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Well-posedness and stability results in a Timoshenko-type system of thermoelasticity of type III with delay

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Abstract. In this paper, we consider a one-dimensional linear thermoelastic system of Timoshenko type with delay, where the heat conduction is given by Green and Naghdi's theory. We establish the well-posedness and the stability of the system for the cases of equal and nonequal speeds of wave propagation.

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1. Introduction

The issue of existence and stability of Timoshenko systems has attracted a great deal of attention in the last decades. From a physical or engineering point of view, Timoshenko theory is an improvement of Euler–Bernoulli theory. Indeed, in the Euler–Bernoulli beam theory, it is assumed that plane cross-sections that are perpendicular to the axis of the beam remain plane and perpendicular to the axis after deformation, which implies that the transverse shear strain is zero. When the rotational inertia and the transverse shear are significant in the beam model, one has to use rather the Timoshenko theory. The transverse vibrations of the beam depend in general on its geometrical properties (its length, size and shape, cross-section, moment of inertia, shear coefficient) and its mechanical properties (density, Young's modulus, modulus of rigidity). To be more precise, we have the following model, which was developed by Timoshenko on [1] in 1921,

$$\begin{aligned} &\rho u_{tt}(x,t) = (K(u_x(x,t)-\varphi(x,t)))_x, \quad \text{in } (0,L)\times(0,+\infty) \\ &I_\rho \varphi_{tt}(x,t) = (EI\varphi_x(x,t))_x + K(u_t(x,t)-\varphi(x,t)), \ \text{ in } (0,L)\times(0,+\infty), \end{aligned} \tag{1.1}$$

together with boundary conditions of the form

$$EI\varphi_x|_{x=0}^{x=L} = 0, \quad (u_x-\varphi)|_{x=0}^{x=L} = 0,$$

where $u(x,t)$ is the transverse displacement, $\varphi(x,t)$ is the rotational angle of the beam, ρ denotes the mass density, I_ρ is the moment of mass inertia, EI is the rigidity coefficient, K is the shear modulus of elasticity, and L is the length of the beam.

Due to a surrounding flow of wind, gas or fluid, the beam is subject to mechanical vibrations. These vibrations are of course undesirable because of their damaging and destructing nature. To reduce these harmful vibrations, several control mechanisms have been designed. This is achieved either by incorporating into the structure a smart material actuator as piezoceramic or by acting inside or at the free edges of the beam. Several researchers employed different types of damping mechanisms to stabilize these systems and to obtain precise rates of decay. For internal or boundary frictional damping, we quote, among others, the work of Kim and Renardy [2], Raposo et al. [3], Soufyane and Wehbe [4], Rivera and Racke [5,6], and Mustafa and Messaoudi [7]. Regarding Timoshenko systems for material with “finite” or “infinite” memory, we refer to Ammar-Khodja et al. [8], Guesmia and Messaoudi [9], and Fernández Sare and Rivera [10].

For stabilization via heat dissipation, Rivera and Racke [11] established several exponential decay results for linear Timoshenko systems coupled with the classical heat equation, in which the heat flux is given by Fourier’s law. Since this theory predicts an infinite speed of heat propagation, to overcome this physical paradox, many theories have emerged. One of which, given by Green and Naghdi [12–14], suggests replacing Fourier’s law by so- called thermoelasticity of type III for heat conduction modeling thermal disturbances as wave-like pulses traveling at finite speed. See [15] for more details.

Taking into account Green and Naghdi’s theory, a Timoshenko system of thermoelasticity of type III of the form

$$\begin{array}{ll} \rho_1\varphi_{tt} - K\left(\varphi_x+\psi\right)_x = 0 & \text{in } (0,\infty)\times(0,1)\,, \\ \rho_2\psi_{tt} - b\psi_{xx} + K\left(\varphi_x+\psi\right) + \beta\theta_x = 0 & \text{in } (0,\infty)\times(0,1)\,, \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - k\theta_{txx} = 0 & \text{in } (0,\infty)\times(0,1)\,, \end{array} \tag{1.2}$$

where φ, ψ, and θ are functions of (x,t) which model the transverse displacement of the beam, the rotation angle of the filament, and the difference temperature respectively, was studied by Messaoudi and Said-Houari [16], and an exponential decay result in the case of equal wave speeds $\left(\frac{K}{\rho_1} = \frac{b}{\rho_2}\right)$ was proved. The case of nonequal speeds $\left(\frac{K}{\rho_1} \neq \frac{b}{\rho_2}\right)$ was studied later by Messaoudi and Fareh [17], and a polynomial decay result was proved for solutions with smooth initial data. A decay result, where a viscoelastic damping of the form $\int_0^t g(t-s)\theta_{xx}(s)\mathrm{d}s$ is acting in the third equation instead of the strong heat dissipation $-k\theta_{txx}$, was also established by Kafini [18].

Time delays arise in many applications because most phenomena naturally depend not only on the present state but also on some past occurrences. In recent years, the control of PDEs with time delay effects has become an active area of research, see for example [19] and references therein.

In many cases, it was shown that delay is a source of instability unless additional conditions or control terms are used, see [20]. Therefore, the stability issue of systems with delay is of theoretical and practical great importance.

For the system of wave equation with locally distributed damping of the form

$$\begin{cases} u_{tt}(x,t) - \triangle u(x,t) + a_0 u_t(x,t) + a u_t(x,t-\tau) = 0, & \text{in } \Omega\times(0,\infty) \\ u(x,t) = 0, \quad x\in\Gamma_0, t>0 \\ \frac{\partial u}{\partial v}(x,t) = 0, \quad x\in\Gamma_1, t>0, \end{cases} \tag{1.3}$$

it is well-known, in the absence of delay ($a = 0, a_0 > 0$), that the system is exponentially stable, see [21]. In the presence of delay ($a > 0$), Nicaise and Pignotti [22] examined (1.3) and proved, under the assumption that the weight of the feedback with delay is smaller than the one without delay ($a < a_0$), that the energy is exponentially stable. For the opposite case, they produced a sequence of delays for which the corresponding solution is instable. The same results were obtained for the case of boundary delay, see also [23] for the treatment to this problem in more general abstract form. When the delay term in (1.3) is replaced by the distributed delay

$$\int_{\tau_1}^{\tau_2} a(s)u_t(x,t-s)\mathrm{d}s,$$

exponential stability results have been obtained in [24] under the condition $\int_{\tau_1}^{\tau_2} a(s)\mathrm{d} < a_0$.

Introducing a delay term in the internal feedback of the thermoelastic system may turn a well-behaved system into a wild one. For instance, contrary to the exponential stability of the classical thermoelastic system without delay, Racke [25] proved that, any constant delay makes the system instable.

In this work, we are concerned with the following Timoshenko system of thermoelasticity of type III with delay of the form

$$\begin{cases} \rho_1\phi_{tt} - K\left(\phi_x+\psi\right)_x + \mu_1\phi_t(x,t) + \mu_2\phi_t(x,t-\tau) = 0 \text{ in } (0,1)\times(0,\infty) \\ \rho_2\psi_{tt} - b\psi_{xx} + K\left(\phi_x+\psi\right) + \beta\theta_{tx} = 0 \quad \text{ in } (0,1)\times(0,\infty) \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{tx} - k\theta_{txx} = 0 \quad \text{in } (0,1)\times(0,\infty), \\ \theta(.,0) = \theta_0,\ \theta_t(.,0) = \theta_1,\ \psi(.,0) = \psi_0,\ \psi_t(.,0) = \psi_1, \\ \phi(.,0) = \phi_0,\ \phi_t(.,0) = \phi_1, \\ \phi_t(x,t-\tau) = f_0(x,t-\tau), \quad t\in(0,\tau), \\ \phi(0,t) = \phi(1,t) = \psi(0,t) = \psi(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0, \quad \forall t\geq 0, \end{cases} \tag{1.4}$$

where $\rho_1, \rho_2, \rho_3, K, b, k, \beta, \gamma, \delta, \mu_1$ are positive constants, μ_2 is a real number, and $\tau > 0$ represents the time delay. We prove, under suitable conditions on the initial data that the energy decays exponentially in the case of equal wave speeds in spite of the existence of the delay. The second part of our result is the case of nonequal speeds which is of much importance because practically or physically the speeds are not necessarily equal. In that case, we prove that the energy decays polynomially.

In [26], the well-posedness and stability of the same system (1.4) without delay and with infinite memory considered in the first or second equation of Timoshenko system was proved, and general decay estimates were obtained depending on the growth of the kernel function at infinity and the wave speeds. In [27], the exponential stability of an abstract hyperbolic system with a discrete time delay and an infinite memory was proved under the assumption that the kernel function converges exponentially to zero and the weight of the delay is small enough. The system considered in [27] is not dissipative due to the fact that the unique considered dissipation is generated by the infinite memory. More results are found in [28,29] and [30].

2. Preliminaries

As in [24], we introduce the new variable

$$z(x,\rho,t) = \phi_t(x,t-\tau\rho), \qquad x\in(0,1),\ \rho\in(0,1),\ t>0.$$

Thus, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad x\in(0,1),\ \rho\in(0,1),\ t>0.$$

So, problem (1.4) is equivalent to

$$\begin{cases} \rho_1\phi_{tt} - K\left(\phi_x+\psi\right)_x + \mu_1\phi_t(x,t) + \mu_2 z(x,1,t) = 0 \text{ in } (0,1)\times(0,\infty) \\ \rho_2\psi_{tt} - b\psi_{xx} + K\left(\phi_x+\psi\right) + \beta\theta_{tx} = 0 \quad \text{ in } (0,1)\times(0,\infty) \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{tx} - k\theta_{txx} = 0 \quad \text{in } (0,1)\times(0,\infty) \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \quad \text{in } (0,\infty)\times(0,1)\times(0,1) \\ \phi(.,0) = \phi_0,\ \phi_t(.,0) = \phi_1,\ z(x,0,t) = \phi_t(x,t), \\ \theta(.,0) = \theta_0,\ \theta_t(.,0) = \theta_1,\ \psi(.,0) = \psi_0,\ \psi_t(.,0) = \psi_1 \\ \phi(0,t) = \phi(1,t) = \psi(0,t) = \psi(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0 \\ z(x,\rho,0) = f_0(x,-\rho\tau), \quad x\in(0,1),\ \rho\in(0,1). \end{cases} \tag{2.1}$$

In order to be able to use Poincaré's inequality for θ , we introduce

$$\overline{\theta}(x,t) = \theta(x,t) - t\int_0^1 \theta_1(x)\,\mathrm{d}x - \int_0^1 \theta_0(x)\,\mathrm{d}x.$$

Then by $(2.1)_3$ we have

$$\int_0^1 \overline{\theta}\,(x,t)\,\mathrm{d}x = 0, \quad \forall t \geq 0.$$

In this case, Poincaré's inequality is applicable for $\overline{\theta}$ and, furthermore, $\left(\phi, \psi, \overline{\theta}, z\right)$ satisfies the same equations and boundary conditions of (2.1). In what follows, we will work with $\overline{\theta}$ but, for convenience, we write θ instead of $\overline{\theta}$.

We will assume that

$$\mu_1 \geq |\mu_2| \tag{2.2}$$

and show the well-posedness of the problem and that this condition is sufficient to prove the uniform decay of the solution energy.

3. Well-posedness of the problem

In this section, we give the existence and uniqueness result for problem (2.1) using the semigroup theory. We will use the following standard $L^2(0,1)$ space with the scalar product and norm denoted by

$$\langle u, v\rangle_{L^2(0,1)} = \int_0^1 u\,v\,\mathrm{d}x, \qquad \|u\|_2^2 = \int_0^1 |u|^2\,\mathrm{d}x,$$

respectively. Introducing the vector function $\mathcal{U}(t) = (\phi, \varphi, \psi, u, \theta, v, z)^T$, where $\varphi = \phi_t,\ u = \psi_t,$ and $v = \theta_t$, system (2.1) can be re-written as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{U}(t) + \mathcal{A}\mathcal{U}(t) = 0, \qquad t > 0, \\ \mathcal{U}(0) = \mathcal{U}_0 = (\phi_0,\ \phi_1,\ \psi_0,\ \psi_1,\ \theta_0,\ \theta_1,\ f_0)^T, \end{cases} \tag{3.1}$$

where the linear operator $\mathcal{A}\colon D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is defined by

$$\mathcal{A}\mathcal{U} = \begin{pmatrix} -\varphi \\ -\dfrac{K}{\rho_1}\left(\phi_x + \psi\right)_x + \dfrac{\mu_1}{\rho_1}\varphi + \dfrac{\mu_2}{\rho_1}z(.,1) \\ -u \\ -\dfrac{b}{\rho_2}\psi_{xx} + \dfrac{K}{\rho_2}\left(\phi_x + \psi\right) + \dfrac{\beta}{\rho_2}v_x \\ -v \\ -\dfrac{\delta}{\rho_3}\theta_{xx} + \dfrac{\gamma}{\rho_3}u_x - \dfrac{k}{\rho_3}v_{xx} \\ \frac{1}{\tau}z_\rho \end{pmatrix}.$$

Next, we introduce

$$\begin{aligned} L^2_\star(0,1) &= \left\{ w \in L^2(0,1) : \int_0^1 w(s)\mathrm{d}s = 0, \right\} \\ H^1_\star(0,1) &= H^1(0,1) \cap L^2_\star(0,1), \\ H^2_\star(0,1) &= \left\{ w \in H^2(0,1) : w_x(0) = w_x(1) = 0 \right\} \end{aligned}$$

and the energy space

$$\mathcal{H} = H_0^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \times L^2(0,1) \times H_\star^1(0,1) \times L_\star^2(0,1) \times L^2\left((0,1), L^2(0,1)\right).$$

For a positive constant ξ satisfying

$$\begin{cases} \gamma\tau\,|\mu_2| < \xi < \gamma\tau\,(2\mu_1 - |\mu_2|) & \text{if } \mu_1 > |\mu_2| \\ \xi = \gamma\tau\,|\mu_2| = \gamma\tau\mu_1 & \text{if } \mu_1 = |\mu_2| \end{cases} \tag{3.1*}$$

we equip $\mathcal{H}$ with the inner product

$$\left\langle \mathcal{U}, \widetilde{\mathcal{U}} \right\rangle_{\mathcal{H}} = \gamma \int_0^1 \left\{ \rho_1 \varphi\widetilde{\varphi} + \rho_2 u\widetilde{u} + K\left(\phi_x + \psi\right)\left(\widetilde{\phi}_x + \widetilde{\psi}\right) + b\psi_x\widetilde{\psi}_x \right\} \mathrm{d}x$$

$$+\beta \int_0^1 \left(\rho_3 v\widetilde{v} + \delta\theta_x\widetilde{\theta}_x\right) \mathrm{d}x + \xi \int_0^1\int_0^1 z(x,\rho)\widetilde{z}(x,\rho)\mathrm{d}\rho\mathrm{d}x.$$

The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \begin{Bmatrix} \mathcal{U} \in \mathcal{H} \mid \phi,\, \psi \in H^2(0,1) \cap H_0^1(0,1), & \theta, v \in H_\star^1(0,1), & \varphi, u \in H_0^1(0,1), \\ \delta\theta + kv \in H_\star^2(0,1), & z, z_\rho \in L^2\left((0,1), L^2(0,1)\right), & z(x,0) = \varphi(x) \end{Bmatrix}.$$

and it is dense in $\mathcal{H}$.

We have the following existence and uniqueness result:

Theorem 3.1. *Assume $\mathcal{U}_0 \in \mathcal{H}$ and (2.2) holds. Then, there exists a unique solution $\mathcal{U} \in (\mathbb{R}^+, \mathcal{H})$ of problem (2.1). Moreover, if $\mathcal{U}_0 \in D(\mathcal{A})$ then $\mathcal{U} \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

Proof. We use the semigroup approach. So, we prove that $\mathcal{A}$ is a maximal monotone operator. First, we prove that $\mathcal{A}$ is monotone. For any $\mathcal{U} \in D(\mathcal{A})$, we have

$$(\mathcal{A}\mathcal{U}, \mathcal{U})_{\mathcal{H}} = \gamma\mu_1 \int_0^1 \varphi^2 \mathrm{d}x + \beta k \int_0^1 v_x^2 \mathrm{d}x + \gamma\mu_2 \int_0^1 \varphi z(.,1)\mathrm{d}x + \frac{\xi}{\tau}\int_0^1\int_0^1 zz_\rho \mathrm{d}\rho\mathrm{d}x. \tag{3.2}$$

By using Young's inequality, the third term in the right-hand side of (3.2) gives

$$-\mu_2 \int_0^1 \varphi z(.,1)\mathrm{d}x \le \frac{|\mu_2|}{2}\int_0^1 \varphi^2\mathrm{d}x + \frac{|\mu_2|}{2}\int_0^1 z^2(.,1)\mathrm{d}x.$$

Also, using integration by parts and the fact that $z(x,0) = \varphi(x)$, the last term in the right-hand side of (3.2) gives

$$\int_0^1\int_0^1 zz_\rho\mathrm{d}\rho\mathrm{d}x = -\frac{1}{2}\int_0^1 \varphi^2\mathrm{d}x + \frac{1}{2}\int_0^1 z^2(.,1)\mathrm{d}x.$$

Consequently, (3.2) yields

$$(\mathcal{A}\mathcal{U}, \mathcal{U})_{\mathcal{H}} \ge \frac{1}{2\tau}\left(\gamma\tau(2\mu_1 - |\mu_2|) - \xi\right)\int_0^1 \varphi^2\mathrm{d}x + \frac{1}{2\tau}\left(\xi - \gamma\tau|\mu_2|\right)\int_0^1 z^2(.,1)\mathrm{d}x + \beta k\int_0^1 v_x^2\mathrm{d}x$$

and by using (3.1*), we get

$$(\mathcal{A}\mathcal{U}, \mathcal{U})_{\mathcal{H}} \ge m_0\left(\int_0^1 \varphi^2\mathrm{d}x + \int_0^1 z^2(.,1)\mathrm{d}x\right) + k\beta\int_0^1 v_x^2\mathrm{d}x,$$

for some constant $m_0 \geq 0$. Thus, $\mathcal{A}$ is monotone. Next, we prove that the operator $I+\mathcal{A}$ is surjective. Given $F=(f_1,f_2,f_3,f_4,f_5,f_6,f_7)^T \in \mathcal{H}$, we prove that there exists a unique $\mathcal{U} \in D(\mathcal{A})$ such that

$$(I+\mathcal{A})\mathcal{U}=F. \tag{3.3}$$

That is,

$$\begin{cases} -\varphi+\phi=f_1 & \text{in } H_0^1(0,1) \\ -K\left(\phi_x+\psi\right)_x+\left(\mu_1+\rho_1\right)\varphi+\mu_2 z(.,1)=\rho_1 f_2 & \text{in } L^2(0,1) \\ -u+\psi=f_3 & \text{in } H_0^1(0,1) \\ -b\psi_{xx}+K\left(\phi_x+\psi\right)+\beta v_x+\rho_2 u=\rho_2 f_4 & \text{in } L^2(0,1) \\ -v+\theta=f_5 & \text{in } H_\star^1(0,1) \\ -\delta\theta_{xx}+\gamma u_x-k v_{xx}+\rho_3 v=\rho_3 f_6 & \text{in } L_\star^2(0,1) \\ z_\rho+\tau z=\tau f_7 & \text{in } L^2\big((0,1),L^2(0,1)\big) \end{cases} \tag{3.4}$$

Using (3.4)$_7$ and the fact that $z(x,0)=\varphi(x)$, we get

$$z(x,\rho)=\varphi(x)e^{-\tau\rho}+\tau e^{-\tau\rho}\int_0^\rho e^{\tau s} f_7(x,s)\mathrm{d}s. \tag{3.5}$$

In order to solve (3.4), we consider the following variational formulation

$$B\left((\phi,\psi,\theta),(\phi_1,\psi_1,\theta_1)\right)=G\left(\phi_1,\psi_1,\theta_1\right), \tag{3.6}$$

where $B:\left[H_0^1(0,1)\times H_0^1(0,1)\times H_\star^1(0,1)\right]^2 \longrightarrow \mathbb{R}$ is the bilinear form defined by

$$\begin{aligned} &B\left((\phi,\psi,\theta),(\phi_1,\psi_1,\theta_1)\right) \\ &\quad=\gamma K\int_0^1\left(\phi_x+\psi\right)\left(\phi_{1x}+\psi_1\right)\mathrm{d}x+\beta\left(\delta+k\right)\int_0^1\theta_x\theta_{1x}\mathrm{d}x+b\gamma\int_0^1\psi_x\psi_{1x}\mathrm{d}x \\ &\qquad+\rho_2\gamma\int_0^1\psi\psi_1\mathrm{d}x+\beta\gamma\int_0^1\theta_x\psi_1\mathrm{d}x+\beta\rho_3\int_0^1\theta\theta_1\mathrm{d}x+\beta\gamma\int_0^1\theta_1\psi_x\mathrm{d}x \\ &\qquad+\gamma\left(\mu_1+\rho_1+\mu_2 e^{-\tau}\right)\int_0^1\phi\phi_1\mathrm{d}x \end{aligned}$$

and $G:\left[H_0^1(0,1)\times H_0^1(0,1)\times H_\star^1(0,1)\right] \longrightarrow \mathbb{R}$ is the linear functional given by

$$\begin{aligned} G\left(\phi_1,\psi_1,\theta_1\right)&=\gamma\rho_1\int_0^1 f_2\phi_1\mathrm{d}x+\gamma\left(\mu_1+\rho_1\right)\int_0^1 f_1\phi_1\mathrm{d}x+\gamma\rho_2\int_0^1 f_4\psi_1\mathrm{d}x \\ &\quad+\gamma\beta\int_0^1 f_{5x}\psi_1\mathrm{d}x+\gamma\rho_2\int_0^1 f_3\psi_1\mathrm{d}x+\beta\rho_3\int_0^1 f_6\theta_1\mathrm{d}x+\beta\rho_3\int_0^1 f_5\theta_1\mathrm{d}x \\ &\quad+\gamma\beta\int_0^1 f_{3x}\theta_1\mathrm{d}x+\beta k\int_0^1 f_{5x}\theta_{1x}\mathrm{d}x-\gamma\tau\mu_2 e^{-\tau}\int_0^1\phi_1\int_0^1 e^{\tau s}f_7(x,s)\mathrm{d}s. \end{aligned}$$

Now, for $V=H_0^1(0,1)\times H_0^1(0,1)\times H_\star^1(0,1)$ equipped with the norm

$$\|\phi,\psi,\theta\|_V^2=\|\left(\phi_x+\psi\right)\|_2^2+\|\phi\|_2^2+\|\psi_x\|_2^2+\|\theta\|_2^2+\|\theta_x\|_2^2,$$

using integration by parts, we have,

$$B\left((\phi,\psi,\theta),(\phi,\psi,\theta)\right)=\gamma K\int_0^1\left(\phi_x+\psi\right)^2\mathrm{d}x+\gamma\left(\mu_1+\rho_1+\mu_2e^{-\tau}\right)\int_0^1\phi^2\mathrm{d}x$$
$$+\beta\left(\delta+k\right)\int_0^1\theta_x^2\mathrm{d}x+b\gamma\int_0^1\psi_x^2\mathrm{d}x+\rho_2\gamma\int_0^1\psi^2\mathrm{d}x$$
$$+\beta\rho_3\int_0^1\theta^2\mathrm{d}x\geq\alpha_0\left\|\phi,\psi,\theta\right\|_V^2,$$

for some $\alpha_0>0$. Thus, B is coercive.
On the other hand, using Cauchy-Schwarz and Poincaré's inequalities, we obtain

$$\begin{aligned}&\left|B\left((\phi,\psi,\theta),(\phi_1,\psi_1,\theta_1)\right)\right|\\&\quad\leq\gamma K\left\|\phi_x+\psi\right\|_2\left\|\phi_{1x}+\psi_1\right\|_2+\gamma b\left\|\psi_x\right\|_2\left\|\psi_{1x}\right\|_2+\gamma\left|\mu_1+\rho_1+\mu_2e^{-\tau}\right|\left\|\phi\right\|_2\left\|\phi_1\right\|_2\\&\qquad+\gamma\rho_2\left\|\psi\right\|_2\left\|\psi_1\right\|_2+\gamma\beta\left\|\theta_x\right\|_2\left\|\psi_1\right\|_2+\beta\left(\delta+k\right)\left\|\theta_x\right\|_2\left\|\theta_{1x}\right\|_2+\beta\rho_3\left\|\theta\right\|_2\left\|\theta_1\right\|_2\\&\qquad+\gamma\beta\left\|\psi_x\right\|_2\left\|\theta_1\right\|_2\\&\quad\leq c\left(\left\|\phi_x+\psi\right\|_2+\left\|\phi\right\|_2+\left\|\psi_x\right\|_2+\left\|\theta\right\|_2+\left\|\theta_x\right\|_2\right)\times\\&\qquad\left(\left\|\phi_{1x}+\psi_1\right\|_2+\left\|\phi_1\right\|_2+\left\|\psi_{1x}\right\|_2+\left\|\theta_1\right\|_2+\left\|\theta_{1x}\right\|_2\right)\\&\quad\leq c\left\|\phi,\psi,\theta\right\|_V\left\|\phi_1,\psi_1,\theta_1\right\|_V.\end{aligned}$$

Similarly

$$\begin{aligned}&\left|G\left(\phi_1,\psi_1,\theta_1\right)\right|\\&\quad\leq c\Big(\left\|f_1\right\|_{H_0^1(0,1)}+\left\|f_2\right\|_2+\left\|f_3\right\|_{H_0^1(0,1)}+\left\|f_4\right\|_2+\left\|f_5\right\|_{H_\star^1(0,1)}+\left\|f_6\right\|_2+\left\|f_7\right\|_{L^2((0,1),L^2(0,1))}\Big)\\&\qquad\times\Big(\left\|\phi_1\right\|_{H_0^1(0,1)}+\left\|\psi_1\right\|_{H_0^1(0,1)}+\left\|\theta_1\right\|_{H_*^1(0,1)}\Big)\\&\quad\leq c\left\|\phi_1,\psi_1,\theta_1\right\|_V.\end{aligned}$$

Consequently, Lax–Milgram lemma guarantees the existence of a unique

$$(\phi,\psi,\theta)\in H_0^1(0,1)\times H_0^1(0,1)\times H_\star^1(0,1)$$

satisfying

$$B\left((\phi,\psi,\theta),(\phi_1,\psi_1,\theta_1)\right)=G\left(\phi_1,\psi_1,\theta_1\right)\quad\forall\left(\phi_1,\psi_1,\theta_1\right)\in V.\tag{3.7}$$

The substitution of ϕ,ψ and θ into $(3.4)_1$,$(3.4)_3$ and $(3.4)_5$ yields

$$(\varphi,u,v)\in H_0^1(0,1)\times H_0^1(0,1)\times H_\star^1(0,1).$$

Moreover, if we take $(\phi_1,\theta_1)\equiv(0,0)\in H_0^1(0,1)\times H_\star^1(0,1)$ in (3.7), we get

$$K\int_0^1\left(\phi_x+\psi\right)\psi_1\mathrm{d}x+b\int_0^1\psi_x\psi_{1x}\mathrm{d}x+\rho_2\int_0^1\psi\psi_1\mathrm{d}x+\beta\int_0^1\theta_x\psi_1\mathrm{d}x$$
$$=\rho_2\int_0^1f_4\psi_1\mathrm{d}x+\beta\int_0^1f_{5x}\psi_1\mathrm{d}x+\rho_2\int_0^1f_3\psi_1\mathrm{d}x.$$

By recalling $(3.4)_3$ and $(3.4)_5$, we arrive at

$$K\int_0^1 (\phi_x+\psi)\,\psi_1 \mathrm{d}x + b\int_0^1 \psi_x\psi_{1x}\mathrm{d}x + \rho_2\int_0^1 \psi\psi_1\mathrm{d}x + \beta\int_0^1 \theta_x\psi_1\mathrm{d}x$$
$$= \rho_2\int_0^1 f_4\psi_1\mathrm{d}x + \beta\int_0^1 (\theta_x - v_x)\psi_1\mathrm{d}x + \rho_2\int_0^1 (\psi-u)\psi_1\mathrm{d}x.$$

Hence, we obtain

$$b\int_0^1 \psi_x\psi_{1x}\mathrm{d}x = \int_0^1 \left[\rho_2 f_4 - K(\phi_x+\psi) - \beta v_x - \rho_2 u\right]\psi_1\mathrm{d}x, \quad \forall \psi_1 \in H_0^1(0,1). \tag{3.8}$$

By noting that

$$\left[\rho_2 f_4 - K(\phi_x+\psi) - \beta v_x - \rho_2 u\right] \in L^2(0,1),$$

then

$$\psi \in H^2(0,1)\cap H_0^1(0,1)$$

and, consequently, (3.8) takes the form

$$b\int_0^1 [-\psi_{xx} + K(\phi_x+\psi) + \beta v_x + \rho_2 u - \rho_2 f_4]\psi_1\mathrm{d}x = 0 \quad \forall \psi_1 \in H_0^1(0,1). \tag{3.9}$$

Therefore, we obtain

$$-\psi_{xx} + K(\phi_x+\psi) + \beta v_x + \rho_2 u = \rho_2 f_4.$$

This gives $(3.4)_4$. Similarly, if we take $(\psi_1,\theta_1)\equiv(0,0)\in H_0^1(0,1)\times H_\star^1(0,1)$ in (3.7), we can show that

$$\phi \in H^2(0,1)\cap H_0^1(0,1)$$

and $(3.4)_2$ are satisfied. Also, if we take $(\phi_1,\psi_1)\equiv(0,0)\in H_0^1(0,1)\times H_0^1(0,1)$ in (3.7), then using $(3.4)_3$ and $(3.4)_5$, we get

$$\delta\theta_{xx} + k v_{xx} = \rho_3 f_6 - \gamma u_x - \rho_3 v \ \text{ in } L_\star^2(0,1),$$

and we conclude that

$$(\delta\theta + kv) \in H^2(0,1).$$

Furthermore, it is obvious from

$$\delta\theta_x + kv_x = \rho_3\int_0^x f_6\mathrm{d}x - \gamma u - \rho_3\int_0^x v\mathrm{d}x,$$

that

$$(\delta\theta_x + kv_x)(0) = (\delta\theta_x + kv_x)(1) = 0.$$

Thus, we get

$$(\delta\theta + kv) \in H_\star^2(0,1).$$

Finally, it follows, from (3.5), that

$$z(x,0)=\varphi(x) \text{ and } z, z_\rho \in L^2\left((0,1), L^2(0,1)\right).$$

Hence, there exists a unique $\mathcal{U}\in D(\mathcal{A})$ such that (3.3) is satisfied. Therefore, $\mathcal{A}$ is a maximal monotone operator. Consequently, the well-posedness result follows from the Hille–Yosida theorem. (see [31]) □

The associated solution energy is given by

$$\begin{aligned} E(t) &= E(t,\phi,\psi,\theta,z) \\ &= \frac{\gamma}{2}\left(\int_0^1 \rho_1\phi_t^2+\rho_2\psi_t^2+K\left|\phi_x+\psi\right|^2+b\psi_x^2\right)\mathrm{d}x \\ &\quad +\frac{\beta}{2}\int_0^1\left(\rho_3\theta_t^2+\delta\theta_x^2\right)\mathrm{d}x+\frac{\xi}{2}\int_0^1\int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x, \end{aligned} \tag{3.10}$$

where, as in (3.1*),

$$\begin{cases} \gamma\tau\left|\mu_2\right|<\xi<\gamma\tau\left(2\mu_1-\left|\mu_2\right|\right) & \text{if} \quad \mu_1>\left|\mu_2\right| \\ \xi=\gamma\tau\left|\mu_2\right|=\gamma\tau\mu_1 & \text{if} \quad \mu_1=\left|\mu_2\right| \end{cases} \tag{3.11}$$

The following lemma shows that the associated energy is decreasing in time.

Lemma 3.1. *Let (ϕ,ψ,θ,z) be the solution of (2.1). Then, for some $C\geq 0$,*

$$E'(t)\leq-\beta k\int_0^1\theta_{tx}^2\mathrm{d}x-C\int_0^1\left(\phi_t^2+z^2(x,1,t)\right)\mathrm{d}x\leq 0. \tag{3.12}$$

Proof. Multiplying equation $(2.1)_1$ by $\gamma\phi_t$, $(2.1)_2$ by $\gamma\psi_t$ and $(2.1)_3$ by $\beta\theta_t$ and integrating over $(0,1)$ and $(2.1)_4$ by $(\xi/\tau)z$ and integrating over $(0,1)\times(0,1)$ with respect to ρ and x summing up, we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^1\gamma\left(\rho_1\phi_t^2+\rho_2\psi_t^2+K\left|\phi_x+\psi\right|^2+b\psi_x^2\right)+\beta\left(\rho_3\theta_t^2+\delta\theta_x^2\right)\right]\mathrm{d}x \\ &\quad+\frac{\xi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x \\ &=-\beta k\int_0^1\theta_{tx}^2\mathrm{d}x-\gamma\mu_1\int_0^1\phi_t^2\mathrm{d}x-\frac{\xi}{\tau}\int_0^1\int_0^1 zz_\rho(x,\rho,t)\mathrm{d}\rho\mathrm{d}x-\gamma\mu_2\int_0^1\phi_t z(x,1,t)\mathrm{d}x. \end{aligned} \tag{3.13}$$

We, now, estimate the last two terms of the right-hand side of (3.13) as follows.

$$\begin{aligned} -\frac{\xi}{\tau}\int_0^1\int_0^1 zz_\rho(x,\rho,t)\mathrm{d}\rho\mathrm{d}x &= -\frac{\xi}{2\tau}\int_0^1\int_0^1\frac{\partial}{\partial\rho}z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x \\ &=\frac{\xi}{2\tau}\int_0^1\left(z^2(x,0,t)-z^2(x,1,t)\right)\mathrm{d}x=\frac{\xi}{2\tau}\left(\int_0^1\phi_t^2\mathrm{d}x-\int_0^1 z^2(x,1,t)\mathrm{d}x\right), \\ -\gamma\mu_2\int_0^1\phi_t z(x,1,t)\mathrm{d}x &\leq\frac{\gamma\left|\mu_2\right|}{2}\left(\int_0^1\phi_t^2\mathrm{d}x+\int_0^1 z^2(x,1,t)\mathrm{d}x\right). \end{aligned}$$

We conclude, then,

$$\frac{\mathrm{d}E(t)}{\mathrm{d}}\leq-\beta k\int_0^1\theta_{tx}^2\mathrm{d}x-\gamma\left(\mu_1-\frac{\xi}{2\tau\gamma}-\frac{\left|\mu_2\right|}{2}\right)\int_0^1\phi_t^2\mathrm{d}x-\gamma\left(\frac{\xi}{2\tau\gamma}-\frac{\left|\mu_2\right|}{2}\right)\int_0^1 z^2(x,1,t)\mathrm{d}x.$$

Using (3.11), we have, for some $C \geq 0$,

$$E'(t) \leq -\beta k \int_0^1 \theta_{tx}^2 \mathrm{d}x - C \int_0^1 \left(\phi_t^2 + z^2(x,1,t)\right) \mathrm{d}x \leq 0.$$

□

4. Decay of solutions

In this section, we state and prove our stability result.

Theorem 4.1. *Suppose that $\mu_1 \geq |\mu_2|$. Then the energy $E(t)$ satisfies $\forall t > 0$,*

$$\begin{aligned} &E(t) \leq CE(0)\, e^{-\alpha t}, && \text{if} \quad \frac{\rho_1}{K} = \frac{\rho_2}{b}, \\ &E(t) \leq C\left(E_1(0) + E_2(0)\right) t^{-1}, && \text{if} \quad \frac{\rho_1}{K} \neq \frac{\rho_2}{b}. \end{aligned} \tag{4.1}$$

In order to prove this result, we introduce various functionals and prove several lemmas.

We note here that the functionals I_1, J and the function $q(x)$, used in Lemma 4.4, were first introduced in [8]. Similarly, the function I_3 was first introduced in [22].

Lemma 4.1. *Let (ϕ, ψ, θ, z) be the solution of (2.1). Then the functional*

$$I_1(t) := \int_0^1 \left(\rho_1 \phi_t \omega + \rho_2 \psi_t \psi\right) \mathrm{d}x$$

satisfies, $\forall \varepsilon_1 > 0$,

$$\begin{aligned} I_1'(t) \leq &\left(-\frac{b}{2} + \varepsilon_1\left(\mu_1 + |\mu_2|\right)\right) \int_0^1 \psi_x^2 \mathrm{d}x + \left(\varepsilon_1 \rho_1 + \frac{\mu_1}{4\varepsilon_1}\right) \int_0^1 \phi_t^2 \mathrm{d}x \\ &+ \left(\rho_2 + \frac{\rho_1}{4\varepsilon_1}\right) \int_0^1 \psi_t^2 \mathrm{d}x + \frac{\beta^2}{2b} \int_0^1 \theta_{tx}^2 \mathrm{d}x + \frac{|\mu_2|}{4\varepsilon_1} \int_0^1 z^2(x,1,t) \mathrm{d}x, \end{aligned} \tag{4.2}$$

where

$$\omega(x,t) = -\int_0^x \psi(y,t) \mathrm{d}y + x \int_0^1 \psi(y,t) \mathrm{d}. \tag{4.3}$$

Proof. By differentiating I_1 and using Eq. (2.1), we conclude that

$$\begin{aligned} I_1'(t) :&= -b \int_0^1 \psi_x^2 \mathrm{d}x + \rho_2 \int_0^1 \psi_t^2 \mathrm{d}x - K \int_0^1 \psi^2 \mathrm{d}x - \beta \int_0^1 \psi \theta_{tx} \mathrm{d}x \\ &+ K \int_0^1 \omega_x^2 \mathrm{d}x + \rho_1 \int_0^1 \phi_t \omega_t \mathrm{d}x - \mu_1 \int_0^1 \phi_t \omega \mathrm{d}x - \mu_2 \int_0^1 \omega z(x,1,t) \omega \mathrm{d}x. \end{aligned}$$

By exploiting the inequalities

$$\int_0^1 \omega_x^2 \mathrm{d}x \leq \int_0^1 \psi^2 \mathrm{d}x \leq \int_0^1 \psi_x^2 \mathrm{d}x$$
$$\int_0^1 \omega_t^2 \mathrm{d}x \leq \int_0^1 \omega_{tx}^2 \mathrm{d}x \leq \int_0^1 \psi_t^2 \mathrm{d}x,$$

and Young's inequality, the result follows. □

Lemma 4.2. *Let (ϕ, ψ, θ, z) be a solution of (2.1). Then the functional*

$$I_2(t) := \rho_2 \rho_3 \int_0^1 \psi_t(x,t) \int_0^x \theta_t(y,t)\, \mathrm{d}y \mathrm{d}x - \delta \rho_2 \int_0^1 \theta_x \psi \mathrm{d}x,$$

satisfies, $\forall \varepsilon_2 > 0$,

$$I_2'(t) \leq -\frac{\rho_2 \gamma}{2} \int_0^1 \psi_t^2 \mathrm{d}x + \varepsilon_2 \int_0^1 \psi_x^2 \mathrm{d}x + \varepsilon_2 \int_0^1 \phi_x^2 \mathrm{d}x + C(\varepsilon_2) \int_0^1 \theta_{tx}^2 \mathrm{d}x.$$

Proof. Using Eq. (2.1) we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_2 \rho_3 \int_0^1 \psi_t(x,t) \int_0^x \theta_t(y,t)\, \mathrm{d}y \mathrm{d}x\right) \\
&= \int_0^1 \rho_2 \psi_t \int_0^x (\delta \theta_{xx} - \gamma \psi_{tx} + k \theta_{txx})\, \mathrm{d}y \mathrm{d}x \\
&\quad + \int_0^1 (b \psi_{xx} - K(\phi_x + \psi) - \beta \theta_{tx}) \int_0^x \rho_3 \theta_t(y,t)\, \mathrm{d}y \mathrm{d}x \\
&= \int_0^1 \rho_2 \psi_t (\delta \theta_x - \gamma \psi_t + k \theta_{tx})\, \mathrm{d}x - \rho_3 K \int_0^1 \psi \int_0^x \theta_t(y,t)\, \mathrm{d}y \mathrm{d}x \\
&\quad - \rho_3 b \int_0^1 \theta_t \psi_x \mathrm{d}x + \rho_3 K \int_0^1 \theta_t \phi \mathrm{d}x + \beta \rho_3 \int_0^1 \theta_t^2 \mathrm{d}x \\
&\quad + \left[\rho_3 \left(\int_0^x \theta_t(y,t)\, \mathrm{d}y\right)(b \psi_x - K\phi - \beta \theta_t)\right]_{x=0}^{x=1}.
\end{aligned}$$

By recalling that θ stands for $\overline{\theta}$, we have

$$\int_0^1 \theta_t(y,t)\, \mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \theta(y,t)\, \mathrm{d}y = 0,$$

consequently

$$\left[\rho_3\left(\int_0^x \theta_t(y,t)\,\mathrm{d}y\right)(b\psi_x - K\phi - \beta\theta_t)\right]_{x=0}^{x=1} = 0.$$

Thus, we obtain

$$\begin{aligned} I_2'(t) = & -\gamma\rho_2\int_0^1 \psi_t^2\mathrm{d}x - \delta\rho_2\int_0^1 \psi\theta_{tx}\mathrm{d}x + k\rho_2\int_0^1 \theta_{tx}\psi_t\mathrm{d}x \\ & -K\rho_3\int_0^1\int_0^x \theta_t(y,t)\,\mathrm{d}y\psi\mathrm{d}x + b\rho_3\int_0^1 \theta_t\psi_x\mathrm{d}x \\ & -K\rho_3\int_0^1 \theta_t\phi\mathrm{d}x + \beta\rho_3\int_0^1 \theta_t^2\mathrm{d}x. \end{aligned}$$

The assertion of the lemma then follows, using Young's and Poincare's inequalities. □

Lemma 4.3. *Let (ϕ,ψ,θ,z) be a solution of (2.1). Then the functional*

$$J(t) := \rho_2\int_0^1 \psi_t(\phi_x+\psi)\,\mathrm{d}x + \rho_2\int_0^1 \psi_x\phi_t\mathrm{d}x$$

satisfies,

$$\begin{aligned} J'(t) \le & [b\phi_x\psi_x]_{x=0}^{x=1} - \frac{K}{2}\int_0^1 (\phi_x+\psi)^2\,\mathrm{d}x + \rho_2\int_0^1 \psi_t^2\mathrm{d}x + \frac{\beta^2}{2K}\int_0^1 \theta_{tx}^2\mathrm{d}x \\ & +\frac{\rho_2\mu_1^2}{2\rho_1}\int_0^1 \phi_t^2\mathrm{d}x + \frac{\rho_2}{\rho_1}\int_0^1 \psi_x^2\mathrm{d}x + \frac{\rho_2\mu_2^2}{2\rho_1}\int_0^1 z^2(x,1,t)\mathrm{d}x \\ & +\left(\frac{\rho_2 K}{\rho_1} - b\right)\int_0^1 \psi_x(\phi_x+\psi)_x\,\mathrm{d}x. \end{aligned} \tag{4.4}$$

Proof. A differentiation of $J(t)$, using (2.1) and integration by parts, gives

$$\begin{aligned} J'(t) = & [b\phi_x\psi_x]_{x=0}^{x=1} + \left(\frac{\rho_2 K}{\rho_1} - b\right)\int_0^1 \psi_x(\phi_x+\psi)_x\,\mathrm{d}x - K\int_0^1 (\phi_x+\psi)^2\,\mathrm{d}x \\ & -\beta\int_0^1 (\phi_x+\psi)\,\theta_{tx}\mathrm{d}x + \rho_2\int_0^1 \psi_t^2\mathrm{d}x - \frac{\rho_2\mu_1}{\rho_1}\int_0^1 \phi_t\psi_x\mathrm{d}x \\ & -\frac{\rho_2\mu_2}{\rho_1}\int_0^1 \psi_x z(x,1,t)\mathrm{d}x. \end{aligned} \tag{4.5}$$

Young's inequality leads to (4.4). □

Next, to handle the boundary terms, appearing in (4.4), we exploit the following function

$$q\,(x) = 2 - 4x, \qquad x \in (0,1)\,.$$

Lemma 4.4. *Let* (ϕ, ψ, θ, z) *be a solution of* (2.1). *Then we have,* $\forall \varepsilon_3 > 0$,

$$\begin{aligned}
[b\phi_x\psi_x]_{x=0}^{x=1} \leq & -\frac{\varepsilon_3}{K}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho_1 q\phi_t\phi_x \mathrm{d}x - \frac{b\rho_2}{4\varepsilon_3}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 q\psi_t\psi_x\mathrm{d}x \\
&+3\varepsilon_3\int_0^1 \phi_x^2\mathrm{d}x + \left(\varepsilon_3 + \frac{3b^2}{4\varepsilon_3} + \frac{b^2}{4\varepsilon_3^3}\right)\int_0^1 \psi_x^2\mathrm{d}x \\
&+\frac{\rho_2 b}{2\varepsilon_3}\int_0^1 \psi_t^2\mathrm{d}x + \frac{K^2}{4}\varepsilon_3\int_0^1 (\phi_x+\psi)^2\,\mathrm{d}x + \frac{\beta^2}{4\varepsilon_3}\int_0^1 \theta_{tx}^2\mathrm{d}x \\
&+\left(\frac{2\rho_1\varepsilon_3}{K} + \frac{\mu_1^2}{\varepsilon_3}\right)\int_0^1 \phi_t^2\mathrm{d}x + \frac{\mu_2^2}{\varepsilon_3}\int_0^1 z^2(x,1,t)\mathrm{d}x.
\end{aligned} \tag{4.6}$$

Proof. By using Young's inequality, we easily see that, $\forall \varepsilon_3 > 0$,

$$[b\phi_x\psi_x]_{x=0}^{x=1} \leq \varepsilon_3\left[\phi_x^2\,(1) + \phi_x^2\,(0)\right] + \frac{b^2}{4\varepsilon_3}\left[\psi_x^2\,(1) + \psi_x^2\,(0)\right]. \tag{4.7}$$

Also,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 b\rho_2 q\psi_t\psi_x\mathrm{d}x = & \frac{b^2}{2}\left[q\psi_x^2\right]_{x=0}^{x=1} - \frac{b^2}{2}\int_0^1 q_x\psi_x^2\mathrm{d}x - \frac{\rho_2 b}{2}\int_0^1 q_x\psi_t^2\mathrm{d}x \\
&-Kb\int_0^1 q\psi_x\,(\phi_x+\psi)\,\mathrm{d}x - \beta b\int_0^1 q\psi_x\theta_{tx}\mathrm{d}x,
\end{aligned} \tag{4.8}$$

which gives

$$\begin{aligned}
\frac{\mathrm{d}}{t}\int_0^1 b\rho_2 q\psi_t\psi_x\mathrm{d}x \leq & -b^2\left[\psi_x^2\,(1) + \psi_x^2\,(0)\right] + 3b^2\int_0^1 \psi_x^2\mathrm{d}x \\
&+2\rho_2 b\int_0^1 \psi_t^2\mathrm{d}x + \varepsilon_3^2K^2\int_0^1 (\phi_x+\psi)^2\,\mathrm{d}x \\
&+\frac{b^2}{\varepsilon_3^2}\int_0^1 \psi_x^2\mathrm{d}x + \beta^2\int_0^1 \theta_{tx}^2\mathrm{d}x.
\end{aligned} \tag{4.9}$$

Similarly, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho_1 q\phi_t\phi_x \mathrm{d}x &\le -K\left[\phi_x^2(1)+\phi_x^2(0)\right]\\ &\quad +(2\varepsilon_3+3K)\int_0^1 \phi_x^2\mathrm{d}x + K\int_0^1 \psi_x^2\mathrm{d}x \\ &\quad +2\rho_1\int_0^1 \phi_t^2\mathrm{d}x + \frac{\mu_1^2}{\varepsilon_3}\int_0^1 \phi_t^2\mathrm{d}x + \frac{\mu_2^2}{\varepsilon_3}\int_0^1 z^2(x,1,t)\mathrm{d}x.\end{aligned} \tag{4.10}$$

A combination of (4.7)–(4.10) then yields the desired result. ☐

Lemma 4.5. *Let* (ϕ,ψ,θ,z) *be a solution of* (2.1). *Then, the functional*

$$\Theta(t) := \int_0^1 \left(\rho_3\theta_t\theta + \frac{k}{2}\theta_x^2 + \gamma\psi_x\theta\right)\mathrm{d}x,$$

satisfies, $\forall \varepsilon_2 > 0$,

$$\Theta'(t) \le -\delta\int_0^1 \theta_x^2\mathrm{d}x + \varepsilon_2\int_0^1 \psi_x^2\mathrm{d}x + \left(\frac{\gamma^2}{4\varepsilon_2}+\rho_3\right)\int_0^1 \theta_t^2\mathrm{d}x. \tag{4.11}$$

Proof. A simple differentiation, using equation $(2.1)_3$ and Young's inequality gives the result. ☐

Lemma 4.6. *Let* (ϕ,ψ,θ,z) *be a solution of* (2.1). *Then the functional*

$$I_3(t) := \int_0^1\int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x,$$

satisfies, for some $m_0 > 0$,

$$I_3'(t) \le -m_0\int_0^1\int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x - \frac{c}{\tau}\int_0^1 z^2(x,1,t)\mathrm{d}x + \frac{1}{\tau}\int_0^1 \phi_t^2\mathrm{d}x. \tag{4.12}$$

Proof. Direct differentiation of I_3, using $(2.1)_4$, gives

$$\begin{aligned} I_3'(t) &= -\frac{2}{\tau}\int_0^1\int_0^1 e^{-2\tau\rho} z z_\rho(x,\rho,t)\mathrm{d}\rho\mathrm{d}x\\ &= -2\int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x - \frac{1}{\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\tau\rho}z^2(x,\rho,t)\right)\mathrm{d}\rho\mathrm{d}x\\ &\le -m_0\int_0^1\int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x - \frac{1}{\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\tau\rho}z^2(x,\rho,t)\right)\mathrm{d}\rho\mathrm{d}x.\end{aligned}$$

Simple integration of the last term gives the result. ☐

Proof of Theorem 4.1. To finalize the proof, we define the Lyapunov functional $\mathcal{L}$ as follows

$$\mathcal{L}(t) := NE(t) + N_1 I_1(t) + N_2 I_2(t) + J(t) + \Theta(t) + I_3(t) + \frac{b\rho_2}{4\varepsilon_3}\int_0^1 q\psi_t\psi_x \mathrm{d}x + \frac{\varepsilon_3}{K}\int_0^1 \rho_1 q\phi_t\phi_x \mathrm{d}x,$$

where N, N_1, N_2 are positive constants to be chosen properly later.

A combination of (3.12), (4.2), (4.4), (4.6), (4.11), (4.12) and using of

$$\int_0^1 \theta_t^2 \mathrm{d}x \leq \int_0^1 \theta_{tx}^2 \mathrm{d}x, \quad \int_0^1 \phi_x^2 \mathrm{d}x \leq 2\int_0^1 (\phi_x+\psi)^2 \mathrm{d}x + 2\int_0^1 \psi_x^2 \mathrm{d}x,$$

we arrive at

$$\begin{aligned}
\mathcal{L}'(t) \leq& \left[-\beta k N + N_1\frac{\beta^2}{2b} + N_2 C(\varepsilon_2)\frac{\beta^2}{4\varepsilon_3} + \frac{\gamma^2}{4\varepsilon_2} + \rho_3\right]\int_0^1 \theta_{tx}^2 \mathrm{d}x \\
&+ \left[N_1\left(-\frac{b}{2} + \varepsilon_1(\mu_1 + |\mu_2|)\right) + 3\varepsilon_2 N_2 + \frac{\rho_1}{\rho_2} + \frac{3b^2}{4\varepsilon_3} + \frac{b^2}{4\varepsilon_3^3} + 7\varepsilon_3\right]\int_0^1 \psi_x^2 \mathrm{d}x \\
&+ \left[-NC + N_1\left(\varepsilon_1\rho_1 + \frac{\mu_1}{4\varepsilon_1}\right) + \frac{\rho_2\mu_1^2}{2\rho_1} + \left(\frac{2\rho_1\varepsilon_3}{K} + \frac{\mu_1^2}{\varepsilon_3}\right) + \frac{1}{2\tau}\right]\int_0^1 \phi_t^2 \mathrm{d}x \\
&+ \left[-\frac{N_2\rho_2\gamma}{2} + N_1\left(\rho_2 + \frac{\rho_1}{4\varepsilon_1}\right) + \rho_2 + \frac{\rho_2 b}{2\varepsilon_3}\right]\int_0^1 \psi_t^2 \mathrm{d}x \\
&+ \left[-\frac{K}{2} + \left(\frac{K^2}{4} + 6\right)\varepsilon_3 + 2N_2\varepsilon_2\right]\int_0^1 (\phi_x+\psi)^2 \mathrm{d}x - \delta\int_0^1 \theta_x^2 \mathrm{d}x \\
&+ \left[-NC - \frac{c}{2\tau} + N_1\frac{|\mu_2|}{4\varepsilon_1} + \frac{\mu_2^2}{\varepsilon_3} + \frac{\rho_2\mu_2^2}{2\rho_1}\right]\int_0^1 z^2(x,1,t)\mathrm{d}x - m_0\int_0^1\int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x \\
&+ \left(\frac{\rho_2 K}{\rho_1} - b\right)\int_0^1 \psi_x(\phi_x+\psi)_x \mathrm{d}x.
\end{aligned} \tag{4.13}$$

At this point, we choose our constants carefully. First, take

$$\varepsilon_3 \leq \frac{K}{2}\left(\frac{K^2}{4} + 6\right)^{-1}, \quad \varepsilon_1 \leq \frac{b}{2(\mu_1 + |\mu_2|)}.$$

Now, select N_1 large enough such that

$$-N_1\left(\frac{b}{2} - \varepsilon_1(\mu_1 + |\mu_2|)\right) + \frac{\rho_1}{\rho_2} + \frac{3b^2}{4\varepsilon_3} + \frac{b^2}{4\varepsilon_3^3} + 7\varepsilon_3 = k_1 < 0,$$

and then N_2 large so that

$$-\frac{N_2\rho_2\gamma}{2} + N_1\left(\rho_2 + \frac{\rho_1}{4\varepsilon_1}\right) + \rho_2 + \frac{\rho_2 b}{2\varepsilon_3} < 0.$$

Next, pick ε_2 so small that

$$k_1 + 3\varepsilon_2 N_2 < 0, \quad -\frac{K}{2} + \left(\frac{K^2}{4} + 6\right)\varepsilon_3 + 2N_2\varepsilon_2 < 0.$$

Finally, choose N so large that

$$\begin{aligned} -\beta k N + N_1 \frac{\beta^2}{2b} + N_2 C(\varepsilon_2) + \frac{\beta^2}{4\varepsilon_3} + \frac{\gamma^2}{4\varepsilon_2} + \rho_3 < 0, \\ -NC + N_1\left(\varepsilon_1\rho_1 + \frac{\mu_1}{4\varepsilon_1}\right) + \frac{\rho_2\mu_1^2}{2\rho_1} + \left(\frac{2\rho_1\varepsilon_3}{K} + \frac{\mu_1^2}{\varepsilon_3}\right) + \frac{1}{2\tau} < 0, \\ -NC - \frac{c}{2\tau} + N_1\frac{|\mu_2|}{4\varepsilon_1} + \frac{\mu_2^2}{\varepsilon_3} + \frac{\rho_2\mu_2^2}{2\rho_1} < 0, \end{aligned}$$

and, further, for some $\beta_1, \beta_2 > 0$, we have

$$\beta_1 E(t) \le \mathcal{L}(t) \le \beta_2 E(t), \qquad t \ge 0. \tag{4.14}$$

Therefore, (4.13) becomes

$$\begin{aligned} \mathcal{L}'(t) &\le -\eta_1 \int_0^1 \left(\theta_t^2 + \theta_{xt}^2 + \psi_x^2 + \psi_t^2 + \phi_t^2 + (\phi_x + \psi)^2\right) \mathrm{d}x \\ &\quad - m_0 \int_0^1\int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x + \left(\frac{\rho_2 K}{\rho_1} - b\right)\int_0^1 \psi_x(\phi_x+\psi)_x\,\mathrm{d}x \\ &\le -C_1 E(t) + \left(\frac{\rho_2 K}{\rho_1} - b\right)\int_0^1 \psi_x(\phi_x+\psi)_x\,\mathrm{d}x, \end{aligned} \tag{4.15}$$

where η_1, C_1 are two positive constants.

Up to this point, we have to distinguish two cases.

Case 1: $\frac{\rho_1}{K} = \frac{\rho_2}{b}$

In this case, (4.15) takes the form

$$\mathcal{L}'(t) \le -C_1 E(t).$$

Using (4.14), we get, for $\alpha = C_1/\beta_2$,

$$\mathcal{L}'(t) \le -\alpha\mathcal{L}(t), \quad \forall t \ge 0. \tag{4.16}$$

A simple integration of (4.16) over $(0,t)$ leads to

$$\mathcal{L}(t) \le \mathcal{L}(0)\, e^{-\alpha t}, \quad \forall t \ge 0.$$

Recalling (4.14), we obtain $(4.1)_1$. □

Case 2: $\frac{\rho_1}{K} \ne \frac{\rho_2}{b}$

This case is more important from the physical point of view, where waves are not necessarily of equal speeds. Let

$$E(t) = E(t,\phi,\psi,\theta,z) = E_1(t)$$

denotes the first-order energy defined in (3.10), and

$$E_2(t) = E(t,\phi_t,\psi_t,\theta_t,z_t)$$

denotes the second-order energy. Then, we have

$$E_2'(t) \leq -\beta k \int_0^1 \theta_{ttx}^2 \mathrm{d}x - C \int_0^1 \left(\phi_{tt}^2 + z_t^2(x,1,t)\right) \mathrm{d}x. \tag{4.17}$$

Let $k_0 = \frac{\rho_2 K}{\rho_1} - b$. The last term in (4.15) can be handled, using $(2.1)_1$ and Young's inequality, as follows.

$$\begin{aligned} k_0 \int_0^1 \psi_x (\phi_x + \psi)_x \mathrm{d}x &= \frac{k_0}{K}\left(\int_0^1 \psi_x \phi_{tt}\mathrm{d}x + \mu_1 \int_0^1 \psi_x \phi_t \mathrm{d}x + \mu_2 \int_0^1 \psi_x z(x,1,t)\mathrm{d}x\right) \\ &\leq \frac{k_0}{K}\left(\frac{1}{4\varepsilon}\int_0^1 \phi_{tt}^2\mathrm{d}x + \varepsilon\int_0^1 \psi_x^2\mathrm{d}x + \varepsilon \int_0^1 \phi_t^2 \mathrm{d}x + \frac{1}{4\varepsilon}\int_0^1 z^2(x,1,t)\mathrm{d}x\right), \end{aligned}$$

for any $\varepsilon > 0$. In addition, from (3.10), we have

$$\frac{\varepsilon k_0}{K}\int_0^1 \left(\phi_t^2 + \psi_x^2\right)\mathrm{d}x \leq \frac{2\varepsilon k_0}{\gamma K}\left(\frac{1}{\rho_1} + \frac{1}{b}\right) E_1(t).$$

Thus, (4.15) becomes

$$\mathcal{L}'(t) \leq \left[-C_1 + \frac{2\varepsilon k_0}{\gamma K}\left(\frac{1}{\rho_1} + \frac{1}{b}\right)\right] E_1(t) + \frac{k_0}{4\varepsilon K}\int_0^1 \left(\phi_{tt}^2 + z^2(x,1,t)\right)\mathrm{d}x. \tag{4.18}$$

By choosing ε small enough, (4.18) becomes

$$\mathcal{L}'(t) \leq -C_2 E_1(t) + \frac{k_0}{4\varepsilon K}\int_0^1 \left(\phi_{tt}^2 + z^2(x,1,t)\right)\mathrm{d}x, \tag{4.19}$$

for some $C_2 > 0$.

If we set

$$F(t) = \mathcal{L}(t) + N_3\left(E_1(t) + E_2(t)\right), \tag{4.20}$$

then it is easy to see that, for large $N_3 > 0$, there exist two positive constants c_1 and c_2 such that

$$c_1\left(E_1(t) + E_2(t)\right) \leq F(t) \leq c_2\left(E_1(t) + E_2(t)\right). \tag{4.21}$$

Exploiting (3.12),(4.17), and (4.19), we obtain

$$\begin{aligned} F'(t) &= \mathcal{L}'(t) + N_3\left(E_1'(t) + E_2'(t)\right) \\ &\leq -C_2 E_1(t) + \frac{k_0}{4\varepsilon K}\int_0^1 \left(\phi_{tt}^2 + z^2(x,1,t)\right)\mathrm{d}x \\ &\quad + N_3\left[-\beta k\int_0^1 \theta_{tx}^2\mathrm{d}x - C\int_0^1 \left(\phi_t^2 + z^2(x,1,t)\right)\mathrm{d}x\right] \\ &\quad + N_3\left[-\beta k\int_0^1 \theta_{ttx}^2\mathrm{d}x - C\int_0^1 \left(\phi_{tt}^2 + z_t^2(x,1,t)\right)\mathrm{d}x\right] \\ &\leq -C_2 E_1(t) + \left(\frac{k_0}{4\varepsilon K} - N_3 C\right)\int_0^1 \phi_{tt}^2\mathrm{d}x + \left(\frac{k_0}{4\varepsilon K} - N_3 C\right)\int_0^1 z^2(x,1,t)\mathrm{d}x. \end{aligned} \tag{4.22}$$

We choose N_3 large enough so that (4.21) remains valid and, further,

$$F'(t) \leq -C_2 E_1(t). \tag{4.23}$$

Integrating (4.23) over $(0, t)$, yields

$$\int_0^t E_1(r)\,dr \leq \frac{1}{C_3}(F(0) - F(t)) \leq \frac{1}{C_3} F(0) \leq \frac{c_2}{C_3}(E_1(0) + E_2(0)), \quad \forall t > 0. \tag{4.24}$$

Using the fact that

$$(tE_1(t))' = tE_1'(t) + E_1(t) \leq E_1(t)$$

and (4.24), we get

$$tE_1(t) \leq \frac{c_2}{C_3}(E_1(0) + E_2(0)),$$

which is the desired result $(4.1)_2$. This completes the proof. □

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