_\lambda$. Therefore, there exist a subsequence $\{u_n\}$ and u_0 in X_λ such that

$$\begin{aligned} u_n &\rightharpoonup u_0 \text{ weakly in } X_\lambda;\\ u_n &\to u_0 \text{ strongly in } L^r_{loc}\left(\mathbb{R}^3\right) \text{ for } 2\leq r<6;\\ u_n(x) &\to u_0(x) \text{ a.e. on } \mathbb{R}^3. \end{aligned}$$

Moreover, $J'_{\lambda,\mu}(u_0)=0$. Now we prove that $u_n \to u_0$ strongly in X_λ. Let $v_n = u_n - u_0$. Then $v_n \rightharpoonup 0$ in X_λ. It follows from the condition $(V2)$ that

$$\begin{aligned} \int_{\mathbb{R}^3} v_n^2 \mathrm{d}x &= \int_{\{V^+\geq b\}} v_n^2 \mathrm{d}x + \int_{\{V^+<b\}} v_n^2 \mathrm{d}x\\ &\leq \frac{1}{\lambda b}\int_{\mathbb{R}^3} \lambda V^+ v_n^2 \mathrm{d}x + \int_{\{V^+<b\}} v_n^2\mathrm{d}x\\ &\leq \frac{1}{\lambda b}\|v_n\|_\lambda^2 + o(1). \end{aligned} \tag{23}$$

Using this and combining the Hölder and Sobolev inequalities, we have

$$\begin{aligned} \int_{\mathbb{R}^3} |v_n|^r \mathrm{d}x &\leq \left(\int_{\mathbb{R}^3} |v_n|^2 \mathrm{d}x\right)^{\frac{6-r}{4}} \left(\int_{\mathbb{R}^3} |v_n|^6 \mathrm{d}x\right)^{\frac{r-2}{4}}\\ &\leq \left[\frac{1}{\lambda b}\int_{\mathbb{R}^3}\left(|\nabla v_n|^2 + \lambda V^+ v_n^2\right)\mathrm{d}x\right]^{\frac{6-r}{4}} \left(\overline{S}^{-6}\left[\int_{\mathbb{R}^3}|\nabla v_n|^2\mathrm{d}x\right]^3\right)^{\frac{r-2}{4}} + o(1)\\ &\leq \left(\frac{1}{\lambda b}\right)^{\frac{6-r}{4}} \overline{S}^{-\frac{3(r-2)}{2}} \|v_n\|_\lambda^r + o(1). \end{aligned} \tag{24}$$

Furthermore, from Lemma 3.2 it follows that

$$J_{\lambda,\mu}(v_n) = J_{\lambda,\mu}(u_n) - J_{\lambda,\mu}(u_0) + o(1) \text{ and } J'_{\lambda,\mu}(v_n) = o(1).$$

Consequently, this together with the condition $(D3)$, (7), (23) and Lemma 2.1, we obtain

$$\begin{aligned} D &\geq c - J_{\lambda,\mu}(u_0)\\ &\geq J_{\lambda,\mu}(v_n) - \frac{1}{4}\left\langle J'_{\lambda,\mu}(v_n), v_n\right\rangle + o(1)\\ &\geq \frac{\mu_0-1}{4\mu_0}\|v_n\|_\lambda^2 - d_0\int_{\mathbb{R}^3} v_n^2\mathrm{d}x + o(1)\\ &\geq \left(\frac{\mu_0-1}{4\mu_0} - \frac{d_0}{\lambda b}\right)\|v_n\|_\lambda^2 + o(1), \end{aligned}$$

which implies that

$$\|v_n\|_\lambda^2 \leq \left(\frac{\mu_0-1}{4\mu_0}-\frac{d_0}{\lambda b}\right)^{-1} D+o(1), \text{ for every } \lambda>\frac{4\mu_0 d_0}{b(\mu_0-1)}.$$

Moreover, by (9), one has

$$\begin{aligned}\int_{\mathbb{R}^3}|v_n|^r\,\mathrm{d}x &\leq \left|\{V^+<b\}\right|^{\frac{6-r}{6}}\overline{S}^{-r}\|v_n\|_\lambda^r \\ &\leq \frac{|\{V^+<b\}|^{\frac{6-r}{6}}}{\overline{S}^r}\left[\left(\frac{\mu_0-1}{4\mu_0}-\frac{d_0}{\lambda b}\right)^{-1} D\right]^{\frac{r}{2}}+o(1). \qquad (25)\end{aligned}$$

Thus, it follows from (24) and (25) that

$$\begin{aligned}o(1) &= \int_{\mathbb{R}^3}\left(|\nabla v_n|^2+V_\lambda v_n^2\right)\mathrm{d}x+\mu\int_{\mathbb{R}^3}K\phi_{K,v_n}v_n^2\mathrm{d}x-\int_{\mathbb{R}^3}f(x,v_n)v_n\mathrm{d}x \\ &\geq \frac{\mu_0-1}{\mu_0}\|v_n\|_\lambda^2-\left(|p^+|_\infty+\epsilon\right)\int_{\mathbb{R}^3}v_n^2\mathrm{d}x-C_\epsilon\int_{\mathbb{R}^3}|v_n|^r\,\mathrm{d}x \\ &\geq \frac{\mu_0-1}{\mu_0}\|v_n\|_\lambda^2-\frac{|p^+|_\infty+\epsilon}{\lambda b}\|v_n\|_\lambda^2-C_\epsilon\left(\int_{\mathbb{R}^3}|v_n|^r\,\mathrm{d}x\right)^{(r-2)/r}\left(\int_{\mathbb{R}^3}|v_n|^r\,\mathrm{d}x\right)^{2/r} \\ &\geq \left(\frac{\mu_0-1}{\mu_0}-\frac{|p^+|_\infty+\epsilon}{\lambda b}\right)\|v_n\|_\lambda^2-\left(\left|\{V^+<b\}\right|^{\frac{6-r}{6}}\overline{S}^{-r}\right)^{(r-2)/r} \\ &\quad\cdot\left[\left(\frac{\mu_0-1}{4\mu_0}-\frac{d_0}{\lambda b}\right)^{-1} D\right]^{(r-2)/2}\left[\left(\frac{1}{\lambda b}\right)^{\frac{6-r}{4}}\overline{S}^{-\frac{3(r-2)}{2}}\right]^{2/r}\|v_n\|_\lambda^2 \\ &\geq \|v_n\|_\lambda^2\cdot\left[\frac{\mu_0-1}{\mu_0}-\frac{|p^+|_\infty+\epsilon}{\lambda b}\right. \\ &\quad\left.-\left(\frac{|\{V^+<b\}|^{\frac{6-r}{6}}}{\overline{S}^r}\right)^{\frac{r-2}{r}}\left[\left(\frac{\mu_0-1}{4\mu_0}-\frac{d_0}{\lambda b}\right)^{-1} D\right]^{(r-2)/2}\left(\left(\frac{1}{\lambda b}\right)^{\frac{6-r}{4}}\overline{S}^{-\frac{3(r-2)}{2}}\right)^{2/r}\right] \\ &\quad+o(1),\end{aligned}$$

since $\left\langle J'_{\lambda,\mu}(v_n), v_n\right\rangle=o(1)$ and $\int_{\mathbb{R}^3}f(x,v_n)v_n\mathrm{d}x\leq(|p^+|_\infty+\epsilon)\int_{\mathbb{R}^3}v_n^2\mathrm{d}x+C_\epsilon\int_{\mathbb{R}^3}|v_n|^r\,\mathrm{d}x$. Therefore, there exists $\Lambda_0=\Lambda(D)\geq\frac{4\mu_0 d_0}{b(\mu_0-1)}$ such that $v_n\to 0$ strongly in X_λ for $\lambda>\Lambda_0$. This completes the proof. □

Now we give the proof of Theorem 1.1 By Proposition 3.3 and $0<\eta\leq c_{\lambda,\mu}\leq c_{0,\mu}(\Omega)$ for all $\lambda\geq\frac{\overline{S}^2}{b}\left|\{V^+<b\}\right|^{-\frac{2}{3}}$, for each $\mu\in(0,m(k))$ there exists

$$\Lambda^*\geq\max\left\{\frac{\overline{S}^2}{b\left|\{V^+<b\}\right|^{\frac{2}{3}}},\frac{4\mu_0 d_0}{b(\mu_0-1)}\right\}>0$$

such that for every $\lambda>\Lambda^*$ and $(C)_{c_{\lambda,\mu}}$-sequence $\{u_n\}$ for $J_{\lambda,\mu}$ on X_λ there exist a subsequence $\{u_n\}$ and $u_\lambda\in X_\lambda$ such that $u_n\to u_\lambda$ strongly in X_λ. Moreover, $J_{\lambda,\mu}(u_\lambda)=c_{\lambda,\mu}$ and $(u_\lambda,\phi_{K,u_\lambda})$ is a nontrivial solution of Eq. $(SP_{\lambda,\mu})$.

4. Proof of Theorem 1.2

Proof. Suppose that u is a nontrivial solution of Eq. $(SP_{\lambda,\mu})$, then

$$\langle J'_{\lambda,\mu}(u),u\rangle=\int_{\mathbb{R}^3}\left(|\nabla u|^2+V_\lambda u^2\right)\mathrm{d}x+\mu\int_{\mathbb{R}^3}K\phi_{K,u}u^2\mathrm{d}x-\int_{\mathbb{R}^3}f(x,u)u\mathrm{d}x=0.$$

(i) By the conditions $(V1)-(V4)$ and $|q|_\infty\overline{S}^{-2}|\Omega|^{\frac{2}{3}}<\frac{\mu_0-1}{\mu_0}$, there exists $b_1>0$ such that

$$|q|_\infty\overline{S}^{-2}\left|\{V^+<b_1\}\right|^{\frac{2}{3}}<\frac{\mu_0-1}{\mu_0},$$

which implies that

$$\begin{aligned}\int_{\mathbb{R}^3}qu^2\mathrm{d}x&\le|q|_\infty\int_{\{V^+<b_1\}}u^2\mathrm{d}x+|q|_\infty\int_{\{V^+\ge b_1\}}u^2\mathrm{d}x\\&\le|q|_\infty\left|\{V^+<b_1\}\right|^{\frac{2}{3}}\overline{S}^{-2}\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x+\frac{|q|_\infty}{\lambda b_1}\int_{\{V^+\ge b_1\}}\lambda V^+u^2\mathrm{d}x\\&<\frac{\mu_0-1}{\mu_0}\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x+\frac{|q|_\infty}{\lambda b_1}\int_{\{V^+\ge b_1\}}\lambda V^+u^2\mathrm{d}x.\end{aligned}\tag{26}$$

Then, by the conditions $(D2)$ and $(D4)$, (7), (9) and (26), for $\lambda>\Lambda_*:=\frac{|q|_\infty\mu_0}{b_1(\mu_0-1)}$, we have

$$\begin{aligned}0=\langle J'_{\lambda,\mu}(u),u\rangle&=\int_{\mathbb{R}^3}\left(|\nabla u|^2+V_\lambda u^2\right)\mathrm{d}x+\mu\int_{\mathbb{R}^3}K\phi_{K,u}u^2\mathrm{d}x-\int_{\mathbb{R}^3}f(x,u)u\mathrm{d}x\\&\ge\frac{\mu_0-1}{\mu_0}\|u\|_\lambda^2-\int_{\mathbb{R}^3}q(x)u^2\mathrm{d}x\\&>\frac{\mu_0-1}{\mu_0}\|u\|_\lambda^2-\left(\frac{\mu_0-1}{\mu_0}\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x+\frac{|q|_\infty}{\lambda b_1}\int_{\{V^+\ge b_1\}}\lambda Vu^2\mathrm{d}x\right)\\&\ge\left(\frac{\mu_0-1}{\mu_0}-\frac{|q|_\infty}{\lambda b_1}\right)\int_{\{V^+\ge b_1\}}\lambda V^+u^2\mathrm{d}x\ge0,\end{aligned}$$

which is a contradiction. Therefore, Eq. $(SP_{\lambda,\mu})$ does not admit any nontrivial solution.
(ii) To proceed, we consider the proof in two separate cases.

Case 1 :$\int_{\mathbb{R}^3}q(x)u^4\mathrm{d}x=0$. By (7) one has

$$\begin{aligned}0=\langle J'_{\lambda,\mu}(u),u\rangle&=\int_{\mathbb{R}^3}\left(|\nabla u|^2+V_\lambda u^2\right)\mathrm{d}x+\mu\int_{\mathbb{R}^3}K\phi_{K,u}u^2\mathrm{d}x-\int_{\mathbb{R}^3}f(x,u)u\mathrm{d}x\\&\ge\frac{\mu_0-1}{\mu_0}\|u\|_\lambda^2+\mu\int_{\mathbb{R}^3}K\phi_{K,u}u^2\mathrm{d}x-\int_{\mathbb{R}^3}q(x)u^4\mathrm{d}x\\&=\frac{\mu_0-1}{\mu_0}\|u\|_\lambda^2+\mu\int_{\mathbb{R}^3}K\phi_{K,u}u^2\mathrm{d}x>0,\end{aligned}$$

which is a contradiction.

Case 2: $\int_{\mathbb{R}^3} q(x)u^4\mathrm{d}x > 0$. We set

$$v = \frac{u}{\left(\int_{\mathbb{R}^3} q(x)u^4\mathrm{d}x\right)^{1/4}}.$$

Clearly, $\int_{\mathbb{R}^3} q(x)v^4\mathrm{d}x = 1$. Then, by the conditions $(D2)$, $(D4)$, (4) and (9), we have

$$\begin{aligned}
0 = \langle J'_{\lambda,\mu}(u), u\rangle &= \int\limits_{\mathbb{R}^3} \left(|\nabla u|^2 + V_\lambda u^2\right)\mathrm{d}x + \mu \int\limits_{\mathbb{R}^3} K\phi_{K,u}u^2\mathrm{d}x - \int\limits_{\mathbb{R}^3} f(x,u)u\mathrm{d}x \\
&\geq \frac{\mu_0 - 1}{\mu_0}\|u\|_\lambda^2 + \frac{1}{\widehat{\mu}_1(q)}\int\limits_{\mathbb{R}^3} K\phi_{K,u}u^2\mathrm{d}x - \int\limits_{\mathbb{R}^3} q(x)u^4\mathrm{d}x \\
&= \frac{\mu_0 - 1}{\mu_0}\left(\int\limits_{\mathbb{R}^3} q(x)u^4\mathrm{d}x\right)^{1/2}\|v\|_\lambda^2 + \frac{1}{\widehat{\mu}_1(q)}\int\limits_{\mathbb{R}^3} q(x)u^4\mathrm{d}x\left(\int\limits_{\mathbb{R}^3} K\phi_{K,v}v^2\mathrm{d}x - \widehat{\mu}_1(q)\right) \\
&> 0,
\end{aligned}$$

which is a contradiction. Therefore, Eq. $(SP_{\lambda,\mu})$ does not admit any nontrivial solution. This completes the proof. □

5. Concentration for nontrivial solutions

In this section, we investigate the concentration of nontrivial solutions and give the proof of Theorem 1.3.

Proof of Theorem 1.3: We follow the argument in [5] (or see [32]). For any sequence $\lambda_n \to \infty$, setting $u_n := u_{\lambda_n}$ are the critical points of $J_{\lambda_n,\mu}$ obtained in Theorem 1.1. Since

$$\begin{aligned}
D_\mu \geq \alpha_{\lambda_n} &= J_{\lambda_n,\mu}(u_n) \\
&\geq \begin{cases} \left(\frac{\mu_0-1}{4\mu_0} - d_0\left|\{V^+ < b\}\right|^{\frac{2}{3}}\overline{S}^{-2}\right)\|u_n\|_{\lambda_n}^2, & \text{if } k=1 \\ \frac{1}{4}\|u_n\|_{\lambda_n}^2, & \text{if } k=3,4, \end{cases}
\end{aligned}$$

we have

$$\|u_n\|_{\lambda_n} \leq C_0, \tag{27}$$

where the constant C_0 is independent of λ_n. Therefore, we may assume that $u_n \rightharpoonup u_0$ weakly in X and $u_n \to u_0$ strongly in $L^r_{loc}(\mathbb{R}^3)$ for $2 \leq r < 6$. By Fatou's Lemma, we have

$$\int\limits_{\mathbb{R}^3} V^+u_0^2\mathrm{d}x \leq \liminf_{n\to\infty}\int\limits_{\mathbb{R}^3} V^+u_n^2\mathrm{d}x \leq \liminf_{n\to\infty}\frac{\|u_n\|_{\lambda_n}^2}{\lambda_n} = 0,$$

which implies that $u_0 = 0$ a.e. in $\mathbb{R}^3 \setminus V^{-1}(0)$ and $u_0 \in H_0^1(\Omega)$ by $(V3)$. Now for any $\varphi \in C_0^\infty(\Omega)$, since $\left\langle J'_{\lambda_n,\mu}(u_n), \varphi\right\rangle = 0$, it is easy to check that

$$\int\limits_{\mathbb{R}^3} \nabla u_0 \cdot \nabla\varphi\mathrm{d}x - \int\limits_{\mathbb{R}^3} V^- u_0\varphi\mathrm{d}x + \mu\int\limits_{\mathbb{R}^3} K\phi_{K,u_0}u_0\varphi\mathrm{d}x = \int\limits_{\mathbb{R}^3} f(x,u_0)\varphi\mathrm{d}x,$$

that is, u_0 is a weak solution of (SP_∞) by the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$.

Now we show that $u_n \to u_0$ in $L^r\left(\mathbb{R}^3\right)$ for $2 \le r < 6$. Otherwise, by Lions vanishing lemma [22] there exist $\delta > 0, R_0 > 0$ and $x_n \in \mathbb{R}^3$ such that

$$\int_{B^3(x_n,R_0)} (u_n - u_0)^2 \,\mathrm{d}x \ge \delta.$$

Moreover, $x_n \to \infty$, hence $\left|B(x_n, R_0) \cap \left\{x \in \mathbb{R}^3 : V^+ < b\right\}\right| \to 0$. By the Hölder inequality, we have

$$\int_{B(x_n,R_0)\cap\{V^+<b\}} (u_n - u_0)^2 \,\mathrm{d}x \to 0.$$

Consequently,

$$\begin{aligned}
\|u_n\|_{\lambda_n}^2 &\ge \lambda_n b \int_{B(x_n,R_0)\cap\{V^+\ge b\}} u_n^2 \mathrm{d}x = \lambda_n b \int_{B(x_n,R_0)\cap\{V^+\ge b\}} (u_n - u_0)^2 \,\mathrm{d}x \\
&= \lambda_n b \left(\int_{B(x_n,R_0)} (u_n - u_0)^2 \,\mathrm{d}x - \int_{B(x_n,R_0)\cap\{V^+<b\}} (u_n - u_0)^2 \,\mathrm{d}x + o(1) \right) \\
&\to \infty,
\end{aligned}$$

which contradicts (27). Therefore, $u_n \to u_0$ in $L^r\left(\mathbb{R}^3\right)$ for $2 \le r < 6$. Thus, by the conditions $(D1)\,,(D2)$ and $u_n \to u_0$ in $L^r\left(\mathbb{R}^3\right)$, we have

$$\int_{\mathbb{R}^3} f(x, u_n)\, u_n \mathrm{d}x \to \int_{\mathbb{R}^3} f(x, u_0)\, u_0 \mathrm{d}x. \tag{28}$$

Now, choose $\epsilon \in (0, \Theta_0 - |p^+|_\infty)$ as in the proof of Lemma 2.4 and use the conditions $(D1)$ and $(D2)$ to get

$$\left| \int_{\mathbb{R}^3} f(x,u)u\mathrm{d}x \right| \le \int_{\mathbb{R}^N} [\left(\left|p^+\right|_\infty + \epsilon\right) u^2 + C_\epsilon \left|u\right|^r]\mathrm{d}x. \tag{29}$$

Since $\left\langle J'_{\lambda_n,\mu}(u_n), u_n \right\rangle = 0$, by (8), (29) and the fact that $u_n \ne 0$, for n large we have

$$\begin{aligned}
\|u_n\|^2 \le \|u_n\|_{\lambda_n}^2 &\le \frac{(|p^+|_\infty + \epsilon) \max\left\{\frac{\overline{S}^2}{b}, |\{V^+ < b\}|^{\frac{2}{3}}\right\}}{\overline{S}^2} \|u_n\|^2 \\
&\quad + C_\epsilon \frac{\left(\max\left\{\frac{\overline{S}^2}{b}, |\{V^+ < b\}|^{\frac{2}{3}}\right\}\right)^{\frac{6-r}{4}}}{\overline{S}^r} \|u_n\|^r \\
&= \frac{|p^+|_\infty + \epsilon}{\min\left\{b, \overline{S}^2 |\{V^+ < b\}|^{-\frac{2}{3}}\right\}} \|u_n\|^2 + \frac{C_\epsilon}{\overline{S}^{\frac{3r-6}{2}} \left(\min\left\{b, \overline{S}^2 |\{V^+ < b\}|^{-\frac{2}{3}}\right\}\right)^{\frac{6-r}{4}}} \|u_n\|^r \\
&= \frac{|p^+|_\infty + \epsilon}{\Theta_0} \|u_n\|^2 + \frac{C_\epsilon}{\overline{S}^{\frac{3r-6}{2}} \Theta_0^{\frac{6-r}{4}}} \|u_n\|^r,
\end{aligned}$$

which implies that

$$\|u_n\| \ge \left(\frac{\overline{S}^{\frac{3r-6}{2}} \Theta_0^{\frac{6-r}{4}} (\Theta_0 - |p^+|_\infty - \epsilon)}{C_\epsilon \Theta_0} \right)^{1/(r-2)} > 0. \tag{30}$$

Moreover,

$$\int_{\mathbb{R}^3} f(x,u_n)u_n \mathrm{d}x = \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} V_{\lambda_n} u_n^2 \mathrm{d}x\right) + \mu \int_{\mathbb{R}^3} K\phi_{K,u_n} u_n^2 \mathrm{d}x$$
$$\geq \frac{\mu_0 - 1}{\mu_0} \|u_n\|_{\lambda_n}^2 \geq \frac{\mu_0 - 1}{\mu_0} \|u_n\|^2 . \tag{31}$$

By (28)–(31), we have

$$\int_{\mathbb{R}^3} f(x,u_0)\, u_0 \mathrm{d}x \geq \frac{\mu_0 - 1}{\mu_0} \left(\frac{\overline{S}^{\frac{3r-6}{2}} \Theta_0^{\frac{6-r}{4}} \left(\Theta_0 - |p^+|_\infty - \epsilon\right)}{C_\epsilon \Theta_0} \right)^{2/(r-2)} > 0,$$

this shows that $u_0 \neq 0$. Finally, we show that $u_n \to u_0$ in X. Since $\left\langle J'_{\lambda_n,\mu}(u_n), u_n \right\rangle = \left\langle J'_{\lambda_n,\mu}(u_n), u_0 \right\rangle = 0$, we have

$$\|u_n\|_{\lambda_n}^2 - \int_{\mathbb{R}^3} V^- u_n^2 \mathrm{d}x + \mu \int_{\mathbb{R}^3} K(x)\phi_{K,u_n} u_n^2 \mathrm{d}x = \int_{\mathbb{R}^3} f(x,u_n)u_n \mathrm{d}x, \tag{32}$$

and

$$\langle u_n, u_0\rangle_{\lambda_n} - \int_{\mathbb{R}^3} V^- u_n u_0 \mathrm{d}x + \mu \int_{\mathbb{R}^3} K(x)\phi_{K,u_n} u_n u_0 \mathrm{d}x = \int_{\mathbb{R}^3} f(x,u_n)u_0 \mathrm{d}x. \tag{33}$$

It is easy to verify that

$$\int_{\mathbb{R}^3} V^- \left(u_n^2 - u_n u_0\right) \mathrm{d}x \to 0 \tag{34}$$

and

$$\int_{\mathbb{R}^3} \left(K(x)\phi_{K,u_n} u_n^2 - K(x)\phi_{K,u_n} u_n u_0\right) \mathrm{d}x \to 0. \tag{35}$$

Indeed, for (34), similar to the proof of (17), we have

$$\left|\int_{\mathbb{R}^3} V^- \left(u_n^2 - u_n u_0\right) \mathrm{d}x\right| = \left|\int_{\mathrm{supp}\ V^-} V^- u_n \left(u_n - u_0\right) \mathrm{d}x\right|$$
$$\leq \left|V^-\right|_\infty \left(\int_{\mathrm{supp}\ V^-} |u_n - u_0|^2 \,\mathrm{d}x\right)^{1/2} |u_n|_2$$
$$\to 0,$$

since $u_n \to u_0$ in $L^2_{loc}\left(\mathbb{R}^3\right)$ and $\{u_n\}$ is bounded in $L^2\left(\mathbb{R}^3\right)$. For (35), if $K(x) \in L^\infty(\mathbb{R}^3)$, then the Hölder inequality and $u_n \to u_0$ in $L^3(\mathbb{R}^3)$ imply that

$$\int_{\mathbb{R}^3} \left(K(x)\phi_{K,u_n} u_n^2 - K(x)\phi_{K,u_n} u_n u_0\right) \mathrm{d}x \leq |K|_\infty |\phi_{u_n}|_6 |u_n|_2 |u_n - u_0|_3 \to 0.$$

If $K(x) \in L^2(\mathbb{R}^3)$, similar to the proof of (2.11) in [32], then (35) holds. Moreover, by (32)–(35), we have

$$\lim_{n\to\infty} \|u_n\|_{\lambda_n}^2 = \lim_{n\to\infty} \langle u_n, u_0\rangle_{\lambda_n} = \lim_{n\to\infty} \langle u_n, u_0\rangle = \|u_0\|^2.$$

On the other hand, weakly lower semi-continuity of norm yields that

$$\|u_0\|^2 \leq \liminf_{n\to\infty} \|u_n\|^2 \leq \limsup_{n\to\infty} \|u_n\|^2 \leq \lim_{n\to\infty} \|u_n\|^2_{\lambda_n}.$$

This shows that $u_n \to u_0$ in X. This completes the proof.

References

1. Alves, C.O., Carrião, P.C., Medeiros, E.S.: Multiplicity of solutions for a class of quasilinear problem in exterior domains with Neumann conditions. Abstr. Appl. Anal. **3**, 251–268 (2004)
2. Ambrosetti, A., Ruiz, D.: Multiple bound states for the Schrödinger–Poisson problem. Commun. Contemp. Math. **10**(3), 39–404 (2008)
3. Azzollini, A.: Concentration and compactness in nonlinear Schrödinger–Poisson system with a general nonlinearity. J. Differ. Equ. **249**, 1746–1763 (2010)
4. Azzollini, A., Pomponio, A.: Ground state solutions for the nonlinear Schrödinger–Maxwell equations. J. Math. Anal. Appl. **345**, 90–108 (2008)
5. Bartsch, T., Pankov, A., Wang, Z.-Q.: Nonlinear Schrödinger equations with steep potential well. Commun. Contemp. Math. **3**, 549–569 (2001)
6. Bartsch, T., Tang, Z.: Multibump solutions of nonlinear Schrödinger equations with steep potential well and indefinite potential. Discrete Contin. Dyn. Syst. **33**, 7–26 (2013)
7. Bartsch, T., Wang, Z.-Q.: Existence and multiplicity results for superlinear elliptic problems on $\mathbb{R}^N$. Commun. Partial Differ. Equ. **20**, 1725–1741 (1995)
8. Benci, V., Fortunato, D.: An eigenvalue problem for the Schrödinger–Maxwell equations. Topol. Methods Nonlinear Anal. **11**, 283–293 (1998)
9. Brezis, H., Lieb, E.H.: A relation between pointwise convergence of functions and convergence functionals. Proc. Am. Math. Soc. **8**, 486–490 (1983)
10. Chen, C.Y., Kuo, Y.C., Wu, T.F.: Existence and multiplicity of positive solutions for the nonlinear Schrödinger–Poisson equations. Proc. R. Soc. Edinb. Sect. A **143**, 745–764 (2013)
11. Cerami, G., Vaira, G.: Positive solutions for some non-autonomous Schrödinger–Poisson systems. J. Differ. Equ. **248**, 521–543 (2010)
12. D'Aprile, T., Mugnai, D.: Solitary waves for nonlinear Klein–Gordon–Maxwell and Schrödinger-Maxwell equations. Proc. R. Soc. Edinb. Sect. **134A**, 893–906 (2004)
13. Ding, Y.H., Szulkin, A.: Bound states for semilinear Schrödinger equations with sign-changing potential. Calc. Var. Partial Differ. Equ. **29**, 397–419 (2007)
14. Ekeland, I.: Convexity Methods in Hamiltonian Mechanics. Springer, Berlin (1990)
15. Figueiredo, D.G.de : Positive Solutions of Semilinear Elliptic Problems, Lecture in Math. vol. 957. Springer, Berlin (1982)
16. Ianni, I.: Sign-changing radial solutions for the Schrödinger–Poisson–Slater problem. Topol. Methods Nonlinear Anal. **41**, 365–385 (2013)
17. Ianni, I., Ruiz, D.: Ground and bound states for a static Schrödinger–Poisson–Slater problem. Commun. Contemp. Math. **14**, 1250003 (2012)
18. Ianni, I., Vaira, G.: On concentration of positive bound states for the Schrödinger–Poisson problem with potentials. Adv. Nonlinear Stud. **8**, 573–595 (2008)
19. Jiang, Y., Zhou, H.: Schrödinger–Poisson system with steep potential well. J. Differ. Equ. **251**, 582–608 (2011)
20. Kikuchi, H.: Existence and stability of standing waves for Schrödinger–Poisson–Slatere equation. Adv. Nonlinear Stud. **7**, 403–437 (2007)
21. Kikuchi, H.: On the existence of solutions for elliptic system related to the Maxwell–Schrödinger equations. Nonlinear Anal. **67**, 1445–1456 (2007)
22. Lions, P.L.: The concentration-compactness principle in the calculus of variations. The locally compact case, Part I. Ann. Inst. H. Poincare Anal. NonLineaire **1**, 109–145 (1984)
23. Ruiz, D.: The Schrödinger–Poisson equation under the effect of a nonlinear local term. J. Funct. Anal. **237**, 655–674 (2006)
24. Ruiz, D.: On the Schrödinger–Poisson–Slater System: behavior of minimizers, radial and nonradial cases. Arch. Ratl. Mech. Anal. **198**, 349–368 (2010)
25. Sato, Y., Tanaka, K.: Sign-changing multi-bump solutions for nonlinear Schrödinger equations with steep potential wells. Trans. Am. Math. Soc. **361**, 6205–6253 (2009)
26. Sun, J.: Infinitely many solutions for a class of sublinear Schrödinger–Maxwell equations. J. Math. Anal. Appl. **390**, 514–522 (2012)

27. Sun, J., Chen, H., Nieto, J.J.: On ground state solutions for some non-autonomous Schrödinger–Poisson systems. J. Differ. Equ. **252**, 3365–3380 (2012)
28. Sun, J., Wu, T.F.: Ground state solutions for an indefinite Kirchhoff type problem with steep potential well. J. Differ. Equ. **256**, 1771–1792 (2014)
29. Ye, Y., Tang, C.: Existence and multiplicity of solutions for Schrödinger–Poisson equations with sign-changing potential. Calc. Var. Partial Differ. Equ. doi:10.1007/s00526-014-0753-6
30. Wang, Z., Zhou, H.: Positive solution for a nonlinear stationary Schrödinger–Poisson system in $\mathbb{R}^3$. Discrete Contin. Dyn. Syst. **18**, 809–816 (2007)
31. Wang, Z., Zhou, H.: Positive solutions for nonlinear Schrödinger equations with deepening potential well. J. Eur. Math. Soc. **11**, 545–573 (2009)
32. Zhao, L., Liu, H., Zhao, F.: Existence and concentration of solutions for the Schrödinger-Poisson equations with steep well potential. J. Differ. Equ. **255**, 1–23 (2013)
33. Zhao, L., Zhao, F.: On the existence of solutions for the Schrödinger–Poisson equations. J. Math. Anal. Appl. **346**, 155–169 (2008)

Juntao Sun
School of Science
Shandong University of Technology
Zibo 255049
People's Republic of China
e-mail: sunjuntao2008@163.com

Tsung-fang Wu
Department of Applied Mathematics
National University of Kaohsiung
Kaohsiung 811, Taiwan
e-mail: tfwu@nuk.edu.tw

(Received: March 4, 2014; revised: October 13, 2014)